

Supplemental Technical Appendix

1 Proofs of Lemmas

Lemma A2: Under Assumptions 2, 3 and 4', there exists some $0 < W_1 < \infty$ such that

$$E \left(T^{-1} \sum_{t=1}^T \|N^{-(1+\alpha)/2} e_t' \Lambda\|^2 \right) = E \left(T^{-1} \sum_{t=1}^T \left\| N^{-(1+\alpha)/2} \sum_{i=1}^N e_{it} \lambda_i \right\|^2 \right) \leq W_1$$

for all N and T .

Proof:

$$\begin{aligned} E \left\| N^{-(1+\alpha)/2} \sum_{i=1}^N e_{it} \lambda_i \right\|^2 &= E \left(\frac{1}{N^{1+\alpha}} \sum_{i=1}^N \sum_{j=1}^N e_{it} e_{jt} \lambda_i' \lambda_j \right) \\ &= \frac{1}{N^{1+\alpha}} \sum_{i=1}^N \sum_{j=1}^N E(e_{it} e_{jt}) E(\lambda_i' \lambda_j) \leq \frac{1}{N^{1+\alpha}} \sum_{i=1}^N \sum_{j=1}^N |\tau_{ij}| \left(E \|\lambda_i\|^2 E \|\lambda_j\|^2 \right)^{\frac{1}{2}} \leq W^2 \end{aligned}$$

where the second line of the equations uses Assumption 3(b), and the last inequality uses Assumptions 2 and 4'(c). **Q.E.D.**

Lemma A3: Under Assumptions 1–3 and 4'–5',

- (i) $\frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2 = O_p(\phi_{NT}^{-2})$.
- (ii) For any $k \leq r$, $V(k, \tilde{F}^k) - V(k, FH^k) = O_p(\phi_{NT}^{-1})$.
- (iii) For any $k < r$, there exists a $\tau_k > 0$ such that $\text{plim} \inf_{N, T \rightarrow \infty} V(k, FH^k) - V(r, F) = \tau_k$
- (iv) For any $k \geq r$, $V(r, \tilde{F}^r) - V(k, \tilde{F}^k) = O_p(\phi_{NT}^{-2})$.

Proof:

- (i) Note that

$$\frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2 \leq \frac{4}{T} \sum_{t=1}^T (a_t + b_t + c_t + d_t)$$

where $a_t = \frac{1}{T^2} \left\| \sum_{s=1}^T \hat{F}_s^k \gamma_N(s, t) \right\|^2$, $b_t = \frac{1}{T^2} \left\| \sum_{s=1}^T \hat{F}_s^k \zeta_{st} \right\|^2$, $c_t = \frac{1}{T^2} \left\| \sum_{s=1}^T \hat{F}_s^k \eta_{st} \right\|^2$, and $d_t = \frac{1}{T^2} \left\| \sum_{s=1}^T \hat{F}_s^k \xi_{st} \right\|^2$.

First, $\frac{1}{T} \sum_{t=1}^T a_t = O_p\left(\frac{1}{T}\right)$. The proof of this result is the same as that of Theorem 1 of Bai and

Ng (2002). Second,

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T b_t &\leq \frac{1}{T^3} \sum_{t=1}^T \left\| \sum_{s=1}^T \hat{F}_s^k \zeta_{st} \right\|^2 = \frac{1}{T^3} \sum_{t=1}^T \left(\sum_{s=1}^T \sum_{u=1}^T \hat{F}_s^{k'} \hat{F}_u^k \zeta_{st} \zeta_{ut} \right) \\
&\leq \left(\frac{1}{T^2} \sum_{s=1}^T \sum_{u=1}^T (\hat{F}_s^{k'} \hat{F}_u^k)^2 \right)^{\frac{1}{2}} \left[\frac{1}{T^2} \sum_{s=1}^T \sum_{u=1}^T \left(\frac{1}{T} \sum_{t=1}^T \zeta_{st} \zeta_{ut} \right)^2 \right]^{\frac{1}{2}} \\
&\leq \left(\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^k\|^2 \right) \left[\frac{1}{T^2} \sum_{s=1}^T \sum_{u=1}^T \left(\frac{1}{T} \sum_{t=1}^T \zeta_{st} \zeta_{ut} \right)^2 \right]^{\frac{1}{2}}
\end{aligned}$$

Since $E \left(T^{-1} \sum_{t=1}^T \zeta_{st} \zeta_{ut} \right)^2 = T^{-2} \sum_{t=1}^T \sum_{v=1}^T E(\zeta_{st} \zeta_{ut} \zeta_{sv} \zeta_{uv}) \leq \max_{s,t} E|\zeta_{st}|^4$ and

$$E|\zeta_{st}|^4 = N^{-2(1-\alpha)} E \left| N^{-(1+\alpha)/2} \sum_{i=1}^N [e_{is} e_{it} - E(e_{is} e_{it})] \right|^4 \leq N^{-2(1-\alpha)} W$$

by Assumption 4'(e), it follows that

$$\frac{1}{T} \sum_{t=1}^T b_t \leq O_p(1) O_p \left(\frac{1}{N^{1-\alpha}} \right)$$

Third,

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T c_t &\leq \frac{1}{T^3} \sum_{t=1}^T \left\| \sum_{s=1}^T \hat{F}_s^k \eta_{st} \right\|^2 = \frac{1}{T^3} \sum_{t=1}^T \left\| \sum_{s=1}^T \hat{F}_s^k F_s' \Lambda' e_t / N \right\|^2 \\
&\leq \left(\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^k\|^2 \right) \left(\frac{1}{T} \sum_{s=1}^T \|F_s\|^2 \right) \frac{1}{T} \sum_{t=1}^T \left(\frac{1}{N^{1-\alpha}} \left\| \frac{\Lambda' e_t}{\sqrt{N^{1+\alpha}}} \right\|^2 \right) = O_p(1) O_p \left(\frac{1}{N^{1-\alpha}} \right)
\end{aligned}$$

by Assumptions 1 and Lemma A2.

Finally,

$$\begin{aligned}
\frac{1}{T} \sum_{t=1}^T d_t &\leq \frac{1}{T^3} \sum_{t=1}^T \left\| \sum_{s=1}^T \hat{F}_s^k \zeta_{st} \right\|^2 = \frac{1}{T^3} \sum_{t=1}^T \left\| \sum_{s=1}^T \hat{F}_s^k e_s' \Lambda F_t / N \right\|^2 \\
&\leq \frac{1}{N^{1-\alpha}} \left(\frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \right) \left(\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^k\|^2 \right) \left(\frac{1}{T} \sum_{s=1}^T \left\| \frac{e_s' \Lambda}{\sqrt{N^{1+\alpha}}} \right\|^2 \right) \\
&= O_p(1) O_p \left(\frac{1}{N^{1-\alpha}} \right)
\end{aligned}$$

by Assumption 1 and Lemma A2. Thus, under Assumptions 1–3 and 4', $\frac{4}{T} \sum_{t=1}^T (a_t + b_t + c_t + d_t) = O_p(T^{-1}) + O_p(N^{-1+\alpha}) = O_p(\phi_{NT}^{-2})$.

(ii) Lemma 2 of Bai and Ng (2002) shows that

$$V(k, \tilde{F}^k) - V(k, FH^k) \leq O_p(1) \left(\frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2 \right)^{\frac{1}{2}} = O_p(\delta_{NT}^{-1}).$$

Under Assumptions 1–3 and 4', $V(k, \tilde{F}^k) - V(k, FH^k)$ is still bounded by $O_p(1) \left(\frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2 \right)^{\frac{1}{2}}$, where the $O_p(1)$ is exactly same as the one in Bai and Ng (2002). To conserve space, we will not repeat the derivation of this $O_p(1)$ term. Since $\frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2$ converges at the rate $O_p(\phi_{NT}^{-2})$ instead of $O_p(\delta_{NT}^{-2})$ as Assumption 4' replaces Assumption 4, $V(k, \tilde{F}^k) - V(k, FH^k) = O_p(\phi_{NT}^{-1})$.

(iii) For $k < r$,

$$\begin{aligned} V(k, FH^k) - V(r, F) &= \frac{1}{NT} \sum_{i=1}^N (F\lambda_i + e_i)' (P_F - P_{FH^k}) (F\lambda_i + e_i) \\ &= \frac{1}{NT} \sum_{i=1}^N \lambda_i' F' (P_F - P_{FH^k}) F \lambda_i + \frac{2}{NT} \sum_{i=1}^N e_i' (P_F - P_{FH^k}) F \lambda_i \\ &\quad + \frac{1}{NT} \sum_{i=1}^N e_i' (P_F - P_{FH^k}) e_i \\ &= I + II + III \end{aligned}$$

Now,

$$\begin{aligned} I &= \text{tr} \left(T^{-1} F' (P_F - P_{FH^k}) F N^{-1} \sum_{i=1}^N \lambda_i \lambda_i' \right) \\ &= \text{tr} \left(\left[\frac{F' F}{T} - \frac{F' F H^k}{T} \left(\frac{H^{k'} F' F H^k}{T} \right)^{-1} \frac{H^{k'} F' F}{T} \right] \frac{1}{N} \sum_{i=1}^N \lambda_i \lambda_i' \right) \end{aligned}$$

Note that $F' F / T \rightarrow_p \Sigma_F$ and $\Lambda' \Lambda / N \rightarrow_p \Sigma_\Lambda$ by Assumptions 1 and 2. To show that term I is positive, it is sufficient to show that $\text{tr}(A \Sigma_\Lambda) > 0$, where $A = \Sigma_F - \Sigma_F H^k \left(H^{k'} \Sigma_F H^k \right)^{-1} H^{k'} \Sigma_F$. Since $\text{rank}(H^{k'}) = k < r$ and $\text{rank}(\Sigma_F) = r$, we have $A \neq 0$. Also, A is positive semi-definite, and Σ_Λ is positive definite, so $\text{tr}(A \Sigma_\Lambda) > 0$.

Also,

$$II = \frac{2}{NT} \sum_{i=1}^N e_i' P_F F \lambda_i - \frac{2}{NT} \sum_{i=1}^N e_i' P_{FH^k} F \lambda_i$$

Consider the second term:

$$\begin{aligned}
& \left| \frac{2}{NT} \sum_{i=1}^N e_i' P_{FH^k} F \lambda_i \right| \\
&= \left| \frac{2}{NT} \sum_{i=1}^N \sum_{t=1}^T e_{it} F_t' H^k \left(\frac{H^{k'} F' F H^k}{T} \right)^{-1} \frac{H^{k'} F' F \lambda_i}{T} \right| \\
&= \left| \frac{2}{NT} \sum_{t=1}^T F_t' H^k \left(\frac{H^{k'} F' F H^k}{T} \right)^{-1} \frac{H^{k'} F' F}{T} \sum_{i=1}^N e_{it} \lambda_i \right| \\
&\leq N^{-(1-\alpha)/2} \left(\frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{T} \sum_{t=1}^T \left\| N^{-(1+\alpha)/2} \sum_{i=1}^N e_{it} \lambda_i \right\|^2 \right)^{\frac{1}{2}} \left\| H^k \left(\frac{H^{k'} F' F H^k}{T} \right)^{-1} \frac{H^{k'} F' F}{T} \right\| \\
&= O_p \left(N^{-(1-\alpha)/2} \right) O_p(1)
\end{aligned}$$

where the $O_p \left(N^{-(1-\alpha)/2} \right)$ term in the last equation follows from Lemma A2, and the $O_p(1)$ term follows from Assumption 1 and the fact that $H^k = O_p(1)$ and $H^{k'} F' F H^k / T$ is positive definite with rank k in the limit. The first term of II is also $O_p \left(N^{-(1-\alpha)/2} \right)$, which can be proved using a very similar argument.

For term III , since $P_F - P_{FH^k}$ is positive semi-definite, it is always non-negative. Thus, $V(k, FH^k) - V(r, F) > 0$ for $k < r$ as N and $T \rightarrow \infty$.

(iv) Consider $V(r, \tilde{F}^r) - V(k, \tilde{F}^k) = \frac{1}{NT} \sum_{i=1}^N X_i' M_{\tilde{F}^r} X_i - \frac{1}{NT} \sum_{i=1}^N X_i' M_{\tilde{F}^k} X_i$. The rank of H^k is r because $k \geq r$. Let H^{k+} be the generalized inverse of H^k such that $H^k H^{k+} = I_r$. For $k \geq r$, we have

$$\begin{aligned}
X_i &= F \lambda_i + e_i \\
&= \tilde{F}^k H^{k+} \lambda_i + e_i - (\tilde{F}^k - FH^k) H^{k+} \lambda_i
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{1}{NT} \sum_{i=1}^N X_i' M_{\tilde{F}^k} X_i &= \frac{1}{NT} \sum_{i=1}^N [e_i - (\tilde{F}^k - FH^k) H^{k+} \lambda_i]' M_{\tilde{F}^k} [e_i - (\tilde{F}^k - FH^k) H^{k+} \lambda_i] \\
&= \frac{1}{NT} \sum_{i=1}^N e_i' M_{\tilde{F}^k} e_i - \frac{2}{NT} \sum_{i=1}^N \lambda_i' H^{k+'} (\tilde{F}^k - FH^k)' M_{\tilde{F}^k} e_i \\
&\quad + \frac{1}{NT} \sum_{i=1}^N [(\tilde{F}^k - FH^k) H^{k+} \lambda_i]' M_{\tilde{F}^k} [(\tilde{F}^k - FH^k) H^{k+} \lambda_i] \\
&= a_k + b_k + c_k
\end{aligned}$$

For term b_k and $k \geq r$, note that $|\text{tr}(J)| \leq p\|J\|$ for any $p \times p$ matrix J , so we have

$$\begin{aligned}
b_k &= \frac{2}{NT} \sum_{i=1}^N \text{tr} \left(H^{k+'} (\tilde{F}^k - FH^k)' e_i \lambda_i' \right) - \frac{2}{NT} \sum_{i=1}^N \text{tr} \left(H^{k+'} (\tilde{F}^k - FH^k)' P_{\tilde{F}^k} e_i \lambda_i' \right) \\
&\leq 2r \|H^{k+}\| \left\| \frac{\tilde{F}^k - FH^k}{\sqrt{T}} \right\| \left\| \frac{1}{\sqrt{TN}} \sum_{i=1}^N e_i \lambda_i' \right\| (1 + \sqrt{k}) \\
&\leq 2(1 + \sqrt{k})r \|H^{k+}\| \left(\frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2 \right)^{\frac{1}{2}} \frac{1}{\sqrt{N^{1-\alpha}}} \left(\frac{1}{T} \sum_{t=1}^T \left\| N^{-(1+\alpha)/2} \sum_{i=1}^N e_{it} \lambda_i \right\|^2 \right)^{\frac{1}{2}} \\
&= O_p(\phi_{NT}^{-1}) O_p \left(N^{-(1-\alpha)/2} \right) \leq O_p(\phi_{NT}^{-2})
\end{aligned}$$

where the last equation follows from Lemmas A2 and A3(i). (We use the fact that $\|P_{\tilde{F}^k}\| = \text{tr}(P_{\tilde{F}^k})^{1/2} = \text{rank}(\tilde{F}^k)^{1/2} = \sqrt{k}$).

For term c_k and $k \geq r$, we use the inequality that $x' M_{\tilde{F}^k} x \leq x' x$, so

$$\begin{aligned}
c_k &\leq \frac{1}{NT} \sum_{i=1}^N [(\tilde{F}^k - FH^k) H^{k+} \lambda_i]' [(\tilde{F}^k - FH^k) H^{k+} \lambda_i] \\
|c_k| &\leq \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \|H^{k+}\|^2 \right) \frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - H^{k'} F_t\|^2 \\
&= O_p(1) O_p(\phi_{NT}^{-2}),
\end{aligned}$$

where the last equation follows from Assumption 2 and A3(i).

For term a_k , note that $a_r - a_k = \frac{1}{NT} \sum_{i=1}^N e_i' P_{\tilde{F}^k} e_i - \frac{1}{NT} \sum_{i=1}^N e_i' P_{\tilde{F}^r} e_i$, so it is sufficient to show that $\frac{1}{NT} \sum_{i=1}^N e_i' P_{\tilde{F}^k} e_i = O_p(\phi_{NT}^{-2})$ for $k \geq r$. Using Assumption 5', we have

$$\begin{aligned}
\frac{1}{NT} \sum_{i=1}^N e_i' P_{\tilde{F}^k} e_i &\leq \frac{1}{G} \sum_{g=1}^G \left(\frac{1}{M_g T} \sum_{i \in g} e_i' P_{\tilde{F}^k} e_i \right) \frac{\max_{g=1, \dots, G} \{M_g\} G}{N} \\
&\leq \frac{1}{G} \sum_{g=1}^G \left(\frac{1}{M_g T} \sum_{i \in g} e_i' P_{\tilde{F}^k} e_i \right) O(1)
\end{aligned}$$

where the last inequality follows from the assumption that $G = O(N^\alpha)$ and $M_g \leq N^{1-\alpha} W$ for all g . Note that $\frac{1}{M_g T} \sum_{i \in g} e_i' P_{\tilde{F}^k} e_i$ is bounded by the sum of the first k largest eigenvalues of $\frac{1}{M_g T} \underline{e}_g' \underline{e}_g$ (see Bai and Ng, 2006b), so it is sufficient to show that $\frac{1}{G} \sum_{g=1}^G \rho_1 \left(\frac{1}{M_g T} \underline{e}_g' \underline{e}_g \right) = O_p(\phi_{NT}^{-2})$. By Assumption 5', we have $\underline{e}_g' \underline{e}_g = Q_g' \varepsilon_g' P_g' P_g \varepsilon_g Q_g$. Thus,

$$\begin{aligned}
\frac{1}{G} \sum_{g=1}^G \rho_1 \left(\frac{1}{M_g T} \underline{e}_g' \underline{e}_g \right) &\leq \frac{1}{G} \sum_{g=1}^G \rho_1(P_g' P_g) \rho_1(Q_g Q_g') \rho_1 \left(\frac{1}{M_g T} \varepsilon_g \varepsilon_g' \right) \\
&= \frac{1}{G} \sum_{g=1}^G \rho_1(P_g' P_g) \rho_1(Q_g' Q_g) \rho_1 \left(\frac{1}{M_g T} \varepsilon_g' \varepsilon_g \right) \leq W^2 \frac{1}{G} \sum_{g=1}^G \rho_1 \left(\frac{1}{M_g T} \varepsilon_g' \varepsilon_g \right)
\end{aligned}$$

where the equation in the second line uses the fact that $\rho_1(Q_g Q'_g)$ and $\rho_1(Q'_g Q_g)$ have identical non-zero eigenvalues (the same result is true for $\rho_1(e'_g e_g)$ and $\rho_1(e_g e'_g)$), and the last inequality follows from Assumption 5'.

Next, we show that $\frac{1}{G} \sum_{g=1}^G \rho_1 \left(\frac{1}{M_g T} \varepsilon'_g \varepsilon_g \right) = O_p(\phi_{NT}^{-2})$. Let $y_g = \lim_{M_g, T \rightarrow \infty} M_g/T$. Yin, Bai and Krishnaiah (1988) shows that $\rho_1 \left(\frac{1}{T} \varepsilon'_g \varepsilon_g \right) \rightarrow_{a.s.} (1 + \sqrt{y_g})^2 \sigma_g^2$ under the assumptions that $\varepsilon_{it, i \in g}$ are iid and have finite 4th moment and that $0 < y_g < \infty$. This result implies that $\rho_1 \left(\frac{1}{M_g T} \varepsilon'_g \varepsilon_g \right) = O_p(1/\min(M_g, T)) = O_p(\phi_{NT}^{-2})$ if $M_g/T \rightarrow y_g$ and $0 < y_g < \infty$. We will argue that the same result holds without assuming identical distribution and $y_g > 0$. First, Jonsson (1985) shows that $\rho_1 \left(\frac{1}{T} \varepsilon'_g \varepsilon_g \right) \rightarrow_{a.s.} (1 + \sqrt{y_g})^2 \sigma_g^2$ if $\varepsilon_{it, i \in g}$ are independent, $E|\varepsilon_{it, i \in g}|^7 \leq W$, $E\varepsilon_{it, i \in g} = 0$ and $0 < y_g < \infty$. This implies $\rho_1 \left(\frac{1}{M_g T} \varepsilon'_g \varepsilon_g \right) = O_p(\phi_{NT}^{-2})$ if we drop the identical distribution assumption but assume uniformly bounded 7th moment of $\varepsilon_{it, i \in g}$. Secondly, we show that the same result holds as long as $0 \leq y_g < \infty$. To see why the same result holds when $y_g = 0$, let $\nu_g = [\varepsilon_g, \epsilon_g]$ be a $T \times M_g^*$ matrix, where ϵ_g is $T \times (M_g^* - M_g)$ containing independent elements with mean zero, $E\epsilon_g^2 = \sigma_g^2$ and uniformly bounded 7th moment. Let $M_g^*/T \rightarrow y_g^* > 0$, so we have

$$\begin{aligned} \rho_1 \left(\frac{1}{T} \varepsilon'_g \varepsilon_g \right) &= \rho_1 \left(\frac{1}{T} \varepsilon_g \varepsilon'_g \right) \leq \rho_1 \left(\frac{1}{T} \varepsilon_g \varepsilon'_g + \frac{1}{T} \epsilon_g \epsilon'_g \right) = \rho_1 \left(\frac{1}{T} \nu_g \nu'_g \right) \\ &= \rho_1 \left(\frac{1}{T} \nu'_g \nu_g \right) \rightarrow_{a.s.} (1 + \sqrt{y_g^*})^2 \sigma_g^2 \end{aligned}$$

Since y_g^* can be arbitrarily close to zero, $\rho_1 \left(\frac{1}{T} \varepsilon'_g \varepsilon_g \right)$ will converge to σ_g^2 , and $\rho_1 \left(\frac{1}{M_g T} \varepsilon'_g \varepsilon_g \right) = O_p(N^{-1+\alpha}) \leq O_p(\phi_{NT}^{-2})$ since $M_g \geq N^{1-\alpha} W_0$.

Finally, we claim that $\rho_1 \left(\frac{1}{M_g T} \varepsilon'_g \varepsilon_g \right)$ is $O_p(\phi_{NT}^{-2})$ uniformly in g . To see this, first note that y_g is uniformly bounded because $N^{1-\alpha} W_0 \leq M_g \leq N^{1-\alpha} W$ for all g . Using the above result, we have $\rho_1 \left(\frac{1}{T} \varepsilon'_g \varepsilon_g \right) \rightarrow_{a.s.} (1 + \sqrt{y_g})^2 \sigma_g^2 \leq W$. Thus, $\frac{1}{G} \sum_{g=1}^G \rho_1 \left(\frac{1}{M_g T} \varepsilon'_g \varepsilon_g \right) = O_p(\phi_{NT}^{-2})$. **Q.E.D.**

Lemma B1:

(i) Under Assumptions 1–4, $\text{plim inf}_{N, T \rightarrow \infty} \delta_{NT}^2 [V(k, \tilde{F}^k) - V(k+1, \tilde{F}^{k+1})] > 0$ for any fixed constant $k \geq r$.

(ii) Under Assumptions 1–3 and 4', $\text{plim inf}_{N, T \rightarrow \infty} \phi_{NT}^2 [V(k, \tilde{F}^k) - V(k+1, \tilde{F}^{k+1})] > 0$ for any fixed constant $k \geq r$.

Proof:

(i) Note that $V(k, \tilde{F}^k) = \text{trace}(XX'/NT) - \sum_{i=1}^k \rho_i(XX'/NT)$. Thus, $V(k, \tilde{F}^k) - V(k+1, \tilde{F}^{k+1}) = \rho_{k+1}(XX'/NT)$. Also, according to the inequality $\rho_{k+1}(XX'/NT) \geq \rho_{k+r+1}(ee'/NT)$ (See Eq. (9) of Onatski (2010)), it is sufficient to show that $\text{plim inf}_{N, T \rightarrow \infty} \delta_{NT}^2 \rho_k(ee'/NT) > 0$ for any fixed constant k .

Note that $\text{rank}(e'e) = \text{rank}(ee') = \text{rank}(P\varepsilon QQ'\varepsilon'P') = \text{rank}(\varepsilon QQ'\varepsilon') = \text{rank}(Q'\varepsilon'\varepsilon Q) = \text{rank}(\varepsilon'\varepsilon) = \text{rank}(\varepsilon\varepsilon')$ because $\text{rank}(PP') = \text{rank}(P) = T$ and $\text{rank}(QQ') = \text{rank}(Q) = N$ by Assumption 5. Since $\text{plim}_{N, T \rightarrow \infty} \rho_k(\varepsilon'\varepsilon/T) = (1 + \lim_{N, T \rightarrow \infty} \sqrt{\frac{N}{T}})^2 \sigma^2 > 0$ by the results of Yin, Bai and Krishnaiah (1988) and Jonsson (1985), we have $\text{plim inf}_{N, T \rightarrow \infty} N \rho_k(e'e/TN) > 0$ due to the fact

that $\text{rank}(e'e/T) = \text{rank}(\varepsilon'\varepsilon/T)$. Similarly, since $\text{plim}_{N,T \rightarrow \infty} \rho_k(\varepsilon\varepsilon'/N) = (1 + \lim_{N,T \rightarrow \infty} \sqrt{\frac{T}{N}})^2 \sigma^2 > 0$, we have $\text{plim inf}_{N,T \rightarrow \infty} T\rho_k(e'e/TN) > 0$ due to the fact that $\text{rank}(ee'/N) = \text{rank}(\varepsilon\varepsilon'/N)$. Combining the two results, we have $\text{plim inf}_{N,T \rightarrow \infty} \min(N, T)$

$\rho_k(ee'/NT) > 0$.

(ii) Similar to part (i), it is sufficient to show that $\text{plim inf}_{N,T \rightarrow \infty} \phi_{NT}^2 \rho_k(ee'/NT) > 0$ for any fixed constant k . First, note that $\text{plim inf}_{N,T \rightarrow \infty} T\rho_k(ee'/NT) > 0$ by Assumption 4'(i) that $\text{rank}(ee'/N) \geq k$ for all N and T .

Also, by Assumption 4'(i), there exist $b_{j_1}, \dots, b_{j_k} \in \{b_1, b_2, \dots, b_m\}$ such that $\text{plim} \rho_k[ww'/(T \sum_{s=1}^k M_{b_{j_s}})] > 0$, where $w = [\bar{e}_{b_{j_1}}, \dots, \bar{e}_{b_{j_k}}]$. Thus, we have

$$\begin{aligned} N^{1-\alpha} \rho_k(ee'/NT) &= N^{1-\alpha} \rho_k \left(\left[\sum_{s=1}^k \bar{e}_{b_{j_s}} \bar{e}'_{b_{j_s}} + \sum_{b \neq b_{j_1}, \dots, b_{j_k}} \bar{e}_b \bar{e}'_b \right] / NT \right) \\ &\geq \rho_k \left(ww' / \left[T \sum_{s=1}^k M_{b_{j_s}} \right] \right) \frac{N^{1-\alpha} \sum_{s=1}^k M_{b_{j_s}}}{N} \\ &\geq \rho_k \left(ww' / \left[T \sum_{s=1}^k M_{b_{j_s}} \right] \right) W_0 \end{aligned}$$

Therefore, $\text{plim inf}_{N,T \rightarrow \infty} N^{1-\alpha} \rho_k(ee'/NT) > 0$ and combining with $\text{plim inf}_{N,T \rightarrow \infty} T\rho_k(ee'/NT) > 0$ will give the desired result. **Q.E.D.**

Lemma C1:

(i) Under Assumptions 1–4, $N^{-1} \sum_{i=1}^N \|T^{-1} e'_i \hat{F}^k\|^2 = O_p \left(\frac{1}{\delta_{NT}^2} \right)$ for $k \leq r$ and $N^{-1} \sum_{i=1}^N \|T^{-1} e'_i \tilde{F}^k\|^2 = O_p \left(\frac{1}{\delta_{NT}^2} \right)$ for $k \leq k_{max}$.

(ii) Under Assumptions 1–4 and 6–7, $\frac{1}{N} \sum_{i=1}^N \|(\tilde{F}^r - FH^r)' e_i / T\|^2 = O_p \left(\frac{1}{\delta_{NT}^4} \right)$

Proof:

(i) Note that

$$\rho_k(XX'/NT) \geq \rho_k(F\Lambda'\Lambda F'/NT) - \rho_1(ee'/NT) \quad (1.1)$$

by Weyl's inequality (see Eq. (7) in Onatski (2010)). Since $\rho_k(F\Lambda'\Lambda F'/NT) = \rho_k(\Lambda'\Lambda F'F/NT) \rightarrow_p \rho_k(\Sigma_\Lambda \Sigma_F) > 0$ for $k \leq r$ and $\rho_1(ee'/NT) = O_p(\delta_{NT}^{-2})$ (See Bai and Ng (2006b)), $\rho_k(XX'/NT)$ will be bounded away from zero asymptotically. This implies that V_{NT}^k is non-singular for $k \leq r$ and $(V_{NT}^k)^{-1} = O_p(1)$, because it is a diagonal matrix consisting of first k eigenvalues of XX'/NT . Also,

$\hat{F}^k = \tilde{F}^k (V_{NT}^k)^{-1}$ by the definition of \tilde{F}^k . Thus, we have

$$\begin{aligned}
\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' \hat{F}^k \right\|^2 &= \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' \tilde{F}^k (V_{NT}^k)^{-1} \right\|^2 \\
&= \frac{1}{N} \sum_{i=1}^N \left\| \left[\frac{1}{T} e_i' (\tilde{F}^k - FH^k) + \frac{1}{T} e_i' FH^k \right] (V_{NT}^k)^{-1} \right\|^2 \\
&\leq \left[\frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' (\tilde{F}^k - FH^k) \right\|^2 + \frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' FH^k \right\|^2 \right] \|(V_{NT}^k)^{-1}\|^2 \\
&\leq \left[\left(\frac{2}{N} \sum_{i=1}^N \sum_{t=1}^T e_{it}^2 \right) \frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^k - F_t H^k\|^2 + \frac{2}{TN} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} e_i' F \right\|^2 \|H^k\|^2 \right] \|(V_{NT}^k)^{-1}\|^2 \\
&= \left[O_p \left(\frac{1}{\delta_{NT}^2} \right) + O_p \left(\frac{1}{T} \right) \right] O_p(1)
\end{aligned}$$

where we use the fact that $N^{-1} \sum_{i=1}^N \left\| 1/\sqrt{T} e_i' F \right\|^2 = O_p(1)$ by Assumption 3(a) and the fact that $\|(V_{NT}^k)^{-1}\| = O_p(1)$ (see Lemma A3 of Bai (2003)).

(ii)

$$\begin{aligned}
&\frac{1}{N} \sum_{i=1}^N \left\| \frac{(\tilde{F}^r - FH^r)' e_i}{T} \right\|^2 \\
&\leq \frac{4}{N} \sum_{i=1}^N \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \hat{F}_s^r \gamma_N(s, t) e_{it} \right\|^2 + \frac{4}{N} \sum_{i=1}^N \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \hat{F}_s^r \zeta_{st} e_{it} \right\|^2 \\
&\quad + \frac{4}{N} \sum_{i=1}^N \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \hat{F}_s^r \eta_{st} e_{it} \right\|^2 + \frac{4}{N} \sum_{i=1}^N \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \hat{F}_s^r \xi_{st} e_{it} \right\|^2 \\
&= 4(I + II + III + IV)
\end{aligned}$$

First, term I can be bounded by

$$\frac{2}{N} \sum_{i=1}^N \left(\left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T (\hat{F}_s^r - H^r F_s) \gamma_N(s, t) e_{it} \right\|^2 + \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T H^r F_s \gamma_N(s, t) e_{it} \right\|^2 \right)$$

The first term is bounded by

$$\frac{1}{T} \left(\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^r - H^r F_s\|^2 \right) \left(\frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T |\gamma_N(s, t)|^2 \right) \left(\frac{2}{NT} \sum_{i=1}^N \sum_{t=1}^T e_{it}^2 \right) = \frac{1}{T} O_p \left(\frac{1}{\delta_{NT}^2} \right) O_p(1)$$

where we use Lemma A1(i) and the fact that $\frac{1}{T} \sum_{t=1}^T \sum_{s=1}^T |\gamma_N(s, t)|^2 \leq W < \infty$ by Lemma 1(i) of

Bai and Ng (2002). The second term can be bounded by

$$\begin{aligned} & \frac{2}{NT^2} \sum_{i=1}^N \left[\left(\frac{1}{T} \sum_{s=1}^T \|F_s\|^2 \right) \frac{1}{T} \sum_{s=1}^T \left(\sum_{t=1}^T \gamma_N(s, t) e_{it} \right)^2 \right] \|H^r\|^2 \\ &= \frac{1}{T^2} \left(\frac{1}{T} \sum_{s=1}^T \|F_s\|^2 \right) \|H^r\|^2 \left(\frac{1}{NT} \sum_{i=1}^N \sum_{s=1}^T \sum_{t=1}^T \sum_{u=1}^T \gamma_N(s, t) \gamma_N(s, u) e_{it} e_{iu} \right) \end{aligned}$$

Note that $T^{-1} \sum_{s=1}^T \|F_s\|^2$ and $\|H^r\|^2$ are both $O_p(1)$. We claim that $(NT)^{-1} \sum_{i=1}^N \sum_{s=1}^T \sum_{t=1}^T \sum_{u=1}^T \gamma_N(s, t) \gamma_N(s, u) e_{it} e_{iu}$ is also $O_p(1)$, because its expectation is bounded by

$$\begin{aligned} & (NT)^{-1} \sum_{i=1}^N \sum_{s=1}^T \sum_{t=1}^T \sum_{u=1}^T |\gamma_N(s, t) \gamma_N(s, u)| E|e_{it} e_{iu}| \\ & \leq (NT)^{-1} \sum_{i=1}^N \sum_{s=1}^T \sum_{t=1}^T \sum_{u=1}^T |\gamma_N(s, t) \gamma_N(s, u)| (E e_{it}^2 E e_{iu}^2)^{\frac{1}{2}} \\ & \leq (NT)^{-1} \sum_{i=1}^N \sum_{s=1}^T \left(\sum_{t=1}^T |\gamma_N(s, t)| \right)^2 W \\ & \leq W^3 \end{aligned}$$

where the inequality in the third line follows from Assumption 4(a) and the last inequality follows from Assumption 6(a). This means that the second term is $O_p\left(\frac{1}{T^2}\right)$, so term I is $O_p\left(\frac{1}{T\delta_{NT}^2}\right)$.

Second, we consider term II , which can be bounded by

$$\frac{2}{N} \sum_{i=1}^N \left(\left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T (\hat{F}_s^r - H^r F_s) \zeta_{st} e_{it} \right\|^2 + \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T H^r F_s \zeta_{st} e_{it} \right\|^2 \right)$$

The first term is bounded by

$$\begin{aligned} & \frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^r - H^r F_s\|^2 \frac{2}{NT} \sum_{i=1}^N \sum_{s=1}^T \left(\frac{1}{T} \sum_{t=1}^T \zeta_{st} e_{it} \right)^2 \\ & \leq \left(\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^r - H^r F_s\|^2 \right) \frac{2}{NT} \sum_{i=1}^N \sum_{s=1}^T \left(\frac{1}{T} \sum_{t=1}^T \zeta_{st}^2 \frac{1}{T} \sum_{t=1}^T e_{it}^2 \right) \\ & = \left(\frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^r - H^r F_s\|^2 \right) \left(\frac{2}{NT} \sum_{i=1}^N \sum_{t=1}^T e_{it}^2 \right) \left(\frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^T \zeta_{st}^2 \right) \\ & = O_p\left(\frac{1}{\delta_{NT}^2}\right) O_p(1) O_p\left(\frac{1}{N}\right) \end{aligned}$$

where $T^{-2} \sum_{s=1}^T \sum_{t=1}^T \zeta_{st}^2 = O_p(N^{-1})$ follows from the fact that

$$\frac{1}{T^2} \sum_{s=1}^T \sum_{t=1}^T \zeta_{st}^2 = \frac{1}{NT^2} \sum_{s=1}^T \sum_{t=1}^T \left(\frac{1}{\sqrt{N}} \sum_{k=1}^N [e_{ks} e_{kt} - E(e_{ks} e_{kt})] \right)^2 = O_p\left(\frac{1}{N}\right)$$

by Assumption 4(e). Now, let $z_t = (1/\sqrt{NT}) \sum_{s=1}^T \sum_{k=1}^N F_s [e_{ks} e_{kt} - E(e_{ks} e_{kt})]$. The second term of term *II* can be written as

$$\begin{aligned}
& \frac{2}{N} \sum_{i=1}^N \left\| \frac{H^r}{\sqrt{NT^3}} \sum_{t=1}^T z_t e_{it} \right\|^2 \\
& \leq \frac{2}{N^2 T} \sum_{i=1}^N \left(\frac{1}{T} \sum_{t=1}^T \|z_t\|^2 \frac{1}{T} \sum_{t=1}^T e_{it}^2 \right) \|H^r\|^2 \\
& = \frac{2}{NT} \left(\frac{1}{T} \sum_{t=1}^T \|z_t\|^2 \right) \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T e_{it}^2 \right) \|H^r\|^2 \\
& = O_p \left(\frac{1}{NT} \right)
\end{aligned}$$

where $T^{-1} \sum_{t=1}^T \|z_t\|^2$ is $O_p(1)$ by Assumption 7(a). Thus, term *II* is $O_p \left(\frac{1}{N\delta_{NT}^2} \right)$.

Next, term *III* can be written as

$$\begin{aligned}
& \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{NT^2} \sum_{t=1}^T \sum_{s=1}^T \hat{F}_s^r F_s' \sum_{k=1}^N \lambda_k e_{kt} e_{it} \right\|^2 \\
& \leq \left\| \frac{1}{T} \sum_{s=1}^T \hat{F}_s^r F_s' \right\|^2 \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{NT} \sum_{t=1}^T \sum_{k=1}^N \lambda_k e_{kt} e_{it} \right\|^2 \\
& \leq \frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^r\|^2 \frac{1}{T} \sum_{s=1}^T \|F_s\|^2 \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{NT} \sum_{t=1}^T \sum_{k=1}^N \lambda_k e_{kt} e_{it} \right\|^2 \\
& = O_p(1) O_p \left(\frac{1}{N\delta_{NT}^2} \right)
\end{aligned}$$

where we use the fact that $N^{-1} \sum_{i=1}^N \left\| (NT)^{-1} \sum_{t=1}^T \sum_{k=1}^N \lambda_k e_{kt} e_{it} \right\|^2 = O_p \left(\frac{1}{N\delta_{NT}^2} \right)$, which is proved below:

$$\begin{aligned}
& \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{NT} \sum_{t=1}^T \sum_{k=1}^N \lambda_k e_{kt} e_{it} \right\|^2 \\
& \leq \frac{2}{N^2 T} \sum_{i=1}^N \left\| \frac{1}{\sqrt{NT}} \sum_{t=1}^T \sum_{k=1}^N \lambda_k [e_{kt} e_{it} - E(e_{kt} e_{it})] \right\|^2 + \frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{NT} \sum_{t=1}^T \sum_{k=1}^N \lambda_k E(e_{kt} e_{it}) \right\|^2 \\
& \leq O_p \left(\frac{1}{NT} \right) + O_p \left(\frac{1}{N^2} \right)
\end{aligned}$$

where the $O_p \left(\frac{1}{NT} \right)$ term follows from Assumption 7(b) and the $O_p \left(\frac{1}{N^2} \right)$ term follows from the fact

that

$$\begin{aligned}
& E \left[\frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{NT} \sum_{t=1}^T \sum_{k=1}^N \lambda_k E(e_{kt} e_{it}) \right\|^2 \right] \\
& \leq \frac{2}{N} \sum_{i=1}^N E \left(\frac{1}{NT} \sum_{t=1}^T \sum_{k=1}^N \|\lambda_k\| |\tau_{ki}| \right)^2 \\
& = \frac{2}{N^3} \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N E (\|\lambda_k\| \|\lambda_j\| |\tau_{ki}| |\tau_{ji}|) \\
& \leq \frac{2}{N^3} \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \left(E \|\lambda_k\|^2 E \|\lambda_j\|^2 \right)^{\frac{1}{2}} |\tau_{ki}| |\tau_{ji}| \\
& \leq \frac{2}{N^3} \sum_{i=1}^N \left(\sum_{k=1}^N |\tau_{ki}| \right)^2 W \leq \frac{2W^3}{N^2}
\end{aligned}$$

by Assumptions 2(a), 4(c) and 6(b).

Finally, term IV can be bounded by

$$\frac{2}{N} \sum_{i=1}^N \left(\left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T (\hat{F}_s^r - H^r F_s) \xi_{st} e_{it} \right\|^2 + \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T H^r F_s \xi_{st} e_{it} \right\|^2 \right)$$

For the first term, we have

$$\begin{aligned}
& \frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T (\hat{F}_s^r - H^r F_s) \xi_{st} e_{it} \right\|^2 \\
& \leq \frac{1}{T} \sum_{s=1}^T \|\hat{F}_s^r - H^r F_s\|^2 \frac{2}{NT} \sum_{i=1}^N \sum_{s=1}^T \left(\frac{1}{T} \sum_{t=1}^T \xi_{st} e_{it} \right)^2 \\
& \leq O_p(\delta_{NT}^{-2}) \frac{2}{NT} \sum_{i=1}^N \sum_{s=1}^T \left(\frac{1}{TN} \sum_{t=1}^T F_t' e_{it} \sum_{k=1}^N \lambda_k e_{ks} \right)^2 \\
& \leq O_p(\delta_{NT}^{-2}) \frac{1}{NT} \left(\frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t' e_{it} \right\|^2 \right) \left(\frac{1}{T} \sum_{s=1}^T \left\| \frac{1}{\sqrt{N}} \sum_{k=1}^N \lambda_k e_{ks} \right\|^2 \right) \\
& = O_p \left(\frac{1}{\delta_{NT}^2 NT} \right) O_p(1)
\end{aligned}$$

where the $O_p(1)$ term follows from Assumptions 3(a) and 7(d). The second term can be written as

$$\begin{aligned}
& \frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T H^r F_s \frac{1}{N} \sum_{k=1}^N \lambda'_k F_t e_{ks} e_{it} \right\|^2 \\
& \leq \frac{2}{N^2 T^2} \sum_{i=1}^N \left\| \frac{1}{\sqrt{NT}} \sum_{s=1}^T \sum_{k=1}^N F_s \lambda'_k e_{ks} \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t e_{it} \right\|^2 \|H^r\|^2 \\
& \leq \frac{2}{NT^2} \left\| \frac{1}{\sqrt{NT}} \sum_{s=1}^T \sum_{k=1}^N F_s \lambda'_k e_{ks} \right\|^2 \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^T F_t e_{it} \right\|^2 \|H^r\|^2 \\
& = \frac{2}{NT^2} O_p(1)
\end{aligned}$$

where the $O_p(1)$ term follows from Assumption 3(a) and 7(c). Thus,

$$I + II + III + IV = O_p\left(\frac{1}{T\delta_{NT}^2}\right) + O_p\left(\frac{1}{N\delta_{NT}^2}\right) + O_p\left(\frac{1}{\delta_{NT}^2 NT}\right) + O_p\left(\frac{1}{NT^2}\right) = O_p\left(\frac{1}{\delta_{NT}^4}\right)$$

Q.E.D.

Lemma C2: Under Assumptions 1–4 and for $k \leq r$,

- (i) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\hat{F}^k} e_j \right| \rightarrow_p 0$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} e'_i M_{\hat{F}^k} e_j \right| \rightarrow_p 0$;
- (ii) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda'_i F' M_{\hat{F}^k} e_j \right| \rightarrow_p 0$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda'_i F' M_{\hat{F}^k} e_j \right| \rightarrow_p 0$.

Proof:

Note that $M_{\tilde{F}^k} = M_{\hat{F}^k}$, so we will show the results with $M_{\tilde{F}^k}$ replaced by $M_{\hat{F}^k}$.

(i)

$$\begin{aligned}
\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\hat{F}^k} e_j \right| & \leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} e'_i e_j \right| + \left| \frac{1}{T} e'_i P_{\hat{F}^k} e_j \right| \right) \\
& = I + II
\end{aligned}$$

For term I ,

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i e_j \right| \\
& = \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} \sum_{t=1}^T e_{it} e_{jt} \right| - \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) + \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right|
\end{aligned}$$

Note that the first term is $O_p(1/\sqrt{T})$ because its expectation can be bounded by

$$\frac{1}{\sqrt{T} \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} E \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T (e_{it} e_{jt} - \tau_{ij,t}) \right| \leq \frac{W}{\sqrt{T}}$$

where the last inequality follows from Assumption 4(f). By Assumption 4(c), the second term can

be bounded by

$$\begin{aligned} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |\tau_{ij}| &\leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{i=1}^N \sum_{j=1}^N |\tau_{ij}| \\ &\leq \frac{NW}{\sum_{b=1}^B M_b^2} = O(N^{-\alpha}) \end{aligned}$$

Next, we consider term II . Since $\hat{F}^{k'} \hat{F}^k / T = I_k$, term II can be bounded by

$$\begin{aligned} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left\| \frac{1}{T} e_i' \hat{F}^k \right\| \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\| &= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \left(\sum_{i \in b} \left\| \frac{1}{T} e_i' \hat{F}^k \right\| \right)^2 \\ &\leq \frac{B}{B \sum_{b=1}^B M_b^2} \sum_{b=1}^B M_b \sum_{i \in b} \left\| \frac{1}{T} e_i' \hat{F}^k \right\|^2 \\ &\leq \frac{B \max_b \{M_b\}}{N^2} \sum_{b=1}^B \sum_{i \in b} \left\| \frac{1}{T} e_i' \hat{F}^k \right\|^2 \\ &= O(1) \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' \hat{F}^k \right\|^2 = O_p \left(\frac{1}{\delta_{NT}^2} \right) \end{aligned}$$

where the inequality in the third line follows from the fact that $B \sum_{b=1}^B M_b^2 \geq (\sum_{b=1}^B M_b)^2 = N^2$ and the last line follows from Assumption 4(g) and Lemma C1(i). Thus,

$$\left(\sum_{b=1}^B M_b^2 \right)^{-1} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e_i' M_{\hat{F}^k} e_j \right| \rightarrow_p 0.$$

For the between-block analog, note that

$$\begin{aligned} &\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} e_i' M_{\hat{F}^k} e_j \right| \\ &\leq \left(\frac{N^2}{N^2 - \sum_{b=1}^B M_b^2} \right) \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{T} e_i' M_{\hat{F}^k} e_j \right| \end{aligned}$$

where $N^2 / (N^2 - \sum_{b=1}^B M_b^2)$ is $O(1)$ because M_b is uniformly $O(N^\alpha)$ by Assumption 4(g). Then it can be shown that $N^{-2} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{T} e_i' M_{\hat{F}^k} e_j \right| \rightarrow_p 0$ using a very similar argument provided above.

(ii) To conserve space, we will only provide the proof of the within-block result because the

result for the between-block analog can be proved using a very similar argument.

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^k} e_j \right| \\
& \leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} \lambda_i' F' e_j \right| + \left| \frac{1}{T} \lambda_i' F' \hat{F}^k \left(\frac{\hat{F}^{k'} \hat{F}^k}{T} \right)^{-1} \frac{1}{T} \hat{F}^{k'} e_j \right| \right) \\
& = I + II
\end{aligned}$$

For term I ,

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' e_j \right| \\
& \leq \frac{1}{\sqrt{T} \sum_{b=1}^B M_b^2} \sum_{b=1}^B \left(\sum_{i,j \in b} \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{i,j \in b} \left\| \frac{1}{\sqrt{T}} F' e_j \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{B}{\sqrt{T} N^2} \sum_{b=1}^B M_b \left(\sum_{i \in b} \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{j \in b} \left\| \frac{1}{\sqrt{T}} F' e_j \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{B \max_b \{M_b\}}{N^2} \left(\sum_{b=1}^B \sum_{i \in b} \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{b=1}^B \sum_{j \in b} \left\| \frac{1}{\sqrt{T}} F' e_j \right\|^2 \right)^{\frac{1}{2}} \\
& = \frac{B \max_b \{M_b\}}{N} \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{1}{\sqrt{T}} F' e_j \right\|^2 \right)^{\frac{1}{2}} \\
& = O(1) O_p(1) O_p \left(\frac{1}{\sqrt{T}} \right)
\end{aligned}$$

where the $O(1)$ term follows from Assumption 4(g), the $O_p(1)$ follows from Assumption 2, and the $O_p \left(\frac{1}{\sqrt{T}} \right)$ term follows from Assumption 3(a). Next, note that term II can be bounded by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \left\| \frac{1}{T} F' \hat{F}^k \right\| \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\| \\
& \leq \frac{1}{\sum_{b=1}^B M_b^2} \left\| \frac{1}{T} F' \hat{F}^k \right\| \left(\sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{b=1}^B \sum_{i,j \in b} \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{B}{N^2} \left\| \frac{1}{T} F' \hat{F}^k \right\| \left(\sum_{b=1}^B M_b \sum_{i \in b} \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{b=1}^B M_b \sum_{j \in b} \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{B \max_b \{M_b\}}{N} \left\| \frac{1}{T} F' \hat{F}^k \right\| \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\|^2 \right)^{\frac{1}{2}} \\
& = O(1) O_p(1) O_p(\delta_{NT}^{-1})
\end{aligned}$$

where $\left\| \frac{1}{T} F' \hat{F}^k \right\|$ is $O_p(1)$ due to the fact that

$$\left\| \frac{1}{T} F' \hat{F}^k \right\| \leq \left(\frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \frac{1}{T} \sum_{t=1}^T \|\hat{F}_t^k\|^2 \right)^{\frac{1}{2}} = O_p(1)$$

and the $O_p(\delta_{NT}^{-1})$ term follows from Lemma C1(i). **Q.E.D.**

Lemma C3:

(i) Under Assumptions 1–4 and 8 and for $k < r$, $\frac{1}{M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |z_{ij}^k|$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} |z_{ij}^k|$ converge to the same limit c_1^k and $c_1^k > 0$.

(ii) Under Assumptions 1–4 and for $k < r$, there exists $c_1^\dagger > 0$ such that

$\text{plim} \inf_{N,T \rightarrow \infty} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |z_{ij}^k| \geq c_1^\dagger$ and $\text{plim} \inf_{N,T \rightarrow \infty} \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} |z_{ij}^k| \geq c_1^\dagger$.

Proof:

(i) First, we show that $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} |z_{ij}^k|$ converges to the same limit c_1^k because

$$\begin{aligned} & \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} |z_{ij}^k| \\ = & \frac{N^2}{\left(N^2 - \sum_{b=1}^B M_b^2\right) N^2} \sum_{i=1}^N \sum_{j=1}^N |z_{ij}^k| - \frac{\sum_{b=1}^B M_b^2}{\left(N^2 - \sum_{b=1}^B M_b^2\right) \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |z_{ij}^k| \end{aligned} \quad (1.2)$$

where the first term converges to c_1^k and the second term converges to zero by Assumption 8(a) and the fact that $\sum_{b=1}^B M_b^2$ is $O(N^{1+\alpha})$.

Next, we show that $c_1^k > 0$. Note that the absolute value is always non-negative, so we will only prove that $c_1^k \neq 0$. Let D be some non-singular matrix such that $\Sigma_F = DD'$. z_{ij}^k can be re-written as

$$\begin{aligned} & \lambda_i' \left(\Sigma_F - \Sigma_F H_0^k \left(H_0^{k'} \Sigma_F H_0^k \right)^{-1} H_0^{k'} \Sigma_F \right) \lambda_j \\ = & \lambda_i' D \left[I_r - D' H_0^k \left(H_0^{k'} D D' H_0^k \right)^{-1} H_0^{k'} D \right] D' \lambda_j \\ = & \lambda_i' D B^k D' \lambda_j \end{aligned}$$

where $B^k \equiv I_r - D' H_0^k \left(H_0^{k'} D D' H_0^k \right)^{-1} H_0^{k'} D$.

Note that $z_{ij}^k = 0$ if there exists a $k \times 1$ vector $\tilde{\lambda}_i$ (or $\tilde{\lambda}_j$) such that $D' \lambda_i = D' H_0^k \tilde{\lambda}_i$ (or $D' \lambda_j = D' H_0^k \tilde{\lambda}_j$), i.e. $\lambda_i = H_0^k \tilde{\lambda}_i$ (or $\lambda_j = H_0^k \tilde{\lambda}_j$) because D is non-singular.

Now, suppose $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} |z_{ij}^k| \rightarrow_p c_1^k = 0$, then it implies that only $o(N^2)$ many $|z_{ij}^k|$ can be non-zero. When both λ_i and λ_j cannot be expressed as $\lambda_i \neq H_0^k \tilde{\lambda}_i$ and $\lambda_j \neq H_0^k \tilde{\lambda}_j$ for any $k \times 1$ vectors $\tilde{\lambda}_i$ and $\tilde{\lambda}_j$, $z_{ij}^k = 0$ only if $B^k D' \lambda_i$ is orthogonal to $B^k D' \lambda_j$. But an r -dimensional vector space has at most r many non-zero $B^k D' \lambda_i$'s such that they are pairwise orthogonal. Thus, it must follow that all λ_i 's can be expressed as $\lambda_i = H_0^k \tilde{\lambda}_i$ except $o(N)$ many λ_i 's. This implies that the rank of $\text{plim}_{N \rightarrow \infty} (\Lambda' \Lambda / N)$ will be $k < r$, which contradicts Assumption 2. Thus, c_1 has to

be positive.

(ii) The proof is similar to that of part (i). Let \tilde{N} denote the number of λ_i 's that cannot be expressed as $\lambda_i = H_0^k \tilde{\lambda}_i$ for some $k \times 1$ vector $\tilde{\lambda}_i$. \tilde{N} has to be proportional to N , otherwise $\Lambda' \Lambda / N$ will have asymptotic rank less than r . This implies that the number of non-zero $|z_{ij}|$ must be proportional to N^2 . Hence, $\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N |z_{ij}|$ will be asymptotically bounded away from zero. Note that the second term in Eq. (1.2) is $o_p(1)$ and that $\sum_{b=1}^B M_b^2$ is $O(N^{1+\alpha})$, Eq. (1.2) implies that $\frac{1}{N^2 - \sum M_b^2} \sum_{i \in b, j \in b', b \neq b'} |z_{ij}|$ is bounded away from zero as N and $T \rightarrow \infty$.

Also, let \tilde{M}_b denote the number of λ_i 's that cannot be expressed as $\lambda_i = H_0^k \tilde{\lambda}_i$ for $i \in b$. Consider a block b that has \tilde{M}_b proportional to N^α . We claim that the number of such blocks has to be proportional to B , otherwise \tilde{N} will be $o(N)$ (recall that $N^{1-\alpha} W_0 \leq B \leq N^{1-\alpha} W$). Thus, the number of non-zero $|z_{ij}|$ for $i, j \in b$ and $b = 1, \dots, B$ has to be proportional to $N^{1+\alpha}$. Thus, $\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i, j \in b} |z_{ij}|$ is bounded away from zero as N and $T \rightarrow \infty$. **Q.E.D.**

Lemma C4: Under Assumptions 1–4 and 6–7 and for $k < r$,

$$\begin{aligned} & \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i, j \in b} \left(\left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^k} F \lambda_j \right| - |z_{ij}| \right) \rightarrow_p 0 \\ & \frac{1}{\sum M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left(\left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^k} F \lambda_j \right| - |z_{ij}| \right) \rightarrow_p 0 \end{aligned}$$

Proof:

To conserve space, we will provide proof of the within-block statement, and the second result on the between-block can be proved similarly.

$$\begin{aligned} & \left| \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i, j \in b} \left(\left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^k} F \lambda_j \right| - |z_{ij}| \right) \right| \\ & \leq \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i, j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^k} F \lambda_j - z_{ij} \right| \\ & \leq \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i, j \in b} \left(\left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^k} F \lambda_j - \frac{1}{T} \lambda_i' F' M_{FH^k} F \lambda_j \right| \right. \\ & \quad \left. + \left| (\lambda_j' \otimes \lambda_i') \text{vec} \left(\frac{1}{T} F' M_{FH^k} F - A_0 \right) \right| \right) \\ & = I + II \end{aligned}$$

For term I ,

$$\begin{aligned}
& \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' (M_{\tilde{F}^k} - M_{FH^k}) F \lambda_j \right| \\
&= \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' \left[(\tilde{F}^k - FH^k) (\tilde{F}^{k'} \tilde{F}^k)^{-1} (\tilde{F}^k - FH^k)' \right. \right. \\
&\quad \left. \left. + (\tilde{F}^k - FH^k) (\tilde{F}^{k'} \tilde{F}^k)^{-1} H^{k'} F' + FH^k (\tilde{F}^{k'} \tilde{F}^k)^{-1} (\tilde{F}^k - FH^k)' \right. \right. \\
&\quad \left. \left. + FH^k \left((\tilde{F}^{k'} \tilde{F}^k)^{-1} - (H^{k'} F' FH^k)^{-1} \right) H^{k'} F' \right] F \lambda_j \right| \\
&= \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |a_{ij} + b_{ij} + c_{ij} + d_{ij}|
\end{aligned}$$

Let $D_k = \tilde{F}^{k'} \tilde{F}^k / T$ and $D_{0,k} = H^{k'} F' FH^k / T$. We will use the fact that $\|D_k^{-1}\| = O_p(1)$ for $k \leq r$, which is proved by Bai and Ng (2002) on page 216. First, term $\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |a_{ij}|$ can be re-written as

$$\begin{aligned}
& \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T^2} \sum_{t=1}^T \lambda_i' F_t (\tilde{F}_t^{k'} - F_t' H^k) \left(\tilde{F}^{k'} \tilde{F}^k / T \right)^{-1} \sum_{s=1}^T (\tilde{F}_s^k - H^{k'} F_s) F_s' \lambda_j \right| \\
&= \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T (\tilde{F}_t^{k'} - F_t' H^k) D_k^{-1} (\tilde{F}_s^k - H^{k'} F_s) F_s' \lambda_j \lambda_i' F_t \right| \\
&\leq \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \|\tilde{F}_t^{k'} - F_t' H^k\|^2 \|\tilde{F}_s^k - H^{k'} F_s\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^T \|F_t\|^2 \|F_s\|^2 \right)^{\frac{1}{2}} \|\lambda_i\| \|\lambda_j\| \|D_k^{-1}\| \\
&= \frac{1}{T} \sum_{t=1}^T \|\tilde{F}_t^{k'} - F_t' H^k\|^2 \frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \|D_k^{-1}\| \\
&= O_p(\delta_{NT}^{-2}) O_p(1)
\end{aligned}$$

where the $O_p(\delta_{NT}^{-2})$ follows from Lemma A1(i) and the $O_p(1)$ term follows from Assumptions 1 and 2. Second, term $\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |b_{ij}|$ is equal to

$$\begin{aligned}
& \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T^2} \sum_{t=1}^T \lambda_i' F_t (\tilde{F}_t^{k'} - F_t' H^k) D_k^{-1} \sum_{s=1}^T H^{k'} F_s F_s' \lambda_j \right| \\
&\leq \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \frac{1}{T^2} \left\| \sum_{t=1}^T F_t (\tilde{F}_t^{k'} - F_t' H^k) \sum_{s=1}^T H^{k'} F_s F_s' \right\| \|D_k^{-1}\| \\
&\leq \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \left\| \frac{1}{T} \sum_{t=1}^T F_t (\tilde{F}_t^{k'} - F_t' H^k) \right\| \frac{1}{T} \sum_{s=1}^T \|F_s\|^2 \|H^k\| \|D_k^{-1}\| \\
&= O_p \left(\frac{1}{\delta_{NT}^2} \right)
\end{aligned}$$

where we use the facts that $\|F'(\tilde{F}^k - FH^k)/T\| \leq \|F'(\tilde{F}^r - FH^r)/T\| = O_p(\delta_{NT}^{-2})$ for $k \leq r$

by Lemma B2 of Bai (2003) and that $\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| = O_p(1)$ by Assumption 2. Using a similar argument, it can be proved that $\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |c_{ij}|$ is also $O_p(\delta_{NT}^{-2})$. Lastly, $\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |d_{ij}|$ is bounded by

$$\begin{aligned} & \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \|F'F/T\|^2 \|H^k\|^2 \|D_k^{-1} - D_{0,k}^{-1}\| \\ &= \begin{cases} O_p\left(\frac{1}{\delta_{NT}}\right) & \text{under Assumptions 1 - 4} \\ O_p\left(\frac{1}{\delta_{NT}^2}\right) & \text{under Assumptions 1 - 4 and 6 - 7} \end{cases} \end{aligned} \quad (1.3)$$

Note that the convergence rate depends on $\|D_k^{-1} - D_{0,k}^{-1}\|$: under Assumptions 1-4, $\|D_k^{-1} - D_{0,k}^{-1}\| = O_p(\delta_{NT}^{-1})$ for $k \leq r$ (see page 216, Bai and Ng (2002)); under Assumptions 1-4 and 6-7, Lemmas B2 and B3 of Bai (2003) imply that $\|D_k - D_{0,k}\| \leq \|(\tilde{F}'\tilde{F}^r - H'F'FH^r)/T\| = O_p(\delta_{NT}^{-2})$ for $k \leq r$, so $\|D_k^{-1} - D_{0,k}^{-1}\| = O_p(\delta_{NT}^{-2})$ because $D_{0,k}$ (for $k \leq r$) converges to a positive definite matrix (see page 216, Bai and Ng (2002)). Thus, combining the above results, we have term $I = \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |a_{ij} + b_{ij} + c_{ij} + d_{ij}| = o_p(1)$.

For term II,

$$\begin{aligned} & \frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| (\lambda_j' \otimes \lambda_i') \text{vec} \left(\frac{1}{T} F' M_{FH^k} F - A_0 \right) \right| \\ & \leq \left(\frac{1}{\sum M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_j' \otimes \lambda_i'\| \right) \left\| \text{vec} \left(\left[\frac{F'F}{T} - \frac{F'FH^k}{T} \left(\frac{H^k F' F H^k}{T} \right)^{-1} \frac{H^k F' F}{T} \right] - A_0 \right) \right\| \\ & \rightarrow_p 0 \end{aligned}$$

because $\frac{F'F}{T} - \frac{F'FH^k}{T} \left(\frac{H^k F' F H^k}{T} \right)^{-1} \frac{H^k F' F}{T} \rightarrow_p A_0$ by the definition of A_0 . **Q.E.D.**

Lemma C5: Under Assumptions 1-4 and 6-7,

- (i) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} e_j \right| = O_p(\delta_{NT}^{-2})$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} e_j \right| = O_p(\delta_{NT}^{-2})$.
- (ii) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} F \lambda_j \right| = O_p(\delta_{NT}^{-2})$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} F \lambda_j \right| = O_p(\delta_{NT}^{-2})$.
- (iii) $\frac{1}{T \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} (|e_i' M_{\tilde{F}^r} e_j| - |e_i' e_j|) = O_p(\delta_{NT}^{-2})$ and $\frac{1}{(N^2 - \sum_{b=1}^B M_b^2) T} \sum_{i \in b, j \in b', b \neq b'} (|e_i' M_{\tilde{F}^r} e_j| - |e_i' e_j|) = O_p(\delta_{NT}^{-2})$.

Proof:

To conserve space, we will provide proof of the within-block results, and the results on the between-block can be proved similarly.

(i) Using the fact that $M_{FH^r}F = 0$ for , we have

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} e_j \right| \\
&= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' (M_{\tilde{F}^r} - M_{FH^r}) e_j \right| \\
&\leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T^2} \lambda_i' F' (\tilde{F}^r - FH^r) D_r^{-1} (\tilde{F}^r - FH^r)' e_j \right| + \left| \frac{1}{T^2} \lambda_i' F' (\tilde{F}^r - FH^r) D_r^{-1} H^r' F' e_j \right| \right. \\
&\quad \left. + \left| \frac{1}{T^2} \lambda_i' F' FH^r D_r^{-1} (\tilde{F}^r - FH^r)' e_j \right| + \left| \frac{1}{T^2} \lambda_i' F' FH^r (D_r^{-1} - D_{0,r}^{-1}) H^r' F' e_j \right| \right) \\
&= I + II + III + IV
\end{aligned}$$

First, term I can be bounded by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \left\| \frac{(\tilde{F}^r - FH^r)' e_j}{T} \right\| \left\| \frac{F'(\tilde{F}^r - FH^r)}{T} \right\| \|D_r^{-1}\| \\
&\leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \left(\frac{1}{T} \sum_{t=1}^T e_{jt}^2 \right)^{\frac{1}{2}} \left(\frac{1}{T} \sum_{t=1}^T \|\tilde{F}^r - FH^r\|^2 \right)^{\frac{1}{2}} \left\| \frac{F'(\tilde{F}^r - FH^r)}{T} \right\| \|D_r^{-1}\|
\end{aligned}$$

By Lemma A1(i) and facts that $\|D_r^{-1}\| = O_p(1)$ and that $\|F'(\tilde{F}^r - FH^r)/T\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$ by Lemma B2 of Bai (2003), the above can be further bounded by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \left(\sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\sum_{b=1}^B \sum_{i,j \in b} \frac{1}{T} \sum_{t=1}^T e_{jt}^2 \right)^{\frac{1}{2}} O_p\left(\frac{1}{\delta_{NT}}\right) O_p\left(\frac{1}{\delta_{NT}^2}\right) O_p(1) \\
&\leq \frac{B \max_b \{M_b\}}{N} \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{NT} \sum_{j=1}^N \sum_{t=1}^T e_{jt}^2 \right)^{\frac{1}{2}} O_p\left(\frac{1}{\delta_{NT}^3}\right) \\
&= O_p\left(\frac{1}{\delta_{NT}^3}\right)
\end{aligned}$$

where the last equation follows from Assumptions 2(a), 4(a) and 4(g). Secondly, term II can be bounded by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \left\| \frac{H^r' F' e_j}{T} \right\| \left\| \frac{F'(\tilde{F}^r - FH^r)}{T} \right\| \|D_r^{-1}\| \\
&\leq \frac{B \max_b \{M_b\}}{N\sqrt{T}} \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{F' e_j}{\sqrt{T}} \right\|^2 \right)^{\frac{1}{2}} \|H^r\| \left\| \frac{F'(\tilde{F}^r - FH^r)}{T} \right\| \|D_r^{-1}\| \\
&= O_p\left(\frac{1}{\delta_{NT}^2 \sqrt{T}}\right)
\end{aligned}$$

by Assumptions 2(a), 3(a) and 4(g) and Lemma B2 of Bai (2003). Next, term *III* can be bounded by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \left\| \frac{(\tilde{F}^r - FH^r)' e_j}{T} \right\| \|H^r\| \|D_r^{-1}\| \left\| \frac{F'F}{T} \right\| \\
& \leq \frac{B \max_b \{M_b\}}{N} \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{(\tilde{F}^r - FH^r)' e_j}{T} \right\|^2 \right)^{\frac{1}{2}} O_p(1) \\
& = O_p \left(\frac{1}{\delta_{NT}^2} \right)
\end{aligned}$$

where the last equation follows from Lemma C1(ii). Finally, term *IV* can be bounded by

$$\begin{aligned}
& \frac{1}{\sqrt{T} \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \left\| \frac{F' e_j}{\sqrt{T}} \right\| \|H^r\|^2 \left\| \frac{F'F}{T} \right\| \|D_r^{-1} - D_{0,r}^{-1}\| \\
& \leq \frac{B \max_b \{M_b\}}{\sqrt{T} N} \left(\frac{1}{N} \sum_{i=1}^N \|\lambda_i\|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{F' e_j}{\sqrt{T}} \right\|^2 \right)^{\frac{1}{2}} \|H^r\|^2 \left\| \frac{F'F}{T} \right\| \|D_r^{-1} - D_{0,r}^{-1}\| \\
& = \frac{1}{\sqrt{T}} O_p(1) O_p \left(\frac{1}{\delta_{NT}} \right)
\end{aligned}$$

where we use the fact that $\|D_r^{-1} - D_{0,r}^{-1}\| = O_p(\delta_{NT}^{-1})$ under Assumptions 1–4, which is proved by Bai and Ng (2002) on page 216. Thus, $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} e_j \right| = O_p(\delta_{NT}^{-2})$.

(ii) Using the fact that $M_{FH^r} F = 0$ for , we have

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} F \lambda_j \right| \\
& = \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' (M_{\tilde{F}^r} - M_{FH^r}) F \lambda_j \right| \\
& \leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T^2} \lambda_i' F' (\tilde{F}^r - FH^r) D_r^{-1} (\tilde{F}^r - FH^r)' F \lambda_j \right| + \left| \frac{1}{T^2} \lambda_i' F' (\tilde{F}^r - FH^r) D_r^{-1} H^r' F' F \lambda_j \right| \right. \\
& \quad \left. + \left| \frac{1}{T^2} \lambda_i' F' F H^r D_r^{-1} (\tilde{F}^r - FH^r)' F \lambda_j \right| + \left| \frac{1}{T^2} \lambda_i' F' F H^r (D_r^{-1} - D_{0,r}^{-1}) H^r' F' F \lambda_j \right| \right) \\
& = I + II + III + IV
\end{aligned}$$

Term *I* can be bounded by

$$\left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \right) \left\| \frac{F'(\tilde{F}^r - FH^r)}{T} \right\|^2 \|D_r^{-1}\| = O_p \left(\frac{1}{\delta_{NT}^4} \right)$$

where we use the fact that $\|F'(\tilde{F}^r - FH^r)/T\| = O_p(\delta_{NT}^{-2})$ by Lemma B2 of Bai (2003). Term *II*

and *III* can be bounded by

$$\left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \right) \left\| \frac{F'(\tilde{F}^r - FH^r)}{T} \right\| \left\| \frac{F'F}{T} \right\| \|H^r\| \|D_r^{-1}\| = O_p \left(\frac{1}{\delta_{NT}^2} \right)$$

where we use Lemma B2 of Bai (2003) again. Term *IV* can be bounded by

$$\left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \right) \left\| \frac{F'F}{T} \right\|^2 \|H^r\|^2 \|D_r^{-1} - D_{0,r}^{-1}\| = O_p \left(\frac{1}{\delta_{NT}^2} \right)$$

where we use the fact that $\|D_r^{-1} - D_{0,r}^{-1}\| = O_p(\delta_{NT}^{-2})$, which is proved in Lemma C4 (see the proof of Eq. (1.3)). Thus, $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} F \lambda_j \right| = O_p(\delta_{NT}^{-2})$.

(iii) Note that $M_{\tilde{F}^k} = M_{\hat{F}^k}$, so we have

$$\begin{aligned} & \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} e_i' M_{\tilde{F}^r} e_j \right| - \left| \frac{1}{T} e_i' e_j \right| \right) \\ & \leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} e_i' M_{\tilde{F}^r} e_j \right| - \left| \frac{1}{T} e_i' e_j \right| \right) \\ & = \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e_i' P_{\tilde{F}^r} e_j \right| \\ & \leq \frac{B}{N^2} \sum_{b=1}^B \left(\sum_{i \in b} \left\| \frac{1}{T} e_i' \hat{F}^r \right\| \right)^2 \\ & \leq \left(\frac{B \max_b M_b}{N} \right) \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' \hat{F}^r \right\|^2 \\ & = O_p(1) O_p \left(\frac{1}{\delta_{NT}^2} \right) \end{aligned}$$

where the last equation follows from Assumption 4(g) and Lemma C1(i). **Q.E.D.**

Lemma C6: Under Assumptions 1–3 and 4', and for $k \leq r$, $N^{-1} \sum_{i=1}^N \left\| T^{-1} e_i' \hat{F}^k \right\|^2 = O_p(\phi_{NT}^{-2})$.

Proof:

The proof of this lemma is similar to that of Lemma C1(i). Again, Eq. (1.1) implies that $\rho_k(XX'/NT)$ is positive for $k \leq r$. V_{NT}^k is non-singular for $k \leq r$ and $(V_{NT}^k)^{-1} = O_p(1)$ because it is a diagonal matrix consisting of first k eigenvalues of XX'/NT . Also, $\hat{F}^k = \tilde{F}^k (V_{NT}^k)^{-1}$ by the

definition of \tilde{F}^k . Thus, we have

$$\begin{aligned}
\frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e'_i \hat{F}^k \right\|^2 &= \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e'_i \tilde{F}^k (V_{NT}^k)^{-1} \right\|^2 \\
&= \frac{1}{N} \sum_{i=1}^N \left\| \left[\frac{1}{T} e'_i (\tilde{F}^k - FH^k) + \frac{1}{T} e'_i FH^k \right] (V_{NT}^k)^{-1} \right\|^2 \\
&\leq \left[\frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{T} e'_i (\tilde{F}^k - FH^k) \right\|^2 + \frac{2}{N} \sum_{i=1}^N \left\| \frac{1}{T} e'_i FH^k \right\|^2 \right] \|(V_{NT}^k)^{-1}\|^2 \\
&\leq \left[\left(\frac{2}{N} \sum_{i=1}^N \sum_{t=1}^T e_{it}^2 \right) \frac{1}{T} \sum_{t=1}^T \|\tilde{F}^k - FH^k\|^2 + \frac{2}{TN} \sum_{i=1}^N \left\| \frac{1}{\sqrt{T}} e'_i F \right\|^2 \|H^k\|^2 \right] \|(V_{NT}^k)^{-1}\|^2 \\
&= \left[O_p \left(\frac{1}{\phi_{NT}^2} \right) + O_p \left(\frac{1}{T} \right) \right] O_p(1)
\end{aligned}$$

where the last equation follows from Lemma A3(i). **Q.E.D.**

Lemma C7: Under Assumptions 1–3 and 4' and for $k \leq r$,

(i) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\tilde{F}^k} e_j \right| = O_p(1)$ and there exists an $\omega_k > 0$, such that

$$\text{plim}_{N,T \rightarrow \infty} \inf \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\tilde{F}^k} e_j \right| = \omega_k.$$

(ii) $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} e'_i M_{\tilde{F}^k} e_j \right| \rightarrow_p 0$.

(iii) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda'_i F' M_{\tilde{F}^k} e_j \right| \rightarrow_p 0$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda'_i F' M_{\tilde{F}^k} e_j \right| \rightarrow_p 0$.

Proof:

(i) First, we show that $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i P_{\tilde{F}^k} e_j \right| = O_p(\phi_{NT}^{-2})$. Note that $\hat{F}^{k'} \hat{F}^k / T = I_k$, so we have

$$\begin{aligned}
\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i P_{\tilde{F}^k} e_j \right| &= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T^2} e'_i \hat{F}^k \hat{F}^{k'} e_j \right| \\
&\leq \frac{B \max_b \{M_b\}}{N^2} \sum_{i=1}^N \left\| \frac{1}{T} e'_i \hat{F}^k \right\|^2 = O_p \left(\frac{1}{\phi_{NT}^2} \right) \quad (1.4)
\end{aligned}$$

where the last equation follows from Lemma C6. Also, note that $M_{\tilde{F}^k} = M_{\hat{F}^k}$, so we have

$$\begin{aligned}
&\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\tilde{F}^k} e_j \right| \\
&\leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} \sum_{t=1}^T (e_{it} e_{jt} - \tau_{ij,t}) \right| + \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| + \left| \frac{1}{T^2} e'_i \hat{F}^k \hat{F}^{k'} e_j \right| \right) \\
&\leq O_p \left(\frac{1}{\sqrt{T}} \right) + O(1) + O_p \left(\frac{1}{\phi_{NT}^2} \right)
\end{aligned}$$

where the $O_p(1/\sqrt{T})$ term follows from Assumption 4'(f) and the $O(1)$ term follows from Assumption 4'(c). Next, we show that $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\hat{F}^k} e_j \right|$ is bounded away from zero in probability.

$$\begin{aligned} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i M_{\hat{F}^k} e_j \right| &\geq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} e'_i e_j \right| - \left| \frac{1}{T} e'_i P_{\hat{F}^k} e_j \right| \right) \\ &= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i e_j \right| + O_p \left(\frac{1}{\phi_{NT}^2} \right) \end{aligned}$$

Note that $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i e_j \right|$ is bounded away from zero in probability, because

$$\begin{aligned} &\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} e'_i e_j \right| \\ &= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} \sum_{t=1}^T e_{it} e_{jt} \right| - \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) + \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \end{aligned}$$

where the first term is $o_p(1)$ by Assumption 4'(f). For the second term, we have

$$\begin{aligned} \text{plim}_{N,T \rightarrow \infty} \inf \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| &\geq \text{plim}_{N,T \rightarrow \infty} \inf \left(\frac{\sum_{b=b_1}^{b_m} M_b^2}{\sum_{b=1}^B M_b^2} \right) \frac{1}{\sum_{b=b_1}^{b_m} M_b^2} \sum_{b \in \{b_1, \dots, b_m\}} \sum_{i,j \in b} \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \\ &\geq \text{plim}_{N,T \rightarrow \infty} \inf \left(\frac{\sum_{b=b_1}^{b_m} M_b^2}{\sum_{b=1}^B M_b^2} \right) \frac{1}{\sum_{b=b_1}^{b_m} M_b^2} \sum_{b \in \{b_1, \dots, b_m\}} M_b^2 W_0 \\ &\geq \text{plim}_{B \rightarrow \infty} \inf \left(\frac{\sum_{b=b_1}^{b_m} M_b^2}{\sum_{b=1}^B M_b^2} \right) W_0 \\ &\geq \text{plim}_{B \rightarrow \infty} \inf \frac{b_m W_0^3}{B W^2} = \frac{c_0^\dagger W_0^3}{W^2} > 0 \end{aligned}$$

by Assumptions 4'(g) and 4'(h) that $M_b \leq W N^\alpha$ for $b = 1, \dots, B$ and $W_0 N^\alpha \leq M_b \leq W N^\alpha$ for $b = b_1, \dots, b_m$ with $\text{plim}_{B \rightarrow \infty} \inf m/B = c_0^\dagger \in (0, 1]$. Thus,

$$\text{plim}_{N,T \rightarrow \infty} \inf \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} e'_i e_j \right| - \left| \frac{1}{T} e'_i P_{\hat{F}^k} e_j \right| \right) > 0,$$

which leads to the desired result.

(ii) Using the fact that $M_{\hat{F}^k} = M_{\tilde{F}^k}$ and $\hat{F}^{k'} \hat{F}^k / T = I_k$, we have

$$\begin{aligned}
& \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} e_i' M_{\tilde{F}^k} e_j \right| \\
& \leq \left(\frac{N^2}{N^2 - \sum_{b=1}^B M_b^2} \right) \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left[\left(\left| \frac{1}{T} e_i' e_j \right| - \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) + \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| + \left| \frac{1}{T^2} e_i' \hat{F}^k \hat{F}^{k'} e_j \right| \right] \\
& \leq O(1) \left[\frac{1}{N^2 \sqrt{T}} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{\sqrt{T}} \sum_{t=1}^T (e_{it} e_{jt} - \tau_{ij,t}) \right| + \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N |\tau_{ij}| + \frac{1}{N} \sum_{i=1}^N \left\| \frac{1}{T} e_i' \hat{F}^k \right\|^2 \right] \\
& \leq O_p \left(\frac{1}{\sqrt{T}} \right) + O \left(\frac{1}{N^{1-\alpha}} \right) + O_p \left(\frac{1}{\phi_{NT}^2} \right)
\end{aligned}$$

where the $O_p \left(\frac{1}{\sqrt{T}} \right)$ follows from Assumption 4'(f), the $O \left(\frac{1}{N^{1-\alpha}} \right)$ term follows from Assumption 4'(c), and the $O_p \left(\frac{1}{\phi_{NT}^2} \right)$ follows from Lemma C6.

(iii) Using the fact that $M_{\tilde{F}^k} = M_{\hat{F}^k}$ and $\hat{F}^{k'} \hat{F}^k / T = I_k$, we have

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^k} e_j \right| \\
& \leq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} \lambda_i' F' e_j \right| + \left| \frac{1}{T^2} \lambda_i' F' \hat{F}^k \hat{F}^{k'} e_j \right| \right) \\
& = I + II
\end{aligned}$$

Term I can be bounded by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \left(\sum_{b=1}^B \sum_{i,j \in b} |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{b=1}^B \sum_{i,j \in b} \left\| \frac{1}{T} F' e_j \right\|^2 \right)^{\frac{1}{2}} \\
& \leq \frac{B \max_b M_b}{N} \left(\frac{1}{N} \sum_{i=1}^N |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{1}{\sqrt{T}} F' e_j \right\|^2 \right)^{\frac{1}{2}} \frac{1}{\sqrt{T}} \\
& = O_p \left(\frac{1}{\sqrt{T}} \right)
\end{aligned}$$

where the last equation follows from Assumptions 2(a), 3(a), and 4'(g). Term II can be bounded

by

$$\begin{aligned}
& \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |\lambda_i| \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\| \left\| \frac{1}{T} F' \hat{F}^k \right\| \\
& \leq \frac{1}{\sum_{b=1}^B M_b^2} \left\| \frac{1}{T} F' \hat{F}^k \right\| \left(\sum_{b=1}^B \sum_{i,j \in b} |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\sum_{b=1}^B \sum_{i,j \in b} \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\|^2 \right)^{\frac{1}{2}} \\
& = \frac{B \max_b M_b}{N} \left\| \frac{1}{T} F' \hat{F}^k \right\| \left(\frac{1}{N} \sum_{i=1}^N |\lambda_i|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{j=1}^N \left\| \frac{1}{T} \hat{F}^{k'} e_j \right\|^2 \right)^{\frac{1}{2}} \\
& = O_p(1) O_p \left(\frac{1}{\phi_{NT}^2} \right)
\end{aligned}$$

where the last equation follows from Lemma C6. The proof of $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^k} e_j \right| = o_p(1)$ can be proved similarly. **Q.E.D.**

Lemma C8: Under Assumptions 1–3 and 4',

(i) For $k < r$, there exists an $\omega_k > 0$, such that

$$\text{plim}_{N,T \rightarrow \infty} \inf \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^k} F \lambda_j \right| \geq \omega_k$$

(ii) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^r} F \lambda_j \right| \rightarrow_p 0$ and $\frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^r} F \lambda_j \right| \rightarrow_p 0$.

Proof:

(i) First note that $M_{\hat{F}^k}$ can be replaced by $M_{\hat{F}^k}$. Now, consider

$$\begin{aligned}
& \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^k} F \lambda_j \right| \\
& = \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left| \lambda_i' \left[\frac{F' F}{T} - \frac{F' \hat{F}^k}{T} \left(\frac{\hat{F}^{k'} \hat{F}^k}{T} \right)^{-1} \frac{\hat{F}^{k'} F}{T} \right] \lambda_j \right|
\end{aligned}$$

First, we will show that the above term is bounded away from zero by contradiction, and the argument is very similar to that of Lemma C3. Suppose not, then $\left[\frac{F' F}{T} - \frac{F' \hat{F}^k}{T} \left(\frac{\hat{F}^{k'} \hat{F}^k}{T} \right)^{-1} \frac{\hat{F}^{k'} F}{T} \right] \lambda_i$ can be non-zero for only $o(N)$ many i 's, which implies that

$$\frac{1}{N} \sum_{i=1}^N \lambda_i' \left[\frac{F' F}{T} - \frac{F' \hat{F}^k}{T} \left(\frac{\hat{F}^{k'} \hat{F}^k}{T} \right)^{-1} \frac{\hat{F}^{k'} F}{T} \right] \lambda_i \rightarrow_p 0 \tag{1.5}$$

However,

$$\frac{1}{N} \sum_{i=1}^N \lambda'_i \left[\frac{F'F}{T} - \frac{F'\hat{F}^k}{T} \left(\frac{\hat{F}^{k'}\hat{F}^k}{T} \right)^{-1} \frac{\hat{F}^{k'}F}{T} \right] \lambda_i = \text{tr} \left(\left[\frac{F'F}{T} - \frac{F'\hat{F}^k}{T} \left(\frac{\hat{F}^{k'}\hat{F}^k}{T} \right)^{-1} \frac{\hat{F}^{k'}F}{T} \right] \frac{1}{N} \sum_{i=1}^N \lambda_i \lambda'_i \right)$$

where $\frac{F'F}{T} - \frac{F'\hat{F}^k}{T} \left(\frac{\hat{F}^{k'}\hat{F}^k}{T} \right)^{-1} \frac{\hat{F}^{k'}F}{T}$ is positive semi-definite and non-zero as N and $T \rightarrow \infty$ because the column rank of \hat{F}^k is $k < r$. By Assumption 2(b), $N^{-1} \sum_{i=1}^N \lambda_i \lambda'_i$ converges to a positive definite matrix in probability, so the above term has to be bounded away from zero as N and $T \rightarrow \infty$, which contradicts Eq. (1.5). Thus, $\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{T} \lambda'_i F' M_{\hat{F}^k} F \lambda_j \right|$ is asymptotically bounded away from zero. For the between-block analog, note that

$$\begin{aligned} & \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} \left| \frac{1}{T} \lambda'_i F' M_{\hat{F}^k} F \lambda_j \right| \\ = & \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i=1}^N \sum_{j=1}^N \left| \frac{1}{T} \lambda'_i F' M_{\hat{F}^k} F \lambda_j \right| - \left(\frac{\sum_{b=1}^B M_b^2}{N^2 - \sum_{b=1}^B M_b^2} \right) \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda'_i F' M_{\hat{F}^k} F \lambda_j \right| \end{aligned}$$

where the second term is $o_p(1)$ and the first term has been shown to be bounded away from zero.

(ii) Using the fact that $M_{FH^r} F = 0$ for , we have

$$\begin{aligned} & \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda'_i F' M_{\tilde{F}^r} F \lambda_j \right| \\ = & \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda'_i F' (M_{\tilde{F}^r} - M_{FH^r}) F \lambda_j \right| \\ \leq & \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T^2} \lambda'_i F' (\tilde{F}^r - FH^r) D_r^{-1} (\tilde{F}^r - FH^r)' F \lambda_j \right| + \left| \frac{1}{T^2} \lambda'_i F' (\tilde{F}^r - FH^r) D_r^{-1} H^r' F' F \lambda_j \right| \right. \\ & \left. + \left| \frac{1}{T^2} \lambda'_i F' FH^r D_r^{-1} (\tilde{F}^r - FH^r)' F \lambda_j \right| + \left| \frac{1}{T^2} \lambda'_i F' FH^r (D_r^{-1} - D_{0,r}^{-1}) H^r' F' F \lambda_j \right| \right) \\ = & I + II + III + IV \end{aligned}$$

Term I can be bounded by

$$\left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \right) \left(\frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \right) \left(\frac{1}{T} \sum_{t=1}^T \|\tilde{F}^r - FH^r\|^2 \right) \|D_r^{-1}\| = O_p \left(\frac{1}{\phi_{NT}^2} \right)$$

where the equation follows from Lemma A3(i) and the fact that $\|D_r^{-1}\| = O_p(1)$, which will be proved below . Term II and III can be bounded by

$$\left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \right) \left(\frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \frac{1}{T} \sum_{t=1}^T \|\tilde{F}^r - FH^r\|^2 \right)^{\frac{1}{2}} \left\| \frac{F'F}{T} \right\| \|H^r\| \|D_r^{-1}\| = O_p \left(\frac{1}{\phi_{NT}} \right)$$

where we use Lemma A3(i) again. Term *IV* can be bounded by

$$\left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \|\lambda_i\| \|\lambda_j\| \right) \left\| \frac{F'F}{T} \right\|^2 \|H^r\|^2 \|D_r^{-1} - D_{0,r}^{-1}\| = O_p \left(\frac{1}{\phi_{NT}} \right)$$

Now, we will prove that $\|D_r^{-1} - D_{0,r}^{-1}\| = O_p(\phi_{NT}^{-1})$. First note that

$$\begin{aligned} \|D_r - D_{0,r}\| &= \frac{1}{T} \left\| \tilde{F}^{r'} \tilde{F}^r - H^{r'} F' F H^r \right\| \\ &\leq \left\| \frac{(\tilde{F}^r - F H^r)' (\tilde{F}^r - F H^r)}{T} \right\| + 2 \left\| \frac{(\tilde{F}^r - F H^r)' F H^r}{T} \right\| \\ &\leq \left\| \frac{(\tilde{F}^r - F H^r)' (\tilde{F}^r - F H^r)}{T} \right\| + 2 \left(\frac{1}{T} \sum_{t=1}^T \|F_t\|^2 \frac{1}{T} \sum_{t=1}^T \|\tilde{F}^r - F H^r\|^2 \right)^{\frac{1}{2}} \|H^r\| \\ &= O_p \left(\frac{1}{\phi_{NT}^2} \right) + O_p \left(\frac{1}{\phi_{NT}} \right) \end{aligned}$$

by Lemma A3(i). Also, note that $D_r = V_{NT}^r \hat{F}^{r'} \hat{F}^r V_{NT}^r / T = (V_{NT}^r)^2$, where V_{NT}^r is a diagonal matrix consisting of the first r non-zero eigenvalues of XX'/NT (see the proof in Lemma C6), so $\|D_r^{-1}\| = O_p(1)$. Also, $\|D_r - D_{0,r}\| = O_p(\phi_{NT}^{-1})$ implies that $D_{0,r}$ is diagonal and non-singular as N and $T \rightarrow \infty$ and that $\|D_{0,r}^{-1}\| = O_p(1)$. Thus, $D_r^{-1} - D_{0,r}^{-1} = D_{0,r}^{-1}(D_{0,r} - D_r)D_r^{-1}$ implies that $\|D_r^{-1} - D_{0,r}^{-1}\| = O_p(\phi_{NT}^{-1})$. Thus, $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \lambda_i' F' M_{\tilde{F}^r} F \lambda_j \right| \rightarrow_p 0$. The result for the between-block analog can be proved using a similar argument. **Q.E.D.**

Lemma D1:

- (i) Under Assumptions 1-5, $|b(k) - b(k+1)| = O_p(\delta_{NT}^{-2})$ for any fixed $k \geq r$.
- (ii) Under Assumptions 1-3 and 4'-5', $|b(k) - b(k+1)| = O_p(\phi_{NT}^{-2})$ for any fixed $k \geq r$.

Proof:

(i)

$$\begin{aligned}
|b(k) - b(k+1)| &= \frac{1}{N^2 - \sum_{b=1}^B M_b^2} \sum_{i \in b, j \in b', b \neq b'} (|X_i' M_{\hat{F}^k} X_j / T| - |X_i' M_{\hat{F}^{k+1}} X_j / T|) \\
&\leq \left(\frac{N^2}{N^2 - \sum_{b=1}^B M_b^2} \right) \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N ||X_i' M_{\hat{F}^k} X_j / T| - |X_i' M_{\hat{F}^{k+1}} X_j / T|| \\
&\leq O_p(1) \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N |X_i' (M_{\hat{F}^k} - M_{\hat{F}^{k+1}}) X_j / T| \\
&\leq O_p(1) \frac{1}{N^2 T} \sum_{i=1}^N \sum_{j=1}^N \|X_i' (M_{\hat{F}^k} - M_{\hat{F}^{k+1}})\| \| (M_{\hat{F}^k} - M_{\hat{F}^{k+1}}) X_j \| \\
&\leq O_p(1) \frac{1}{N^2 T} \left[\sum_{i=1}^N (X_i' (M_{\hat{F}^k} - M_{\hat{F}^{k+1}}) X_i)^{\frac{1}{2}} \right]^2 \\
&\leq O_p(1) \frac{1}{NT} \sum_{i=1}^N X_i' (M_{\hat{F}^k} - M_{\hat{F}^{k+1}}) X_i = O_p(\delta_{NT}^{-2})
\end{aligned}$$

where we use the fact that $(M_{\hat{F}^k} - M_{\hat{F}^{k+1}})^2 = M_{\hat{F}^k} - M_{\hat{F}^{k+1}}$ and Lemma A1(iv).

(ii) Given Lemma A3(iv), the result of part (ii) can be proved using an argument similar to that of part (i). **Q.E.D.**

Lemma D2:

(i) Under Assumptions 1–4 and 6–7, $\text{Prob}(a(r) < a(k)) \rightarrow 1$ for $k < r$ as N and $T \rightarrow \infty$.

(ii) Under Assumptions 1–3, 4' and 9, $\text{plim inf}_{N, T \rightarrow \infty} \tilde{a}(k) > 0$ for $k \leq r$ and $\tilde{a}(k)/b(k) = O_p(1)$ for $k < r$.

Proof:

(i) First, we show that $\text{Prob}(a(k) > a(r)) \rightarrow 1$ for $k < r$ as N and $T \rightarrow \infty$. For $k < r$, we have

$$\begin{aligned}
a(k) &= \frac{1}{T \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i, j \in b} |\lambda_i' F' M_{\hat{F}^k} F \lambda_j + \lambda_i' F' M_{\hat{F}^k} e_j + e_i' M_{\hat{F}^k} F \lambda_j + e_i' M_{\hat{F}^k} e_j| \\
&\geq \frac{1}{T \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i, j \in b} (|\lambda_i' F' M_{\hat{F}^k} F \lambda_j| - |\lambda_i' F' M_{\hat{F}^k} e_j + e_i' M_{\hat{F}^k} F \lambda_j + e_i' M_{\hat{F}^k} e_j|) \\
&= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i, j \in b} \left(\left| \frac{1}{T} \lambda_i' F' M_{\hat{F}^k} F \lambda_j \right| - |z_{ij}^k| + |z_{ij}^k| \right) + o_p(1) \\
&= \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i, j \in b} |z_{ij}^k| + o_p(1)
\end{aligned}$$

where the $o_p(1)$ term follows from Lemmas C2 and C4. Note that $\text{plim inf}_{N, T \rightarrow \infty} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i, j \in b} |z_{ij}^k| \geq c_1^\dagger > 0$ by Lemma C3(ii), so $a(k)$ is bounded away from zero in probability. Also, it is straightforward that $a(r) = o_p(1)$ by Lemmas C2 and C5(ii). Therefore, $\text{Prob}(a(k) > a(r)) \rightarrow 1$ for $k < r$ as N and $T \rightarrow \infty$.

(ii) Recall that $a(k) = O_p(1)$ for $k \leq r$, that $b(k)$ is bounded away from zero in probability for $k < r$, that $a(r)$ is bounded away from zero in probability, and that $b(r) \rightarrow_p 0$. We just need to prove that $a(k)$ is bounded away from zero for $k < r$ as N and T go to infinity. By the definition of $a(k)$, we have

$$\begin{aligned}
a(k) &= \frac{1}{T \sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \lambda_i' F' M_{\hat{F}^k} F \lambda_j + \lambda_i' F' M_{\hat{F}^k} e_j + e_i' M_{\hat{F}^k} F \lambda_j + e_i' e_j - e_i' P_{\hat{F}^k} e_j \right| \\
&\geq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| \frac{1}{T} \lambda_i' F' M_{F H^k} F \lambda_j + \frac{1}{T} e_i' e_j \right| \right. \\
&\quad \left. - \left| \frac{1}{T} \lambda_i' F' (P_{F H^k} - P_{\hat{F}^k}) F \lambda_j + \frac{1}{T} \lambda_i' F' M_{\hat{F}^k} e_j + \frac{1}{T} e_i' M_{\hat{F}^k} F \lambda_j - \frac{1}{T} e_i' P_{\hat{F}^k} e_j \right| \right) \\
&\geq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| - \left| \frac{1}{T} \lambda_i' F' M_{F H^k} F \lambda_j - z_{ij}^k \right| - \left| \frac{1}{T} (e_i' e_j - \sum_{t=1}^T \tau_{ij,t}) \right| \right) + o_p(1) \tag{1.6} \\
&\geq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| + o_p(1) \tag{1.7}
\end{aligned}$$

where the $o_p(1)$ term in (1.6) follows from Lemma C7(iii) and Eq. (1.4) in the Supplemental Technical Appendix, and the $o_p(1)$ term in (1.7) follows from Assumptions 1 and 4'(f). By Assumption 9, we derive $\text{plim inf}_{N,T \rightarrow \infty} a(k) > 0$ for $k < r$. Also, recall that $a(r)$ is bounded away from zero. By the definition of $\tilde{a}(k)$, we have $\text{plim inf}_{N,T \rightarrow \infty} \tilde{a}(k) > 0$ for $k \leq r$. Also, recall that $a(k) = O_p(1)$ for $k \leq r$ and $b(k)$ is bounded away from zero as N and $T \rightarrow \infty$ for $k < r$, so we have $\tilde{a}(k) = O_p(1)$ and $\tilde{a}(k)/b(k) = O_p(1)$ for $k < r$. **Q.E.D.**

2 Preliminary Conditions for Assumptions 8 and 9

In this section, we provide some sufficient conditions for Assumption 8 to hold. These conditions are not likely to be necessary, so Assumption 8 can be true under conditions even weaker than those considered below.

Recall that $H_0^k \equiv \text{plim} H^k$, $z_{ij}^k \equiv \lambda_j' \otimes \lambda_i' \text{vec}(A_0^k)$, and $y_{ij,T} = \sqrt{T} \left(\left| \frac{1}{T} e_i' e_j \right| - \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right)$, where $A_0^k = \Sigma_F - \Sigma_F H_0^k \left(H_0^{k'} \Sigma_F H_0^k \right)^{-1} H_0^{k'} \Sigma_F$.

Lemma E1: Under Assumptions 1–4, Assumption 8(a) holds if λ_i^k 's are independently distributed and the following condition holds:

$$\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} E |z_{ij}^k| \rightarrow c_1^k \text{ and } \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N E |z_{ij}^k| \rightarrow c_1^k \text{ as } N \rightarrow \infty.$$

Proof:

By Assumption 2(a) and the definition of z_{ij}^k , there exists $W' > 0$ such that

$$E |z_{ij}^k|^2 \leq E \|\lambda_i \lambda_j'\|^2 \|A_0^k\|^2 \leq \max_i E \|\lambda_i\|^4 \|A_0^k\|^2 \leq W \|A_0^k\|^2 = W' < \infty.$$

Define $q_b^k = \sum_{i,j \in b} (|z_{ij}^k| - E|z_{ij}^k|)$. Thus, it follows that

$$\begin{aligned}
& E \left[\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} (|z_{ij}^k| - E|z_{ij}^k|) \right]^2 = E \left(\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B q_b^k \right)^2 \\
&= \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B \sum_{b'=1}^B E q_b^k q_{b'}^k \\
&= \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B \sum_{b'=1}^B E q_b^k E q_{b'}^k - \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B \left(E q_b^k \right)^2 + \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B E [(q_b^k)^2] \\
&= I + II + III
\end{aligned}$$

where we use the fact that q_b^k and $q_{b'}^k$ are independent for $b \neq b'$. It is clear that $I = \left[\left(\sum_{b=1}^B M_b^2 \right)^{-1} \sum_{b=1}^B E q_b^k \right]^2 = 0$ and $II = 0$ because $E q_b^k = 0$. In what follows, we show that $III \rightarrow 0$. Let $v_{ij}^k \equiv |z_{ij}^k| - E|z_{ij}^k|$.

$$\begin{aligned}
III &= \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B E \left[\sum_{i,j \in b} (|z_{ij}^k| - E|z_{ij}^k|) \right]^2 \\
&= \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B E \left(\sum_{i \in b} \sum_{j \in b} \sum_{\ell \in b} \sum_{m \in b} v_{ij}^k v_{\ell m}^k \right) \\
&= \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B E \left(\sum_{i,j,\ell,m \in b, i,j \neq \ell,m} v_{ij}^k v_{\ell m}^k + \sum_{i,j,\ell,m \in b, i=\ell, j \neq m} v_{ij}^k v_{\ell m}^k + \sum_{i,j,\ell,m \in b, i=\ell, j=m} v_{ij}^k v_{\ell m}^k \right) \\
&= a_1^k + a_2^k + a_3^k
\end{aligned}$$

It is clear that $a_1^k = 0$ because $E(v_{ij}^k v_{\ell m}^k) = E v_{ij}^k E v_{\ell m}^k = 0$ for $i \neq \ell$, $i \neq m$, $j \neq \ell$, and $j \neq m$. Also, the number of summands in a_2^k is at most proportional to $M_b(M_b - 1)(M_b - 2)$, and the number of summands in a_3^k is at most proportional to M_b^2 . Note that

$$E|v_{ij}^k v_{\ell m}^k| \leq \max_{i,j} E|v_{ij}^k|^2 = \max_{i,j} E \left| |z_{ij}^k| - E|z_{ij}^k| \right|^2 \leq \max_{i,j} \left[E|z_{ij}^k|^2 + 3 \left(E|z_{ij}^k| \right)^2 \right] \leq 4W'$$

Since M_b is uniformly $O(N^\alpha)$ and $N^{1-\alpha}W_0 \leq B \leq N^{1-\alpha}W$, $a_1^k + a_2^k + a_3^k = \left(\sum_{b=1}^B M_b^2 \right)^{-2} O(N^{1-\alpha}) O(N^{3\alpha}) = O(N^{-1})$. Thus, $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} (|z_{ij}^k| - E|z_{ij}^k|) \rightarrow_{m.s.} 0$, which implies that $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |z_{ij}^k| \rightarrow_p c_1^k$. Using a similar argument, it is not difficult to show that $E \left[N^{-2} \sum_{i=1}^N \sum_{j=1}^N (|z_{ij}^k| - E|z_{ij}^k|) \right]^2 = O(N^{-1})$. Thus, $N^{-2} \sum_{i=1}^N \sum_{j=1}^N |z_{ij}^k| \rightarrow_p c_1^k$. **Q.E.D.**

Lemma E2: Under Assumptions 1–4, Assumption 8(b) holds if the following conditions hold.

- (1) All pairs of e_i and e_j are independent except for $O(N)$ many pairs.
- (2) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} E y_{ij,T} \rightarrow c_2$ and $\frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N E y_{ij,T} \rightarrow c_2$ as $N \rightarrow \infty$.

Note that the first condition implies $O(N)$ many pairs of e_i and e_j can have non-zero cross-sectional correlation. It strengthens Assumption 4(c) but still allows for weak cross-sectional correlation of the idiosyncratic shocks.

Proof:

Note that $E|y_{ij,T}|^2 \leq W$ for all i, j , and T by Assumption 4(f). To simplify the notation, we drop the subscript “ T ” in $y_{ij,T}$.

$$\begin{aligned} & E \left[\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} (y_{ij} - Ey_{ij}) \right]^2 \\ &= \left(\sum_{b=1}^B M_b^2 \right)^{-2} \sum_{b=1}^B \sum_{b'=1}^B \sum_{i \in b} \sum_{j \in b} \sum_{\ell \in b'} \sum_{m \in b'} (Ey_{ij}y_{\ell m} - Ey_{ij}E_{\ell m}) \end{aligned} \quad (2.1)$$

Note that $Ey_{ij}y_{\ell m} - Ey_{ij}E_{\ell m} = 0$ if y_{ij} is independent of $y_{\ell m}$. Now, we consider the number of pairs that consist of dependent y_{ij} and $y_{\ell m}$. Note that y_{ij} and $y_{\ell m}$ are not independent if e_i is not independent of e_ℓ for $i \in b$ and $\ell \in b'$. For b and $b' = 1, \dots, B$ and $i \in b$ and $\ell \in b'$, there are at most pairs of $i \in b$ and $\ell \in b'$ such that e_i is not independent of e_ℓ . Given the choice of $i \in b$ and $\ell \in b'$, there are $O(N^{2\alpha})$ possible choices for e_j and e_m where $j \in b$ and $m \in b'$. Therefore, Eq. (2.1) contains at most $O(N^{1+2\alpha})$ many summands, $Ey_{ij}y_{\ell m} - Ey_{ij}E_{\ell m}$, in which y_{ij} is not independent of $y_{\ell m}$. Since $\left(\sum_{b=1}^B M_b^2\right)^{-2} = O(N^{-2-2\alpha})$ and $E|y_{ij}|^2 \leq W$ for all i and j , Eq. (2.1) is equal to $O(N^{-1})$. Thus, $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} (y_{ij} - Ey_{ij}) \rightarrow_{m.s.} 0$, which implies that $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} y_{ij} \rightarrow_p c_2$. Using a similar argument, it is not difficult to show that $E \left[N^{-2} \sum_{i=1}^N \sum_{j=1}^N (y_{ij} - Ey_{ij}) \right]^2 = O(N^{-1})$. Thus, $N^{-2} \sum_{i=1}^N \sum_{j=1}^N y_{ij} \rightarrow_p c_2$. **Q.E.D.**

Before providing the sufficient conditions for Assumption 9, we define the following notations. Let $\dim(Y)$ denote the dimension of a $p \times p$ matrix Y , i.e. $\dim(Y) = p$. Let $\sum(Y)$ denote the sum of all entries of Y . Let $\{z_{ij}^k\}_{i,j \in b}$ denote the matrix of z_{ij}^k corresponding to the b^{th} block and let $\{\frac{1}{T} \sum_{t=1}^T \tau_{ij,t}\}_{i,j \in b}$ denote the matrix of $\frac{1}{T} \sum_{t=1}^T \tau_{ij,t}$ corresponding to the b^{th} block.

Lemma E3: Under Assumptions 1–3 and 4', Assumption 9 holds if ANY of the following conditions holds. For any $k < r$,

- (1) Either $\text{plim inf}_{N,T \rightarrow \infty} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} z_{ij}^k > 0$ or $\text{plim inf}_{N,T \rightarrow \infty} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} > 0$.
- (2) $\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} |z_{ij}^k| - \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \neq 0$ with probability approaching 1.
- (3) For $b \in \{b_1, b_2, \dots, b_m\}$, the b^{th} block $\{\frac{1}{T} \sum_{t=1}^T \tau_{ij,t}\}_{i,j \in b}$ has a principal submatrix Y_b with $\dim(Y_b) \geq N^\alpha c^*$ and $\dim(Y_b)^{-2} \sum(Y_b) \geq W_0$, and $\text{plim inf}_{B \rightarrow \infty} m/B = \check{c}$, where c^* and $\check{c} \in (0, 1]$.
- (4) For $b \in \{b_1, b_2, \dots, b_m\}$, the b^{th} block $\{z_{ij}^k\}_{i,j \in b}$ has a principal submatrix Z_b^k with $\dim(Z_b^k) \geq N^\alpha c^*$ and $\dim(Z_b^k)^{-2} \sum(Z_b^k) \geq W_0$, and $\text{plim inf}_{B \rightarrow \infty} m/B = \check{c}$, where c^* and $\check{c} \in (0, 1]$.

Proof:

(1) Note that both $\sum_{b=1}^B \sum_{i,j \in b} z_{ij}^k$ and $\sum_{b=1}^B \sum_{i,j \in b} \frac{1}{T} \sum_{t=1}^T \tau_{ij,t}$ are non-negative. Hence,

$$\begin{aligned} & \text{plim}_{N,T \rightarrow \infty} \inf \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) \\ & \geq \text{plim}_{N,T \rightarrow \infty} \inf \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right) > 0 \end{aligned}$$

(2) This is straightforward because

$$\frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) \geq \left| \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(|z_{ij}^k| - \left| \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) \right|.$$

(3) Let $i, j \in Y_b$ denote that $\frac{1}{T} \sum_{t=1}^T \tau_{ij,t}$ is an entry from the principal submatrix Y_b of the b^{th} block.

$$\begin{aligned} \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b=1}^B \sum_{i,j \in b} \left(\left| z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right| \right) & \geq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b \in \{b_1, \dots, b_m\}} \sum_{i,j \in Y_b} \left(z_{ij}^k + \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \right) \\ & \geq \frac{1}{\sum_{b=1}^B M_b^2} \sum_{b \in \{b_1, \dots, b_m\}} \sum(Y_b) \\ & \geq \frac{\min_{b \in \{b_1, \dots, b_m\}} \dim(Y_b)^2 m}{\sum_{b=1}^B M_b^2} \left[\frac{1}{m} \sum_{b \in \{b_1, \dots, b_m\}} \frac{1}{\dim(Y_b)^2} \sum(Y_b) \right] \\ & \geq \frac{(N^\alpha c^*)^2 m}{BN^{2\alpha} W^2} W_0 \end{aligned} \tag{2.2}$$

where we use the fact that $\sum_{b \in \{b_1, \dots, b_m\}} \sum_{i,j \in Y_b} z_{ij}^k$ is non-negative and Assumption 4'(g). Since $W < \infty$, $W_0 > 0$, $c^* > 0$ and $\text{plim} \inf_{B \rightarrow \infty} m/B = \check{c} > 0$, (2.2) is bounded away from zero.

(4) This part can be proved using the same argument of part (3). **Q.E.D.**

Remark: Conditions (3) and (4) are weaker version of condition (1). To see this, take condition (3) as an example. Let $m = B$ and $Y_b =$ the b^{th} block for all b , so condition (3) means $M_b^{-2} \sum_{i,j \in b} \frac{1}{T} \sum_{t=1}^T \tau_{ij,t} \geq W_0$ for all b , which implies condition (1). Note that all principal submatrices of $\{z_{ij}^k\}_{i,j \in b}$ and $\{\frac{1}{T} \sum_{t=1}^T \tau_{ij,t}\}_{i,j \in b}$ are positive semidefinite, but Conditions (3) and (4) only impose restrictions on some of these principal submatrices. Hence, Conditions (3) and (4) are general enough to allow different structures in the covariance matrices $\{z_{ij}^k\}_{i,j \in b}$ and $\{\frac{1}{T} \sum_{t=1}^T \tau_{ij,t}\}_{i,j \in b}$.

3 Additional Monte Carlo Results

First, we decrease the signal-to-noise ratio to see how these estimators behave. We generate the data using DGP1 with $\mu = \sqrt{72r/13}$ so that $R^2 = \text{E}[\text{trace}(ee')]/\text{E}[\text{trace}(XX')] = 25\%$. Table 5 summarizes the results with $k_{\text{max}} = 10$. When $\tau = 0.25$, all estimators tend to underestimate for small N and T , but LUB 's estimates are closer to the truth than others. This is different

from the results in Table 1, where all of the estimators give rather precise estimates for $\tau = 0.25$ and $R^2 = 50\%$. When $\tau = 0.5$, Table 5 shows that IC_{p1} tends to overestimate, but ED , MPC and LUB tend to underestimate. In specific, both ED and MPC become very conservative, even if N and T are reasonably large. LUB 's performance is better than that of ED and MPC in the sense that the mean and mode are much closer to the truth when $N = 400$. The results in Table 5 is informative. All estimators perform well with low signal-to-noise ratio when there is no cross-sectional correlation. However, even some weak correlation ($\tau = 0.25$) can deteriorate the performance of all estimators when data is relatively noisy. As the correlation becomes even stronger, it is more difficult to distinguish the information in the factors and the noises in finite samples. It turns out that LUB is the most robust estimator when the signal-to-noise ratio is low¹.

Also, we examine the robustness of these estimators to the choice of k_{max} with $R^2 = 50\%$. Table 6 reports the means and modes of different estimators when $k_{max} = 15$. Note that this setup is challenging for MPC and LUB because the theory requires that $B \gg k_{max}$, but we have $k_{max} > B = 10$ when $N = 100$. Panel A of Table 6 summarizes the results under DGP1. It is clear that IC_{p1} is not robust to the choice of k_{max} . Also, ED is also not robust to k_{max} under DGP1 when the sample sizes are small. The performance of MPC and LUB is reasonably good under DGP1. In panel B, we report the results under the completely wrong block structure considered in the first subcase of DGP2. It turns out that MPC frequently overestimates the number of factors when $N = 100$ and $k_{max} = 15$. \overline{MPC} performs better than MPC but still tends to overestimate when the error terms are persistent. As N increases, however, the sensitivity of MPC and \overline{MPC} to the choice of k_{max} becomes weaker. The Panel C of Table 6 presents the means and modes of different estimators under the imperfect block structure of DGP2. The performance of ED does not change very much relative to the results in Table 3. It still tends to underestimate in terms of either the mean or the mode of the estimates. MPC performs similarly to what has been shown in Table 3, except for the cases where $N = 100$ and $T = 500$. Note that the theory requires k_{max} to be a fixed number and B to diverge. When $N = 100$, $B = 10$ and $k_{max} = 15$, the theoretical requirement that $k_{max} \ll B$ does not hold. This explains why MPC may not perform well for large k_{max} in samples with small N . It is also notable that as N increases to 225, the performance of MPC becomes much better. Finally, we summarize the performance of LUB when $k_{max} = 15$. In general, LUB 's performance does not change too much relative to the case where $k_{max} = 10$. Hence, it is more accurate than other estimators and less sensitive to the choice of k_{max} .

¹We also conduct simulations with $R^2 = 25\%$ under DGP2. The results in subcase 1 of DGP2 have very similar pattern to those derived under DGP1 with $\tau = 0.25$, and results in subcase 2 of DGP2 have very similar pattern to those derived under DGP1 with $\tau = 0.5$. To save the space, we do not report these results in the paper.

Table 4 (full version): Means and modes of estimators for different values of r

$$N = 100, T = 200 \text{ and } B = 10.$$

$$k_{max} = 10 \text{ and } E[\text{trace}(ee')] = E[\text{trace}(F\Lambda'\Lambda F')]$$

r	1	2	3	4	5	6	7
DGP1: $\tau = 0.25$							
IC_{p1}	1.01 (1)	2.02 (2)	3.02 (3)	4.01 (4)	5.01 (5)	5.97 (6)	6.75 (7)
ED	1.19 (1)	2.02 (2)	3.00 (3)	4.00 (4)	4.79 (5)	4.33 (6)	2.54 (1)
MPC	1.00 (1)	2.00 (2)	3.00 (3)	4.00 (4)	5.00 (5)	5.90 (6)	6.49 (7)
LUB	1.00 (1)	2.00 (2)	3.00 (3)	4.00 (4)	5.00 (5)	5.98 (6)	6.75 (7)
DGP1: $\tau = 0.5$							
IC_{p1}	10.00 (10)	10.00(10)	10.00 (10)	10.00 (10)	10.00 (10)	10.00 (10)	10.00 (10)
ED	1.07 (1)	2.00 (2)	2.77 (3)	1.43 (1)	1.07 (1)	1.13 (1)	1.08 (1)
MPC	1.00 (1)	2.00 (2)	3.00 (3)	3.90 (4)	4.25 (4)	4.00 (4)	3.15 (3)
LUB	1.00 (1)	2.00 (2)	3.00 (3)	3.98 (4)	4.69 (5)	4.62 (5)	3.65 (4)
DGP1: $\tau = 0.75$							
IC_{p1}	10.00 (10)	10.00(10)	10.00 (10)	10.00 (10)	10.00 (10)	10.00 (10)	10.00 (10)
ED	1.03 (1)	1.96 (2)	1.11 (1)	1.00 (1)	1.01 (1)	1.05 (1)	1.03 (1)
MPC	1.00 (1)	2.00 (2)	2.90 (3)	2.99 (3)	2.33 (2)	1.51 (1)	1.14 (1)
LUB	1.00 (1)	2.00 (2)	2.98 (3)	3.52 (4)	2.92 (3)	1.90 (1)	1.32 (1)
DGP2 subcase (1): $J = 3, \omega = 0.2, \text{ and } \varphi = 0.3$							
IC_{p1}	2.28 (1)	3.24 (2)	4.26 (4)	5.17 (5)	6.12 (6)	7.00 (6)	7.77 (7)
ED	1.03 (1)	2.02 (2)	3.02 (3)	3.97 (4)	4.20 (5)	2.77 (1)	1.53 (1)
MPC	4.46 (6)	4.80 (6)	5.09 (6)	5.48 (6)	5.95 (5)	6.47 (6)	6.94 (7)
\overline{MPC}	2.05 (1)	3.00 (2)	3.96 (3)	4.83 (4)	5.67 (5)	6.38 (6)	6.92 (7)
LUB	1.32 (1)	2.32 (2)	3.30 (3)	4.30 (4)	5.15 (5)	5.56 (6)	5.06 (6)
DGP2 subcase (2): $J = 8, \omega = 0.2, \text{ and } \varphi = 0.3$							
IC_{p1}	9.93 (10)	10.00(10)	10.00 (10)	10.00 (10)	10.00 (10)	10.00 (10)	10.00 (10)
ED	2.57 (1)	2.15 (2)	1.56 (1)	1.12 (1)	1.05 (1)	1.02 (1)	1.02 (1)
MPC	1.01 (1)	2.02 (2)	2.98 (3)	3.36 (4)	2.71 (2)	1.71 (1)	1.18 (1)
LUB	1.00 (1)	2.00 (2)	2.96 (3)	3.38 (4)	2.63 (2)	1.69 (1)	1.24 (1)

Note: DGP1 follows the exact block structure and τ controls the within-block cross-sectional correlation. DGP2 subcase (1) follows the completely wrong block structure, and DGP2 subcase (2) follows the imperfect block structure. ω and J control the strength and pervasiveness of the cross-sectional correlation, respectively, and φ controls the serial correlation of the error terms. The numbers outside (inside) the parentheses are the means (modes) of the estimators in 2000 Monte Carlo replications.

Table 5: Means and modes of different estimators in 2000 Monte Carlo replications of DGP1 with $R^2 = 25\%$

$$k_{max} = 10 \text{ and } E[\text{trace}(ee')] = 3E[\text{trace}(F\Lambda' \Lambda F')]$$

τ	N	B	T	IC_{p1}	ED	MPC	LUB
0	100	10	100	1.11 (1)	3.08 (1)	4.12 (4)	3.71 (4)
0	100	10	200	1.47 (1)	4.91 (5)	4.78 (5)	4.80 (5)
0	100	10	500	2.19 (2)	5.00 (5)	4.99 (5)	5.00 (5)
0	225	15	100	1.54 (1)	5.00 (5)	4.91 (5)	4.39 (5)
0	225	15	200	3.58 (4)	5.00 (5)	5.00 (5)	5.00 (5)
0	225	15	500	4.96 (5)	5.00 (5)	5.00 (5)	5.00 (5)
0	400	20	100	2.02 (2)	5.00 (5)	5.00 (5)	4.60 (5)
0	400	20	200	4.74 (5)	5.00 (5)	5.00 (5)	5.00 (5)
0	400	20	500	5.00 (5)	5.00 (5)	5.00 (5)	5.00 (5)
0.25	100	10	100	1.27 (1)	1.14 (1)	1.49 (1)	1.67 (1)
0.25	100	10	200	1.90 (2)	1.08 (1)	1.81 (1)	2.27 (2)
0.25	100	10	500	2.89 (3)	1.02 (1)	2.16 (2)	2.98 (3)
0.25	225	15	100	1.80 (2)	1.34 (1)	3.00 (3)	3.75 (4)
0.25	225	15	200	4.06 (4)	2.44 (1)	3.23 (3)	4.75 (5)
0.25	225	15	500	4.99 (5)	4.69 (5)	3.80 (4)	4.99 (5)
0.25	400	20	100	2.27 (2)	2.61 (1)	4.07 (4)	4.75 (5)
0.25	400	20	200	4.88 (5)	4.74 (5)	4.39 (4)	5.00 (5)
0.25	400	20	500	5.00 (5)	5.00 (5)	4.87 (5)	5.00 (5)
0.5	100	10	100	9.79 (10)	1.01 (1)	1.01 (1)	1.01 (1)
0.5	100	10	200	10.00 (10)	1.00 (1)	1.00 (1)	1.01 (1)
0.5	100	10	500	10.00 (10)	1.00 (1)	1.00 (1)	1.01 (1)
0.5	225	15	100	9.02 (10)	1.05 (1)	1.08 (1)	1.33 (1)
0.5	225	15	200	10.00 (10)	1.05 (1)	1.01 (1)	1.58 (1)
0.5	225	15	500	10.00 (10)	1.09 (1)	1.00 (1)	1.93 (1)
0.5	400	20	100	5.22 (5)	1.08 (1)	1.50 (1)	2.58 (3)
0.5	400	20	200	10.00 (10)	1.12 (1)	1.24 (1)	3.66 (4)
0.5	400	20	500	10.00 (10)	1.66 (1)	1.07 (1)	4.71 (5)

Note: DGP1 follows the exact block structure and τ controls the within-block cross-sectional correlation. The numbers outside (inside) the parentheses are the means (modes) of the estimators in 2000 Monte Carlo replications.

Table 6: Robustness of different estimators to the choice of k_{max} , $k_{max} = 15$

Panel A: DGP1, the exact block structure

τ	N	B	T	IC_{p1}	ED	MPC	LUB	τ	N	B	T	IC_{p1}	ED	MPC	LUB
0.25	100	10	100	4.20 (5)	3.34 (5)	4.37 (5)	4.97 (5)	0.75	100	10	100	14.39 (14)	14.72 (15)	2.67 (1)	2.57 (2)
0.25	100	10	200	4.68 (5)	6.14 (5)	4.61 (5)	5.00 (5)	0.75	100	10	200	14.68 (15)	14.99 (15)	2.69 (1)	2.92 (3)
0.25	100	10	500	4.94 (5)	12.36 (14)	4.97 (5)	5.00 (5)	0.75	100	10	500	14.88 (15)	15.00 (15)	3.71 (1)	3.59 (4)
0.25	225	15	100	4.63 (5)	4.35 (5)	4.85 (5)	5.00 (5)	0.75	225	15	100	15.00 (15)	2.12 (1)	3.29 (5)	4.73 (5)
0.25	225	15	200	4.99 (5)	4.87 (5)	4.91 (5)	5.00 (5)	0.75	225	15	200	15.00 (15)	2.25 (1)	3.12 (5)	4.95 (5)
0.25	225	15	500	5.05 (5)	4.90 (5)	4.98 (5)	5.00 (5)	0.75	225	15	500	15.00 (15)	2.34 (1)	3.18 (5)	5.00 (5)
0.25	400	20	100	4.81 (5)	4.88 (5)	4.97 (5)	5.00 (5)	0.75	400	20	100	15.00 (15)	2.81 (1)	3.74 (5)	4.99 (5)
0.25	400	20	200	5.00 (5)	5.01 (5)	5.00 (5)	5.00 (5)	0.75	400	20	200	15.00 (15)	3.37 (5)	3.54 (5)	5.00 (5)
0.25	400	20	500	5.03 (5)	5.00 (5)	5.00 (5)	5.00 (5)	0.75	400	20	500	15.00 (15)	3.97 (5)	3.52 (5)	5.00 (5)

Panel B: DGP2, the completely wrong block structure, $J = 3$

φ	N	B	T	IC_{p1}	ED	MPC	\overline{MPC}	LUB	φ	N	B	T	IC_{p1}	ED	MPC	\overline{MPC}	LUB
0	100	10	100	5.13 (5)	3.21 (5)	10.23 (11)	5.13 (5)	5.60 (5)	0	225	15	100	5.00 (5)	5.01 (5)	6.21 (5)	5.00 (5)	5.09 (5)
0	100	10	200	5.47 (5)	4.52 (5)	10.23 (11)	5.45 (5)	5.39 (5)	0	225	15	200	5.00 (5)	5.01 (5)	5.09 (5)	5.00 (5)	5.01 (5)
0	100	10	500	6.65 (5)	4.97 (5)	9.50 (10)	6.46 (5)	5.07 (5)	0	225	15	500	5.00 (5)	5.00 (5)	5.00 (5)	5.00 (5)	5.00 (5)
0.3	100	10	100	5.61 (5)	2.64 (1)	10.50 (12)	5.60 (5)	5.60 (5)	0.3	225	15	100	5.00 (5)	5.00 (5)	7.15 (5)	5.00 (5)	5.16 (5)
0.3	100	10	200	6.13 (6)	4.21 (5)	10.42 (11)	6.11 (5)	5.50 (5)	0.3	225	15	200	5.00 (5)	5.01 (5)	5.35 (5)	5.00 (5)	5.05 (5)
0.3	100	10	500	7.78 (6)	4.94 (5)	9.69 (11)	7.42 (5)	5.10 (5)	0.3	225	15	500	5.00 (5)	5.01 (5)	5.00 (5)	5.00 (5)	5.00 (5)
0.6	100	10	100	12.95 (15)	1.45 (1)	11.25 (12)	11.00 (12)	5.24 (5)	0.6	225	15	100	10.95 (10)	3.38 (5)	9.96 (11)	9.45 (10)	4.99 (5)
0.6	100	10	200	12.18 (15)	2.68 (1)	10.88 (12)	10.60 (12)	5.86 (5)	0.6	225	15	200	8.24 (8)	5.00 (5)	7.11 (8)	6.98 (7)	5.26 (5)
0.6	100	10	500	12.70 (15)	4.64 (5)	10.25 (11)	10.14 (11)	5.37 (5)	0.6	225	15	500	6.48 (6)	5.01 (5)	5.14 (5)	5.14 (5)	5.03 (5)

Panel C: DGP2, the imperfect block structure, $J = 0.8\sqrt{N}$

J	φ	N	B	T	IC_{p1}	ED	MPC	LUB	J	φ	N	B	T	IC_{p1}	ED	MPC	LUB
8	0	100	10	100	12.59 (13)	1.63 (1)	2.74 (2)	2.55 (2)	12	0	225	15	100	15.00 (15)	1.11 (1)	4.06 (4)	4.47 (5)
8	0	100	10	200	13.48 (14)	2.60 (1)	5.50 (2)	2.91 (3)	12	0	225	15	200	15.00 (15)	1.13 (1)	3.35 (3)	4.75 (5)
8	0	100	10	500	14.06 (14)	4.55 (1)	14.20 (14)	3.56 (4)	12	0	225	15	500	15.00 (15)	1.25 (1)	3.18 (3)	4.92 (5)
8	0.3	100	10	100	12.90 (13)	1.57 (1)	2.75 (2)	2.46 (2)	12	0.3	225	15	100	15.00 (15)	1.12 (1)	4.10 (4)	4.42 (5)
8	0.3	100	10	200	13.62 (14)	2.27 (1)	3.70 (2)	2.72 (2)	12	0.3	225	15	200	15.00 (15)	1.13 (1)	3.35 (3)	4.71 (5)
8	0.3	100	10	500	14.12 (14)	3.95 (1)	13.73 (14)	3.40 (4)	12	0.3	225	15	500	15.00 (15)	1.20 (1)	3.16 (3)	4.91 (5)
8	0.6	100	10	100	14.80 (15)	1.34 (1)	2.51 (1)	2.04 (1)	12	0.6	225	15	100	15.00 (15)	1.10 (1)	4.28 (5)	4.05 (5)
8	0.6	100	10	200	14.46 (15)	1.59 (1)	2.54 (1)	2.33 (1)	12	0.6	225	15	200	15.00 (15)	1.10 (1)	3.41 (3)	4.56 (5)
8	0.6	100	10	500	14.45 (14)	2.61 (1)	9.08 (14)	3.07 (3)	12	0.6	225	15	500	15.00 (15)	1.10 (1)	3.10 (3)	4.88 (5)

Note: In these experiments, we set $E[\text{trace}(ee')] = E[\text{trace}(FA'\Lambda F')]$. The numbers outside (inside) the parentheses are the means (modes) of the estimators in 2000 Monte Carlo replications.