A proof of the set-theoretic version of the salmon conjecture

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Abstract

We show that the irreducible variety of $4 \times 4 \times 4$ complex valued tensors of border rank at most 4 is the zero set of polynomial equations of degree 5 (the Strassen commutative conditions), of degree 6 (the Landsberg-Manivel polynomials), and of degree 9 (the symmetrization conditions).

Key words: rank of tensors, border rank of tensors, the salmon conjecture.

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1 Introduction

In this paper we identify $\mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$ with the space of tensors $T = [t_{i,j,k}]_{i=j=k}^{m,n,l} \in \mathbb{C}^{m \times n \times l}$, where we choose the standard bases in $\mathbb{C}^m, \mathbb{C}^n, \mathbb{C}^l$, unless stated otherwise. Let $V_r(m, n, l) \subseteq \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$ be the variety of tensors of border rank at most $r$. The border rank of a tensor $T \neq 0$ is $r$ if $T$ is the limit of a sequence of rank $r$ tensors, and there does not exist a sequence of tensors, $(T_i)$, such that the limit of $(T_i)$ is $T$ and the rank $T_i < r$ for all $i \in \mathbb{N}$. The projectivization of $V_r(m, n, l)$ is the $r$th secant variety of $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{l-1}$.

In 2007, Elizabeth Allman posed the problem of determining the ideal $I_4(4, 4, 4)$ generated by all polynomials vanishing on $V_4(4, 4, 4)$ [2]. Allman offered a prize of a freshly-caught smoked Copper river salmon for the solution, and thus, the problem is colloquially called the salmon problem. Conjecture 3.24 in [8] states that $I_4(4, 4, 4)$ is generated by polynomials of degree 5 and 9. A first nontrivial step in characterizing $V_4(4, 4, 4)$ is to characterize $V_4(3, 3, 4)$. In [6], Landsberg and Manivel show that $V_4(3, 3, 4)$ satisfies a set of polynomial equations of degree 6 which are not in the ideal generated by the equations of degree 5 from the original conjecture. (See also [7, Remark 5.7] and [3].) Hence the revised version of the salmon conjecture (a term coined in [3]), states that $I_4(4, 4, 4)$ is generated by polynomials of degree 5, 6 and 9 [10, §2]. This in particular implies the set-theoretic version of the salmon conjecture: $V_4(4, 4, 4)$ is the zero set of homogeneous polynomials of degree 5, 6 and 9.
It is shown theoretically in [5] that $V_4(4, 4, 4)$ is cut out by polynomials of degree 5, 9 and 16. In [5, Theorem 4.5], it is shown that $V_4(3, 3, 4)$ is cut out by polynomials of degrees 9 and 16. The degree 9 equations follow from the observation in [5] that the four frontal slices of $X \in V_4(3, 3, 4)$, which are four $3 \times 3$ matrices, are symmetrizable by multiplication on the left and by multiplication on the right by nonzero matrices $L, R \in \mathbb{C}^{3 \times 3}$ respectively. The existence of nonzero matrices $L$ and $R$ is equivalent to the vanishing of all $9 \times 9$ minors of two corresponding $12 \times 9$ matrices whose entries are linear in the entries of $X$ [5, Lemma 4.3]. We call this set of polynomials the symmetrization conditions.

One can choose $L$ and $R$ such that their entries are polynomials of degree 8 in the entries of $X$. The degree 16 equations in [5] are a result of the condition

$$LR^\top = R^\top L = \frac{\text{tr}(LR^\top)}{3}I_3.$$  \hspace{1cm} (1.1)

The degree 16 equations are used only in the case A.I.3 of the proof of Theorem 4.5 of [5].

In [6], Landsberg and Manivel give an algorithm to construct polynomials of degree 6, referred here as the LM-polynomials, that vanish on $V_4(3, 3, 4)$ but are not in the ideal generated by the known polynomials of degree 5. In [3], Bates and Oeding explicitly construct a basis of the these degree 6 polynomials which consist of ten linearly independent polynomials. Using methods from numerical algebraic geometry, Bates and Oeding give numerical confirmation that $V_4(3, 3, 4)$ is the zero set of a set of polynomials of degree 6 and 9 [3], where the degree 6 polynomials are the LM-polynomials.

The aim of this paper is to show that $V_4(3, 3, 4)$ is cut out by polynomials of degree 6 and 9. This is done by showing that in Case A.I.3 of [5, Proof of Theorem 4.5] the use of polynomials of degree 16 can be eliminated by use of the LM-polynomials. More precisely we show that any $3 \times 3 \times 4$ tensor $X = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4}$ whose four frontal slices are of the form

$$X_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} & 0 \\ x_{2,1,k} & x_{2,2,k} & 0 \\ 0 & 0 & x_{3,3,k} \end{bmatrix}, \quad k = 1, 2, 3, 4, \quad (1.2)$$

has border rank at most four if and only if the ten basis LM-polynomials vanish on $X$.

As we will see later, a tensor $X \in \mathbb{C}^{3 \times 3 \times 4}$ of the form (1.2) has border rank at most four if and only if either the four matrices $\begin{bmatrix} x_{1,1,k} & x_{1,2,k} \\ x_{2,1,k} & x_{2,2,k} \end{bmatrix}, k = 1, 2, 3, 4$ are linearly dependent or $x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. Note that the condition that the above four $2 \times 2$ matrices are linearly dependent is equivalent to the vanishing of the polynomial

$$f(X) = \det \begin{bmatrix} x_{1,1,1} & x_{1,2,1} & x_{2,1,1} & x_{2,2,1} \\ x_{1,1,2} & x_{1,2,2} & x_{2,1,2} & x_{2,2,2} \\ x_{1,1,3} & x_{1,2,3} & x_{2,1,3} & x_{2,2,3} \\ x_{1,1,4} & x_{1,2,4} & x_{2,1,4} & x_{2,2,4} \end{bmatrix}. \quad (1.3)$$

Computer-aided calculations show that the restrictions of the ten basis LM-polynomials to $X$ of the form (1.2) are the polynomials

$$x_{3,3,k}x_{3,3,l}f(X) \text{ for } 1 \leq k \leq l \leq 4. \quad (1.4)$$
Hence $\mathcal{X}$ has a border rank at most four if and only if the ten basis LM-polynomials vanish on $\mathcal{X}$. Combining this with the results in [5] we deduce the set-theoretic version of the salmon conjecture.

We summarize briefly the content of the paper. In §2 we restate the characterization of $V_4(3, 3, 4)$ given in [5, Theorem 4.5]. In §3 we show that the use of polynomials of degree 16 in the proof of [5, Theorem 4.5] can be replaced by the use of the LM-polynomials. In §4 we summarize briefly the characterization of $V_4(4, 4, 4)$ as the zero set of polynomials of degree 5, 6 and 9.

## 2 A characterization of $V_4(3, 3, 4)$

We now state [5, Theorem 4.5] which characterizes $V_4(3, 3, 4)$. Let $\mathcal{X} = [x_{i,j,k}]_{i,j,k=1}^{3,3,4} \in \mathbb{C}^{3 \times 3 \times 4}$. The four frontal slices of $\mathcal{X}$ are denoted as the matrices $X_k = [x_{i,j,k}]_{i,j=1}^{3,3} \in \mathbb{C}^{3 \times 3}, k = 1, 2, 3, 4$. Assume that $\mathcal{X} \in V_4(3, 3, 4)$. A special case of [5, Lemma 4.3] claims that there exist nontrivial matrices $L, R \in \mathbb{C}^{3 \times 3} \setminus \{0\}$ satisfying the conditions

\[
\begin{align*}
LX_k - X_k^T L^T &= 0, \quad k = 1, \ldots, 4, \quad L \in \mathbb{C}^{3 \times 3}, \quad \text{(2.1)} \\
X_kR - R^T X_k^T &= 0, \quad k = 1, \ldots, 4, \quad R \in \mathbb{C}^{3 \times 3}. \quad \text{(2.2)}
\end{align*}
\]

These are the symmetrization conditions.

If the entries of $R$ and $L$ are viewed as the entries of two vectors with 9 coordinates each, then the systems (2.1) and (2.2) are linear homogeneous equations with coefficient matrices $C_L(\mathcal{X}), C_R(\mathcal{X}) \in \mathbb{C}^{12 \times 9}$ respectively. (Observe that for any $A \in \mathbb{C}^{3 \times 3}$ the matrix $A - A^\top$ is skew symmetric, which has, in general, 3 free parameters.) The entries of $C_L(\mathcal{X}), C_R(\mathcal{X})$ are linear functions in the entries of $\mathcal{X}$. For a generic $\mathcal{X} \in V_4(3, 3, 4)$, rank $C_L(\mathcal{X}) = \text{rank } C_R(\mathcal{X}) = 8$ [5]. Hence we can express the entries of $L$ and $R$ in terms of corresponding 8 × 8 minors of $C_L(\mathcal{X}), C_R(\mathcal{X})$ respectively. There are a finite number of ways to express $L$ and $R$ in this way, and some of these expressions may be zero matrices. Nonetheless, the entries of $L$ and $R$ are polynomials of degree 8 in the entries of $\mathcal{X}$. If rank $C_L(\mathcal{X}) = \text{rank } C_R(\mathcal{X}) = 8$ then it is necessary that the condition (1.1) holds for every expression of $L$ and $R$ [5]. Furthermore, if rank $C_L(\mathcal{X}) < 8$ then each possible expression of $L$ in terms of 8 × 8 minors of $C_L(\mathcal{X})$ is a zero matrix, and a similar statement holds for $R$, so (1.1) holds trivially.

Thus, the characterization of $V_4(3, 3, 4)$ is given by [5, Theorem 4.5].

**Theorem 2.1** $\mathcal{X} = [x_{i,j,k}]_{i=j=k=1}^{3,3,4} \in \mathbb{C}^{3 \times 3 \times 4}$ has border rank at most 4 if and only if the following conditions holds.

1. Let $X_k := [x_{i,j,k}]_{i=j=1}^{3} \in \mathbb{C}^{3 \times 3}, k = 1, \ldots, 4$ be the four frontal slices of $\mathcal{X}$. Then the ranks of $C_L(\mathcal{X}), C_R(\mathcal{X})$ are less than 9. (These are degree 9 equations.)

2. Let $R, L$ be solutions of (2.1) and (2.2) respectively given by 8 × 8 minors of $C_L(\mathcal{X}), C_R(\mathcal{X})$. Then (1.1) holds. (These are degree 16 equations.)

The proof of Theorem 2.1 in [5] consists of discussing a number of cases. The degree 16 polynomial conditions (1.1) are used only in the case A.I.3. In the next section we show how to prove the theorem in the case A.I.3 using only the ten basis LM-polynomials of degree 6.
3 The case A.I.3 of [5, Theorem 4.5]

Suppose \( \mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4} \) and there exist two nonzero matrices \( L, R \in \mathbb{C}^{3 \times 3} \) such that (2.1)–(2.2) hold. The case A.I.3 assumes that \( L \) and \( R \) are rank one matrices. The degree 16 equations yield that \( LR^\top = R^\top L = 0 \), thus, the remainder of the proof of [5, Theorem 4.5] in the case A.I.3 resolves the case where \( LR^\top = R^\top L = 0 \). Therefore, to eliminate the use of polynomial conditions of degree 16 we need to show the following.

Claim 3.1 Let \( \mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4} \). Let \( R, L \in \mathbb{C}^{3 \times 3} \) be rank one matrices satisfying the conditions (2.1)–(2.2) respectively. Suppose furthermore that either \( LR^\top \neq 0 \) or \( R^\top L \neq 0 \). If the ten LM-polynomials vanish on \( \mathcal{X} \) then \( \mathcal{X} \in V_4(3,3,4) \).

In the rest of this section we prove Claim 3.1. Assume that \( L = uv^\top, R = xy^\top \).

The following claim is straightforward.

\[
uv^\top A \text{ is symmetric if and only if } v^\top A = bu^\top \text{ for some } b \in \mathbb{C}, \quad (3.1)
\]

\[
Axy^\top \text{ is symmetric if and only if } Ax = cy \text{ for some } c \in \mathbb{C}. \quad (3.2)
\]

By changing bases in two copies of \( \mathbb{C}^3 \) we can assume that \( u = v = e_3 = (0,0,1)^\top \).

(Changes of bases do not affect the vanishing condition of either \( LR^\top \) or \( R^\top L \) [5].)

Let \( P, Q \in GL(3, \mathbb{C}) \) such that

\[
P^\top e_3, Q^\top e_3 \in \text{span}(e_3). \quad (3.3)
\]

Then if \( A \in \mathbb{C}^3 \times \mathbb{C}^3 \) such that (3.1) and (3.2) hold, \( e_3 e_3^\top (PAQ) \) is symmetric. Observe next that \( PAQ(Q^{-1}x)(Py)^\top \) is also symmetric. Thus we need to analyze what kind of vectors can be obtained from two nonzero vectors \( x, y \) by applying \( Q^{-1}x, Py \), where \( P, Q \) satisfy (3.3). By letting \( Q_1 := Q^{-1} \) we see that \( Q_1 \) satisfies the same conditions \( Q \) in (3.3). Hence \( Q_1, P \) have the zero pattern

\[
\begin{bmatrix}
* & * & *
\end{bmatrix}
\]

(3.4)

Lemma 3.2 Let \( y \in \mathbb{C}^3 \setminus \{0\} \). If \( e_3^\top y \neq 0 \) then there exists \( P \in GL(3, \mathbb{C}) \) of the form (3.4) such that \( Py = e_3 \). If \( e_3^\top y = 0 \) then there exists \( P \in GL(3, \mathbb{C}) \) of the form (3.4) such that \( Py = e_2 \).

Proof. Assume first that \( e_3^\top y \neq 0 \). Let \( f = (f_1, f_2, f_3)^\top, g = (0, g_2, g_3)^\top \in \mathbb{C}^3 \setminus \{0\} \) such that \( f^\top y = g^\top y = 0 \). Then \( f_1 g_2 \neq 0 \). Hence there exists \( P \in GL(3, \mathbb{C}) \) of the form (3.4), whose first and second rows are \( f^\top, g^\top \) respectively, such that \( Py = e_3 \).

Suppose now that \( e_3^\top y = 0 \). Hence there exists \( P = P_1 \oplus [1], P_1 \in GL(2, \mathbb{C}) \) such that \( Py = e_2 \).

Corollary 3.3 Let \( A \in \mathbb{C}^{3 \times 3} \) and assume that \( LA \) and \( AR \) are symmetric matrices for some rank one matrices \( L, R \in \mathbb{C}^{3 \times 3} \). Then there exists \( P, Q \in GL(3, \mathbb{C}) \) such that by replacing \( A, L, R \) by \( A_1 := PAQ, L_1 := Q^\top LP^{-1}, R_1 = Q^{-1}RP^\top \) we can assume \( L_1 = e_2 e_3^\top \) and \( R_1 \) has one of the following 4 forms

\[
e_3 e_2^\top, e_3 e_2^\top, e_2 e_3^\top, e_2 e_2^\top. \quad (3.5)
\]
To prove Claim 3.1 we need to consider the first three choices of $R_1$ in (3.5) since the last choice implies $LR^\top = R^\top L = 0$. Note that by changing the first two indices in $\mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4}$ we need to consider only the first two choices of $R_1$ in (3.5).

### 3.1 The case $L = R = e_3 e_4^\top$

In the remainder of this section we say that a tensor $T \in \mathbb{C}^{m \times n \times l}$ is represented as a tensor $T' = [t'_{i,j,k}] \in \mathbb{C}^{m' \times n' \times l'}$ if the following condition hold. There exist bases in $\mathbb{C}^m, \mathbb{C}^n, \mathbb{C}^l$ such that the tensor $T$ is represented by the tensor $\hat{T} = [\hat{t}_{i,j,k}] \in \mathbb{C}^{m \times n \times l}$, where the following conditions hold. First $\hat{t}_{i,j,k} = \hat{t}_{i,j,k}$ for $i = 1, \ldots, m', j = 1, \ldots, n', k = 1, \ldots, l'$. Second $\hat{t}_{i,j,k} = 0$ if $\hat{t}_{i,j,k}$ is not a coordinate of $T'$. Clearly, $\text{rank } T = \text{rank } T'$, $\text{brank } T = \text{brank } T'$.

Let $X_1, X_2, X_3, X_4 \in \mathbb{C}^{3 \times 3}$ be the four frontal sections of $\mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4}$. Assume that (2.1)–(2.2) hold. Then each $X_k$ has the form of (1.2). (This is the case discussed in [5, (4.7)].)

Using Mathematica, we took the ten basis LM-polynomials available in the ancillary material of [3, deg6_salmon.txt] and let $x_{1,3,k} = 0, x_{2,3,k} = 0, x_{3,1,k} = 0, x_{3,2,k} = 0$ for $k = 1, 2, 3, 4$. The resulting polynomials had 24 terms. We then factored $f(\mathcal{X})$ from these restricted polynomials. This symbolic computations shows that the restriction of the ten basis LM-polynomials to $\mathcal{X}$ satisfying (2.1)–(2.2) are the polynomials given in (1.4). Therefore, by the result of Landsberg-Manivel [6], if $\mathcal{X} \in V_4(3, 3, 4)$ then all polynomials in (1.4) vanish on $\mathcal{X}$.

Vice versa, suppose that all polynomials in (1.4) vanish on $\mathcal{X}$. Let

$$Y_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} \\ x_{2,1,k} & x_{2,2,k} \end{bmatrix}, \quad k = 1, 2, 3, 4,$$

be the projection of the four frontal sections of $\mathcal{X}$ given by (1.2) on $\mathbb{C}^{2 \times 2}$. Then $f(\mathcal{X}) = 0$ if and only if $Y_1, Y_2, Y_3, Y_4$ are linearly dependent. Decompose the tensor $\mathcal{X}$ to a sum $\mathcal{Y} + \mathcal{Z}$. The four frontal sections of $\mathcal{Y}$ are block diagonal matrices $\text{diag}(Y_k, 0), k = 1, 2, 3, 4$ and the four frontal sections of $\mathcal{Z}$ are $\text{diag}(0, 0, x_{3,3,k}), k = 1, 2, 3, 4$.

Assume first that the polynomial $f(\mathcal{X})$ given by (1.3) vanishes in $\mathcal{X}$. Since $Y_1, Y_2, Y_3, Y_4$ are linearly dependent, it follows the tensor $\mathcal{Y}$ is represented as a $2 \times 2 \times 3$ tensor.

A particular case of [4, Theorem 3.1] tells us

$$\text{brank } T \leq \min(n, 2m) \text{ for any } T \in \mathbb{C}^{2 \times m \times n} \text{ where } 2 \leq m \leq n$$

Hence the border rank of $\mathcal{Y}$ is at most 3. (It is straightforward to show that any three dimensional subspace of $\mathbb{C}^{2 \times 2}$ is spanned by 3 rank one matrices. Hence [5, Theorem 2.1] implies that rank $\mathcal{Y} \leq 3$.) Clearly rank $\mathcal{Z} \leq 1$. Therefore rank $\mathcal{X} \leq 4$. (More precisely rank $\mathcal{X} \leq 4$.)

Assume now that $f(\mathcal{X}) \neq 0$. Since the ten polynomials in (1.4) vanish on $\mathcal{X}$ it follows that $x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. So $\mathcal{Z} = 0$. In this case $\mathcal{X}$ is represented a $2 \times 2 \times 4$ tensor. Hence, by (3.7), its border rank is at most 4. (More precisely, [5, Theorem 2.1] implies that rank $\mathcal{Y} \leq 4$.)
3.2 The case \( L = e_3e_3^\top, R = e_3e_2^\top \)

Let \( X_1, X_2, X_3, X_4 \in \mathbb{C}^{3 \times 3} \) be the four frontal sections of \( \mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4} \). Assume that (2.1)–(2.2) hold. This means that our tensor \( \mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4} \) has the following zero entries \( x_{1,3,k} = x_{3,1,k} = x_{3,2,k} = x_{3,3,k} = 0 \) for \( k = 1, 2, 3, 4 \). So our tensor is represented a \( 2 \times 3 \times 4 \) tensor and hence, by (3.7), its border rank is at most 4.

4 The defining polynomials of \( V_4(4, 4, 4) \)

In this section we state for the reader’s convenience the defining equations of \( V_4(4, 4, 4) \). We briefly repeat the arguments in [5] by replacing the degree 16 polynomial equations with the degree 6 polynomial equations. Let \( \mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{4 \times 4 \times 4} \). For each \( l \in \{1, 2, 3\} \) we fix \( i_l \) while we let \( i_p, i_q = 1, 2, 3, 4 \) where \( \{p, q\} = \{1, 2, 3\} \setminus \{l\} \). In this way we obtain four \( l \)-sections \( X_{1,l}, \ldots, X_{4,l} \in \mathbb{C}^{4 \times 4} \). (Note that \( X_{k,3} = [x_{i,j,k}]_{i=j=1}^{4}, k = 1, 2, 3, 4 \) are the four frontal sections of \( \mathcal{X} \).) Denote by \( \mathbf{X}_l = \text{span}(X_{1,l}, \ldots, X_{4,l}) \subset \mathbb{C}^{4 \times 4} \) the \( l \)-section subspace corresponding to \( \mathcal{X} \). For each \( l \in \{1, 2, 3\} \) we define the following linear subspaces of polynomials of degrees 5, 6, 9 respectively in the entries of \( \mathcal{X} \). The defining polynomials could be any basis in each of these linear subspaces.

We first describe the Strassen commutative conditions [9]. (These conditions where rediscovered independently in [1].) Take \( U_1, U_2, U_3 \in \mathbf{X}_l \). View \( U_i = \sum_{j=1}^4 u_{j,i} X_{j,l} \) for \( i = 1, 2, 3 \). So the entries of each \( X_{j,l} \) are fixed scalars and \( u_{j,i}, i = 1, 2, 3, j = 1, 2, 3, 4 \) are viewed as variables. Let \( \text{adj} \, U_2 \) be the adjoint matrix of \( U_2 \). Then the Strassen commutative conditions are

\[
U_1(\text{adj} \, U_2)U_3 - U_3(\text{adj} \, U_2)U_1 = 0
\]

Since the values of \( u_{j,i}, i = 1, 2, 3, j = 1, 2, 3, 4 \) are arbitrary, we regroup the above condition for each entry as a polynomial in \( u_{j,i} \). The coefficient of each monomial in the \( u_{j,i} \) variables is a polynomial of degree 5 in the entries of \( \mathcal{X} \) and must be equal to zero. The set of all such polynomials of degree 5 span a linear subspace, and we can choose any basis in this subspace.

The degree 6 and 9 polynomial conditions are obtained in a a slightly different way. Let \( P = [p_{ij}], Q = [q_{ij}] \in \mathbb{C}^{4 \times 4} \) be matrices with entries viewed as variables. View \( PX_{k,l}Q, k = 1, 2, 3, 4 \) as the four frontal slices of the \( 4 \times 4 \times 4 \) tensor \( \mathcal{X}(P, Q, l) = [x_{i,j,k}(P, Q, l)]_{i,j,k=1}^{3,3,4} \). Now \( \mathcal{Y} \) must satisfy the degree 6 polynomial conditions of Landsberg-Manivel and the degree 9 symmetrization conditions. Since the entries of \( P, Q \) are variables, this means that the coefficients of the monomials in the variables \( p_{ij}, q_{ij}, i, j = 1, 2, 3, 4 \) must vanish identically. This procedure gives rise to 10 polynomial conditions of degree 6 [6], which are linearly independent, and 440 polynomial conditions of degree 9 [5], which may be linearly dependent. Using appropriate software one may reduce the number of linearly independent conditions of degree 9.

The zero set of the above polynomials of degrees 5, 6 and 9 defines \( V_4(4, 4, 4) \).
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References


