



Hydrodynamics of charge fluctuations and balance functions

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We apply stochastic hydrodynamics to the study of charge-density fluctuations in QCD matter undergoing Bjorken expansion. We find that the charge-density correlations are given by a time integral over the history of the system, with the dominant contribution coming from the QCD crossover region where the change of susceptibility per entropy, $\chi T/s$, is most significant. We study the rapidity and azimuthal angle dependence of the resulting charge balance function using a simple analytic model of heavy-ion collision evolution. Our results are in agreement with experimental measurements, indicating that hydrodynamic fluctuations contribute significantly to the measured charge correlations in high-energy heavy-ion collisions. The sensitivity of the balance function to the value of the charge diffusion coefficient D allows us to estimate the typical value of this coefficient in the crossover region to be rather small, of the order of $(2\pi T)^{-1}$, characteristic of a strongly coupled plasma.

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I. INTRODUCTION

Event-by-event fluctuations and two particle correlations [1] in high-energy heavy-ion collision experiments provide valuable information about the collective dynamics: thermal and transport properties of the hot and dense QCD matter. Much recent effort has been devoted to the measurement and understanding of the correlations observed at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC). In this paper we focus on charge-dependent correlations. The suppression of the event-by-event fluctuations of the net (electric) charge has been proposed as a signature of the quark-gluon plasma (QGP) formation [2,3] and studied experimentally [4,5]. More differential measures of charge fluctuations, such as the azimuthal and rapidity dependence of charge-dependent correlations, have also attracted interest. The difference between like-sign and unlike-sign correlations, often expressed as balance functions [6–9], have also been studied experimentally [10–13].

In this paper, we apply relativistic stochastic hydrodynamics [14] to study the balance functions in heavy-ion collisions. Hydrodynamic equations describe the evolution of conserved quantities such as energy, momentum, and charge averaged over a statistical thermodynamic ensemble. Fluctuations around static equilibrium can be described using the fluctuation-dissipation theorem. In order to describe fluctuations around a *nonstatic* solution of the hydrodynamic equations (such as, e.g., Bjorken expansion) one can introduce local noise into the hydrodynamic equations, as has been proposed by Landau and Lifshitz [15] but has only recently been applied in the context of relativistic heavy-ion collisions [14]. The hydrodynamic evolution of such local noise naturally leads to observable particle correlations.

As emphasized in Ref. [14] (see also references therein), the hydrodynamic fluctuations are not the only source of observed correlations. Other sources include initial-state fluctuations, fluctuations induced by rare hard processes (jets), and final-state (freeze-out) fluctuations. These contributions remain

important in the case of charge correlations. However, the fact that the initial state in the ultrarelativistic heavy-ion collisions is dominated by saturated glue carrying no electric charge might suppress the contribution of initial-state fluctuations to charge correlations we discuss here compared to entropy fluctuations discussed in Ref. [14].

Under the conditions of the boost-invariant 1+1-dimensional Bjorken expansion [16], the effect of stochastic baryon number current at nonzero mean baryon density was studied in Ref. [14] without considering diffusion. Here we consider the effect of diffusion and a stochastic charge current at zero mean charge density. The analytical simplicity of the Bjorken solution allows us to understand in detail the mechanisms at work while using a phenomenologically reasonable description of a heavy-ion collision. As a step towards adequately addressing azimuthal dependence of correlations we introduce transverse expansion on top of the Bjorken solution using the standard “blast wave” model.

The paper is organized as follows: In Sec. II, we briefly review hydrodynamics with noise. We linearize the stochastic hydrodynamic equations (around the Bjorken solution at zero charge density) and analytically solve them. We find the charge correlations emerging as a superposition of contributions of past local noise sources which have diffused over the time separating the source and the observation. Successive contributions cancel each other unless the quantity $\chi T/s$ (more precisely $\chi T\tau$) is changing with time. Thus we find the dominant contribution coming from the crossover region of the QCD phase diagram where the effective degrees of freedom change from those of the quark-gluon plasma to those of the hadron gas. In Sec. II E, we use the lattice QCD data [17,18] to obtain the dependence of susceptibility per entropy $\chi T/s$ on temperature which determines the magnitude of the charge correlations. We apply these results to a simple semianalytical model of expansion with Cooper-Frye freeze-out and make an example comparison with experimental data from RHIC in Sec. III. We conclude with a discussion in Sec. IV.

II. HYDRODYNAMIC FLUCTUATIONS

A. Hydrodynamics and noise

Hydrodynamics describes the slow evolution of conserved quantities such as energy, momentum, and conserved charges. In the case of QCD the charge could be the baryon number, electric charge, or strangeness. At top energies at RHIC and LHC most particles in the final state are pions, which carry only electric charge. Therefore in this work we shall focus on electric charge fluctuations. The extension to other conserved charges such as baryon number or strangeness should be straightforward. The five hydrodynamic equations of motion are the conservation equations for energy momentum and charge

$$\nabla_\mu(T^{\mu\nu}) = 0, \quad \partial_\mu(\sqrt{-g}J^\mu) = 0. \quad (1)$$

Here ∇_μ denotes the covariant derivative with respect to the background metric $g_{\mu\nu}$ and $g \equiv \det[g_{\mu\nu}]$ —we shall only consider flat space time but use curvilinear (Bjorken) coordinates. Fluctuations are described by adding stochastic noise terms $S^{\mu\nu}$ and I^μ , as explained in Ref. [19] or, in the relativistic context, in Ref. [14] as follows:

$$\begin{aligned} T^{\mu\nu} &= T_{\text{ideal}}^{\mu\nu} + \Delta T^{\mu\nu} + S^{\mu\nu}, \\ J^\mu &= nu^\mu + \Delta J^\mu + I^\mu. \end{aligned} \quad (2)$$

Here n and u^μ are the charge density and fluid velocity, $T_{\text{ideal}}^{\mu\nu}$ is the stress-energy tensor for an ideal fluid, and $\Delta T^{\mu\nu}$ and ΔJ^μ are dissipative (gradient) corrections to stress and current. The dissipative correction to the current to the first order in gradients is given by

$$\Delta J^\mu = \sigma T \Delta^\mu \left(\frac{\mu}{T} \right), \quad (3)$$

where σ is the charge conductivity, μ is the chemical potential, and $\Delta^\mu \equiv \Delta^{\mu\nu} \partial_\nu$ ($\Delta^{\mu\nu} \equiv u^\mu u^\nu - g^{\mu\nu}$) is the spatial derivative in the local rest frame of the fluid (whose four-velocity is u^μ). The Einstein relation

$$D = \frac{\sigma}{\chi}, \quad (4)$$

where χ is the electric charge susceptibility

$$\chi \equiv \left(\frac{\partial n}{\partial \mu} \right)_T. \quad (5)$$

The hydrodynamic equations (1) are nonlinear. However, in the domain of applicability of hydrodynamics these equations can be linearized [15] in the perturbations around a given solution of the (nonlinear) deterministic equations of motion, i.e., Eqs. (1) without noise. Such a linearized approach is sufficient to study two-point correlations which are the subject of this paper.

To find two-point correlation functions of hydrodynamic variables we need to know the two-point correlation functions of the noise. One-point functions vanish by definition of the noise. The fluctuation-dissipation theorem determines the magnitude of the two-point correlator as

follows:

$$\langle I^\mu(x) \rangle = 0, \quad (6)$$

$$\langle I^\mu(x) I^\nu(y) \rangle = 2\sigma T \Delta^{\mu\nu} \delta(x - y), \quad (7)$$

where σ and T are functions of space and time given by the solution of the deterministic (without noise) hydrodynamic equations (1). The correlators of $S^{\mu\nu}$ are written down in Ref. [14], but we will not need them in this work.

Generalization to nonlinear treatment of fluctuations is an interesting problem, potentially relevant for the study of higher-point correlations or fluctuations near a critical point. Although linearized treatment is sufficient for the purposes of this paper, it is worth keeping in mind the issues involved in the nonlinear generalization. The most obvious issue is that the noise would become multiplicative since the magnitude of its correlator in Eq. (7) would be a function of the fluctuating hydrodynamic variables. The formal definition in Eq. (7) would have to be supplemented by a prescription (e.g., Ito or Stratonovich) to resolve the well-known equal-time product ambiguity (see, e.g., Ref. [20]). The nonlinearities also give rise to short-distance singularities [21] reminiscent of the ultraviolet divergences in quantum field theories. Such issues do not arise in the linearized treatment and we leave them outside of the scope of this paper.

B. Bjorken expansion and linear perturbations

We shall use the well-known boost-invariant Bjorken solution [16] of the hydrodynamic (1) as the background for the linearized fluctuation analysis. It is most convenient to describe the Bjorken flow in the coordinates $(\tau, \vec{x}_\perp, \eta)$ defined as

$$\tau \equiv \sqrt{t^2 - z^2}, \quad (8)$$

$$\eta \equiv \tanh^{-1} \left(\frac{z}{t} \right). \quad (9)$$

The Bjorken time τ is invariant under boosts along the z axis while Bjorken rapidity η shifts by a constant (the boost rapidity). The liquid undergoing boost-invariant expansion is locally at rest in these coordinates

$$\bar{u}^\mu(x) = \{1, \vec{0}, 0\}. \quad (10)$$

while the energy (or entropy) density is a function of τ , which can be found by solving an ordinary differential equation.

We denote the background quantities with an overbar and consider small perturbations to entropy density, flow velocity, and charge density expressed as $\rho \equiv \delta s/\bar{s}$, δu^x , δu^y , δu^η , and δn ,

$$\begin{aligned} \varepsilon(\tau, \vec{x}_\perp, \eta) &= \bar{\varepsilon}(\tau) + \bar{T}(\tau) \bar{s}(\tau) \rho(\tau, \vec{x}_\perp, \eta) \\ &\quad + \bar{\mu}(\tau) \delta n(\tau, \vec{x}_\perp, \eta), \end{aligned} \quad (11)$$

$$u^\mu(\tau, \vec{x}_\perp, \eta) = \{1, \delta \vec{u}_\perp(\tau, \vec{x}_\perp, \eta), \delta u^\eta(\tau, \vec{x}_\perp, \eta)\}, \quad (12)$$

$$n(\tau, \vec{x}_\perp, \eta) = \bar{n}(\tau) + \delta n(\tau, \vec{x}_\perp, \eta), \quad (13)$$

where, as in Eq. (10), we are working in the Bjorken coordinates. The quantity δu^τ vanishes at linear order due to the unit norm constraint $u_\mu u^\mu = 1$.

In general, the fluctuations of the charge and the energy density mix in Eq. (1). However, in the special case of zero background net charge density ($\bar{n} = 0$) or, equivalently, zero chemical potential ($\bar{\mu} = 0$) the fluctuations of the charge density δn separate, at linear order considered here, from the fluctuations of entropy density and flow velocity. Since we are going to study only fluctuations of charge density, this simplifies our task considerably. For top-energy RHIC collisions and at LHC the chemical potential is very small compared to relevant microscopic (QCD) scale and the approximation $\bar{\mu} = 0$ is useful. Since, as far as charge correlations are concerned, we can ignore entropy and flow velocity fluctuations, we shall no longer distinguish between quantities such as \bar{s} and s or \bar{T} and T .

The stochastic charge diffusion equation in Eq. (1) becomes

$$\partial_\tau J^\tau + \frac{J^\tau}{\tau} + \partial_\eta J^\eta + \vec{\nabla}_\perp \cdot \vec{J}_\perp = 0. \quad (14)$$

Since for the fluid locally at rest (10) the only derivatives in $\Delta^{\mu\nu} \nabla_\nu (\mu/T)$ are spatial and since T depends on τ only, we can simplify Eq. (3) for ΔJ^μ ,

$$\Delta J^\mu = \sigma \Delta^\mu \mu = D \Delta^\mu n, \quad (15)$$

which is Fick's law of diffusion. Substituting this into Eq. (14), we find

$$\frac{1}{\tau} \partial_\tau (\tau \delta n) - D \left[\nabla_\perp^2 + \frac{1}{\tau^2} \partial_\eta^2 \right] \delta n = -\nabla_i I^i - \nabla_\eta I^\eta. \quad (16)$$

We use Latin indices i, j, \dots to denote the two transverse directions.

To facilitate the analysis of azimuthal correlations it is useful to decompose the noise current in the transverse plane as¹

$$I^i = \tau \nabla_j [g^{ij} I_S - \epsilon^{ij} I_V]. \quad (17)$$

Only I_S will contribute to Eq. (16). In order to solve Eq. (16), we express \vec{x}_\perp in polar coordinates r and ϕ and use a Fourier-Bessel transformation for δn , I^η , and I_S , which we define, for any function f , as

$$f(\tau, r, \phi, \eta) = \sum_n \frac{e^{in\phi}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_\eta}{2\pi} e^{ik_\eta \eta} \int_0^\infty dk_\perp k_\perp \times J_n(k_\perp r) \tilde{f}_n(\tau, k_\perp, k_\eta), \quad (18)$$

$$\tilde{f}_n(\tau, q_\perp, q_\eta) = \int_0^{2\pi} e^{-in\phi} d\phi \int_{-\infty}^{\infty} e^{-iq_\eta \eta} d\eta \int_0^\infty r dr \times J_n(q_\perp r) f(\tau, r, \phi, \eta). \quad (19)$$

Inverting the transformation requires the closure relation

$$\int_0^\infty r dr J_n(k_\perp r) J_n(q_\perp r) = \frac{\delta(k_\perp - q_\perp)}{k_\perp}. \quad (20)$$

¹We are splitting a two-component vector into the gradient of a scalar, I_S , and a divergenceless two-vector, which, in two dimensions can also be written in terms of a scalar I_V .

C. Solution and correlations

The charge-density fluctuation at a time τ_f sourced by the hydrodynamic noise I is given by (upon Fourier-Bessel transform)

$$\delta \tilde{n}_n(\tau_f, k_\perp, k_\eta) = \frac{1}{\tau_f} \int_{\tau_0}^{\tau_f} d\tau [(\tau k_\perp)^2 \tilde{I}_{S,n} - i\tau k_\eta \tilde{I}_n^\eta] e^{-H(\tau_f, \tau, k_\perp, k_\eta)}, \quad (21)$$

where we defined

$$H(\tau_f, \tau, k_\perp, k_\eta) \equiv \int_\tau^{\tau_f} d\tau' D(\tau') \left(\frac{k_\eta^2}{\tau'^2} + k_\perp^2 \right). \quad (22)$$

The coefficients of k_\perp^2 and k_η^2 in Eq. (22) are the squared diffusion distances in the x_\perp and η directions. It is easy to see that by considering the equation $dl^2 = Ddt$ for the diffusion (random walk) distance dl in time dt in a locally comoving frame. The length element in Bjorken coordinates is $dl^2 = \tau^2 d\eta^2 + dx_\perp^2$ and the time element is $dt = d\tau$. Thus the diffusion distance squared in the rapidity direction is given by $(\Delta\eta)^2 = \int d\tau D/\tau^2$ and in the transverse direction by $(\Delta x_\perp)^2 = \int d\tau D$.

To determine the charge-density correlations, one needs Fourier-Bessel transform of the noise correlators in Eq. (6),

$$\langle \tilde{I}_n^\eta(\tau, k_\perp, k_\eta) \tilde{I}_m^{*\eta}(\tau', q_\perp, q_\eta) \rangle = \frac{2\sigma T}{\tau^3} \delta(\tau - \tau') \hat{\delta}_{nm}(k, q), \quad (23)$$

$$\langle \tilde{I}_{S,n}(\tau, k_\perp, k_\eta) \tilde{I}_{S,m}^*(\tau', q_\perp, q_\eta) \rangle = \frac{2\sigma T}{\tau^3 k_\perp^2} \delta(\tau - \tau') \hat{\delta}_{nm}(k, q). \quad (24)$$

Here we introduced a shorthand

$$\hat{\delta}_{nm}(k, q) \equiv (2\pi)^2 \delta_{n,m} \delta(k_\eta - q_\eta) \frac{\delta(k_\perp - q_\perp)}{k_\perp}. \quad (25)$$

Using Eqs. (21), (23), and (24) we find for the charge-density correlations (at equal time τ_f)

$$\begin{aligned} \langle \delta \tilde{n}_n(\tau_f, k_\perp, k_\eta) \delta \tilde{n}_m^*(\tau_f, q_\perp, q_\eta) \rangle \\ = \hat{\delta}_{nm}(k, q) \frac{1}{\tau_f^2} \int_{\tau_0}^{\tau_f} \frac{2\sigma(\tau) T(\tau)}{\tau} [\tau^2 k_\perp^2 + k_\eta^2] e^{-2H(\tau_f, \tau, k_\perp, k_\eta)} d\tau. \end{aligned} \quad (26)$$

With the aid of Eq. (4), Eq. (26) can be written as

$$\begin{aligned} \langle \delta \tilde{n}_n(\tau_f, k_\perp, k_\eta) \delta \tilde{n}_m^*(\tau_f, q_\perp, q_\eta) \rangle \\ = \hat{\delta}_{nm}(k, q) \frac{1}{\tau_f^2} \int_{\tau_0}^{\tau_f} \chi(\tau) T(\tau) \tau \frac{d}{d\tau} e^{-2H(\tau_f, \tau, k_\perp, k_\eta)} d\tau. \end{aligned} \quad (27)$$

Finally, performing an integration by parts, we find

$$\begin{aligned} \langle \delta \tilde{n}_n(\tau_f, k_\perp, k_\eta) \delta \tilde{n}_m^*(\tau_f, q_\perp, q_\eta) \rangle \\ = \hat{\delta}_{mn}(k, q) \frac{1}{\tau_f} [\chi_f T_f - s_f \tilde{A}(\tau_f, k_\perp, k_\eta)], \end{aligned} \quad (28)$$

where we defined a dimensionless function in Fourier-Bessel space as

$$\tilde{A}(\tau_f, k_\perp, k_\eta) \equiv \frac{1}{s_f \tau_f} \left[\chi_0 T_0 \tau_0 e^{-2H(\tau_f, \tau_0, k_\perp, k_\eta)} + \int_{\tau_0}^{\tau_f} d\tau e^{-2H(\tau_f, \tau, k_\perp, k_\eta)} \frac{d(\chi T \tau)}{d\tau} \right]. \quad (29)$$

Note that

$$\tilde{A}(\tau_f, k_\perp = 0, k_\eta = 0) = \frac{\chi_f T_f}{s_f}. \quad (30)$$

This, according to Eq. (28), implies the vanishing of $\langle \delta n(x) \delta n(y) \rangle$ integrated over all space [this is trivially seen also in Eq. (26)], which is a consequence of charge conservation.

It is useful to rewrite the integral over τ in Eq. (29) in terms of an integral over temperature T , which is straightforward. It is also useful to rewrite the integral in the definition of H in Eq. (22) in the same way. For that we need to know $dT/d\tau$. We shall neglect viscous corrections and use the ideal hydro equation $\tau s = \text{const}$, or

$$\frac{dT}{d\tau} + \frac{T v_s^2(T)}{\tau} = 0, \quad (31)$$

where $v_s^2 \equiv \partial P / \partial \varepsilon$ is the speed of sound (at zero charge density). Then

$$H(T_f, T, k_\perp, k_\eta) = \int_{T_f}^T \left[\frac{D(T') s(T')}{\tau_f s_f} k_\eta^2 + \frac{s_f \tau_f D(T')}{s(T')} k_\perp^2 \right] \frac{dT'}{v_s^2(T') T'}. \quad (32)$$

For convenience, we define dimensionless quantities $\hat{D}(T) \equiv DT$ and $\hat{s}(T) \equiv s(T)/T^3$. Then $H(T_f, T, k_\perp, k_\eta)$ can be written as

$$H(T_f, T, k_\perp, k_\eta) = \frac{k_\eta^2}{\tau_f s_f} \int_{T_f}^T \frac{\hat{D}(T') \hat{s}(T') T'}{v_s^2(T')} dT' + s_f \tau_f k_\perp^2 \int_{T_f}^T \frac{\hat{D}(T')}{v_s^2(T') \hat{s}(T') (T')^5} dT', \quad (33)$$

where the function \hat{s} can be taken directly from lattice data (see Sec. II E). The speed of sound (at $n = 0$) can also be found from \hat{s} ,

$$v_s^2(T) = \left(3 + \frac{d \ln \hat{s}}{d \ln T} \right)^{-1}. \quad (34)$$

Combining Eq. (29) and Eq. (33), we find

$$\tilde{A}(\tau_f, k_\perp, k_\eta) = \left[\frac{\chi_0 T_0}{s_0} e^{-2H(T_f, T_0, k_\perp, k_\eta)} - \int_{T_f}^{T_0} dT \frac{d}{dT} \left(\frac{\chi T}{s} \right) e^{-2H(T_f, T, k_\perp, k_\eta)} \right]. \quad (35)$$

D. Discussion and interpretation

Equation (28) along with the definitions Eqs. (33) and (35) describe the charge-density correlation due to stochastic noise and is the main result of this section.

The correlator in the square brackets in Eq. (28) naturally separates into a local part (the first term, which is independent of k and is thus a δ function in position space) and the nonlocal part [the second term, given by Eq. (35), which vanishes at large k].

For a Boltzmann gas (which is a good approximation at freeze-out) we can identify the local term with the equilibrium self-correlation, which exists even in a noninteracting gas. On general grounds, the correlation function of a gas of particles in equilibrium is expected to have such a δ -function term [15, 19],

$$\langle \delta n(\vec{x}_1) \delta n(\vec{x}_2) \rangle = \bar{n} \delta^3(\vec{x}_1 - \vec{x}_2) + \dots \quad (36)$$

where “ \dots ” denotes the correlations from interactions. This δ function is not due to a correlation between *two* particles, as it is present even in a free gas. It is a trivial manifestation of statistical *fluctuations* in the gas, a reflection of the fact that particles are trivially correlated with themselves (see Sec. 116 of Ref. [19]). In a free Boltzmann gas $\bar{n} = \chi T$, which is exactly the factor appearing in Eq. (28). The factor of τ_f^{-1} in Eq. (28) is the volume Jacobian factor, $1/\sqrt{-g}$, for the δ function in Bjorken coordinates. Because experimental measures count only *two*-particle correlations it is necessary to separate the self-correlation term before comparing with the data. The separation of such a self-correlation term has been also discussed in Ref. [9] but not in Refs. [14, 22].

We now turn to the nonlocal contribution to the correlator in Eq. (35). We note that the dimensionless quantity $\chi T/s$ (charge susceptibility per entropy) and its T dependence plays an important role. The first term in \tilde{A} is a three-dimensional negative Gaussian with width (in position space) given by the diffusion distance over the entire expansion history (since τ_0) and a magnitude controlled by the initial value of $\chi_0 T_0/s_0$. If $\chi T/s$ were constant, which would be the case in a conformal theory, and is approximately the case in high-temperature QGP, all nontrivial correlations resulting from the diffusion history of hydrodynamical fluctuations would be contained in this negative Gaussian.

The second term in Eq. (29) is due to the change of $\chi T/s$. This term is a superposition of many Gaussians with different widths and amplitudes. Because this contribution clearly requires the (charge-carrying) constituents of the plasma to change, its main contribution comes from the QCD crossover region. We also find that this term gives the dominant contribution to the charge correlations in heavy-ion collisions. It is, therefore, essential for our calculation to know $\chi T/s$ throughout the history of a heavy-ion collision, especially in the crossover region, which is the subject of the next section.

E. The temperature dependence of susceptibility per entropy

The behavior of entropy density s and charge susceptibility χ as a function of temperature is easy to understand qualitatively and semiquantitatively. In the crossover region the QCD matter undergoes a smooth transition from the hadron gas to the QGP state. This leads to a significant increase in the number of degrees of freedom (liberation of color), i.e., growth of s/T^3 . Although the number of charged degrees of freedom also increases, their average charge is smaller in QGP, and as a result the growth of χ/T^2 is only moderate. The

growth of s/T^3 is much more significant (due to the gluons) and as a result the dimensionless ratio $\chi T/s$ decreases with temperature in the crossover region.

To make the above description more quantitative we can estimate $\chi T/s$ in the QGP by considering ideal massless gases of gluons and quarks (3 massless flavors). Although this approximation is only valid for asymptotically high T , it is sufficient for our illustrative purposes. A straightforward calculation leads to

$$\text{QGP: } \frac{s}{T^3} = \frac{19\pi^2}{9}; \quad \frac{\chi}{T^2} = \frac{2}{3}; \quad \frac{\chi T}{s} = \frac{6}{19\pi^2}. \quad (37)$$

For a rough estimate of these quantities in the hadron gas phase we can take ideal gas of massless pions, for which we find

$$\text{Pion gas: } \frac{s}{T^3} = \frac{2\pi^2}{15}; \quad \frac{\chi}{T^2} = \frac{1}{3}; \quad \frac{\chi T}{s} = \frac{5}{2\pi^2}. \quad (38)$$

We see that an approximately 16-fold increase of s/T^3 in QGP relative to the pion gas overwhelms the only 2-fold increase of χ/T^2 , leading to a significant decrease of $\chi T/s$.

These simple estimates are in qualitative and semiquantitative agreement with lattice QCD calculations [17,23–26]

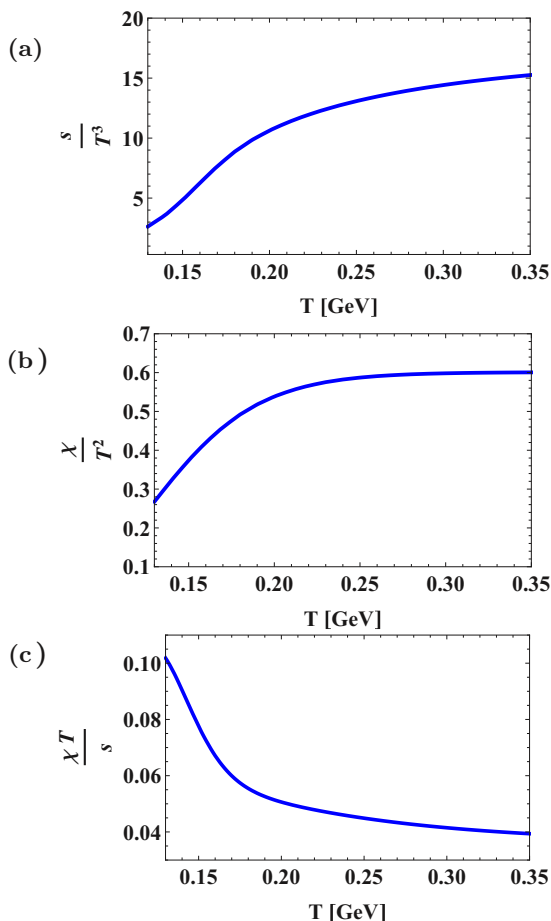


FIG. 1. (Color online) The temperature dependence (in units of T) of entropy (s/T^3), charge susceptibility (χ/T^2) taken from the lattice data [17,18], and the resulting charge susceptibility per entropy ($\chi T/s$) which we use in this paper.

which show that both entropy density and electric charge susceptibility change significantly in the crossover region. Figure 1 shows our attempt to extract the ratio $\chi T/s$ from these lattice results. In our exploratory analysis we shall ignore statistical or systematic errors on these data and use equation of state shown in Fig. 1 in our computations.

III. TOWARDS COMPARISON WITH EXPERIMENT

With the lattice data on electric susceptibility χ , the entropy density s , and the charge diffusion coefficient² D , one can use the results of Sec. II B to determine the spatial correlations of the net charge due to hydrodynamic fluctuations. These position space correlations need to be translated into particle momentum space correlations which are measured experimentally in a heavy-ion collision. To achieve this goal we need to address several important issues.

A. Partial chemical equilibrium

The lattice equation of state, discussed in Sec. II E, describes QCD matter in full thermal and chemical equilibrium. Although this is a reasonable approximation during much of the expansion history, it breaks down after chemical freeze-out. The rate of reactions responsible for chemical equilibration (inelastic collisions) is too slow in the hadronic phase to maintain chemical equilibrium. However, the thermal (kinetic) equilibrium is maintained until later times. In the intermediate region between the chemical and kinetic freeze-out the matter can be described using the so-called partial chemical equilibrium (PCE) equation of state [27].

Rather than using the PCE equation of state we shall use a simpler approach, based on the observation in Ref. [28] that the PCE equation of state expressed as pressure versus energy density is very similar to the full equilibrium (FE) equation of state. Since hydrodynamic equations involve the equation of state $P(\varepsilon)$, their solution under PCE should be similar to their solution under FE.³ The difference is manifested when we ask what the temperature is, given a value of the energy density, i.e., at a given point in time in the expansion history. Thus the actual kinetic freeze-out temperature T_{kf} , which determines the final (observed) momentum spectra of the particles, differs from the temperature, T_{hf} which would correspond to the final energy density in the full equilibrium equation of state. The results of Ref. [28] suggest that for the kinetic freeze-out temperature $T_{\text{kf}} \approx 100$ MeV the reasonable choice of the corresponding temperature at which the FE equation of state gives the same energy density is around

²Lattice data on the charge conductivity or the diffusion coefficient D is subject to more uncertainties due to the analytic continuation from imaginary to real time and will be discussed at the end of this section and in Sec. IV.

³To maintain simplicity and transparency of our results we make an assumption that the same is true for the charge susceptibility χT . The charge susceptibility was not discussed in Ref. [28], and it would be interesting to test this natural assumption by applying the same methods to calculate χT . Deviations from this assumption can be included in our approach if necessary.

$T_{\text{hf}} \approx 130$ MeV. In this approach one ends hydrodynamic evolution at a final temperature (hydrodynamic freeze-out) T_{hf} and implements the freeze-out procedure with the momentum spectra of particles determined by T_{kf} [28].

Additional conservation laws emerging under PCE are reflected in the appearance of corresponding chemical potentials. For our study we will need to use a chemical potential for the total pion number (the sum of the numbers of π^+ and π^-), μ_π . The value of this chemical potential has also been estimated in Ref. [28] in the range of $\mu_\pi \sim 80$ MeV. We shall use the above values for T_{kf} , T_{hf} , and μ_π in our calculations of the balance functions below.

B. Transverse expansion

Since significant contribution to the balance function comes from the crossover region, we must also take into account the fact that, due to finite transverse size of the colliding nuclei, the expansion does not remain purely Bjorken. Radial flow becomes significant at times τ of order the initial radius of the nucleus R_0 and the expansion approaches isotropic 3D expansion at later times. As a result the entropy density drops much faster, approximately as τ^{-3} during this later period, as opposed to τ^{-1} during the 1D Bjorken period [29,30].

Since analytical treatment of the full 3D expansion is beyond our reach, we consider a simple idealized approximation by assuming that the 3D stage of the expansion is short enough (due to fast drop in entropy and, thus, temperature) that the diffusion during this period can be neglected. We shall refer to this picture as the sudden transverse expansion approximation. In this idealized picture of the collision the expansion follows the 1D Bjorken solution until a point in time which we denote τ_{1D} , upon which it undergoes sudden transverse expansion and freezes out shortly thereafter at time τ_f with a pattern of flow given by the blast-wave ansatz.

We determine the charge correlator at time τ_{1D} (instead of τ_f) using Eq. (28) and Eq. (29) derived under conditions of the 1D Bjorken expansion

$$\begin{aligned} & \langle \delta \tilde{n}_n(\tau_{1D}, k_\perp, k_\eta) \delta \tilde{n}_m^*(\tau_{1D}, q_\perp, q_\eta) \rangle \\ &= \hat{\delta}_{mn}(k, q) \frac{s_{1D}}{\tau_{1D}} \left[\left(\frac{\chi T}{s} \right)_{1D} - \tilde{A}(\tau_{1D}, k_\perp, k_\eta) \right], \end{aligned} \quad (39)$$

where $\tilde{A}(\tau_{1D}, k_\perp, k_\eta)$ is given by Eq. (35) with τ_f replaced with τ_{1D} . We then should treat Eq. (39) as the initial condition for the period of the ‘‘sudden’’ 3D expansion. This fast expansion is essentially adiabatic and thus the ratio n/s is conserved. Since the entropy density drops from s_{1D} to s_f during this period, the charge density must also drop by the same factor. This means the charge correlator at time τ_f obeys

$$\langle \delta \tilde{n}_n \delta \tilde{n}_m^* \rangle_f = \frac{s_f^2}{s_{1D}^2} \langle \delta \tilde{n}_n \delta \tilde{n}_m^* \rangle_{1D}, \quad (40)$$

where $\langle \delta \tilde{n}_n \delta \tilde{n}_m^* \rangle_{1D}$ is given by Eq. (39). Thus

$$\begin{aligned} \langle \delta \tilde{n}_n \delta \tilde{n}_m^* \rangle_f &= \hat{\delta}_{mn}(k, q) \frac{s_f}{\tau_f} \frac{s_f \tau_f}{s_{1D} \tau_{1D}} \\ &\times \left[\left(\frac{\chi T}{s} \right)_{1D} - \tilde{A}(\tau_{1D}, k_\perp, k_\eta) \right]. \end{aligned} \quad (41)$$

The first (local in position space) term in Eq. (41) contains the contribution of the self-correlation which we need to subtract. As discussed in Sec. IID, this self-correlation term is given by $\hat{\delta}_{mn}(k, q)(\chi T/\tau)_f$. Thus we write the charge-density correlations at freeze-out as

$$\langle \delta \tilde{n}_n \delta \tilde{n}_m^* \rangle_f \equiv \hat{\delta}_{mn}(k, q) \left[\frac{\chi T_f}{\tau_f} - \frac{s_f}{\tau_f} \tilde{A}_f(k_\perp, k_\eta) \right], \quad (42)$$

which defines two-particle hydrodynamic correlator $\tilde{A}_f(k_\perp, k_\eta)$ at freeze-out. Comparing Eq. (41) and Eq. (42), we find

$$\begin{aligned} \tilde{A}_f(k_\perp, k_\eta) &= \frac{s_f \tau_f}{s_{1D} \tau_{1D}} \tilde{A}(\tau_{1D}, k_\perp, k_\eta) \\ &+ \left(\frac{\chi T}{s} \right)_f - \frac{s_f \tau_f}{s_{1D} \tau_{1D}} \left(\frac{\chi T}{s} \right)_{1D} \end{aligned} \quad (43)$$

and use it to calculate the balance function later in this section.

From Eq. (43) we see that the density-density correlations built during the 1D Bjorken expansion period are diluted due to the transverse expansion by a factor $(s_f \tau_f)/(s_{1D} \tau_{1D})$ which would be equal to 1 if the system continued pure 1D expansion until freeze-out. Furthermore, the correlator \tilde{A}_f contains a local term, independent of k , because the last two terms in Eq. (43) do not cancel. This is the contribution of the noise from the period of the sudden transverse expansion. It is represented by a δ function in position space because the noise is local and we neglected diffusion during this short time, which would otherwise broaden the δ function.

C. Cooper-Frye freeze-out

In order to compare our results with experimental measurements, we need to translate the hydrodynamic correlations in position space into correlations in the kinematic (momentum) space of the observed particles. For this purpose we use the standard Cooper-Frye prescription for pions,

$$\frac{dN_Q}{dyd\phi} = \frac{1}{(2\pi)^3} \int p_\perp dp_\perp \int d\sigma_\mu p^\mu f_Q(\vec{x}, \vec{p}), \quad (44)$$

where $f_Q = \exp\{Q\mu/T_{\text{hf}} + \mu_\pi/T_{\text{kf}} - p_\mu u^\mu/T_{\text{kf}}\}$ is the equilibrium distribution function for pions carrying charge Q (equal to ± 1) in the Boltzmann approximation.⁴

We have also defined kinematic rapidity as $y \equiv \tanh^{-1}(p^z/E)$ and kinematic azimuthal angle as $\phi \equiv \tan^{-1}(p^y/p^x)$ and denoted the freeze-out hypersurface normal four-vector as $d\sigma_\mu$. The p_\perp integration range is determined by experimental p_\perp cuts. We choose an isochronous freeze-out condition⁵ at $\tau = \tau_f$, thus

$$d\sigma_\mu p^\mu = \tau_f m_\perp d^2x_\perp d\eta \cosh(y - \eta), \quad (45)$$

⁴The factors $1/T$ accompanying μ and μ_π in f_Q reflect the definitions of these chemical potentials. While μ is defined in terms of the FE equation of state used in hydrodynamics, the potential μ_π accounts for the pion excess at kinetic freeze-out due to PCE.

⁵For net charge correlations at zero chemical potential this is equivalent to isothermal freeze-out because fluctuations of temperature do not mix with charge fluctuations.

where $m_{\perp} \equiv \sqrt{p_{\perp}^2 + m^2}$, with m being the rest mass of the pion.

Since we are interested in the effect of the hydrodynamic fluctuations, we expand the distribution function to linear order in fluctuations of temperature, chemical potential, and fluid velocity. If the average of the net chemical potential $\bar{\mu}$ is 0, then only the chemical potential fluctuation survives in the difference between particles and antiparticles,

$$\delta \frac{dN_{\text{net}}}{dyd\phi} = \frac{2\tau_f}{(2\pi)^3 T_{\text{hf}}} \int m_{\perp}^2 dm_{\perp} \int d^2x_{\perp} \int d\eta \times \delta\mu(\tau_f, \vec{x}_{\perp}, \eta) \cosh(y - \eta) f_0(\vec{x}, \vec{p}), \quad (46)$$

where f_0 is the Boltzmann distribution function at $\mu = 0$ and

$$N_{\text{net}} \equiv N_+ - N_-. \quad (47)$$

Fluctuations of chemical potential are related to those of the charge density by $\delta n = \chi \delta\mu$.

D. Blast wave

As we already discussed in Sec. III B, finiteness of the transverse size of the system leads to transverse expansion. We shall describe the transverse flow velocity profile $v_r(r)$ using transverse rapidity $\kappa(r)$

$$v_r(r) = \frac{u^r}{u^{\tau}} \equiv \tanh \kappa(r). \quad (48)$$

The distribution function f_0 can be then written as [31]

$$f_0(\vec{x}, \vec{p}) = \exp\{\hat{\mu}_{\pi} + \hat{p}_{\perp} \cos(\phi - \psi) \sinh \kappa_f(r) - \hat{m}_{\perp} \cosh(y - \eta) \cosh \kappa_f(r)\}, \quad (49)$$

where $\kappa_f(r)$ describes the radial flow profile at kinetic freeze-out and ψ is the position space azimuthal angle characterizing the direction of the radius-vector \vec{x} , and we introduced convenient dimensionless variables as follows:

$$\hat{\mu}_{\pi} = \mu_{\pi}/T_{\text{kf}} \quad \hat{m}_{\perp} = m_{\perp}/T_{\text{kf}}, \quad \hat{p}_{\perp} = p_{\perp}/T_{\text{kf}}. \quad (50)$$

We apply the standard blast-wave approach, i.e., we specify the radial flow profile $\kappa(r)$ at freeze-out by hand (as a linear function of r) and limit the transverse size of the system: $r < R$. Such an approach is known to provide a reasonable approximation to single particle observables computed using a full hydrodynamic solution which includes transverse expansion [32].

Finally, we have

$$\delta \frac{dN_{\text{net}}}{dyd\phi} = \frac{\tau_f T_{\text{kf}}^3 R^2}{\chi_{\text{hf}} T_{\text{hf}}} \int d^2\vec{x}_{\perp} \int_{-\infty}^{\infty} d\eta \delta n(\tau_f, \vec{x}_{\perp}, \eta) F(\vec{x}, \vec{p}), \quad (51)$$

where we introduced the function

$$F(\vec{x}, \vec{p}) \equiv \frac{1}{4\pi^3 R^2} \int \hat{m}_{\perp}^2 d\hat{m}_{\perp} \cosh(y - \eta) f_0(\vec{x}, \vec{p}) \Theta(R - r), \quad (52)$$

which acts as a kernel of the transformation from the position variables \vec{x} to kinematic variables \vec{p} . We normalized F in such a way that its Fourier-Bessel transform is dimensionless (see below).

To proceed, we introduce Fourier-Bessel expansions for both δn and F in Eq. (51). Due to azimuthal and boost invariance [and integration over m_{\perp} in Eq. (52)] the function $F(\vec{x}, \vec{p})$ depends only on three arguments: r and the differences $\phi - \psi$, and $y - \eta$. We define the Fourier-Bessel transform with respect to these three variables as $\tilde{F}_n(k_{\perp}, k_y)$ in terms of which we find, substituting Eq. (18) and using the closure relation (20)

$$\delta \frac{dN_{\text{net}}}{dyd\phi} = \frac{\tau_f T_{\text{kf}}^3 R^2}{\chi_{\text{hf}} T_{\text{hf}}} \int_{-\infty}^{\infty} \frac{dk_y e^{ik_y y}}{2\pi} \sum_n \frac{e^{in\phi}}{2\pi} \int_0^{\infty} k_{\perp} dk_{\perp} \times \delta \tilde{n}_n(\tau_f, k_{\perp}, k_y) \tilde{F}_n(k_{\perp}, k_y). \quad (53)$$

For a given transverse flow profile at freeze-out $\kappa_f(r)$ in Eq. (48) we can obtain an expression for $\tilde{F}_n(k_{\perp}, k_y)$ by substituting Eq. (49) into Eq. (52) and integrating over variables $(\phi - \psi)$ and $(y - \eta)$ in the definition of the Fourier-Bessel transform Eq. (19)

$$\tilde{F}_n(k_{\perp}, k_y) = -\frac{e^{i\hat{\mu}_{\pi}}}{\pi^2} \int \hat{m}_{\perp}^2 d\hat{m}_{\perp} \int_0^1 \hat{r} d\hat{r} J_n(\hat{k}_{\perp} \hat{r}) \times \mathcal{I}_n(\hat{p}_{\perp} \sinh \kappa_f) \mathcal{K}'_{ik_y}(\hat{m}_{\perp} \cosh \kappa_f), \quad (54)$$

where \mathcal{I} is a modified Bessel function, \mathcal{K}' is the derivative of a modified Bessel function with respect to its argument and we used convenient dimensionless variables defined in Eq. (50) as well as

$$\hat{k}_{\perp} = k_{\perp} R \quad \text{and} \quad \hat{r} = r/R. \quad (55)$$

It is also useful to note that the average value of the total number of charged pions

$$N_{\text{ch}} \equiv N_+ + N_- \quad (56)$$

per unit rapidity and azimuthal angle given by

$$\left\langle \frac{dN_{\text{ch}}}{dyd\phi} \right\rangle = \frac{2\tau_f}{(2\pi)^3} \int m_{\perp}^2 dm_{\perp} \int_{-\infty}^{\infty} d\eta \int d^2x_{\perp} \times \cosh(y - \eta) f_0(\vec{x}, \vec{p}) \Theta(R - r) \quad (57)$$

can be also expressed as

$$\left\langle \frac{dN_{\text{ch}}}{dyd\phi} \right\rangle = \tau_f T_{\text{kf}}^3 R^2 \tilde{F}_0(0, 0). \quad (58)$$

E. Particle correlations and balance function

Finally, to determine the particle correlations, we multiply two fluctuations given by Eq. (53), average over events, and express the correlator $\langle \delta \tilde{n}_n \delta \tilde{n}_m \rangle$ using Eq. (28), with the self-correlation subtracted. The δ functions in $\hat{\delta}_{mn}$ ensure that the result is only a function of the rapidity difference $\Delta y \equiv y_2 - y_1$ and angular difference $\Delta\phi = \phi_2 - \phi_1$ and we find

$$\begin{aligned} & \left\langle \delta \frac{dN_{\text{net}}}{dy_1 d\phi_1} \delta \frac{dN_{\text{net}}}{dy_2 d\phi_2} \right\rangle \\ &= -\left(\frac{T_{\text{kf}}^3}{\chi_{\text{hf}} T_{\text{hf}}} \right)^2 s_f \tau_f R^2 \int_{-\infty}^{\infty} \frac{dk_y e^{ik_y \Delta y}}{2\pi} \sum_n \frac{e^{in\Delta\phi}}{2\pi} \int_0^{\infty} \hat{k}_{\perp} d\hat{k}_{\perp} \\ & \quad \times \tilde{A}_f(k_{\perp}, k_y) \tilde{F}_n(k_{\perp}, k_y) \tilde{F}_n^*(k_{\perp}, k_y), \end{aligned} \quad (59)$$

where $k_{\perp} = \hat{k}_{\perp}/R$ as in Eq. (55).

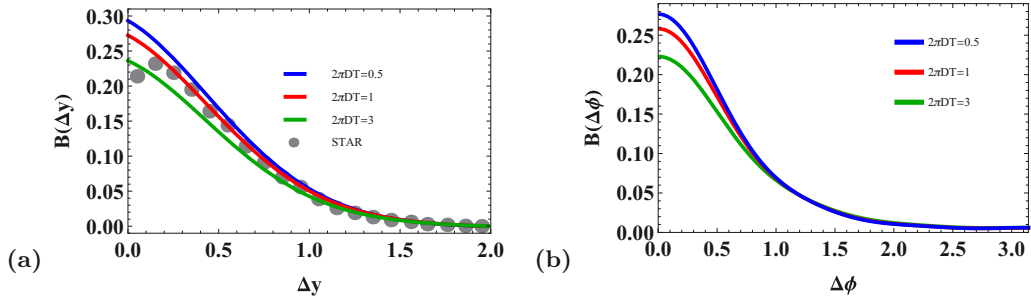


FIG. 2. (Color online) (Left) Rapidity projection of the pion balance function for different values of the diffusion coefficient. To compare with the experimental data from STAR [12] we applied momentum cuts $0.2 \text{ GeV} \leq p_{\perp} \leq 0.6 \text{ GeV}$, $|y| < 1$ and efficiency correction 80%. (Right) azimuthal projection of the pion balance function for $0.2 \text{ GeV} \leq p_{\perp} \leq 2 \text{ GeV}$.

When $\langle N_{+} \rangle = \langle N_{-} \rangle$, the correlator in Eq. (59) is related to the balance function defined in Refs. [6,7] by

$$B(\Delta y, \Delta \phi) \equiv - \left\langle \delta \frac{dN_{\text{net}}}{dy_1 d\phi_1} \delta \frac{dN_{\text{net}}}{dy_2 d\phi_2} \right\rangle \left\langle \frac{dN_{\text{ch}}}{dy d\phi} \right\rangle^{-1}. \quad (60)$$

This relationship is derived in the appendix. Finally, putting Eqs. (60), (59), and (58) together, we find the following expression for the balance function:

$$B(\Delta y, \Delta \phi) = \frac{T_{\text{kt}}^3 s_f}{\chi_{\text{hf}}^2 T_{\text{hf}}^2 \tilde{F}_0(0,0)} \int_{-\infty}^{\infty} \frac{dk_y e^{ik_y \Delta y}}{2\pi} \sum_n \frac{e^{in\Delta\phi}}{2\pi} \int_0^{\infty} \hat{k}_{\perp} d\hat{k}_{\perp} \times \tilde{F}_n(k_{\perp}, k_y) \tilde{F}_n^*(k_{\perp}, k_y) \tilde{A}_f(k_{\perp}, k_y). \quad (61)$$

We use Eq. (61) with $\tilde{A}_f(k_{\perp}, k_y)$ given by Eq. (43) to calculate the balance functions in the next section.

We can calculate the rapidity and the azimuthal projections of the balance function

$$B(\Delta y) = \int_{-\pi}^{\pi} d\Delta\phi B(\Delta y, \Delta\phi), \quad (62)$$

$$B(\Delta\phi) = \int_{-\infty}^{\infty} d\Delta y B(\Delta y, \Delta\phi).$$

Integration over $\Delta\phi$ is equivalent to only considering the $n = 0$ moment in the summation of Eq. (61), while integration over Δy is equivalent to setting $k_y = 0$ instead of performing an integral over k_y .

F. Results

In order to illustrate the typical shape, width, and magnitude of the balance function arising due to the hydrodynamic fluctuations we calculate this function using our semianalytical model of expansion described above. For central collisions at top RHIC energies, we set the time when expansion stops being purely one dimensional to $\tau_{1D} = 7 \text{ fm}$ and the corresponding temperature to $T_{1D} = 150 \text{ MeV}$. We set the initial temperature to $T_0 = 350 \text{ MeV}$. The hydro freeze-out temperature (see Sec. III A) is taken to be $T_{\text{hf}} = 130 \text{ MeV}$ [28]. We use the lattice data on entropy density $s(T)$ [17] and electric charge susceptibility $\chi(T)$ [18] as in Fig. 1. We set the blast-wave transverse flow profile to be linear $v_r = \frac{3}{2} \langle \beta \rangle r / R$

with $\langle \beta \rangle = 0.6$ and maximum radius $R = 12 \text{ fm}$ at $\tau_f = 12 \text{ fm}$ [28,33,34].

In Fig. 2, we show the sensitivity of the balance function to the charge diffusion coefficient, taking the dimensionless combination DT to be constant, with other parameters fixed. In particular, we see that, for chosen values of parameters, the data favor small values of the diffusion coefficient, $2\pi DT \sim 1$, which is characteristic of a strongly coupled medium (short mean free path). Clearly, our semiquantitative analysis is not sufficient to pin down the value of the diffusion coefficient with adequate precision, due to the balance function's sensitivity to parameters which we fixed by hand (using typical values obtained in numerical hydro simulations). However, our results are indicative of the typical resolution one could achieve if a more realistic numerical hydrodynamic simulation were to be used instead of our simplified analytical model. We leave such quantitative investigations to future work.

IV. CONCLUSIONS AND DISCUSSION

We showed that intrinsic hydrodynamic noise induces correlations of charge fluctuations which are observable and typically quantified in terms of the charge balance functions. We have shown how to calculate the noise contribution to the balance function and applied our method to a semianalytical model of hydrodynamic expansion. The balance functions we obtain are in reasonable agreement with experiments and our results suggest that a more realistic calculation may allow one to determine or constrain the charge diffusion coefficient D . Our semiquantitative analysis indicates that a small value of D , characteristic of the strongly coupled medium, is favored by the data.

Two main observations characterize the effect of the hydrodynamic noise and diffusion on the charge balance functions. We find that the magnitude of the balance function receives the most significant contribution from the time interval during the expansion where the charge susceptibility per entropy $\chi T/s$ changes most. The rapidity width of the balance function is determined by the diffusion distance that the (originally local) correlation induced by noise propagates during the time from its origin to the freeze-out time.⁶

⁶The azimuthal width of the balance function is also sensitive to diffusion but is strongly affected by the radial flow.

It is easy to understand that a change of the system's thermodynamic state is necessary to produce a nonlocal correlation. Indeed, in a static medium the correlations must be *local* (on hydrodynamic scale) despite diffusion. This requires that the contributions from successive time intervals cancel each other in a static medium, leaving eventually only the (local) contribution from the most recent time. We found that such cancellations could also occur in a medium undergoing boost-invariant longitudinal expansion as long as $\chi T\tau$ is constant (which is the same as $\chi T/s$ being constant up to small viscous corrections). In general, however, the expansion leads to nonlocal correlations which carry the memory of the expansion.

One can think of this picture as the hydrodynamic description of the mechanism of the suppression of charge fluctuations proposed and analyzed in Refs. [2,3,35]. Indeed the D measure, D_m , introduced in Ref. [2], is related to the balance function as (see the appendix)

$$D_m \equiv 4 \frac{\langle (\delta N_{\text{net}})^2 \rangle}{\langle N_{\text{ch}} \rangle} = 4 \left[1 - \int_{-\infty}^{\infty} B(\Delta y) d\Delta y \right]. \quad (63)$$

Therefore a positive balance function corresponds to suppression of net charge fluctuations ($D_m < 4$). The balance function provides differential phase-space information on the distribution of the anticorrelation responsible for the suppression. Moreover, the positivity of the balance function can be seen as a direct consequence of the fact that $\chi T/s$ is smaller in QGP, i.e., $d(\chi T/s)/dT < 0$ [see Eq. (35)], which is the starting point of the argument in [2,3].

One can also view this hydrodynamic picture as effectively representing the qualitative microscopic mechanism of charge balancing described in Refs. [6,7,9,36]. The advantage of hydrodynamic description is that it does not need to rely on existence of quasiparticles. This is especially important because both quark and hadron quasiparticle descriptions must break down in the crossover region, and this is the region responsible for the major contribution to the balance function. Our approach allows quantitative description of these phenomena from first principles, i.e., from the (lattice) equation of state and information on kinetic coefficients, within a universal hydrodynamic formalism.

One of the many simplifying assumptions in our semianalytic calculation has been the assumption that dimensionless combination DT is temperature independent. It is, perhaps, the easiest assumption to relax, provided information of the temperature dependence of the diffusion coefficient D was available. Unlike the entropy and charge susceptibility which, being static thermodynamic quantities, can be reliably measured on the lattice, the diffusion coefficient is a property of the real-time low-frequency response, which the Euclidean time lattice calculation has well-known difficulties accessing. With this caveat, it would be still interesting to extract the temperature dependence of the charge diffusion coefficient from the existing lattice data.

As a first exploratory step we attempted to combine recent lattice data on electric conductivity [37] with the electric susceptibility χ data from Ref. [18]. Using the relation $D = \sigma/\chi$ we can then plot the temperature dependence of the

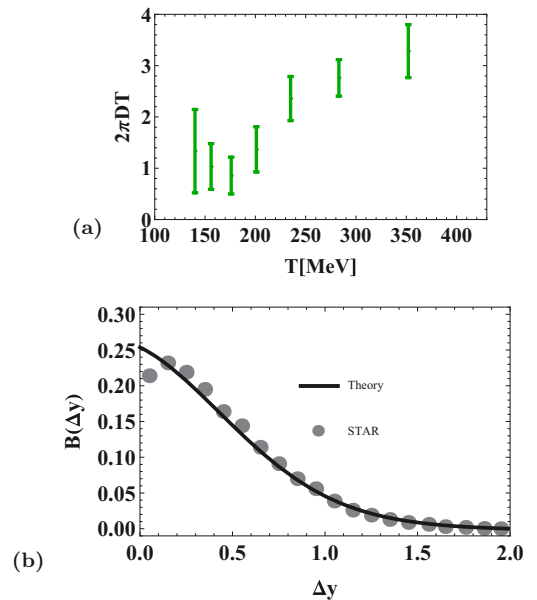


FIG. 3. (Color online) (a) The temperature dependence of the electric charge diffusion coefficient in units of $(2\pi T)^{-1}$ obtained by combining the lattice data from Refs. [18,37]. (b) The balance function using the same parameters as for Fig. 2 but with the temperature-dependent $2\pi DT$ taken from the lattice data.

diffusion coefficient or its dimensionless combination $2\pi DT$ (see Fig. 3).

Despite large error bars one can see that lattice results suggest that the diffusion coefficient D is indeed of order $1/2\pi T$ in the crossover region, where we now know most of the contribution to the balance function comes from. This is consistent with the results of our comparison with experimental data in Fig. 2.

Taking the lattice data as given (and ignoring the error bars) we can also calculate the balance function using our semianalytic model. The result plotted in Fig. 3 shows a reasonable agreement with the data.

An important improvement of our approach can be achieved by implementing a more realistic pattern of radial flow, replacing the blast-wave and sudden transverse expansion approximation. An approach based on the analytic solution proposed by Gubser *et al.* [38,39] is tempting. However, the limitation of this approach to a conformal equation of state is too restrictive for our purpose, since the major contribution to the balance function comes from the nonconformal (crossover) region. A fully numerical hydrodynamic simulation with stochastic noise will, of course, enable a quantitative comparison with experiment. It would also allow extension of our results to noncentral (azimuthally asymmetric) collisions.

A natural application of the stochastic hydrodynamic approach is to fluctuations near the QCD critical point [40], as has been already initiated by Ref. [22]. The conductivity σ and susceptibility χ diverge at the critical point, leading to the expected increase of the charge fluctuations at the critical point. Such fluctuations are important signatures of the critical point in heavy-ion collisions [41–43] and understanding the effect of the time evolution [44,45] and expansion on these signatures

is important for obtaining quantitative predictions. However, such an application requires extension of the formalism to nonzero baryon density where energy and flow velocity fluctuations now mix with charge fluctuations [22].

Another possible future direction is the examination of the balance functions of kaons or protons [9], where the fluctuation of strangeness or baryon number may become important. It should be straightforward to generalize our work to multiple conserved charges. We leave these topics for future studies.

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APPENDIX: A REVIEW OF BALANCE FUNCTIONS

For completeness we review here the definitions and properties of the balance functions [6,7].

To define the balance function we divide the phase space occupied by particles produced in a heavy-ion collision into (infinitesimally) small cells. For the purpose of this paper we consider cells in rapidity y and azimuthal angle ϕ (but integrated over transverse momentum) and denote the coordinates of the cell $\Gamma = (y, \phi)$ and the volume of the cell $d\Gamma = dy d\phi$. We denote the number of particles of charge $a = +, -$ in a cell as $dN^a(\Gamma)$ and its event average $\langle dN^a(\Gamma) \rangle$. Since $\langle dN^a(\Gamma) \rangle \sim d\Gamma$ is infinitesimally small, the probability of finding more than one particle in a cell is negligible ($\mathcal{O}(d\Gamma^2)$ for two or more particles) and the average $\langle dN^a(\Gamma) \rangle \ll 1$ is also the probability to find a particle of charge a in the cell.

The conditional probability of finding a particle of charge b in another cell Γ_2 given a particle of charge a in the cell Γ_1 can be found as $\langle dN^b(\Gamma_2) dN^a(\Gamma_1) \rangle / \langle dN^a(\Gamma_1) \rangle$ which is easy to understand keeping in mind that $dN^a(\Gamma)$ is either 0 or (rarely) 1. The balance function defined on a pair of cells is given by the following:

$$B(\Gamma_2, \Gamma_1) = \frac{1}{2} \sum_{a=+,-} \frac{\langle dN_2^{-a} dN_1^a \rangle - \langle dN_2^a dN_1^a \rangle}{d\Gamma_2 \langle dN_1^a \rangle} = \frac{\langle n_2^{-a} n_1^a \rangle - \langle n_2^a n_1^a \rangle}{\langle n_1^a \rangle}, \quad (\text{A1})$$

where we used a shorthand $dN_i^a \equiv dN^a(\Gamma_i)$ and introduced density per phase space volume $n \equiv dN/d\Gamma$. The balance function measures a difference in conditional probabilities of finding a particle of the opposite charge $-a$ versus the same charge a in the cell Γ_2 given a particle of the charge a in cell Γ_1 . This probability is proportional to the volume $d\Gamma_2$ of the cell and is infinitesimally small, while its ratio to $d\Gamma_2$, as in Eq. (A1), is finite.

Since we are considering a case when $\mu = 0$, we can use $\langle n_1^- \rangle = \langle n_1^+ \rangle$ to simplify Eq. (A1) as follows:

$$B(\Gamma_2, \Gamma_1) = -\frac{\langle (n_2^+ - n_2^-)(n_1^+ - n_1^-) \rangle}{2\langle n_1^+ \rangle} = -\frac{\langle n_2^{\text{net}} n_1^{\text{net}} \rangle}{\langle n_1^{\text{ch}} \rangle}. \quad (\text{A2})$$

Since $n^{\text{net}} = dN_{\text{net}}/dyd\phi$ and $N_{\text{net}} = \delta N_{\text{net}}$ ($\langle N_{\text{net}} \rangle = 0$), this gives us Eq. (60) used in the text.

One also defines the balance function as a function of the phase-space displacement $\Delta\Gamma \equiv \Gamma_2 - \Gamma_1 = (y_2 - y_1, \phi_2 - \phi_1)$ by summing in Eq. (A1) over all cells Γ_1 and Γ_2 separated by $\Delta\Gamma$. To obtain a finite result for infinitely many infinitesimally small cells ($d\Gamma_i \rightarrow 0$) we multiply by $d\Gamma_1 d\Gamma_2$. We can then write this summation as an integral,

$$B(\Delta\Gamma) = \frac{1}{\int d\Gamma} \int d\Gamma_2 \int d\Gamma_1 \delta(\Gamma_2 - \Gamma_1 - \Delta\Gamma) B(\Gamma_2, \Gamma_1). \quad (\text{A3})$$

The normalization factor is chosen in such a way that the result tends to a finite limit with increasing total phase-space volume ($\int d\Gamma$).

The expression in Eq. (A3) simplifies in the case of azimuthal and boost ($\Gamma \rightarrow \Gamma + \Delta\Gamma$) invariance. Since in this case the balance function $B(\Gamma_2, \Gamma_1)$ can only depend on $\Delta\Gamma$ we find from Eq. (A3), simply,

$$B(\Delta\Gamma) = B(\Gamma + \Delta\Gamma, \Gamma), \quad (\text{A4})$$

for any Γ .

The derivation above assumes that the rapidity acceptance window is infinite: $y \in (-\infty, \infty)$, or more precisely, is much larger than the rapidity range of the balance function $B(\Delta y, \Delta\phi)$. In practice, the rapidity interval has a finite width Y . Still assuming boost invariance, but integrating in Eq. (A3) over the finite rapidity window of width Y we find the balance function in a finite rapidity acceptance,

$$B(\Delta\Gamma)_Y = B(\Delta\Gamma)_\infty \frac{Y - \Delta y}{Y}, \quad (\text{A5})$$

where we used $\int dy = Y$ and $\int dy_2 \int dy_1 \delta(y_2 - y_1 - \Delta y) = Y - \Delta y$.

To express the D measure D_m [2] in terms of the balance function we substitute $N_{\text{net}} = \int d\Gamma n^{\text{net}}(\Gamma)$ and $N_{\text{ch}} = \int d\Gamma n^{\text{ch}}(\Gamma)$ into the definition

$$D_m \equiv 4 \frac{\langle (\delta N_{\text{net}})^2 \rangle}{\langle N_{\text{ch}} \rangle} = \frac{4}{\int d\Gamma} \int d\Gamma_2 \int d\Gamma_1 \frac{\langle n_2^{\text{net}} n_1^{\text{net}} \rangle}{\langle n^{\text{ch}} \rangle}. \quad (\text{A6})$$

The integrand is $-B(\Gamma_2, \Gamma_1)$ as given by Eq. (A3), except for $\Gamma_1 = \Gamma_2$, when Eq. (A3) does not apply (we have only defined $B(\Gamma_2, \Gamma_1)$ for $\Gamma_1 \neq \Gamma_2$). We can calculate the contribution from the cells $\Gamma_1 = \Gamma_2$ to Eq. (A6) separately. We note that since dN^a takes (most of the time) values 0 or 1, $(dN^a)^2 = dN^a$ and thus $\langle (dN^+ - dN^-)^2 \rangle = \langle dN^+ + dN^- \rangle$, or $\langle \delta N_{\text{net}}^2 \rangle = \langle \delta N_{\text{ch}} \rangle$, up to terms of order $\mathcal{O}(d\Gamma)^2$. Therefore, since $n = dN/d\Gamma$, the integrand in Eq. (A6) for $\Gamma_1 = \Gamma_2$ is $\langle (n^{\text{net}})^2 \rangle / \langle n^{\text{ch}} \rangle = 1/d\Gamma$. Summation over all cells with $\Gamma_1 = \Gamma_2$ gives therefore a contribution to D_m equal to (up to infinitesimally small terms $\mathcal{O}(d\Gamma)$), $4(\int d\Gamma)^{-1} \int d\Gamma d\Gamma_1/d\Gamma = 4$. This is the value of D_m for completely uncorrelated particles. Adding the contributions from $\Gamma_1 \neq \Gamma_2$ we find therefore

$$D_m = 4 \left[1 - \left(\int d\Gamma \right)^{-1} \int d\Gamma_1 \int d\Gamma_2 B(\Gamma_2, \Gamma_1) \right] = 4 \left[1 - \int d(\Delta\Gamma) B(\Delta\Gamma) \right], \quad (\text{A7})$$

where we used Eq. (A3) for the last equality.

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