Limit $F$-Signature Functions of Diagonal Hypersurfaces

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THESIS

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For Latalia - clear eyes, full hearts, can’t lose.
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# Table of Contents

1 Introduction

2 Background
   2.1 The Representation Ring and the Work of Han and Monsky 7
   2.2 The $F$-signature of Diagonal Hypersurfaces 11
   2.3 Limits of Hilbert-Kunz Multiplicity for Diagonal Hypersurfaces 12
   2.4 Limits of $F$-signature for Diagonal Hypersurfaces 13
   2.5 $F$-signature of Pairs 14

3 Limit $F$-Signature Functions for Diagonal Hypersurfaces 16
   3.1 The Two-Variable Case 16
   3.2 The General Case 21
   3.3 Example: The Two Variable Case (Again) 39
   3.4 Example: $x_1^2 + \cdots + x_n^2$ 40
   3.5 Example: Diagonal du Val Singularities 54
   3.6 Further Questions 57

Cited Literature 59

Appendix 61

Vita 63
Summary

Let $p$ be a prime and $R = \mathbb{F}_p[x_1, \ldots, x_n]$. For integers $d_1, \ldots, d_n$ with $2 \leq d_1 \leq \cdots \leq d_n$, let $f$ be the diagonal hypersurface $x_1^{d_1} + \cdots + x_n^{d_n}$. In this thesis, we consider the continuous functions $\psi_{f \mod p}(t)$, which are defined on points of the form $\frac{a}{p^m}$ as

$$\psi_{f \mod p}(\frac{a}{p^m}) := 1 - \frac{1}{p^{mn}} \dim_{\mathbb{F}_p}(R/(x_1^{p^m}, \ldots, x_n^{p^m}, f^a)).$$

We study the behavior of this family as $p$ goes to infinity. The main results of the thesis are

1. As $p \to \infty$, these functions limit to an explicit piecewise polynomial function $\psi(t)$.

2. The left and right derivatives of the functions converge to the left and right derivatives of $\psi(t)$. In particular, the derivatives converge at all but finitely many explicit points.
Let $p$ be a prime, and let $(R, \mathfrak{m}, k)$ be a complete local integral domain of characteristic $p$ and dimension $d$ with $k$ perfect, and let $I$ be an $\mathfrak{m}$-primary ideal of $R$. The main tool in the study of such rings is the Frobenius homomorphism $F : R \rightarrow R$ defined by $F(r) = r^p$ for all $r \in R$. It has long been a goal of researchers to define invariants of $R$ which leverage Frobenius to say something about the singularities of the ring $R$. The most well-known such invariant, essentially going back to Kunz in [8] although explicitly defined by Monsky in [9], is the *Hilbert-Kunz multiplicity*, which is defined as
\[ e_{HK}(R, I) := \lim_{m \to \infty} \frac{1}{p^m} f \left( \frac{R}{I^{[p^m]}} \right), \]

where, if \( I \) is generated by elements \( y_1, \ldots, y_n \), \( I^{[p^m]} \) is the ideal generated by \( y_1^{p^m}, \ldots, y_n^{p^m} \). This limit was shown to exist in [9], although its behavior is still not well understood. It characterizes regularity in the sense that \( e_{HK}(R, \mathfrak{m}) = 1 \) if and only if \( R \) is regular (this is due to Watanabe and Yoshida in [16]).

One can also consider the \( F \)-signature, a separate but related invariant of \( R \). Since \( F \) is a homomorphism from \( R \) to itself, one can consider the iterated Frobenius map \( F^m \), which is just the composition of \( F \) with itself \( m \) times. Viewing \( R \) as a module over itself via iterated Frobenius (denoted \( F^{m*}R \)), we say that \( R \) is \( F \)-finite if \( F^{m*}R \) is module-finite over \( R \). In this case, we can decompose \( F^{m*}R \) as \( R^{\oplus a_m} \oplus M \), where \( a_m \) is the maximum number of free summands of \( R \) that one can pull off from \( F^{m*}R \). Here \( a_m \) is the so-called \( m^{th} \) \( F \)-splitting number of \( R \).

In analogy to how the Hilbert-Kunz multiplicity was defined, we can study the asymptotic behavior of the \( a_m \) as \( m \to \infty \), giving rise to the \( F \)-signature

\[ s(R) := \lim_{m \to \infty} \frac{1}{p^m} a_m. \]

The study of the asymptotic behavior of the \( a_m \) was first considered by Smith and Van den Bergh in [14], and the \( F \)-signature was shown to exist for Gorenstein rings by Huneke and Leuschke in [7]. It was shown to exist in general by Tucker in [15]. Like the Hilbert-Kunz multiplicity, the \( F \)-signature is known to characterize regularity - it is 1 if and only if \( R \) is regular, and it is positive if and only if \( R \) is
These two invariants are in fact related. In [7], Huneke and Leuschke show that in the case where $R$ is Gorenstein, the $F$-signature can be expressed as a difference of two Hilbert-Kunz multiplicities (see Theorem 6 for the precise statement). More generally, Polstra and Tucker have shown ([12]) that

$$s(R) = \inf \left( e_{HK}(I) - e_{HK}((I, x)) \right),$$

where the infimum is taken over all ideals $I$ of $R$ such that $R/I$ has finite length, and all elements $x \in R$ such that $(I : x) = \mathfrak{m}$.

For a good introduction to both of these invariants, Huneke’s survey [6] gives complete proofs of the major results in the theory as well as some nice historical details. Recently, Polstra and Tucker ([12]) have given a unified treatment of the two invariants and, in particular, are able to show that both limits exist simultaneously.

The main object of study of this thesis is an extension of these ideas introduced by Blickle, Schwede, and Tucker in [1] and [2] - the $F$-signature function of a pair $(R, \mathfrak{a}^t)$, where $R$ is a regular local $F$-finite ring, $\mathfrak{a}$ is a nonzero ideal in $R$, and $t \geq 0$ is a real parameter. At this level of generality, this function is defined as

$$s(R, \mathfrak{a}^t) := \lim_{m \to \infty} \frac{1}{p^{md}} \ell_R(R/(\mathfrak{m}^{[p^m]} : \mathfrak{a}^{[tp^m]})).$$

However, when $\mathfrak{a} = (f)$ is principal, this can be computed as a single length at points of the form $t = \frac{a}{p^m}$ - namely, $s(R, f^{a/p^m}) = \frac{1}{p^{md}} \ell_R(R/(\mathfrak{m}^{[p^m]} : f^a))$. Blickle,
Schwede, and Tucker show that this function is convex (and so, in particular, continuous) and so since points of the form $\frac{a}{p^m}$ are dense on the positive real line, these lengths completely determine the function. For notational ease, we will henceforth write $\psi_f \mod p(t) = s(R, f^t)$ when $R$ and $f$ are fixed. Using the short exact sequence

$$0 \to R/(m^{[p^m]} : f^a) \to R/m^{[p^m]} \to (R/(m^{[p^m]} + f^a)) \to 0$$

along with the fact that $R$ is regular and taking lengths, after dividing by $\frac{1}{p^m}$ we get that

$$\psi_f \mod p\left(\frac{a}{p^m}\right) = 1 - \frac{1}{p^m} \ell_R(R/(m^{[p^m]} + f^a)).$$

We will denote the rightmost term by $\phi_{f \mod p}\left(\frac{a}{p^m}\right)$.

Blickle, Schwede, and Tucker further show that this function is related to a number of more well-studied characteristic-$p$ invariants. First, the $x$-intercept of the function is the $F$-pure threshold of the hypersurface $f$. Furthermore, the derivatives of $\psi_f \mod p$ recover the Hilbert-Kunz multiplicity and $F$-signature of $R/(f)$. More precisely, $\partial_{-}\psi_f \mod p(1) = -s(R/f)$ and $\partial_{+}\psi_f \mod p(0) = -e_{HK}(R/f)$.

Blickle, Schwede and Tucker ask ([2] Question 4.8) if, for fixed $f$, these functions exhibit any sort of uniform behavior as $p \to \infty$. More precisely, if $f$ has integer coefficients, for sufficiently large $p$, none of the coefficients of $f$ die mod $p$, and so it is natural to ask how these functions behave as $p$ grows large. One can also ask if this limiting behavior is related to characteristic 0 invariants of the hypersurface defined by $f$. For instance, if a limiting function exists, it necessarily has $x$-
intercept at the log canonical threshold of the hypersurface $f$, based on the well-known fact that the limit of the $F$-pure thresholds of a hypersurface equals the log canonical threshold. The existence of the limits of the Hilbert-Kunz multiplicity and $F$-signature is less well-understood, and one might hope that understanding such a limiting function might shed some light on this problem.

The main results of this thesis give the first broad class of examples where the limiting behavior of these functions is understood. Namely, if $p$ is a prime, $R = \mathbb{F}_p[x_1, \ldots, x_n]$, $d_1, \ldots, d_n$ are integers with $2 \leq d_1 \leq \cdots \leq d_n$, and $f$ is the diagonal hypersurface $x_1^{d_1} + \cdots + x_n^{d_n}$, we study the limiting behavior as $p \to \infty$ of the functions $\psi_{f \mod p}(t)$. The Hilbert-Kunz functions of diagonal hypersurfaces were studied by Han and Monsky in [5]. Using results from this paper, Gessel and Monsky studied the limiting behavior of the Hilbert-Kunz multiplicity in [4]. Mimicking both of these results, the author showed analogous results in his undergraduate thesis [13] for the $F$-signature of diagonal hypersurfaces. In this thesis, we extend many of these results to the $F$-signature function.

First, we show

**Main Theorem 1:** (Theorem 15) The sequence of $F$-signature functions $\{\psi_{f \mod p}\}$ of $f$ modulo each prime $p > 0$ converges uniformly to a piecewise polynomial limit $F$-signature function $\psi = \lim_{p \to \infty} \psi_{f \mod p}$ given by

$$\psi(t) = 1 - \frac{d_1 \cdots d_n}{2^n \cdot n!} \left(C_0(t) + 2 \sum_{\lambda \in \mathbb{Z}_{\geq 1}} C_{\lambda}(t)\right)$$
where $C_\lambda(t)$ for $\lambda \in \mathbb{Z}_{\geq 0}$ is the piecewise polynomial function

$$C_\lambda(t) = \sum_{i_0 \cdots i_n} \left( i_0 t + \frac{i_1}{d_1} + \cdots + \frac{i_n}{d_n} - 2\lambda \right)^n$$

with the sum taken over all choices of $i_0, \ldots, i_n \in \{\pm 1\}$ with $i_0 t + \frac{i_1}{d_1} + \cdots + \frac{i_n}{d_n} - 2\lambda \geq 0$.

In particular, this is an explicit combinatorially-defined piecewise polynomial function. Furthermore, we show that the derivatives of these functions actually converge to the derivative of the limit:

**Main Theorem 2:** (Theorem 16) With all notation as in the previous theorem, we have that for any $t$, the sequence of derivatives $\partial_- \psi_{f \mod p}(t)$ converges to $\partial_- \psi(t)$ and $\partial_+ \psi_{f \mod p}(t)$ converges to $\partial_+ \psi(t)$. In particular, for all but finitely many explicit points, both sequences converge to $\psi'(t)$.

Using this result, we recover many results from [4] and [13] about the limiting behavior of the Hilbert-Kunz multiplicity and $F$-signature for diagonal hypersurfaces. We also relate this limit function to the log canonical threshold of diagonal hypersurfaces in the case where the sum of the reciprocals of the powers appearing (and hence the log canonical threshold) is less than one, indicating that there is some hope to recover characteristic 0 information from the limiting behavior of these functions in characteristic $p$. Finally, we compute several classes of explicit examples.
Background

Everything in this chapter is a previously published result necessary for understanding the discussion that follows.

THE REPRESENTATION RING AND THE WORK OF HAN AND MONSKY

The content of this section is from Some Surprising Hilbert-Kunz Functions by Han and Monsky ([5]). In this paper they use an object called the representation ring to study the Hilbert-Kunz functions of diagonal hypersurfaces. We briefly survey the key results as well as those that are used in what follows.

Define an $\mathbb{F}_p$-object to be a finitely generated $\mathbb{F}_p[T]$-module on which some
power of $T$ acts as multiplication by 0 (i.e. $T$ acts nilpotently on the module). If $M$ and $N$ are $\mathbb{F}_p$-objects, then so are $M \bigoplus N$ and $M \otimes N$ (where the $T$-action on the tensor product is given by $T(m \otimes n) := Tm \otimes n + m \otimes Tn$). We take the representation ring $\Gamma$ to be the collection of formal differences $M - N$ of $\mathbb{F}_p$-objects modulo the equivalence relation $M - N \sim M' - N'$ if $M \bigoplus N'$ is isomorphic to $M' \bigoplus N$. The direct sum and tensor power operations described above for $\mathbb{F}_p$-objects induce an addition and multiplication on $\Gamma$, and so it is indeed a ring with additive identity 0 and multiplicative identity the equivalence class of the $\mathbb{F}_p$-object $\mathbb{F}_p[T]/T$. Han and Monsky distinguish two bases of $\Gamma$ for computational purposes: $\{\delta_i := \mathbb{F}_p[T]/T^i \}_{i \geq 1}$ and $\{\lambda_i = (-1)^i(\delta_{i+1} - \delta_i) \}_{i \geq 0}$.

The key reason for considering the representation ring is that certain length computations can be recast in terms of multiplication of elements of $\Gamma$ followed by the application of relatively simple functions. To this end, Han and Monsky spend much of [5] proving multiplication formulas in $\Gamma$. Luckily we need only one of these for what follows in Theorem 14 ([5] Theorem 2.15).

**Theorem 1.** If $i$ and $j$ are nonnegative integers with $i + j \leq p$, then

$$\delta_i \delta_j = \sum_{a=1}^{\min(i,j)} \delta_{i+j+1-2a}.$$ 

In order to compute lengths, Han and Monsky consider the function $\alpha(M) := \dim_{\mathbb{F}_p}(M/TM)$, which we can generalize to $\alpha_k(M) := \dim_{\mathbb{F}_p}(M/T^kM)$. 

8
Example 2.

\[ \alpha_k(\delta_i) = \begin{cases} k & k < i \\ i & k \geq i \end{cases} \]

To relate this to length computations, Han and Monsky consider the function

\[ D_{F_p}(k_1, \ldots, k_n) := \dim_{F_p}\left( F_p[x_1, \ldots, x_n]/(x_1^{k_1}, \ldots, x_n^{k_n}, x_1 + \cdots + x_n) \right), \]

which is the same thing as \( \alpha_1(\delta_{k_1} \cdots \delta_{k_n}) \). This is because if \( T \) acts as multiplication by \( x_i \) on \( \delta_{k_i} = F_p[x_i]/x_i^{k_i} \), then \( T \) acts as multiplication by \( x_1 + \cdots + x_n \) on \( \prod \delta_{k_i} = F_p[x_1, \ldots, x_n]/(x_1^{k_1}, \ldots, x_n^{k_n}) \) by the way we defined the action of \( T \) on tensor products of \( F_p \)-objects. Thus it is clear that we have

\[ \alpha_0\left( \prod \delta_{k_i} \right) = D_{F_p}(k_1, \ldots, k_n). \]

To generalize to the setting where \( T \) is acting as multiplication by a larger power of \( x \) on \( F_p[x]/x^a \), we can consider \( \delta_x^b \) to be the class of the \( F_p \)-object \( F_p[x]/x^a \) where \( T \) acts by multiplication by \( x^b \). Then a product of such elements \( \delta_x^{a_1} \cdots \delta_x^{a_n} \) is the class of the \( F_p \)-object \( F_p[x_1, \ldots, x_n]/(x_1^{a_1}, \ldots, x_n^{a_n}) \) where \( T \) acts as multiplication by the diagonal hypersurface \( x_1^{b_1} + \cdots + x_n^{b_n} \). Thus understanding the decompositions of such objects in the representation ring can tell us something about the Hilbert-Kunz multiplicity and \( F \)-signature, as these are exactly the type of length computations that one needs to make when computing these invariants. We also remark that if \( b|a \), say \( a = bc \), then \( \delta_x^b = b\delta_c \), where the \( b \) copies are generated by \( 1, x, x^2, \ldots, x^{b-1} \).
We will need the following result from [5] (Theorem 2.20). This is the key result used from [5] in Gessel and Monsky’s calculation of the limit of Hilbert-Kunz multiplicities in [4], and forms the basis for all of the limit arguments in what follows.

**Theorem 3.** For \( k_1, \ldots, k_m \) positive integers with each \( k_i \leq p \) and \( -m + \Sigma k_i \) even, take \( \gamma = \frac{k_1 + \cdots + k_m - m}{2} \) and write \( \Pi \frac{(1-x^{k_i})}{1-x} = \Sigma c_i x^i \). Then

\[
D_p(k_1, \ldots, k_m) = \sum_{\lambda \in \mathbb{Z}} c_{\gamma-p\lambda}
\]

The theorem is proved via induction on \( m \) and various decompositions in \( \Gamma \). The upshot of this is that we are able to compute lengths as some explicit combinatorial sum, which is what makes the computations in what follows feasible.

The main result of Han and Monsky is to give an explicit formula for the Hilbert-Kunz function of diagonal hypersurfaces and, in particular, to show that the Hilbert-Kunz multiplicity of these hypersurfaces is rational.

**Theorem 4.** Fix \( 2 \leq d_1 \leq \cdots \leq d_n \). Let \( R = \mathbb{F}_p[x_1, \ldots, x_n]/(\sum x_i^{d_i}) \). Then if \( n > 2 \), the Hilbert-Kunz function of \( R \) is given by \( e_m(R) = Cp^{(n-1)m} + \Delta_m \), where \( C \) is some positive rational number and \( \Delta_m = O(p^{(n-3)m}) \).

In particular, from this result one immediately obtains

**Theorem 5.** With \( R \) as above, the Hilbert-Kunz multiplicity of \( R \) is rational.

They prove these theorems by studying the multiplicative structure of the representation ring \( \Gamma \) and proving a functional equation for \( D_{\mathbb{F}_p} \).
The $F$-signature of Diagonal Hypersurfaces

In the author's undergraduate thesis [13], the $F$-signature function of diagonal hypersurfaces was studied à la Han and Monsky using the following theorem of Huneke and Leuschke from [7]:

**Theorem 6.** Let $R$ be a Noetherian, $F$-finite, Gorenstein ring of dimension $d$ with perfect residue field. Let $I$ be an ideal generated by a system of parameters of $R$, and let $\Delta$ be a generator of the socle of $R/I$. Then we have that

$$e_m(R, I) - e_m(R, (I, \Delta)) = a_m(R).$$

Normalizing both sides by dividing by $p^{md}$ and taking the limit of both sides as $m \to \infty$, we obtain that

$$e_{HK}(R, I) - e_{HK}(R, (I, \Delta)) = s(R).$$

In the case of diagonal hypersurfaces, we can take $I = (x_2, \ldots, x_n)$ and $\Delta = x_1^{d_1-1}$. Then the first Hilbert-Kunz multiplicity is just $d_1$, and we are able to mimic the techniques of Han and Monsky to study the second Hilbert-Kunz multiplicity in detail. We obtained the following results:

**Theorem 7.** Fix $2 \leq d_1 \leq \cdots \leq d_n$. Let $R = \mathbb{F}_p[x_1, \cdots, x_n]/(\sum x_i^{d_i})$. Then if $n > 2$, the $F$-splitting numbers of $R$ are given by $a_m(R) = d_1 p^{(n-1)m} - (C p^{(n-1)m} + \Delta_m)$, where $C$ is some positive rational number and $\Delta_m = O(p^{(n-3)m})$.

We then immediately obtain
Theorem 8. With $R$ as above, the $F$-signature of $R$ is rational.

Limits of Hilbert-Kunz Multiplicity for Diagonal Hypersurfaces

In their paper [4], Gessel and Monsky consider the limit as $p \to \infty$ of the Hilbert-Kunz multiplicity of diagonal hypersurfaces using Theorem 3. Namely, they observe that one can approximate the Hilbert-Kunz multiplicity by an explicit sum of (characteristic-independent) binomial coefficients to within a factor of $\frac{1}{p}$, hence showing convergence of the Hilbert-Kunz multiplicity as $p \to \infty$.

More precisely, they show the following result: for $\lambda$ a nonnegative integer, define

$$C_\lambda := \sum (\epsilon_1 \epsilon_2 \cdots \epsilon_n) \left( \frac{\epsilon_1}{d_1} + \frac{\epsilon_2}{d_2} + \cdots + \frac{\epsilon_n}{d_n} - 2\lambda \right)^{n-1},$$

where the sum runs over all $\epsilon_i \in \{1, -1\}$ such that $\frac{\epsilon_1}{d_1} + \frac{\epsilon_2}{d_2} + \cdots + \frac{\epsilon_n}{d_n} > 2\lambda$.

Theorem 9. Let $d = \prod d_i$. Then the limit of the Hilbert-Kunz multiplicities of $R$ as $p \to \infty$ is given by the formula

$$\frac{d}{2^{n-1}(n-1)!} \left( C_0 + 2 \sum_{\lambda \geq 1} C_\lambda \right).$$

They also compute an explicit family of examples:

Theorem 10. If $2 = d_1 = \cdots = d_n$, then the limit of the Hilbert-Kunz multiplicity of $R$ as $p \to \infty$ is $1 + \left( \text{the coefficient of } z^{n-1} \text{ in the power series expansion of } \sec(z) + \tan(z) \right)$.

Their proof for this theorem involves much power-series trickery. We give a
different proof below, although Gessel and Monsky remark in the introduction to their paper that their original proof of this result involved Eulerian polynomials, so it is possible that what they originally did is in a similar spirit to what we do below.

**Limits of F-signature for Diagonal Hypersurfaces**

The author’s undergraduate thesis ([13]) also contains a proof of the analogous result for the F-signature. In this case the limiting sums are: for \( \lambda \) a nonnegative integer, define

\[
C'_\lambda := \sum (\epsilon_1\epsilon_2...\epsilon_n) \left( \epsilon_1\left( \frac{d_1-1}{d_1} \right) + \frac{\epsilon_2}{d_2} + ... + \frac{\epsilon_n}{d_n} - 2\lambda \right)^{n-1},
\]

where the sum runs over all \( \epsilon_i \in \{1, -1\} \) such that \( \epsilon_1\left( \frac{d_1-1}{d_1} \right) + \frac{\epsilon_2}{d_2} + ... + \frac{\epsilon_n}{d_n} > 2\lambda \).

We then have the following theorem:

**Theorem 11.** Let \( d = \prod d_i \). Then the limit of the F-signature of \( R \) as \( p \to \infty \) is given by the formula

\[
d_1 - \frac{d}{2^{n-1}(n-1)!} \left( C'_0 + 2 \sum_{\lambda \geq 1} C'_\lambda \right).
\]

Interestingly, we do not (obviously) recover this theorem below: in fact, we obtain what seems to be a different formula for the limit F-signature of diagonal hypersurfaces. Notably, this formula has a distinguished power \( (d_1) \), whereas the formula given below has no such distinguished power. Although they must

13
actually be the same expression, it is not clear to us how to show this.

\textit{F-signature of Pairs}

Suppose that \((R, \mathfrak{m}, k)\) is a regular \(F\)-finite normal \(d\)-dimensional domain and \(f \in R\) is a nonzero element. In [1] and [2], Blickle, Schwede, and Tucker studied a function which is defined in analogy to both the \(F\)-signature and the Hilbert-Kunz multiplicity, the \(F\)-signature of pairs. In the case of a hypersurface \(f\), the function can be written on points of the form \(a^{p^m}\) as

\[
\psi \left( \frac{a}{p^m} \right) = \frac{1}{p^m} \dim_F \left( \frac{R}{(\mathfrak{m}^{[p^m]}, f^a)} \right).
\]

Blickle, Schwede, and Tucker show a number of nice properties about the function \(\psi\) in fixed characteristic. In particular, they prove the following theorem.

\textbf{Theorem 12.} The function \(\psi(t)\) is convex on \([0, \infty)\). In particular, \(\psi(t)\) is Lipschitz continuous.

Note that from the short exact sequence

\[
0 \to R/(\mathfrak{m}^{[p^m]} : f^k) \to R/\mathfrak{m}^{[p^m]} \to R/(\mathfrak{m}^{[p^m]} + f^k) \to 0,
\]

we can write \(\psi(t) = 1 - \phi(t)\), where we define \(\phi \left( \frac{a}{p^m} \right) = \frac{1}{p^m} \dim_F \left( \frac{R}{(\mathfrak{m}^{[p^m]}, f^a)} \right)\).

This easy change will often be incorporated in the proofs that follow. Moreover, this introduces a connection with the notion of so-called \(p\)-fractals, studied by Monsky and Teixara in [10] and [11].
The result that will allow us to recover many results above is the following:

**Theorem 13.** The left derivative of \( s(R, f^t) \) exists at \( t = 1 \) and equals \(-s(R/f)\).

The right derivative of \( s(R, f^t) \) exists at \( t = 0 \), and equals \(-e_{HK}(R/f)\).

Thus if one can understand the limiting behavior of the \( F \)-signature of pairs functions and their derivatives, one can recover results about both limit \( F \)-signature and limit Hilbert-Kunz multiplicity, with this theorem providing the link.
Everything in this chapter is new.

**The Two-Variable Case**

As a warm-up and to give an idea of how we initially thought about computing these limits, we will prove the limiting formula for the two-variable case by hand using the representation ring. The idea of this proof is that for every prime $p$, we can explicitly compute the function $\psi_f \mod p$ at points of the form $\frac{k}{p}$ for all
integers $0 \leq k \leq p$. While for a fixed prime this does not tell us much about the function, as $p \to \infty$ these points “fill in” the interval $[0, 1]$, allowing us to say something about the functions in the limit.

**Theorem 14.** Suppose that $a \leq b$ are positive integers, and let $f = x^a + y^b$. Then the sequence of $F$-signature functions $\{\psi_{f \mod p}\}$ of $f$ modulo each prime $p > 0$ converges uniformly to the piecewise-polynomial limit $F$-signature function

$$\psi(t) = \lim_{p \to \infty} \psi_{f \mod p} = \begin{cases} 
1 - at & 0 \leq t \leq \frac{1}{a} - \frac{1}{b} \\
\frac{ab}{4}(t - \frac{1}{a} - \frac{1}{b})^2 & \frac{1}{a} - \frac{1}{b} \leq t \leq \frac{1}{a} + \frac{1}{b} \\
0 & \frac{1}{a} + \frac{1}{b} \leq t \leq 1 
\end{cases}.$$

**Proof.** For each prime $p$, we can write $p = ab\ell_p + r_p$ with $0 \leq r_p < ab$. On points of the form $k_p$ we can compute $\psi_{f \mod p}(\frac{k_p}{p})$ as $1 - \frac{1}{p} \dim_{\mathbb{F}_p}(\mathbb{F}_p[x, y]/(x^p, y^p, f^k))$.

Consider the $\mathbb{F}_p$-objects $M_p = \mathbb{F}_p[x, y]/(x^p, y^p)$ and $N_p = \mathbb{F}_p[x, y]/(x^p, y^p, f^k)$, where $T$ acts by multiplication by $f$ on both $M_p$ and $N_p$. Then for any nonnegative integer $k$ we have a short exact sequence

$$0 \to \left(\frac{x^{a\ell_p}, y^{a\ell_p}, f^k}{(x^p, y^p, f^k)}\right) \to \frac{M_p}{T^k M_p} \to \frac{N_p}{T^k N_p} \to 0.$$

Taking dimensions, we see that

$$|\dim_{\mathbb{F}_p} \frac{M_p}{T^k M_p} - \dim_{\mathbb{F}_p} \frac{N_p}{T^k N_p}| = \dim_{\mathbb{F}_p}(\frac{x^{a\ell_p}, y^{a\ell_p}, f^k}{(x^p, y^p, f^k)}) < C_p,$$

The last step comes from a trick that we will use frequently throughout the re-
remainder of the thesis: Namely, the quotient ideal \((x^{ab\ell_p}, y^{ab\ell_p}, f^k)/(x^p, y^p, f^k)\) is equal to the quotient ideal \((x^{ab\ell_p}, y^{ab\ell_p}, f^k)/(x^p, y^p, f^k)\), which has dimension over \(\mathbb{F}_p\) bounded by \(\dim_{\mathbb{F}_p}((x^{ab\ell_p}, y^{ab\ell_p}, f^k)/(x^p, y^p))\) (since a smaller ideal is being killed).

Now this quotient ideal has an explicit \(\mathbb{F}_p\)-basis given by monomials of the form \(x^{\beta_1}y^{\beta_2}\) where at least one of the \(\beta_i \geq ab\ell_p\) and each of \(\beta_1, \beta_2 < p\). Explicitly, there are \(p^2 - (ab\ell_p)^2 = 2abr\ell_p + r^2\) such terms, which is \(O(p)\). Thus we obtain that \(\dim_{\mathbb{F}_p}((x^{ab\ell_p}, y^{ab\ell_p}, f^k)/(x^p, y^p)) < C_p\) for some constant \(C\), as was stated.

Dividing through by \(p^2\), we see that \(\psi_f \mod p\) is approximated to within \(\frac{C}{p}\) by \(1 - \frac{1}{p^2} \dim_{\mathbb{F}_p}(\frac{N_p}{T_N\mathbb{F}_p})\) for all \(p\). In particular, as \(p \to \infty\) these converge, and so it is enough to study \(\dim_{\mathbb{F}_p}(\frac{N_p}{T_N\mathbb{F}_p})\).

Now we observe that \(N_p\) is exactly the \(\mathbb{F}_p\)-object \(ab\delta_{b\ell_p}\delta_{a\ell_p}\), since \(\delta_{ab\ell_p/a}\) decomposes into \(a\) copies of \(\delta_{b\ell_p}\), and similarly for \(\delta_{ab\ell_p/b}\). So we have by Theorem 1 that \(N_p = ab\delta_{b\ell_p}\delta_{a\ell_p}\) decomposes as

\[
ab\delta_{a\ell_p}\delta_{b\ell_p} = ab \sum_{k=1}^{a\ell_p} \delta_{(a+b)\ell_p+1-2k} = ab(\delta_{b\ell_p-a\ell_p+1} + \delta_{b\ell_p-a\ell_p+3} + \cdots + \delta_{b\ell_p+a\ell_p-3} + \delta_{b\ell_p+a\ell_p-1}).
\]

The point of this decomposition is that there are now no products of \(\delta_i\) terms appearing at all, and so applying the \(\alpha_k\) functions from Han and Monsky is quite easy - we just use Example 2. We should also remark that this is where this line of proof gets much harder for more variables: all of the decompositions of the type of that Han and Monsky prove in [5] only involve products of two \(\delta_i\) terms. While this would in principle be enough to decompose the product of arbitrarily many
terms, the combinatorics become quite cumbersome.

Fix \( t \in [0, 1] \), and for each \( p \) choose integers \( k_p \) such that \( \frac{k_p}{p} \to t \) (one should probably be thinking of either \( \lceil pt \rceil \) or \( \lfloor pt \rfloor \) here). We treat each interval separately.

If \( t \in [0, \frac{1}{a} - \frac{1}{b}] \), we can always choose \( k_p \leq b \ell_p - a \ell_p + 1 \), and so we can compute

\[
\frac{1}{p^2} \alpha_{k_p}(ab \delta_{a \ell_p} \delta_{b \ell_p}) = \frac{ab}{p^2} \alpha_{k_p} \left( \sum_{k=1}^{a \ell_p} \delta_{(a+b) \ell_p+1-2k} \right) = \frac{ab}{p^2} (k_p a \ell_p).
\]

The second equality here is by Example 2: since \( k_p \) is less than every index appearing in the sum, \( \alpha_{k_p} \) applied to any \( \delta_i \) appearing in the sum is just \( k_p \), and so since there are exactly \( a \ell_p \) terms appearing in the sum, we get the desired equality. We can rewrite this as \( a^2 b (\ell_p)(k_p) \), which limits to \( a^2 b (\frac{1}{ab})(t) = at \) as \( p \to \infty \) (by the definitions of \( k_p \) and \( \ell_p \)). So the functions \( \psi_f \mod p(t) \) limit to \( 1 - at \) on \([0, \frac{1}{a} - \frac{1}{b}]\).

If \( t \in [\frac{1}{a} - \frac{1}{b}, \frac{1}{a} + \frac{1}{b}] \), we can choose \( k_p \) to be between \( b \ell_p - a \ell_p + 1 \) and \( b \ell_p + a \ell_p - 1 \) as well as expressible as \( k_p = (a+b) \ell_p + 1 - 2m \) for some integer \( m \) (this is just a choice of the parity of \( k_p \) and so will not affect anything in the limit). With these choices made, we can compute

\[
\frac{1}{p^2} \alpha_{k_p}(ab \delta_{a \ell_p} \delta_{b \ell_p}) = \frac{ab}{p^2} \alpha_{k_p} \left( \sum_{k=1}^{m} \delta_{(a+b) \ell_p+1-2k} + \sum_{k=m+1}^{a \ell_p} \delta_{(a+b) \ell_p+1-2k} \right).
\]

The first sum consists of terms which have index larger than \( k_p \), and so \( \alpha_{k_p} \) of these terms is again \( k_p \). The second sum consists of terms which have index smaller than
\( k_p \), and so \( \alpha_{k_p} \) of these terms is the index of the term. So we have that

\[
\frac{1}{p^2} \alpha_{k_p}(ab\delta_{\ell_p}\delta_{\ell_p}) = \frac{ab}{p^2} \left( \sum_{k=1}^{m} k_p + \sum_{k=m+1}^{\alpha_{k_p}} ((a+b)\ell_p + 1 - 2k) \right).
\]

The right hand side can then be explicitly computed as

\[
\frac{ab}{p^2} (mk_p + (a\ell_p - m)(a+b)(\ell_p) + (a\ell_p - m) - (a\ell_p)(a\ell_p + 1) + m(m + 1)).
\]

Observing that \( \frac{m}{p} = \frac{(a+b)\ell_p + 1 - k_p}{2p} \) limits to \( \frac{a+b}{2ab} - \frac{t}{2} \) as \( p \to \infty \), we see that (after distributing \( \frac{1}{p^2} \) to each term) this expression limits to

\[
ab \left( \left( \frac{a+b}{2ab} - \frac{t}{2} \right)(t) + \frac{a^2 + ab}{(ab)^2} - \frac{a+b}{ab} \left( \frac{a+b}{2ab} - \frac{t}{2} \right) - \frac{a^2}{(ab)^2} + \left( \frac{a+b}{2ab} - \frac{t}{2} \right)^2 \right)
\]

which simplifies to

\[
-\frac{abt^2}{4} + \left( \frac{a}{2} + \frac{b}{2} \right)t + 1 - \frac{(a+b)^2}{4ab}
\]

So we can conclude that the functions \( \psi_f \mod p(t) \) limit to

\[
1 - \left( -\frac{abt^2}{4} + \left( \frac{a}{2} + \frac{b}{2} \right)t + 1 - \frac{(a+b)^2}{4ab} \right) = \frac{abt^2}{4} - \left( \frac{a}{2} + \frac{b}{2} \right)t + \frac{(a+b)^2}{4ab}
\]

\[
= \frac{ab}{4} \left( t - \frac{1}{a} - \frac{1}{b} \right)^2
\]

on \( [\frac{1}{a} - \frac{1}{b}, \frac{1}{a} + \frac{1}{b}] \).

Finally, we deal with the case \( t \in [\frac{1}{a} + \frac{1}{b}, 1] \). In this case, we can choose
\[ k_p > b\ell_p + a\ell_p - 1 \text{ for every } p. \] Then we can compute

\[
\frac{1}{p^2} \alpha_{k_p}(ab\delta_{a\ell_p}\delta_{b\ell_p}) = \frac{ab}{p^2} \alpha_{k_p} \left( \sum_{k=1}^{a\ell_p} \delta_{(a+b)\ell_p+1-2k} \right) = \frac{ab}{p^2} \left( \sum_{k=1}^{a\ell_p} ((a+b)\ell_p + 1 - 2k) \right)
\]

since \( k_p \) is greater than each index appearing in the expansion. Via explicit computation, this equals

\[
\frac{ab}{p^2} ((a + b)\ell_p(a\ell_p) + a\ell_p - (a\ell_p)(a\ell_p + 1)) = \frac{(ab)^2\ell_p^2}{p^2},
\]

which limits to 1 as \( p \to \infty \). So the functions \( \psi_{f \mod p}(t) \) limit to \( 1 - 1 = 0 \) on \( [\frac{1}{a} + \frac{1}{b}, 1] \), completing the proof.

Again, while one could in principle do this sort of analysis for more variables, the combinatorics that go into decomposing products of more than two \( \delta_i \) become unwieldy. Luckily, there is another way.

**The General Case**

Let \( R = \mathbb{F}_p[x_1, \ldots, x_n] \). Fix integers \( 2 \leq d_1 \leq \cdots \leq d_n \), and take \( f = x_1^{d_1} + \cdots + x_n^{d_n} \).

Let \( d = \prod d_i \). For each prime \( p \), write \( p = ld + r \), where \( 0 \leq r < d \).

**Theorem 15.** Let \( f = x_1^{d_1} + \cdots + x_n^{d_n} \). Then the sequence of \( F \)-signature functions \( \{ \psi_{f \mod p} \} \) of \( f \) modulo each prime \( p > 0 \) converges uniformly to a piecewise
polynomial limit $F$-signature function $\psi = \lim_{p \to \infty} \psi_f \mod p$ given by

$$\psi(t) = 1 - \frac{d_1 \cdots d_n}{2^n \cdot n!} \left( C_0(t) + 2 \sum_{\lambda \in \mathbb{Z}_{\geq 1}} C_\lambda(t) \right)$$

where $C_\lambda(t)$ for $\lambda \in \mathbb{Z}_{\geq 0}$ is the piecewise polynomial function

$$C_\lambda(t) = \sum (\epsilon_0 \cdots \epsilon_n) \left( \epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \right)^n$$

with the sum taken over all choices of $\epsilon_0, \ldots, \epsilon_n \in \{\pm 1\}$ with $\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \geq 0$.

It is useful to remark that since $\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda$ changes signs precisely at $-\epsilon_0 (\epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda)$, the set of points in $[0, 1]$ where $\psi(t)$ changes is precisely $\{\pm \frac{1}{d_1} \pm \cdots \pm \frac{1}{d_n} \pm 2\lambda\} \cap [0, 1]$ for $\lambda \in \mathbb{Z}$.

It is also helpful to remark that this sum is always finite: a crude bound is given by the fact that $\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda$ is no bigger than $1 + \frac{n}{2}$ since all of the $d_i$ are at least 2 and the sum is maximized when all of the $\epsilon_i$ are positive. Thus only finitely many $C_\lambda(t)$ are nonzero - if $\lambda$ is bigger than $\frac{1}{2} + \frac{n}{4}$, $\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda$ is guaranteed to be negative regardless of the choices of $\epsilon_i$, and so $C_\lambda(t)$ is identically 0 in this case. Moreover, each $C_\lambda(t)$ is a finite sum consisting of at most (again, crudely) $2^{n+1}$ terms (given by all of the possible choices of the $\epsilon_i$ from $\{\pm 1\}$). Thus this entire sum is finite for fixed $f$.

**Proof.** We proceed by a series of reductions that will allow us to directly apply
Theorem 3. We first recall that on the dense set \( \{ \frac{a}{p^c} \} \), \( 1 - \psi_f \mod p(\frac{a}{p^c}) \) agrees with

\[
\phi_p\left(\frac{a}{p^c}\right) := \frac{1}{p^cn} \dim_{\mathbb{F}_p}(R/(m^{[p^c]}, f^a)).
\]

So it is enough to show that the \( \phi_p(t) \) converge to \( \phi(t) := 1 - \psi(t) \). We show that this convergence holds pointwise. Uniform convergence follows from the fact that a sequence of continuous convex functions converging pointwise on an interval converges uniformly. Fix \( t \in [0, 1] \). For each prime \( p \), choose \( a_p \in \mathbb{Z} \) with \( 0 \leq a_p \leq p \) such that \( a_p \to t \) as \( p \to \infty \). More precisely, we can choose \( a_p \) to be either \( \lceil pt \rceil \) or \( \lfloor pt \rfloor \), with which one we choose depending on a parity condition that will be made explicit shortly. Then for each \( p \), we have that

\[
|\phi_p(t) - \phi(t)| \leq |\phi_p(t) - \phi_p\left(\frac{a_p}{p}\right)| + |\phi_p\left(\frac{a_p}{p}\right) - \phi(t)| \leq \frac{C}{p} + |\phi_p\left(\frac{a_p}{p}\right) - \phi(t)|
\]

where the last inequality comes from the fact that \( \phi_p \) is Lipschitz (with a constant \( C \) that we can take independent of \( p \)) and we have (by our choice of the \( a_p \)) that \( |t - \frac{a_p}{p}| \leq \frac{1}{p} \). So it is enough to understand the convergence of the \( \phi_p\left(\frac{a_p}{p}\right) \) to \( \phi(t) \).

We next make another reduction: consider (for any integer \( 0 \leq a \leq p \)) the function \( \tilde{\phi}_p\left(\frac{a}{p}\right) := \frac{1}{p^c} \dim_{\mathbb{F}_p}(R/(x_1^{dt}, \ldots, x_n^{dt}, f^a)) \). We have a short exact sequence

\[
0 \to \frac{(x_1^{dt}, \ldots, x_n^{dt}, f^a)}{(x_1^p, \ldots, x_n^p, f^a)} \to \frac{R}{(m^{[p]}, f^a)} \to \frac{R}{(x_1^{dt}, \ldots, x_n^{dt}, f^a)} \to 0
\]

and so

\[
|\phi_p\left(\frac{a}{p}\right) - \tilde{\phi}_p\left(\frac{a}{p}\right)| \leq \frac{1}{p^a} \dim_{\mathbb{F}_p}\left(\frac{(x_1^{dt}, \ldots, x_n^{dt}, f^a)}{(x_1^p, \ldots, x_n^p, f^a)}\right) \leq \frac{C}{p}.
\]

23
Here this last inequality follows from the fact that
\[
\dim_{\mathbb{F}_p} \left( \frac{(x_1^{d\ell}, \ldots, x_n^{d\ell}, f_a)}{(x_1^p, \ldots, x_n^p, f_a)} \right) \leq \dim_{\mathbb{F}_p} \left( \frac{(x_1^{d\ell}, \ldots, x_n^{d\ell})}{(x_1^p, \ldots, x_n^p)} \right),
\]
which has an explicit \( \mathbb{F}_p \)-basis given by monomials of the form \( x_1^{\beta_1} \cdots x_n^{\beta_n} \) where for each \( i \) we have that \( \beta_i < p \), and at least one of the \( \beta_i \geq d\ell \). Since there are \( p^n - (d\ell)^n = O(p^{n-1}) \) such monomials, dividing through by \( p^n \) gives the desired inequality. Thus we have reduced to showing the convergence of \( \tilde{\phi}_p \left( \frac{a_p}{p} \right) \) to \( \phi(t) \).

Now, let \( M \) be the \( \mathbb{F}_p \)-object \( R/(x_1^{d\ell}, \ldots, x_n^{d\ell}) \) with \( T \) acting as multiplication by \( f \). Then we have
\[
\tilde{\phi}_p \left( \frac{a_p}{p} \right) = \frac{1}{p^n} \dim_{\mathbb{F}_p}(R/(x_1^{d\ell}, \ldots, x_n^{d\ell}, f^{a_p})) = \frac{1}{p^n} \alpha_{a_p}(M).
\]

Take \( e_i \) to be \( \frac{d}{d_i} \) for each \( 1 \leq i \leq n \). Then \( M \) is the \( F \)-object \( d\delta_{e_1\ell} \cdots \delta_{e_n\ell} \) (since \( \delta_{d\ell/d_i} \) decomposes into \( d_i \) copies of \( \delta_{e_i\ell} \)) and so we compute that
\[
\alpha_{a_p}(M) = d \left( \dim_{\mathbb{F}_p} \left( \mathbb{F}_p[x_1, \ldots, x_n]/(x_1^{e_1\ell}, \ldots, x_n^{e_n\ell}, (\Sigma x_i)^{a_p}) \right) \right)
= d \left( \dim_{\mathbb{F}_p} \left( \mathbb{F}_p[x_1, \ldots, x_n, y]/(x_1^{e_1\ell}, \ldots, x_n^{e_n\ell}, y^{a_p}, y + \Sigma x_i) \right) \right)
= \alpha_1(d\delta_{e_1\ell} \cdots \delta_{e_n\ell} \delta_{a_p})
= dD_p(a_p, e_1\ell, \ldots, e_n\ell)
\]
Thus we have that \( \tilde{\phi} \left( \frac{a_p}{p} \right) = \frac{d}{p^n} D_p(a_p, e_1\ell, \ldots, e_n\ell) \). By introducing a new variable we are able to reduce to the case considered by Han and Monsky, and hence can
use Theorem 3, following the approach of Gessel and Monsky in [4]. We have
that, if we write

\[(1 - x)^{-n-1}(1 - x^{a_p})\Pi(1 - x^{e_i\ell}) = \Sigma c_i x^i,\]

then \(D_p(a_p, e_1\ell, \cdots, e_n\ell) = \sum_{\lambda \in \mathbb{Z}} c_{\gamma - p\lambda},\) where \(\gamma = \frac{-(n+1)+a_p+\Sigma e_i\ell}{2}\). This is where the choice of \(a_p\) matters - we choose \(a_p = \lceil pt \rceil\) or \(\lfloor pt \rfloor\) so that \(\gamma\) is an integer (equivalently, so that the parity condition required for Theorem 3 is satisfied, or such that this is an even-degree polynomial and hence has a “middle” term). Now \((1 - x)^{-n-1}(1 - x^{a_p})\Pi(1 - x^{e_i\ell})\) is a polynomial whose coefficients are symmetric about the “middle” coefficient of \(c_\gamma\), and so it is enough to understand the coefficients \(c_{\gamma - p\lambda}\) for \(\lambda \geq 0\), as \(c_{\gamma - \lambda p} = c_{\gamma + \lambda p}\).

Note that given a subset of the multiset \(\{a_p, e_1\ell, \cdots, e_n\ell\}\) with sum \(S\) and size \(u\), we get a contribution of \((-1)^u x^S\) to the expansion of the product \((1 - x^{a_p})\Pi(1 - x^{e_i\ell})\). Furthermore, by the generalized binomial theorem,

\[(1 - x)^{-n-1} = \sum_{k=0}^{\infty} \binom{n+k}{n} x^k,\]

and so for each subset of the multiset \(\{a_p, e_1\ell, \cdots, e_n\ell\}\) with sum \(S\) and size \(u\), we get a contribution of \((-1)^u \binom{n+\gamma-S}{n}\) to the coefficient \(c_\gamma\). If \(e_i\ell\) appears in the subset, then

\[
\gamma - e_i\ell = \frac{-(n+1)+a_p+e_1\ell+\cdots+e_i\ell+\cdots+e_n\ell}{2} - e_i\ell
= \frac{-(n+1)+a_p+e_1\ell+\cdots-e_i\ell+\cdots+e_n\ell}{2}.
\]
Thus for each subset of the multiset \( \{a_p, e_1\ell, \ldots, e_n\ell\} \) of sum \( S \), we can write \( n + \gamma - S \) as \( n + \frac{1}{2}(-n - 1 + \epsilon_0 a_p + \epsilon_1 e_1\ell + \cdots + \epsilon_n e_n\ell) \), where \( \epsilon_i \in \{\pm 1\} \) and the terms with \( \epsilon_i = -1 \) are precisely the terms appearing in the subset. The contribution of this term to \( c_\gamma \) is

\[
(-1)^u \binom{n + \gamma - S}{n} = (-1)^u \binom{n + \gamma - S}{n} = (-1)^u \binom{n + \frac{1}{2}(\epsilon_0 a_p + \epsilon_1 e_1\ell + \cdots + \epsilon_n e_n\ell - n - 1)}{n}.
\]

But since \((-1)^u = \prod \epsilon_i \) (each term included contributes a factor of \(-1\)) and a choice of the \( \epsilon_i \) is equivalent to a choice of a subset of the multiset, we get that

\[
c_\gamma = \sum (\epsilon_0 \cdots \epsilon_n) \left( n + \frac{1}{2}(\epsilon_0 a_p + \epsilon_1 e_1\ell + \cdots + \epsilon_n e_n\ell - n - 1) \right),
\]

where this sum is taken over all choices of \( \epsilon_i \in \{\pm 1\} \).

More generally, by an analogous argument,

\[
c_{\gamma - \lambda p} = \sum (\epsilon_0 \cdots \epsilon_n) \left( n + \frac{1}{2}(\epsilon_0 a_p + \epsilon_1 e_1\ell + \cdots + \epsilon_n e_n\ell - n - 1) - \lambda p \right).
\]

Write \( \omega_\lambda(p) = n + \frac{1}{2}(\epsilon_0 a_p + \epsilon_1 e_1\ell + \cdots + \epsilon_n e_n\ell - n - 1) - \lambda p \) (we will suppress the dependence on the \( \epsilon_i \) in the notation). Consider \( \lim_{p \to \infty} \frac{\omega_\lambda(p)}{p} \). This is equal (just by dividing each term by \( p \) and recalling how we defined \( a_p, \) the \( \epsilon_i, \) and \( \ell \)) to

\[
\frac{1}{2}(\epsilon_0 t + \epsilon_1 \frac{1}{a_1} + \cdots + \epsilon_n \frac{1}{a_n} - 2\lambda).
\]

If this limit is negative, then \( \omega_\lambda(p) \) is also eventually
always negative, and so in the limit \( \binom{\omega(p)}{n} = 0 \). Otherwise,

\[
\frac{1}{p^n} \binom{\omega(p)}{n} = \frac{1}{n!p^n} (\omega(p)(\omega(p) - 1) \cdots (\omega(p) - n + 1) \cdot p^n(p - 1)(p - 2) \cdots (p - n + 1))\]

which limits to \( \frac{1}{2^n n!} (\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda)^n \) as \( p \to \infty \). So we see that for all \( \lambda \geq 0 \),

\[
\lim_{p \to \infty} \frac{1}{p^n} c_{\gamma - \lambda p} = \frac{1}{2^n n!} C_{\lambda}(t),
\]

and so

\[
\lim_{p \to \infty} \tilde{\phi} \left( \frac{a_p}{p} \right) = \lim_{p \to \infty} \frac{d}{p^n} D_p(a_p, e_1 \ell, \cdots, e_n \ell) = \lim_{p \to \infty} \frac{d}{p^n} \sum_{\lambda \in \mathbb{Z}} c_{\gamma - p\lambda} = \frac{d}{2^n n!} (C_0(t) + 2 \sum_{\lambda \geq 1} C_\lambda(t)),
\]

and so

\[
\lim_{p \to \infty} \psi_f^{\mod p}(t) = \psi(t),
\]
as desired.

\[\square\]

Not only do we have uniform convergence of the \( F \)-signature function, but the derivatives actually converge to the derivative of the limit function as well.

**Theorem 16.** With all notation as in the previous theorem, we have that for any \( t \), the sequence of derivatives \( \partial_- \psi_f^{\mod p}(t) \) converges to \( \partial_- \psi(t) \) and \( \partial_+ \psi_f^{\mod p}(t) \) converges to \( \partial_+ \psi(t) \). In particular, for all but finitely many explicit points, both sequences converge to \( \psi'(t) \).
First, we need a series of lemmas. The lemmas are necessary to show that the analogous method of replacing $\phi_p(t)$ by $\tilde{\phi}_p(t)$ works for the derivatives as well. The proof is more difficult here because one seems to lose a factor of $p$ in the denominator of the analogous bound when we compute the derivatives if one tries to do naive bounding tricks. However, the lemmas are enough to give us the necessary bound.

**Lemma 17.** For any ring $R$, any ideal $I$ of $R$, and any element $f$ of $R$, for any positive integers $a$ and $b$, we have that

$$(I, f^{a+b}) : f^a = ((I : f^a), f^b).$$

**Proof.** First, if $x$ is an element of the lefthand side of the equality, we can write $xf^a = i + yf^{a+b}$, where $i \in I$. Rearranging, we have that $f^a(x - yf^b) \in I$, and so $x - yf^b \in (I : f^a)$, and hence $x \in ((I : f^a), f^b)$. This shows that the lefthand side is included in the righthand side.

On the other hand, if $x$ is an element of the righthand side, we can write $x = y + zf^b$ for $y \in (I : f^a)$. So $xf^a = yf^a + zf^{a+b} \in (I, f^{a+b})$. Thus $x$ is in the lefthand side, and we are done.

Next, we prove the second of the necessary lemmas to beat the naive method. Again, take $p = dl + r$, with $0 \leq r < d$. 


28
Lemma 18. For any integer \( a \) with \( 2 \leq a \leq p \), we have that

\[
\dim_{\mathbb{F}_p} \left( \frac{R}{(x_1^p, \ldots, x_n^p, f^a) : f^{a-2}} \right) - \dim_{\mathbb{F}_p} \left( \frac{R}{(x_1^{d\ell}, \ldots, x_n^{d\ell}, f^a) : f^{a-2}} \right) = O(p^{n-2}).
\]

Proof. Applying Lemma 17, the lefthand side is

\[
\left( \dim_{\mathbb{F}_p} \left( \frac{R}{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2} \right) - \dim_{\mathbb{F}_p} \left( \frac{R}{(x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2} \right) \right),
\]

which is equal to

\[
\dim_{\mathbb{F}_p} \left( \frac{(x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2}{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2} \right)
\]

via the short exact sequence

\[
0 \rightarrow \frac{(x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2}{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2} \rightarrow \frac{R}{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2} \rightarrow \frac{R}{(x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2} \rightarrow 0.
\]

Now consider \( \delta := x_1^r \cdot \ldots \cdot x_n^r \). Observe that \( ((x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2) \) is a subset of \( ((x_1^p, \ldots, x_n^p) : f^{a-2}, f^2) \) : \( \delta \), since if \( x \) is in \( (x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2 \), \( \delta x f^{a-2} \) is in \( (x_1^p, \ldots, x_n^p) \), and hence \( x \) is in the second ideal. Thus

\[
\dim_{\mathbb{F}_p} \left( \frac{(x_1^{d\ell}, \ldots, x_n^{d\ell}) : f^{a-2}, f^2}{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2} \right) \leq \dim_{\mathbb{F}_p} \left( \frac{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2}{(x_1^p, \ldots, x_n^p) : f^{a-2}, f^2} \right) : \delta.
\]

For any ideal \( I \) of \( R \) we have an exact sequence

\[
0 \rightarrow (I : f)/(I) \rightarrow R/I \rightarrow R/I \rightarrow R/(I, f) \rightarrow 0
\]
where the middle map is given by multiplication by $f$. Taking dimensions gives that $\dim((I : f)/I) = \dim(R/(I, f))$. Applying this to the above scenario, we obtain that the righthand term of the inequality is equal to

$$\dim_{F_p}(R/((x_1^p, \cdots, x_n^p) : f^{a-2}, f^2, \delta)),$$

which is in turn less than or equal to

$$\dim_{F_p}(R/((x_1^p, \cdots, x_n^p, f^2, \delta))).$$

So we have now reduced to showing that $\dim_{F_p}(R/((x_1^p, \cdots, x_n^p, f^2, \delta))) = O(p^n)$. Let $I = (x_1^p, \cdots x_n^p, f^2)$. We have a filtration

$$(I, \delta) \subset (I, x_1^r \cdots x_n^{r-1}) \subset \cdots \subset (I, x_1) \subset R$$

where at each step we systematically decrease the power of the last remaining $x_i$ in the second generator by one. Note that there are $(r + 1)n$ inclusions in this sequence, a number which is independent of $p$. For successive terms, we have that $\dim_{F_p}\left(\frac{(I, x_1^r \cdots x_n^{r-1})}{(I, x_1^r \cdots x_n^r)}\right) = \dim_{F_p}\left(\frac{R/((I, x_1^r \cdots x_n^{r-1}) : x_1^r \cdots x_i^r)}{R/((I, x_1^{r-1} \cdots x_i^r, x_i)\} \right)$, which is less than $\dim_{F_p}((R/I, x_i)) = \dim_{F_p}(R/((x_1^p, \cdots x_n^p, x_i, f^2)))$. Now $R/((x_1^p, \cdots x_n^p, x_i)$ has an explicit $F_p$-basis given by monomials of the form $x_1^{\beta_1} \cdots x_i^{\beta_i-1} x_{i+1}^{\beta_{i+1}} \cdots x_n^{\beta_n}$ where each $\beta_j$ is between 0 and $p$. Thus this has $F_p$-dimension $O(p^{n-1})$. Now the largest $d_j$ remaining (either $d_n$ or $d_{n-1}$) in the image of $f$ in this quotient ring appears in $f^2$ with exponent $2d_j$ and in every other term containing $x_j$ in $f^2$, $x_j$
appears to a power which is not larger than $2d_j$. Thus by killing $f^2$ we can write all monomials of our basis so that $\beta_j \leq 2d_j$, which is a constant independent of $p$. Thus there are $O(p^{n-2})$ monomials in a basis of the ring after killing $f^2$, and so the $\mathbb{F}_p$-dimension of this ring is $O(p^{n-2})$. Since this holds for each successive quotient in the filtration and the number of such quotients is independent of $p$, the difference of dimensions we care about is also $O(p^{n-2})$, as desired.

The next lemma will actually show how our derivatives converge. The idea is that if we choose pairs of points cleverly to the left of the point at which we desire to calculate the derivative (and we can show something about how these converge) we can say something about the derivative itself. We will use this on the intervals where the limit function $\psi(t)$ is defined by a single polynomial equation.

**Lemma 19.** Let $\mu_p : [a, b] \rightarrow [0, 1]$ be a sequence of decreasing convex functions indexed by primes $p$ converging to a $C^1$ function $\mu$. Furthermore, suppose that for fixed $t$ and for each $p$ there exist $a < \alpha_p < \beta_p < t < \gamma_p < \delta_p < b$ such that the difference quotients $\frac{\mu_p(\beta_p) - \mu_p(\alpha_p)}{\beta_p - \alpha_p}$ and $\frac{\mu_p(\delta_p) - \mu_p(\gamma_p)}{\delta_p - \gamma_p}$ both converge to $\mu'(t)$ as $p \rightarrow \infty$. Then $\partial_- \mu_p(t)$ and $\partial_+ \mu_p(t)$ converge to $\mu'(t)$ as $p \rightarrow \infty$.

**Proof.** Since $\mu_p$ is decreasing and convex, for each $p$ we have

$$\partial_+ \mu_p(t) \leq \frac{\mu_p(\delta_p) - \mu_p(t)}{\delta_p - t} \leq \frac{\mu_p(\delta_p) - \mu_p(\gamma_p)}{\delta_p - \gamma_p}.$$ 

Similarly, we have

$$\partial_- \mu_p(t) \geq \frac{\mu_p(t) - \mu_p(\alpha_p)}{t - \alpha_p} \geq \frac{\mu_p(\beta_p) - \mu_p(\alpha_p)}{\beta_p - \alpha_p}.$$
Combining these inequalities with the fact that \( \partial_- \mu_p(t) \leq \partial_+ \mu_p(t) \) (again since the \( \mu_p \) are decreasing and convex), we get that

\[
\frac{\mu_p(\beta_p) - \mu_p(\alpha_p)}{\beta_p - \alpha_p} \leq \partial_- \mu_p(t) \leq \partial_+ \mu_p(t) \leq \frac{\mu_p(\delta_p) - \mu_p(\gamma_p)}{\delta_p - \gamma_p}.
\]

Letting \( p \) go to infinity gives the desired result.

\( \square \)

We are now ready to prove Theorem 16.

**Proof of Theorem 16.** First, consider \( t \) in the interior of one of the intervals where \( \psi(t) \) is defined by a single polynomial expression. Let \( a_p \) be either \( \lceil pt \rceil \) or \( \lfloor pt \rfloor \) as above, again making the choice so that \( \frac{-(n+1)+a_p+\Sigma_i t}{2} \) is an integer. We wish to apply Lemma 19 to \( \psi_p(t) \) with \( \alpha_p = \frac{a_p-4}{p}, \beta_p = \frac{a_p-2}{p}, \gamma_p = \frac{a_p+2}{p}, \) and \( \delta_p = \frac{a_p+4}{p}. \)

In order to do this, we need to check that the difference quotients limit to the correct derivative. We check this for the \( \alpha_p \) and \( \beta_p \) case, with the other side being handled identically. We first introduce some new terms that will more easily allow us to obtain bounds:

\[
| \frac{\psi_p(\beta_p) - \psi_p(\alpha_p)}{\beta_p - \alpha_p} - \psi'(t) | = | p^2 \left( \psi_p \left( \frac{a_p - 2}{p} \right) - \psi_p \left( \frac{a_p - 4}{p} \right) \right) - \psi'(t) |
\]

\[
= | p^2 \left( \phi_p \left( \frac{a_p - 4}{p} \right) - \phi_p \left( \frac{a_p - 2}{p} \right) \right) - \psi'(t) |
\]

\[
\leq | p^2 \left( \phi_p \left( \frac{a_p - 4}{p} \right) - \phi_p \left( \frac{a_p - 2}{p} \right) \right) |
\]

\[
+ | p^2 \left( \tilde{\phi}_p \left( \frac{a_p - 4}{p} \right) - \tilde{\phi}_p \left( \frac{a_p - 2}{p} \right) \right) - \psi'(t) |
\]

32
We argue separately that each of the terms after the inequality goes to 0. For the first term, from the short exact sequence

\[
0 \to \frac{R}{(x_1^p, \cdots, x_n^p, f_{ap-2}) : f_{ap-4}} \to \frac{R}{(x_1^p, \cdots, x_n^p, f_{ap-2})} \to \frac{R}{(x_1^p, \cdots, x_n^p, f_{ap-4})} \to 0
\]

and the analogous short exact sequence where the powers of \(p\) are replaced by powers of \(d\ell\), taking lengths gives that the first term is equal to

\[
\frac{1}{2p^{n-1}} \left( \dim_{\mathcal{F}_p} \left( \frac{R}{(x_1^p, \cdots, x_n^p, f_{ap-2}) : f_{ap-4}} \right) - \dim_{\mathcal{F}_p} \left( \frac{R}{(x_1^{d\ell}, \cdots, x_n^{d\ell}, f_{ap-2}) : f_{ap-4}} \right) \right).
\]

By Lemma 18, the difference is \(O(p^{n-2})\), and so this whole term goes to 0 as \(p \to \infty\), as desired.

So we have now reduced to showing that

\[
\frac{p}{2} \left( \bar{\phi}_p \left( \frac{a_p - 4}{p} \right) - \bar{\phi}_p \left( \frac{a_p - 2}{p} \right) \right) \to \psi'(t)
\]

as \(p \to \infty\). We can proceed as in the proof of Theorem 15. Since \(a_p\) satisfies the parity condition of Theorem 3, so do \(a_p - 2\) and \(a_p - 4\), and so we can compute the \(\bar{\phi}\) terms using Theorem 3. Since all three of \(\frac{a_p}{p}\), \(\frac{a_p - 2}{p}\), and \(\frac{a_p - 4}{p}\) have the same limit as \(p \to \infty\), precisely the same binomial terms will contribute in the limit. That is, for choices of \(\epsilon_0, \cdots, \epsilon_n\), the signs of the limit as \(p \to \infty\) of \(\frac{1}{p^n} (n + \frac{1}{2}(\epsilon_0(a_p - 4) + \epsilon_1 \ell + \cdots + \epsilon_n \ell - n - 1) - \lambda p)\) and \(\frac{1}{p^n} (n + \frac{1}{2}(\epsilon_0(a_p - 2) + \epsilon_1 \ell + \cdots + \epsilon_n \ell - n - 1) - \lambda p)\)
agree. So we can compute \( \frac{d}{2p^{n-1}} \left( \sum_{\lambda \in \mathbb{Z}} \left( \sum_{\epsilon_0 \cdots \epsilon_n} \left( \binom{\omega(\lambda) - 2\epsilon_0}{n} - \binom{\omega(\lambda) - \epsilon_0}{n} \right) \right) \right) \)

where again we write \( \omega(\lambda) = n + \frac{1}{2} (\epsilon_0 a_p + \epsilon_1 e_1 + \cdots + \epsilon_n e_n - n - 1) - \lambda p \).

Note that the \( 2\epsilon_0 \) and \( \epsilon_0 \) terms just come from the remaining part of \( a_p - 4 \) and \( a_p - 2 \), respectively, after we rewrite in terms of \( \omega(\lambda) \).

If \( \epsilon_0 = 1 \),

\[
\left( \binom{\omega(\lambda) - 2\epsilon_0}{n} - \binom{\omega(\lambda) - \epsilon_0}{n} \right) = -\left( \binom{\omega(\lambda) - 2}{n - 1} \right).
\]

If \( \epsilon_0 = -1 \),

\[
\left( \binom{\omega(\lambda) - 2\epsilon_0}{n} - \binom{\omega(\lambda) - \epsilon_0}{n} \right) = \left( \binom{\omega(\lambda) + 1}{n - 1} \right).
\]

In either case,

\[
\frac{1}{p^{n-1}} \left( \binom{\omega(\lambda) - 2\epsilon_0}{n} - \binom{\omega(\lambda) - \epsilon_0}{n} \right)
\]

limits as \( p \to \infty \) to

\[
\frac{-\epsilon_0}{2^{n-1}(n-1)!} (\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda)^{n-1}.
\]

To be more precise, we can bound the difference of these two terms by \( \frac{C}{p} \) for some constant \( C \) independent of \( t \). First, note that \( |\frac{\epsilon_0 a_p}{p} - \epsilon_0 t| < \frac{\Omega}{p} \) by how we chose \( a_p \). Similarly, each \( |\frac{\epsilon_i e_i t}{p} - \frac{\epsilon_i}{d_i}| < \frac{\Omega}{p} \). Now for any constant \( \zeta \) bounded by, say,
2n, we have that
\[
\left| \frac{\omega_\lambda(p) + \zeta}{p} - \frac{-\epsilon_0}{2}(\epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda) \right| < \frac{c}{p}
\]
for some \( c \) independent of \( t \), by what was just said. Finally, since every summand appearing in both terms is bounded in absolute value by some constant \( C' \) depending on \( f \), we can take products and conclude that
\[
\left| \frac{1}{p^{n-1}} \left( \left( \frac{\omega_\lambda(p) - 2\epsilon_0}{n} \right) \right) - \left( \frac{\omega_\lambda(p) - \epsilon_0}{n} \right) \right| < \frac{C}{p}
\]
independent of \( t \), as desired.

Finally, we can conclude that \( \lim_{p \to \infty} \frac{d}{2} \left( \tilde{\phi}_p \left( \frac{a_p - 4}{p} \right) - \tilde{\phi}_p \left( \frac{a_p - 2}{p} \right) \right) \) is equal to
\[
-\frac{d}{2^n(n-1)!} \sum_{\lambda \in \mathbb{Z}} \sum (\epsilon_0^2 \epsilon_1 \cdots \epsilon_n) \left( \epsilon_0 t + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \right)^{n-1}.
\]
But this is precisely \( \phi'(t) \), and so we are done.

We can now finish with the case where \( t \) is a boundary point for one of the pieces of \( \psi \). We handle the left derivative \( \partial_- \psi(t) \), with the right derivative being dealt with analogously. Since the \( \psi_f \mod p \) are convex, the left derivative functions
∂∗ψf mod p are left continuous. So we have that

\[
\partial_\ast \psi_f \mod p(t) = \lim_{s \to t^-} \partial_\ast \psi_f \mod p(s) \\
\leq \lim_{s \to t^-} (\partial_\ast \psi(s) + \frac{C}{p}) \\
= \partial_\ast \psi(t) + \frac{C}{p}
\]

where the inequality comes from the fact that \( \psi \) is defined by a single polynomial function in some neighborhood to the left of \( t \), and we have seen that on such a piece we can bound the difference of \( \partial_\ast \psi_f \mod p \) and \( \partial_\ast \psi(s) \) by a factor of \( \frac{C}{p} \).

Similarly, we have that \( \partial_\ast \psi(t) - \frac{C}{p} \leq \partial_\ast \psi_f \mod p(t) \). Letting \( p \to \infty \) gives the desired result.

\[\square\]

With our major results in hand, we can begin to study some properties and examples of these limiting functions. First, we show that one can recover characteristic 0 information about the hypersurface \( f \) from the limiting function \( \psi(t) \).

**Corollary 20.** Let \( f = x_1^{d_1} + \cdots + x_n^{d_n} \) where \( \text{LCT}(f) = \sum \frac{1}{d_i} \leq 1 \), and let \( \psi \) be the corresponding limit \( F \)-signature function \( \psi = \lim_{p \to \infty} \psi_f \mod p \). Then for all sufficiently small \( 0 < \epsilon \ll 1 \) and \( t \in [\text{LCT}(f) - \epsilon, \text{LCT}(f)] \), we have

\[
\psi(t) = \frac{d}{2^{n-1} \cdot n!} (\text{LCT}(f) - t)^n.
\]

**Proof.** First, we observe that since \( \sum \frac{1}{d_i} \leq 1 \), the \( C_\lambda(t) \) are identically 0 on \([0, 1]\) for all \( \lambda \geq 1 \). So \( \psi(t) = 1 - \frac{d}{2^{n-1} \cdot n!} C_0 \). Since \( \psi_f \mod p \) vanish above the F-pure
threshold \( fpt(f_p) \) for each \( p \) and \( LCT(f) \geq fpt(f_p) \) for all sufficiently large \( p \), we must have that the limit function \( \psi \) is identically 0 on \( [LCT(f), 1] \). We next note that the only difference between \( C_0 \) on \( [LCT(f), 1] \) and \( [LCT(f) - \epsilon, LCT(f)] \) is that the term \((-1)^n(t - LCT(f))^n\) becomes \( -(t + LCT(f))^n\). Now if \( \mu \) is a piecewise polynomial function, let \( \mu_{[a, b]} \) denote the algebraic expression defining \( \mu \) on the interval \([a, b]\). Then we have that

\[
\psi_{[LCT(f) - \epsilon, LCT(f)]} = \psi_{[LCT(f) - \epsilon, LCT(f)]} - \psi_{[LCT(f), 1]}
\]

\[
= \frac{d}{2^n n!} ((C_0)_{[LCT(f), 1]} - (C_0)_{[LCT(f) - \epsilon, LCT(f)]})
\]

\[
= \frac{d}{2^n n!} ((-1)^n(t - LCT(f))^n + (t + LCT(f))^n)
\]

\[
= \frac{d}{2^{n-1} n!} (LCT(f) - t)^n,
\]

as desired.

Next, our results give a new proof of Theorem 9.

**Corollary 21.** Fix \( f \) as above, and let \( e_{HK,p}(f) \) be the Hilbert-Kunz multiplicity of \( f \) in characteristic \( p \). Then

\[
\lim_{p \to \infty} e_{HK,p}(f) = \frac{d}{2^{n-1}(n-1)!} \left( C_0 + 2 \sum_{\lambda \in \mathbb{Z}_{\geq 1}} C_\lambda \right)
\]

where \( C_\lambda \) for \( \lambda \in \mathbb{Z}_{\geq 0} \) is

37
\[ C_\lambda = \sum (\epsilon_1 \cdots \epsilon_n) \left( \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \right)^{-1} \]

with the sum taken over all choices of \( \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\} \) with \( \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \geq 0 \).

**Proof.** We observe that
\[
\partial_+ C_\lambda(0) = n \sum \epsilon_0^2(\epsilon_1 \cdots \epsilon_n) \left( \epsilon_0 * 0 + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \right)^{-1} = 2nC_\lambda,
\]
where the 2 comes from the fact that each term appears twice (for \( \epsilon_0 = 1 \) and \( \epsilon_0 = -1 \)). So
\[
\lim_{p \to \infty} e_{HK,p}(f) = -\lim_{p \to \infty} \partial_+ \psi_{f_{\text{mod}p}}(0) = -\psi'(0) = \frac{d}{2^n n!} \left( C_0''(0) + 2 \sum_{\lambda \in \mathbb{Z} > 0} C_\lambda''(0) \right),
\]
which is precisely \( \frac{d}{2^n (n-1)!} (C_0 + 2 \sum_{\lambda \in \mathbb{Z} > 0} C_\lambda) \), as desired. \( \square \)

This also gives an analogous statement for the limit \( F \)-signature. However, it seems to be a legitimately different formula from the one obtained in Theorem 11.

**Corollary 22.** Fix \( f \) as above, and let \( s_p(f) \) be the \( F \)-signature of \( f \) in characteristic \( p \). Then
\[
\lim_{p \to \infty} s_p(f) = \frac{d}{2^n (n-1)!} \left( C_0'' + 2 \sum_{\lambda \in \mathbb{Z} \geq 1} C_\lambda'' \right).
\]
where $C''_{\lambda}$ for $\lambda \in \mathbb{Z}_{\geq 0}$ is

$$C''_{\lambda} = \sum (\epsilon_1 \cdots \epsilon_n) \left( \epsilon_0 * 1 + \epsilon_1 \frac{1}{d_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \right)^{n-1}$$

with the sum taken over all choices of $\epsilon_0, \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ with $\epsilon_0 + \epsilon_1 \frac{1}{a_1} + \cdots + \epsilon_n \frac{1}{d_n} - 2\lambda \geq 0$.

Proof. This follows directly by plugging 1 into the expression for the derivative of $\psi(t)$.

Again, we wish to remark that this (on its face) appears to be a different formula from that obtained in Theorem 11. In particular, no $d_i$ is distinguished here.

**Example: The Two Variable Case (Again)**

We give a different proof of Theorem 14 using explicit computation and the formula from Theorem 15.

Let $f = x^a + y^b$ for $2 \leq a \leq b$. Since $\frac{1}{a} + \frac{1}{b} \leq 1$, the only nonzero $C_{\lambda}(t)$ term is $C_0(t)$. On $[0, \frac{1}{a} - \frac{1}{b}]$, this is given by

$$\left( t + \frac{1}{a} + \frac{1}{b} \right)^2 - \left( -t + \frac{1}{a} + \frac{1}{b} \right)^2 - \left( t + \frac{1}{a} - \frac{1}{b} \right)^2 + \left( -t + \frac{1}{a} - \frac{1}{b} \right)^2 = \frac{8}{b} t.$$

So $\psi(t)$ on $[0, \frac{1}{a} - \frac{1}{b}]$ is given by $1 - \frac{ab}{8} \cdot \frac{8}{b} t = 1 - at$. 
On \([\frac{1}{a} - \frac{1}{b}, \frac{1}{a} + \frac{1}{b}]\), \(C_0(t)\) is given by

\[
\left(t + \frac{1}{a} + \frac{1}{b}\right)^2 - \left(-t + \frac{1}{a} + \frac{1}{b}\right)^2 - \left(t + \frac{1}{a} - \frac{1}{b}\right)^2 - \left(t - \frac{1}{a} + \frac{1}{b}\right)^2 = -2t^2 + \left(\frac{4}{a} + \frac{4}{b}\right)t - \left(\frac{2}{a^2} - \frac{4}{ab} + \frac{2}{b^2}\right),
\]

and so \(\psi(t)\) on \([\frac{1}{a} - \frac{1}{b}, \frac{1}{a} + \frac{1}{b}]\) is given by

\[
1 - \frac{ab}{8} \left(-2t^2 + \left(\frac{4}{a} + \frac{4}{b}\right)t - \left(\frac{2}{a^2} - \frac{4}{ab} + \frac{2}{b^2}\right)\right) = \frac{ab}{4} t^2 - \frac{a + b}{2} t + \frac{a}{4b} + \frac{1 + b}{4a}
\]

\[
= \frac{ab}{4} \left(t - \frac{1}{a} - \frac{1}{b}\right)^2
\]

Finally, on \([\frac{1}{a} + \frac{1}{b}, 1]\), \(C_0(t)\) is given by

\[
\left(t + \frac{1}{a} + \frac{1}{b}\right)^2 + \left(t - \frac{1}{a} - \frac{1}{b}\right)^2 - \left(t + \frac{1}{a} - \frac{1}{b}\right)^2 - \left(t - \frac{1}{a} + \frac{1}{b}\right)^2 = 8 \frac{1}{ab}
\]

and so \(\psi(t)\) is equal to \(1 - \frac{ab}{8} \frac{8}{ab} = 0\) on \([\frac{1}{a} + \frac{1}{b}, 1]\). Thus we have recovered the result of Theorem 14, as desired.

**Example:** \(x_1^2 + \cdots + x_n^2\)

The example of Gessel and Monsky indicates that in the limit for sums of squares, Bernoulli and Euler numbers should show up. This is obtained from the classical identities

- \(\sec(x) = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} x^{2k}\) where \(E_{2k}\) is the \((2k)^{th}\) Euler number.
\[ \tan(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (2^{k+1} - 1) B_{2k+2}}{(2k+2)!} x^{2k+1} \text{ where } B_{2k+2} \text{ is the } (2k+2)^{th} \text{ Bernoulli number.} \]

It turns out that the correct generalization for polynomials for our purposes are the so-called Euler polynomials \( E_n(t) \), which are defined in terms of a power series expansion as
\[ \frac{2e^{xt}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!}. \]

First, we collect some relevant facts about these polynomials in a lemma from [3] and [17].

**Lemma 23.**  
\( E_n(t) \) is a degree \( n \) polynomial with leading coefficient 1.

- \( E_n(t + 1/2) + E_n(t - 1/2) = 2(t - 1/2)^n \)
- \( \frac{d}{dx} E_k(t) = k E_{k-1}(t) \)
- \( E_k(0) = -\frac{2}{k} (2^{k+1} - 1) B_{k+1} \)
- \( E_k(1) = -E_k(0) \)
- \( E_k(\frac{1}{2}) = 2^{-k} E_k \)

After much combinatorial wrangling, we obtain the following theorem.

**Theorem 24.** Let \( f = x_1^2 + \cdots + x_n^2 \).

- If \( n \) is even, the limit function on \([0, 1]\) is given by
\[ \psi(t) = 1 - t - \frac{(-1)^{\frac{n}{2}} 2^{n-1}}{n!} E_n(t), \]
where $E_n(t)$ is the $n$th Euler polynomial.

- If $n$ is odd, the limit function on $[0, \frac{1}{2}]$ is given by
  \[
  \psi(t) = 1 - t - \frac{(-1)^{(n-1)/2}2^{n-1}}{n!}E_n(t + \frac{1}{2}).
  \]

- If $n$ is odd, the limit function on $[\frac{1}{2}, 1]$ is given by
  \[
  1 - t + \frac{(-1)^{(n-1)/2}2^{n-1}}{n!}E_n(t - \frac{1}{2}).
  \]

First, we need the following combinatorial lemma:

**Lemma 25.** Given a polynomial $g(t)$, let $\Delta[g](t)$ be defined as $\Delta[g](t) = g(t) - g(t + 1)$. Similarly, let $\Delta^m[g](t) = \Delta(\Delta(\cdots(\Delta[g](t))))$, where here $\Delta$ is being applied $m$ times. Then for any $m$ and $n$,

\[
\Delta^m[t^n](t) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} (t+j)^n.
\]

- For any integer $\ell$,
  \[
  \sum_{j=0}^{m} (-1)^j \binom{m}{j} (\ell+j)^n = \begin{cases} 
    0 & n < m \\
    (-1)^n n! & n = m
  \end{cases}.
  \]

**Proof.** Induct on $m$ for fixed $n$. If $m = 1$, we have that

\[
\Delta[t^n](t) = t^n - (t+1)^n = \sum_{j=0}^{1} (-1)^j \binom{1}{j} (t+j)^n,
\]

42
as desired. Now

\[
\Delta^{m+1}[t^n](t) = \Delta(\Delta^m [t^n](t))
\]

\[= \Delta\left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (t+j)^n\right)(t)\]

\[= \sum_{j=0}^{m} (-1)^j \binom{m}{j} (t+j)^n - \sum_{j=0}^{m} (-1)^j \binom{m}{j} (t+j+1)^n\]

\[= t^n + \left(\sum_{j=1}^{m} (-1)^j \left(\binom{m}{j} + \binom{m}{j-1}\right) (t+j)^n\right) + (t+m+1)^n\]

\[= t^n + \left(\sum_{j=1}^{m} (-1)^j \binom{m+1}{j} (t+j)^n\right) + (t+m+1)^n\]

\[= \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (t+j)^n.\]

as desired. The second statement follows immediately from the fact that $\Delta[g](t)$ has degree strictly less than the degree of $g$. In particular, $\Delta^n[t^n](t)$ is the constant polynomial $(-1)^n n!$, and for $m > n$, $\Delta^m[t^n](t)$ is the constant polynomial 0. Substituting any $\ell$ for $t$ into the expression for these polynomials derived above gives the desired result.

\[\square\]

We can now prove Theorem 24.

Proof. First, we outline the strategy. In each equality appearing in the theorem, both the left-hand and right-hand sides are polynomials of degree $n$. So it is enough to show that these polynomials and their derivatives agree at some collection of $n + 1$ points, as $n + 1$ points uniquely determine a degree $n$ polynomial.
We begin with the case where \( n \) is even. Because \( n \) is even, all expressions of the form \( \epsilon_1 + \cdots + \epsilon_n - 2\lambda \) are integers. Thus \( \psi(t) \) is indeed given by a single polynomial on \([0, 1]\). We will show that all of the even-order derivatives of this polynomial and \( \left( 1 - t - \frac{(-1)^{n/2-1} n!}{2^{n-1}} E_n(t) \right) \) agree at both \( t = 0 \) and \( t = 1 \). This gives the requisite \( n + 1 \) conditions, since the \( n^{th} \) derivative is a constant and so we only get one condition from it.

From Theorem 15, the explicit expression for \( 1 - \psi(t) \) is given by

\[
\mu(t) := \frac{1}{n!} \sum_{m=n/2}^{n} (-1)^{n-m} \cdot \binom{n}{m} \cdot \left( \begin{array}{l}
\sum_{i=0}^{[(m-n/2)/2]} (t + m - n/2 - 2 \cdot i)^n \\
+ \sum_{i=1}^{[(m-n/2)/2]} (t + m - n/2 - 2 \cdot i)^n \\
- \sum_{i=0}^{[(m-n/2-1)/2]} (-t + m - n/2 - 2 \cdot i)^n \\
- \sum_{i=1}^{[(m-n/2-1)/2]} (-t + m - n/2 - 2 \cdot i)^n
\end{array} \right) .
\]

The \( \binom{n}{m} \) factor comes from the fact that this is the number of ways to choose \( \epsilon_1, \cdots, \epsilon_n \) so that \( \epsilon_1 + \cdots + \epsilon_n = m - \frac{n}{2} \). The first two interior sums correspond to the terms of \( \psi(t) \) where \( \epsilon_0 = 1 \) and the second two interior sums correspond to the terms where \( \epsilon_0 = -1 \). The ranges on these sums are precisely the \( i \) for which the interior expression is nonnegative for all \( t \in [0, 1] \).

We need to see that this expression equals

\[
\nu(t) := \left( t + \frac{(-1)^{n/2} \cdot 2^{n-1}}{n!} \cdot E_n(t) \right) .
\]

We first check that the leading terms of the two polynomials agree. This is equivalent to showing that \( \mu^{(n)}(t) = \nu^{(n)}(t) \). Since the leading coefficient of any
Euler polynomial is 1, the leading coefficient of \( \nu(t) \) is \( \frac{(-1)^{n/2} 2^{(n-1)}}{n!} \).

In the expression for \( \mu(t) \), whenever \((m - \frac{n}{2} - 2 \cdot i)\) is nonzero, the \( t^n \) term of \((t + m - n/2 - 2 \cdot i)^n\) is cancelled by the \( t^n \) term of \( -(-t + m - n/2 - 2 \cdot i)^n\). Thus the only terms which contribute to the leading coefficient of \( \mu(t) \) are those which occur when \((m - n/2 - 2 \cdot i) = 0\) - that is, when \( i = \frac{1}{2}(m - \frac{n}{2}) \). In particular, we need \( m - \frac{n}{2} \) even. When \( m = n/2 \), we get one such term (from the first sum in the expression for \( \mu(t) \) with \( i = 0 \)). Otherwise, such a term appears twice, once in the first sum and once in the second sum in the expression for \( \mu(t) \). We now distinguish between the cases when \( \frac{n}{2} \) is even or odd. If \( \frac{n}{2} \) is even, the leading term of \( \mu(t) \) is given by

\[
\frac{1}{n!} \left( \binom{n}{n/2} + 2 \sum_{j=n/4+1}^{n/2} \binom{n}{2j} \right) = \frac{1}{n!} \left( \sum_{j=0}^{n/2} \binom{n}{2j} \right) = \frac{2^{n-1}}{n!}.
\]

If \( \frac{n}{2} \) is odd, the leading term is given by

\[
\frac{-1}{n!} \left( \binom{n}{n/2} + 2 \sum_{j=(n+2)/4}^{(n-2)/2} \binom{n}{2j+1} \right) = \frac{-1}{n!} \left( \sum_{j=0}^{(n-2)/2} \binom{n}{2j+1} \right) = \frac{-2^{n-1}}{n!}.
\]

In both of these expressions we are using the fact that \( \binom{n}{k} = \binom{n}{n-k} \) to rewrite one copy of each term in the sum with coefficient 2, and then rewrite the whole expression as a single sum. But this is just the sum of \( \binom{n}{k} \) for \( k \) even in the first case and \( k \) odd in the second case, both of which equal \( 2^{n-1} \). So in either case, this agrees with the leading term of \( \nu(t) \), as desired.

Now consider \( \mu(0) \). The first and third sums in the expression for \( \mu(0) \) cancel,
as do the second and fourth sums, and so \(\mu(0) = 0\). Similarly, since \(E_n(0) = 0\) for \(n\) even and \(n \geq 2\) (since the Bernoulli numbers \(B_n\) are 0 for \(n\) odd and greater than 1), we see that \(\nu(0) = 0\) and so \(\mu\) and \(\nu\) agree at 0. The exact same argument shows that all order \(k\) derivatives of both \(\mu\) and \(\nu\) with \(k\) even and between 2 and \(n - 2\) agree at \(t = 0\).

At \(t = 1\), the argument is slightly more interesting. \(\nu(1) = 1\), and for \(k\) even with \(0 \leq k \leq n - 2\), \(\nu^{(k)}(1) = 0\). So we need to see that \(\mu(1) = 1\) and \(\mu^{(k)}(1) = 0\) for \(k\) even.

Substituting \(t = 1\) into the explicit expression for \(\mu(t)\), we can see that the third and fourth sums are offset exactly by an additive factor of 2 from the first two sums, and so the only terms that don’t cancel in this whole expression are the \(i = 0\) and \(i = 1\) terms in the first sum. Thus \(\mu(1)\) collapses to

\[
\frac{1}{n!} \left( (-1)^{n-n/2} \binom{n}{n/2} \right) + \sum_{m=n/2+1}^{n} (-1)^{(n-m)} \binom{n}{m} \cdot \left( (m - \frac{n}{2} + 1)^n + (m - \frac{n}{2} - 1)^n \right).
\]

Now using the fact that \(n\) is even and so \((-1)^n = 1^n\), we can rewrite this as

\[
\frac{(-1)^n}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( j - \frac{n}{2} + 1 \right)^n,
\]

where the terms with \(m < \frac{n}{2}\) are the second terms in the above sum. Now by Lemma 25, this is \(\frac{(-1)^n n!}{n!} = 1\).

The proof for the derivatives of \(\mu(t)\) at \(t = 1\) is, \textit{mutatis mutandis}, the same.
For $k$ even with $2 \leq k \leq n-2$, $\mu^{(k)}(1)$ collapses to

$$
\frac{1}{(n-k)!} \left( -1 \right)^{n-n/2} \binom{n}{n/2} + \sum_{m=n/2+1}^{n} (-1)^{(n-m)} \cdot \binom{n}{m} \cdot \left( (m - \frac{n}{2} + 1)^{n-k} - (m - \frac{n}{2} - 1)^{n-k} \right),
$$

which rearranges via an analogous argument to

$$
\frac{(-1)^n}{n!} \sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \left( j - \frac{n}{2} + 1 \right)^{n-k},
$$

which is 0 by Lemma 25.

So we have seen that all even order (between 0 and $n-2$) derivatives of $\mu(t)$ and $\nu(t)$ agree at both $t = 0$ and $t = 1$ and their leading terms agree, and so these polynomials are identical. This proves the theorem when $n$ is even.

Now let $n$ be odd, with $t \in [0, \frac{1}{2}]$. The explicit expression for $1 - \psi(t)$ in this case is

$$
\mu(t) := \frac{1}{n!} \sum_{m=(n+1)/2}^{n} (-1)^{(n-m)} \cdot \binom{n}{m} \cdot \left( \sum_{i=0}^{\lfloor (m-n/2)/2 \rfloor} (t + m - n/2 - 2 \cdot i)^n - \sum_{i=1}^{\lfloor (m-n/2)/2 \rfloor} (-t + m - n/2 - 2 \cdot i)^n \right)
$$
We need to see that this equals

\[
\nu(t) := \left( t + \frac{(-1)^{(n-1)/2} \cdot 2^{(n-1)}}{n!} \cdot E_n(t + 1/2) \right).
\]

We first check that the leading coefficients agree. Again, since every Euler polynomial has leading coefficient 1, the leading coefficient of \(\nu(t)\) is \(\frac{(-1)^{(n-1)/2} \cdot 2^{(n-1)}}{n!}\). So we need to show that this is the leading coefficient of \(\mu(t)\) as well. We observe that each term in every sum appearing in the expression for \(\mu(t)\) contributes precisely 1 to the leading coefficient (since \(n\) is odd), so the leading coefficient is given by

\[
\frac{1}{n!} \sum_{m=(n+1)/2}^{n} \left( (-1)^{n-m} \binom{n}{m} (2 + 4\lfloor (m - n/2)/2 \rfloor) \right).
\]

We consider separately the cases where \(\frac{n-1}{2}\) is odd or even. In either case we will expand this using the familiar facts that \(\binom{n}{m} = \binom{n-1}{m-1}\) and \(\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}\).

If \(\frac{n-1}{2}\) is even, this collapses to

\[
\frac{2}{n!} \left( \binom{n-1}{(n-1)/2} + \sum_{j=(n+3)/4}^{(n-1)/2} \binom{n-1}{2j} \right) = \frac{2}{n!} \sum_{j=0}^{(n-1)/2} \binom{n-1}{2j} = \frac{2}{n!} 2^{n-2} = \frac{2^{n-1}}{n}.
\]

To see how this collapse works, we show it explicitly for \(n = 5\): The above sum (without the factor of \(\frac{1}{5!}\)) is
\[\sum_{m=3}^{5} (-1)^{5-m} \binom{5}{m} (2 + 4 \lfloor \frac{1}{2} (m - \frac{5}{2}) \rfloor) = 2 \left( \binom{5}{3} - \binom{5}{4} \right) + 6 \binom{5}{5} \]
\[= 2 \left( \binom{5}{4} + \binom{4}{2} - \binom{4}{4} - \binom{4}{3} \right) + 6 \binom{4}{4} \]
\[= 2 \binom{4}{2} + 4 \binom{4}{4} \]
\[= 2 \left( \binom{4}{0} + \binom{4}{2} + \binom{4}{4} \right) \]
\[= 16 \]

This should hopefully elucidate how we achieved the above equality.

In the other case where \( \frac{n-1}{2} \) is odd,

\[\frac{1}{n!} \sum_{m=(n+1)/2}^{n} \left( -1 \right)^{n-m} \binom{n}{m} (2 + 4 \lfloor (m - n/2)/2 \rfloor) \]
collapses to

\[\frac{-2}{n!} \left( \binom{n-1}{(n-1)/2} \right) + 2 \sum_{j=(n+1)/4}^{(n-3)/2} \binom{n-1}{2j+1} \]
\[= \frac{-2}{n!} \sum_{j=0}^{(n-3)/2} \binom{n-1}{2j+1} \]
\[= \frac{-2}{n!} 2^{n-2} \]
\[= \frac{-2^{n-1}}{n!} \cdot \]

In either case, this is the desired equality, and so the leading terms of \( \mu(t) \) and \( \nu(t) \) agree.
At $t = 0$, by Lemma 23, $\nu(0) = 0$ because $E_n(1/2) = 2^{-n}E_n$, and $E_n$ is 0 for $n$ odd. Similarly, for $k$ even between 2 and $n - 1$, $\nu^{(k)}(0) = 0$. But $\mu^{(k)}(0)$ is also 0 for $k$ even between 2 and $n - 1$ just by the obvious symmetries in the expression given above.

Next, we want to take odd order derivatives of both terms and substitute $t = 1/2$. First, $\nu^{(1)}(1/2) = 1 + \left(-1\right)^{(n-1)/2} \frac{n}{(n-1)!}E_{n-1}(1)$, which is 1 since $(n - 1)$ is even and so $E_{n-1}(x)$ has a root at 1. Furthermore, for $k$ odd between 3 and $n - 2$, we have that $\nu^{(k)}(1/2) = 0$ by the same reasoning. So we need to show that for $k$ odd, $\mu^{(1)}(1/2) = 1$ and $\mu^{(k)}(1/2) = 0$ for $3 \leq k \leq n - 2$.

Fix such a $k$. Substituting 1/2 into the explicit expression for $\mu^{(k)}(t)$, it collapses to

$$\frac{1}{(n-k)!} \sum_{m=(n+1)/2}^{n} (-1)^{(n-m)} \binom{n}{m} \cdot \left(\sum_{j=1}^{1/2+m-n/2} j^{n-k} + \left(\sum_{j=1}^{1/2+m-n/2-1} j^{n-k}\right)\right),$$

where the first sum in the interior comes from the terms with $\epsilon_0 = 1$, and the second sum comes from the terms with $\epsilon_0 = -1$.

But

$$\sum_{m=(n+1)/2}^{n} (-1)^{(n-m)} \binom{n}{m} \cdot (1/2 + m - n/2)^{n-k}$$

$$= \sum_{m=(n+1)/2}^{n} (-1)^{(n-m)} \binom{n}{m} \cdot (1/2 + m - n/2 - 1)^{n-k}$$

by an application of Lemma 25 using the fact that $k$ is positive and $n - k$ is even, since we can rewrite the second sum as $- \sum_{m=0}^{(n-1)/2} (-1)^{n-m} \binom{n}{m} (1/2 + m - n/2)^{n-k}.$
and so these two expressions must be equal since the difference of the first and second is 0. So our expression for $\mu^{(k)}(1/2)$ becomes

$$\frac{2}{(n - k)!} \sum_{m=(n+1)/2}^{n} (-1)^{(n-m)} \binom{n}{m} \left( \sum_{j=1}^{1/2+m-n/2-1} j^{n-k} \right).$$

Again using the identities that $\binom{n}{m} = \binom{n-1}{m-1}$ and $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for $1 \leq k \leq n - 1$, this becomes

$$\frac{2}{(n - k)!} \sum_{m=(n+1)/2}^{n-1} (-1)^{(m-n-1)} \binom{n-1}{m} \left( m - \frac{n-1}{2} \right)^{n-k}.$$ 

But

$$\sum_{m=(n+1)/2}^{n-1} (-1)^{(m-n-1)} \binom{n-1}{m} \left( m - \frac{n-1}{2} \right)^{n-k}$$

$$= \frac{1}{2} \sum_{m=0}^{n-1} (-1)^{(m-n-1)} \binom{n-1}{m} \left( m - \frac{n-1}{2} \right)^{n-k},$$

which by Lemma 25 is $\frac{(n-1)!}{2}$ if $k = 1$ and 0 for $k > 1$. So we conclude that $\mu^{(k)}(1/2) = 1$ if $k = 1$ and 0 for $k$ odd between 3 and $n - 2$. We have shown that $\mu(t)$ and $\nu(t)$ have the same leading coefficient, have kth derivatives agreeing at 0 when $k$ is even between 0 and $n - 1$ and at $1/2$ when $k$ is odd between 1 and $n - 2$. This gives the requisite $n + 1$ conditions for these polynomials to be equal, and so we have proved the case when $n$ is odd and $t \in [0, \frac{1}{2}]$.

Finally, we deal with the case when $n$ is odd, with $t \in [\frac{1}{2}, 1]$. Let $\mu_2(t)$ be the polynomial expression for $1 - \psi(t)$ on this interval, and let $\mu_1(t)$ be the expression
for $1 - \psi(t)$ on the interval $[0, \frac{1}{2}]$ (i.e. the $\mu(t)$ just discussed). Examining the expression given in Theorem 15, we see that there are three differences between $\mu_2(t)$ and $\mu_1(t)$: first, if $(-t + \frac{1}{2})^n$ appears in the $C_0(t)$ term of $\mu_1(t)$, it is replaced by $(t - \frac{1}{2})^n$ in the $C_0(t)$ term of $\mu_2(t)$. Secondly, any time $(t + \frac{3}{2})^n$ appears in a $C_\lambda(t)$, we get an additional term of $(t - \frac{1}{2})^n$ in $C_{\lambda+1}(t)$. Finally, if $(-t + \frac{1}{2})^n$ appears in a $C_\lambda(t)$ term of $\mu_1(t)$ for $\lambda \geq 1$, it is deleted in the same $C_\lambda(t)$ term of $\mu_2(t)$.

We now observe that since $E_n(t + \frac{1}{2}) + E_n(t - \frac{1}{2}) = 2(t - \frac{1}{2})^n$, we have that

$$
\frac{(-1)^{(n-1)/2}2^{n-1}}{n!} \left( E_n(t + \frac{1}{2}) + E_n(t - \frac{1}{2}) \right) = \frac{(-1)^{(n-1)/2}2^{n-1}}{n!} \left( t - \frac{1}{2} \right)^n.
$$

So since we know that $\mu_1(t) = t + \frac{(-1)^{(n-1)/2}2^{n-1}}{n!} E_n(t + \frac{1}{2})$, if we can see that $\mu_1(t) - \mu_2(t) = \frac{(-1)^{(n-1)/2}2^{n-1}}{n!} (t - \frac{1}{2})^n$, we can conclude that

$$
\mu_2(t) = t - \frac{(-1)^{(n-1)/2}2^{n-1}}{n!} E_n \left( t - \frac{1}{2} \right),
$$

which is the desired result.

We deal with the cases where $\frac{n-1}{2}$ is even and odd separately. If $\frac{n-1}{2}$ is even, in $\mu_1(t)$, $(-t + 1/2)^n$ appears with coefficient $-\left( \frac{n}{n-1-2j} \right)$ in $C_j(t)$ for $0 \leq j \leq \frac{n-1}{4}$. In $\mu_2(t)$, $(t - \frac{1}{2})^n$ appears with coefficient $-\left( \frac{n}{n-1} \right)$ in $C_0(t)$, and with coefficient $-\left( \frac{n}{n-1} \right)$ in $C_k(t)$ for $1 \leq k \leq \frac{n-1}{4}$. So

$$
\mu_1(t) - \mu_2(t) = \frac{2}{n!} \sum_{j=0}^{(n-1)/2} \binom{n}{j} \left( t - \frac{1}{2} \right)^n = \frac{2^n}{n!} \left( t - \frac{1}{2} \right)^n,
$$

52
as desired.

If $\frac{n-1}{2}$ is odd, in $\mu_1(t)$, $(-t + 1/2)^n$ appears with coefficient $\left(\frac{n-1}{2}\right)$ in $C_j(t)$ for $0 \leq j \leq \frac{n-3}{4}$. In $\mu_2(t)$, $(t - \frac{1}{2})^n$ appears with coefficient $\left(\frac{n}{n-1}\right)$ in $C_0(t)$, and with coefficient $\left(\frac{n}{n-1} \frac{n}{2+k}\right)$ in $C_k(t)$ for $1 \leq k \leq \frac{n+1}{4}$. So

$$\mu_1(t) - \mu_2(t) = -\frac{2}{n!} \sum_{j=0}^{(n-1)/2} \left(\begin{array}{c} n \\ j \end{array}\right) n \left(t - \frac{1}{2}\right)^n = \frac{-2n}{n!} n \left(t - \frac{1}{2}\right)^n,$$

again as desired.

So in either case, we have shown the desired statement. This finishes the proof of the theorem.

\[\square\]

Next, we show that this recovers Theorem 10, which was proved by Gessel and Monsky using power series methods, although they make a remark that their first (unincluded) proof of the result used “Eulerian polynomials”:

**Corollary 26.** If $2 = d_1 = \cdots = d_n$, then the limit of the Hilbert-Kunz multiplicity of $R$ as $p \to \infty$ is $1 + (\text{the coefficient of } z^{n-1} \text{ in the power series expansion of } \sec(z) + \tan(z))$.

**Proof.** We first treat the case when $n$ is even. By the convergence of the derivatives of the limit functions, we have that

$$\lim_{p \to \infty} e_{HK}(f) = -\lim_{p \to \infty} \partial^+ \psi_p(0) = -\psi'(0).$$
By the previous theorem, \(-\psi'(0) = 1 + \frac{(-1)^n 2^{n-1}}{(n!)^2} E_n'(0)\). By Lemma 23,

\[ 1 + \frac{(-1)^n 2^{n-1}}{n!} (nE_{n-1}(0)) = 1 + \frac{(-1)^n 2^{n-1}}{(n+1)!} nE_{n-1}(0) = 1 + \frac{(-1)^n 2^n}{(n+1)!} (2^n - 1) B_n. \]

Writing \(n - 1 = 2j + 1\), we have that this expression (in terms of \(j\)) is \(1 + \frac{(-1)^{2j+2}}{(2j+2)!} (2^{2j+2} - 1) B_{2j+2}\). By Lemma 23 again, this is \((1 + the (n-1)^{st} coefficient of sec(z) + tan(z))\), as desired. For \(n\) odd, we need to show that \(1 + \frac{(-1)^n E_{n-1}}{(n-1)!}\) equals \(-\psi'(0) = 1 + \frac{(-1)^{n-1} 2^{n-1}}{(n-1)!} (E_{n-1}(\frac{1}{2}))\), but this is immediate from Lemma 23 again. Thus we have recovered the desired formulas.

\[ \square \]

**Example: Diagonal du Val Singularities**

The rational surface singularities fall into two infinite families (\(A_n\) and \(D_n\)) and three other examples (\(E_6\), \(E_7\), and \(E_8\)). Of these, \(E_6\), \(E_8\), and the \(A_n\) can be written as diagonal hypersurfaces. We explicitly write down their limit \(F\)-signature functions. All of these are obtained via explicit calculation. We further note that these rings are known to have constant (in terms of \(p\)) Hilbert-Kunz multiplicity and \(F\)-signature as long as \(p\) is at least 7 ([16]), so the right derivative at 0 of the limiting function gives us the Hilbert-Kunz multiplicity of the singularity in every sufficiently large characteristic. Similarly, the left derivative of the limiting function at 1 tells us the \(F\)-signature of the singularity in every sufficiently large characteristic.

- \(A_{n-1}\) is given by \(f = x^2 + y^2 + z^n\) for \(n \geq 2\). The explicit form for the
limiting piecewise polynomial function is

\[
\psi(t) = \begin{cases} 
\frac{n}{3} t^3 - \frac{2n-1}{n} t + 1 & 0 \leq t \leq \frac{1}{n} \\
-t^2 + 2t + \frac{3n^2+1}{3n^2} & \frac{1}{n} \leq t \leq 1 - \frac{1}{n} \\
-\frac{n}{3} t^3 + nt^2 - \frac{n^2+1}{n} t + \frac{n^2+3}{3n} & 1 - \frac{1}{n} \leq t \leq 1
\end{cases}
\]

We note that this recovers the facts that \( e_{HK}(R/(f)) = 2 - \frac{1}{n} \) and \( s(R/(f)) = \frac{1}{n} \), since

\[
e_{HK}(R/(f)) = -\psi'(0) = 2 - \frac{1}{n}
\]

and

\[
s(R/(f)) = -\psi'(1) = -\left( -n + 2n - n - \frac{1}{n} \right) = \frac{1}{n}.
\]

- \( E_6 \) is given by \( f = x^2 + y^3 + z^4 \). The limiting piecewise polynomial function here is

\[
\psi(t) = \begin{cases} 
2t^3 - \frac{47}{23} t + 1 & 0 \leq t \leq \frac{1}{12} \\
-t^3 + \frac{1}{4} t^2 - \frac{95}{48} t + \frac{1729}{1728} & \frac{1}{12} \leq t \leq \frac{5}{12} \\
-\frac{3}{2} t^2 - \frac{5}{2} t + \frac{103}{96} & \frac{5}{12} \leq t \leq \frac{7}{12} \\
-t^3 + \frac{13}{4} t^2 - \frac{169}{48} t + \frac{2107}{1728} & \frac{7}{12} \leq t \leq \frac{11}{12} \\
-2t^3 + 6t^2 - \frac{145}{24} t + \frac{49}{24} & \frac{11}{12} \leq t \leq 1
\end{cases}
\]
This recovers the facts that $e_{HK}(R/(f)) = 2 - \frac{1}{24}$ and $s(R/(f)) = \frac{1}{24}$, since

$$e_{HK}(R/(f)) = -\psi'(0) = \frac{47}{24}$$

and

$$s(R/(f)) = -\psi'(1) = -\left(-6 + 12 - \frac{145}{24}\right) = \frac{1}{24}.$$

- $E_8$ is given by $f = x^2 + y^3 + z^5$. The limiting piecewise polynomial function here is

$$\psi(t) = \begin{cases}
\frac{5}{2}t^3 - \frac{239}{120}t + 1 & 0 \leq t \leq \frac{1}{30} \\
\frac{5}{4}t^3 + \frac{1}{8}t^2 - \frac{479}{240}t + \frac{21601}{21600} & \frac{1}{30} \leq t \leq \frac{11}{30} \\
\frac{3}{2}t^2 - \frac{5}{2}t + \frac{637}{600} & \frac{11}{30} \leq t \leq \frac{19}{30} \\
-\frac{5}{4}t^3 + \frac{31}{8}t^2 - \frac{901}{240}t + \frac{29791}{21600} & \frac{19}{30} \leq t \leq \frac{29}{30} \\
-\frac{5}{2}t^3 + \frac{15}{2}t^2 - \frac{901}{120}t + \frac{301}{120} & \frac{29}{30} \leq t \leq 1
\end{cases}.$$

This recovers the facts that $e_{HK}(R/(f)) = 2 - \frac{1}{120}$ and $s(R/(f)) = \frac{1}{120}$, since

$$e_{HK}(R/(f)) = -\psi'(0) = \frac{239}{120}$$

and

$$s(R/(f)) = -\psi'(1) = -\left(-\frac{15}{2} + 15 - \frac{901}{120}\right) = \frac{1}{120}.$$
Further Questions

There is still much that one could do in the framework discussed here. We include a few questions in the vein of what was considered in this thesis that we think it would be interesting to know the answers to.

**Question 1** *How flexible is the representation ring setup for computing these lengths?*

The multiplicative structure in Han and Monsky’s representation ring is rigged to make diagonal hypersurfaces easy to study. However, one could study the decompositions of \( F_p \)-objects on which \( T \) is not acting as multiplication by a diagonal hypersurface. These still could be expressed in terms of the \( \delta_i \) basis of \( \Gamma \). How hard is it to understand these decompositions? In a slightly different direction, one could ostensibly define the action of \( T \) on tensor products of \( F_p \)-objects differently, which would give rise to a multiplicative structure in the representation ring. Are there analagous hypersurfaces that can be studied in this context? It should be remarked that the only result necessary for all of the limiting results appears relatively early in [5], and so it seems plausible to recover similar results in other contexts.

**Question 2** *What other combinatorially interesting examples can one compute in this framework?*

It should be clear from the statement and proof of Theorem 24 that there is some nontrivial combinatorial content to these limits when families are involved. It would be an interesting question to understand this better. There are a wealth
of potential examples that one could compute.
Cited Literature


[4] Ira M. Gessel and Paul Monsky. The limit as $p \to \infty$ of the hilbert-kunz multiplicity of $\sum x_i^{d_i}$, 2010.


The following Sage code computes the limit in Theorem 15. It is based on Python code written by Joe Ingram for the author’s undergraduate thesis to compute the limits in Theorem 11.

import itertools
import math
import functools
import operator

R.<t> = QQ[]

def prod(seq):
    return functools.reduce(operator.mul,seq,1)

def boundaries(d,s):
    all_sums = set()
    all_sums.add(0)
    all_sums.add(1)
    for l in range(s):
        for ep in itertools.product([-1,1],repeat=s):
            total = -2*l
            for i in range(s):
                total+=ep[i]*(1/d[i])
            if total<1 and total>-1:
                all_sums.add(abs(total))
    boundaries = list(all_sums)
    return(sorted(boundaries))
def clambda(d, s, lam, left, right):
    c = 0
    for ep in itertools.product([-1, 1], repeat=s+1):
        summ = -2*lam
        for i in range(0, s):
            summ += (ep[i]) * (1/d[i])
        if (summ + (ep[s] * right) >= 0) and (summ + (ep[s] * left) >= 0):
            inner = pow(summ + ep[s] * t, s)
            c += prod(ep) * inner
    return(c)

def main():
    i = raw_input('Enter Exponents Separated by Spaces: ')
    d = map(int, i.split())
    s = len(d)
    b = boundaries(d, s)
    bound = s + 1
    coeff = (prod(d)) / (pow(2, s) * math.factorial(s))
    for i in range(len(b) - 1):
        c = [0] * (bound + 1)
        left = b[i]
        right = b[i + 1]
        for lam in range(0, bound + 1):
            c[lam] = clambda(d, s, lam, left, right)
        print('On ' + str([left, right]) + ': ', 1 - coeff * (c[0] + 2 * sum(c[1:])))
main()
Vita

Education

- **PhD, Mathematics**
  - University of Illinois at Chicago
  - Completion Date: Summer 2018
  - Advisor: Kevin Tucker

- **AB, Mathematics, Magna Cum Laude**
  - Princeton University, Spring 2013
  - Senior Thesis: *The F-Signature of Diagonal Hypersurfaces*
  - Senior Thesis Advisor: Kevin Tucker

Awards

- Spring 2017: Received funding through the department’s NSF RTG Grant.
- Fall 2015: Received funding through the department’s NSF RTG Grant.
- 2013-2014: Received first year funding through the department’s NSF RTG Grant.

Talks Given
Fall 2017: UIC Graduate Theoretical Computer Science Seminar: P vs. NP: Polynomial Hierarchy, Independence, and Strengthenings

Spring 2016: UIC Graduate Algebraic Geometry Seminar: Introduction to Seshadri Constants

Spring 2016: UIC RTG Examples in Algebraic Geometry Seminar: Failure of Vanishing Theorems in Positive Characteristic

Fall 2015: UIC Graduate Algebraic Geometry Seminar: Hilbert-Kunz Multiplicity and the F-Signature

Spring 2015: UIC Graduate Algebraic Geometry Seminar: Introduction to Local Cohomology

Fall 2014: UIC Graduate Algebraic Geometry Seminar: An Introduction to Frobenius Splitting in Commutative Algebra

Fall 2014: UIC RTG Seminar on Representation Theory: Analyzing the Structure and Representations of an Arbitrary Semisimple Lie Algebra

Work Experience

Fall 2017: Teaching Assistant Coordinator (UIC): Teaching assistant for the first-year graduate student course on teaching. Performed teaching evaluations for new TAs in the Mathematics Department.

Fall 2016: Teaching Assistant (UIC): Taught two problem sessions for MATH 180 (Calculus 1).

Spring 2016: Teaching Assistant (UIC): Taught two problem sessions for MATH 181 (Calculus 2).

Spring 2015: Teaching Assistant (UIC): Taught two problem sessions for MATH 180 (Calculus 1).
• Fall 2014: Teaching Assistant (UIC): Taught three problem sessions for MATH 180 (Calculus 1).

• Fall 2011-Spring 2013: Grader (Princeton): I graded for MAT 103 (Calculus I) for one semester and MAT 202 (Linear Algebra with Applications) for three semesters.

Volunteer Experience

• UIC Graduate Algebraic Geometry Seminar, Organizer for Academic Year 2015-2016