Topics in the Nonlinear Geometry of Banach Spaces.

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Contribution of Authors:

The work in this thesis was done entirely by myself, with the exception of Chapter 5, which is based on a joint work with Andrew Swift (see [BrSw]).

Contents

Chapter 1

Summary.

In this chapter, we introduce the main definitions regarding the nonlinear geometry of Banach spaces, discuss some of the main problems in the field and its main results, as well as give a detailed description of the main results contained in this dissertation. The Banach space theory notation and terminology used here is standard (see [AlKa] for example), and we assume that the reader is familiarized with it throughout this chapter. For an overview of Banach space theory, we direct the reader to Chapter 2, where the necessary background and notation will be recalled.

This dissertation is mainly based on four papers. *Coarse and uniform embeddings* (see [Br2] or Chapter 3), *On weaker notions of nonlinear embeddings between Banach spaces* (see [Br4] of Chapter 4), and *Asymptotic structure and coarse Lipschitz geometry of Banach spaces* (see [Br3] or Chapter 6) were written by the author of this dissertation, while *Coarse embeddings into superstable spaces* (see [BrSw] of Chapter 5) is a joint work with Andrew Swift.

1.1 Basic definitions.

Recently, there has been a significant increase in the study of Banach spaces as metric spaces. For that, instead of studying linear isomorphisms and embeddings between Banach spaces, we look at a Banach space $(X, \|\cdot\|)$ as a metric space endowed with the metric $\|\cdot - \cdot\|$, and study embeddings and equivalences given by different notions of nonlinear maps. The fundamental question regarding the nonlinear geometry of Banach spaces is to understand to which extent the metric structure of a Banach space can enlighten us regarding its linear structure. As it turns out, in many instances, the metric structure of some Banach spaces can completely determine their linear structure. Those concepts have been used in many different areas, and have many applications, e.g., in topology (see [NoYu]), geometric group theory (see [Gr]), and computer science (see [OstR]).

Let (M, d) and (N, ∂) be metric spaces, and consider a map $f : (M, d) \to (N, \partial)$. We define the *modulus of continuity* of f as

$$\omega_f(t) = \sup\{\partial(f(x), f(y)) \mid d(x, y) \leqslant t\}, \tag{1.1.1}$$

and the *expansion modulus* of f as

$$\rho_f(t) = \inf\{\partial(f(x), f(y)) \mid d(x, y) \geqslant t\}, \tag{1.1.2}$$

for all $t \geqslant 0$. So, we have that

$$\rho_f(d(x, y)) \leqslant \partial(f(x), f(y)) \leqslant \omega_f(d(x, y)),$$

for all $x, y \in M$. The map f is *uniformly continuous* if $\lim_{t\to 0_+} \omega_f(t) = 0$, and it is easy to see that the inverse $f^{-1} : f(M) \to M$ exists and is uniformly continuous if and only if $\rho_f(t) > 0$, for all $t > 0$. We call f a *uniform embedding* if both f and f^{-1} are uniformly continuous, and we call f a *uniform equivalence* if f is a surjective uniform embedding. The map f is called *coarse* if $\omega_f(t) < \infty$, for all $t \geqslant 0$, and *expanding* if $\lim_{t\to\infty} \rho_f(t) = \infty$. If f is both expanding and coarse, f is called a *coarse embedding*. A coarse embedding f which is also *cobounded*, i.e., $\sup_{y\in N} \partial(y, f(M)) < \infty$, is called a *coarse equivalence*. If f is both a coarse and a uniform embedding, we call f a *strong embedding*.

We call f *Lipschitz* if there exists some $L \geqslant 0$ such that $\omega_f(t) \leqslant Lt$, for all $t \geqslant 0$, and we call f L-Lipschitz if we want to specify the constant L. It is easy to see that, if there exists $L > 0$ such that $\rho_f(t) \geqslant Lt$, then $f^{-1} : f(M) \to M$ exists and it is Lipschitz. If f and f^{-1} are Lipschitz, then f is a *Lipschitz embedding*. A surjective Lipschitz embedding is called a *Lipschitz isomorphism*. The map f is called *coarse Lipschitz* if there exists $L \geqslant 0$ such that $\omega_f(t) \leqslant Lt + L$, for all $t \geqslant 0$. If f is coarse Lipschitz and there exists $L > 0$ such that $\rho_f(t) \geqslant L^{-1}t - L$, for all $t \geqslant 0$, then f is called a *coarse Lipschitz embedding*. If f is a cobounded coarse Lipschitz embedding, then f is a *coarse Lipschitz equivalence*.

Notice that a coarse (resp. coarse Lipschitz) function does not need to be continuous, and a coarse (resp. coarse Lipschitz) equivalence does not need to be either injective or surjective. However, if $f : M \to N$ is a coarse (resp. coarse Lipschitz) equivalence, then there exists a coarse (resp. coarse Lipschitz) equivalence $g : N \to M$ such that

$$\sup_{x \in M} d(x, g(f(x))) < \infty \quad \text{and} \quad \sup_{y \in N} \partial(y, f(g(y))) < \infty. \tag{1.1.3}$$

Indeed, as f is a coarse (resp. coarse Lipschitz) equivalence, let $\varepsilon = \sup_{y \in N} \partial(y, f(M)) < \infty$, and let us define $g : N \to M$ as follows. For each $y \in N$, pick $x_y \in M$ such that $\partial(y, f(x_y)) \leqslant 2\varepsilon$, and set $g(y) = x_y$. It is easy to check that g is a coarse (resp. coarse Lipschitz) equivalence and that (1.1.3) holds. A coarse (resp. coarse Lipschitz) map $g : N \to M$ satisfying (1.1.3) is called a *coarse inverse* (resp. *coarse Lipschitz inverse*) of f. In fact, a coarse map $f : M \to N$ is a coarse (resp. coarse Lipschitz) equivalence if and only if f has a coarse (resp. coarse Lipschitz) inverse.

The next two simple propositions are very important for the understanding of the different notions of embeddings and equivalences above. See [Ka2], Lemma 1.4 and Proposition 1.5.

Proposition 1.1.1. *Let X be a Banach space, M be a metric space, and consider a map $f : X \to M$. Then the following are equivalent.*

(i) f is a coarse map, and

(ii) f is a coarse Lipschitz map.

In particular, if f is uniformly continuous, then f is both coarse and coarse Lipschitz.

Proof. It is enough to show that if there exists $t_0 > 0$ such that $\omega_f(t_0) < \infty$, then there exists $L > 0$ such that $\omega_f(t) < Lt + L$, for all $t > 0$. Fix such $t_0 > 0$. Let $x, y \in X$, and fix $n \in \mathbb{N}$ such that $(n-1)t_0 \leqslant \|x - y\| < nt_0$. Then there exist $x_0, \dots, x_n \in X$ so that $x_0 = x$, $x_n = y$ and $\|x_i - x_{i+1}\| \leqslant t_0$, for all $i \in \{0, \dots, n-1\}$. Hence, we have that

$$\|f(x) - f(y)\| \leqslant \sum_{i=0}^{n-1} \|f(x_i) - f(x_{i+1})\| \leqslant n \cdot \omega_f(t_0) \leqslant \frac{\omega_f(t_0)}{t_0}\|x - y\| + \omega_f(t_0).$$

□

Let (M, d) be a metric space and $a, b > 0$. A subset $A \subset M$ is called an (a, b)-*net* if $d(x, A) < b$, for all $x \in M$, and $d(x, y) \geqslant a$, for all $x, y \in A$, with $x \neq y$. A subset $A \subset M$ is called a *net* if it is an (a, b)-net for some $a, b > 0$. By Zorn's lemma, every metric space has an (a, a)-net, for all $a > 0$.

Proposition 1.1.2. *Let X and Y be infinite dimensional Banach spaces. Then the following are equivalent.*

(i) X is coarsely equivalent to Y,

(ii) X is coarse Lipschitz equivalent to Y, and

(iii) any net of X is Lipschitz equivalent to any net of Y.

Moreover, all the conditions above hold if

(iv) X is uniformly equivalent to Y.

Proof. (i)⇔(ii). As (ii) clearly implies (i), we only need to show that (i) implies (ii). Let $f : X \to Y$ be a coarse equivalence. By Proposition 1.1.1, f is coarse Lipschitz. As f

is cobounded, we only need to show that $\rho_f(t)$ is bounded below by an affine map. As discussed above, f has a coarse inverse $g : Y \to X$. In particular, there exists $C > 0$ such that $\|x - g(f(x))\| \leqslant C$, for all $x \in X$. By Proposition 1.1.1, there exists $L > 0$ such that $\omega_g(t) \leqslant Lt + L$, for all $t > 0$. Hence, we have that

$$\begin{aligned} L\|f(x) - f(y)\| + L &\geqslant \|g(f(x)) - g(f(y))\| \\ &\geqslant \|x - y\| - \|x - g(f(x))\| - \|y - g(f(y))\| \\ &\geqslant \|x - y\| - 2C. \end{aligned}$$

for all $x, y \in X$. So, f is a coarse Lipschitz equivalence.

(ii)$\Rightarrow$(iii). Let $f : X \to Y$ be a coarse Lipschitz equivalence. So, there exists $L > 0$ such that $\rho_f(t) \geqslant L^{-1}t - L$ and $\omega_f(t) \leqslant Lt + L$, for all $t > 0$. By Proposition 10.22 of [BenLi], any two given nets in an infinite dimensional Banach space are Lipschitz equivalent to each other. Hence, it is enough to show that there exists a net in X which is Lipschitz equivalent to a net in Y. Let $N \subset X$ be an $(2L^2, 2L^2)$-net. Then, $f(N)$ is a net in Y and $f \restriction N : N \to f(N)$ is a Lipschitz equivalence.

(iii)$\Rightarrow$(ii). Let $N \subset X$ and $M \subset Y$ be nets and $f : N \to M$ be a Lipschitz equivalence. In particular, there exists $L > 0$ such that for all $x \in X$, there exists $y \in N$ with $\|x - y\| \leqslant L$. Hence, we can pick a map $\varphi : X \to N$ such that $\|x - g(x)\| \leqslant L$, for all $x \in X$. It easily follows that $f \circ \varphi : X \to Y$ is a coarse Lipschitz equivalence. □

Remark 1.1.3. The terminologies above are still not completely established in the literature. For example, in [Ro4] coarse maps are called “bornologous”, and in [Ka4], the author refers to coarse maps as “coarsely continuous”. As coarse maps are not continuous, and as we are interested in studying coarse maps which are also continuous, we prefer a different terminology. Also, we should mention that, in geometric group theory, coarse Lipschitz embeddings are usually called “quasi-isometries”.

1.2 Main questions.

1.2.1 Relation with the linear structure.

As mentioned above, one of the main questions regarding the nonlinear geometry of Banach spaces is to which extent the existence of certain kinds of nonlinear embeddings (resp. equivalences) between Banach spaces is enough to give us information regarding the linear structure of the Banach spaces. Precisely, the following general question is a central problem when dealing with nonlinear embeddings between Banach spaces.

Problem 1.2.1. Let $\mathcal{P}$ and $\mathcal{P}'$ be two classes of Banach spaces and $\mathcal{E}$ be a kind of nonlinear embedding between Banach spaces. If a Banach space X $\mathcal{E}$-embeds into a Banach space Y in $\mathcal{P}$, does it follow that X is in $\mathcal{P}'$?

For example, if a separable Banach space X coarse Lipschitz embeds into a super-reflexive Banach space, then X is also super-reflexive (this follows from Proposition 1.6 of [Ka2] and Theorem 2.4 of [Ka2], but it was first proved for uniform equivalences in [Ri], Theorem 1A). Another example was given by M. Mendel and A. Naor in [MeN2] (Theorem 1.9 and Theorem 1.11), where they showed that if a Banach space X either coarsely or uniformly embeds into a Banach space Y with cotype q and nontrivial type, then X has cotype $q+\varepsilon$, for all $\varepsilon>0$.

If we look at nonlinear equivalences between Banach spaces, the following is a central problem in the theory.

Problem 1.2.2. Let X be a Banach space and $\mathcal{E}$ be a kind of nonlinear equivalence between Banach spaces. If a Banach space Y is $\mathcal{E}$-equivalent to X, what can we say about the isomorphism type of Y? More precisely:

(i) Is the linear structure of X determined by its $\mathcal{E}$-structure, i.e., if a Banach space Y is $\mathcal{E}$-equivalent to X, does it follow that Y is linearly isomorphic to X?

(ii) Let $\mathcal{P}$ be a class of Banach spaces. If Y is $\mathcal{E}$-equivalent to X, does is follow that Y is linearly isomorphic to $X\oplus Z$, for some Banach space Z in $\mathcal{P}$?

Along those lines, it was shown in [JoLiS] (Theorem 2.1) that the coarse (resp. uniform) structure of ℓ_p completely determines its linear structure, for any $p \in (1,\infty)$. For $p = 1$, we do not even know if the Lipschitz structure of ℓ_1 determines its linear structure. N. Kalton and N. Randrianarivony proved in [KaRa] (Theorem 5.4) that, for any $p_1, \ldots, p_n \in (1,\infty)$ with $2 \notin \{p_1, \ldots, p_n\}$, the linear structure of $\ell_{p_1} \oplus \ldots \oplus \ell_{p_n}$ is determined by its coarse (resp. uniform) structure (see also [JoLiS], Theorem 2.2). This problem is still open if $2 \in \{p_1, \ldots, p_n\}$.

Let T denote the Tsirelson space introduced by T. Figiel and W. Johnson in [FiJo]. For each $p \in [1,\infty)$, let T^p be the p-convexification of T (see Subsection 6.1.5 for definitions). W. Johnson, J. Lindenstrauss and G. Schechtman addressed Problem 1.2.2(ii) above by proving the following (see [JoLiS], Theorem 5.8): suppose that either $1 < p_1 < \ldots < p_n < 2$ or $2 < p_1 < \ldots < p_n$ and set $X = T^{p_1} \oplus \ldots \oplus T^{p_n}$, then a Banach space Y is coarsely equivalent (resp. uniformly equivalent) to X if and only if Y is linearly isomorphic to $X \oplus \bigoplus_{j \in F} \ell_{p_j}$, for some $F \subset \{1, \ldots, n\}$.

1.2.2 Conceptual problems.

These notions of embeddings are fundamentally very different. Indeed, while coarse and coarse Lipschitz embeddings deal with the large scale geometry of the metric spaces concerned, uniform embeddings only deal with their local (uniform) structure. However, despite this conceptual difference, their actual differences are still not completely understood. M. Ribe proved the following important result in 1984 (see [Ri], Theorem 1).

Theorem 1.2.3. ***(M. Ribe, 1984)*** *Let $q > 1$ and $(p_n)_{n=1}^\infty$ be a sequence such that $\lim_n p_n = 1$ and $p_n > 1$, for all $n \in \mathbb{N}$. Then $(\oplus_n L_{p_n})_{\ell_q}$ is uniformly equivalent to $(\oplus_n L_{p_n})_{\ell_q} \oplus L_1$. In particular, there are separable Banach spaces which are uniformly equivalent but are not linearly isomorphic. Moreover, reflexivity is not stable under uniform equivalences.*

On the other hand, it was not until 2012 that N. Kalton was able to show that there are coarsely equivalent separable Banach spaces (i.e., with Lipschitz equivalent nets) which are not uniformly equivalent. Precisely, N. Kalton proved the following two results (see [Ka4], Theorem 8.8 and Theorem 8.9).

Theorem 1.2.4. ***(N. Kalton, 2012)*** *Let X be a asymptotically uniformly smooth Banach space and $(Y_n)_{n=1}^{\infty}$ be a sequence of Banach spaces whose unit balls uniformly embed into a reflexive space. If there exists a coarse Lipschitz embedding $X \to (\oplus_n Y_n)_{\ell_1}$ which is also uniformly continuous, then X is reflexive*

Theorem 1.2.5. ***(N. Kalton, 2012)*** *There exists sequence of Banach spaces $(Y_n)_{n=1}^{\infty}$, with $Y_n \cong \ell_1$, for all $n \in \mathbb{N}$, such that $(\oplus_n Y_n)_{\ell_1}$ is coarsely equivalent to $(\oplus_n Y_n)_{\ell_1} \oplus c_0$. In particular, there exist separable Banach spaces which are coarsely equivalent but are not uniformly equivalent to each other.*

Although Theorem 1.2.5 settles that the concepts of coarse and uniform equivalences are distinct in the Banach space setting, it remains widely open whether the existence of those embeddings are equivalent in the Banach space setting. Precisely, the following problem remains open.

Problem 1.2.6. Let X and Y be Banach spaces. Are the following equivalent?

(i) X coarsely embeds into Y.

(ii) X uniformly embeds into Y.

(iii) X strongly embeds into Y.

In [Ran], N. Randrianarivony has shown that a Banach space coarsely embeds into a Hilbert space if and only if it uniformly embeds into a Hilbert space. In [Ka3], N. Kalton showed that the same also holds for embeddings into ℓ_∞ (Theorem 5.3). C. Rosendal made some improvements on the problem above by showing that if X uniformly embeds into Y, then X simultaneously uniformly and coarsely embeds into $\ell_p(Y)$, for any $p \in [1, \infty)$ (see [Ro4], Theorem 2). In particular, if X uniformly embeds into ℓ_p, then X simultaneously coarsely and uniformly embeds into ℓ_p. On the other hand, A. Naor had recently proven that there exist separable Banach spaces X and Y, and a Lipschitz map f from a net $N \subset X$ into Y such that

$$\sup_{x \in N} \|F(x) - f(x)\| = \infty,$$

for all uniformly continuous maps $F : X \to Y$ (see [N], Remark 2). Such result suggests that it may not be true (or at least not easy to show) that X uniformly embeds into Y, given that X coarsely embeds into Y.

1.3 Coarse and uniform embeddings.

In Chapter, 3 we study the relation between coarse embeddings (resp. coarse equivalences) and uniform embeddings (resp. uniform equivalences) between Banach spaces as well as some properties shared by those notions. We are specially interested in narrowing down the difference between those concepts, and we show that, in many cases, the real difference between a coarse and a uniform embedding is in the uniform continuity of the map, but not in its continuity or in the uniform continuity of its inverse. For example, we prove the following.

Theorem 1.3.1. *Let X be a Banach space and Y be a minimal Banach space.*

(i) If X uniformly embeds into Y, then X simultaneously coarsely and uniformly embeds into Y.

(ii) If X coarsely embeds into Y, then X simultaneously coarsely and homeomorphically embeds into Y by a map with uniformly continuous inverse.

Therefore, Theorem 1.3.1 can be seen as a strengthening of C. Rosendal's result about uniform embeddings into ℓ_p mentioned above. In order to prove Theorem 1.3.1(ii), we study how to approximate coarse maps $(M, d) \to (E, \|\cdot\|)$ by continuous coarse maps and what kind of properties of the original coarse map we can preserve. More precisely, in Section 3.3, we prove Theorem 1.3.2 below, which is a strengthening of Theorem 4.1 of [Du].

Let E be a vector space, and let $A \subset E$. Then we denote the convex hull of A by $\text{conv}(A)$.

Theorem 1.3.2. *Let (M, d) be a metric space, and let $A \subset M$ be a closed subspace. Let E be a normed space, and let $\varphi : M \to E$ be a map such that $\varphi \restriction A$ is continuous. Then, for all $\delta > 0$, there exists a continuous map $\Phi : M \to conv(\varphi(M))$ such that $\Phi \restriction A = \varphi \restriction A$ and*

$$\sup_{x \in M} \|\varphi(x) - \Phi(x)\| \leqslant \inf_{s>0} \omega_\varphi(s) + \inf_{s>0} \omega_{\varphi\restriction A}(s) + \delta.$$

In particular, if φ is coarse (resp. coarse embedding), so is Φ.

As a corollary of Theorem 1.3.2, get the following.

Corollary 1.3.3. *Let X be a Banach space and $A \subset X$ be a closed subset. If there exists a coarse retraction $X \to A$, then there exists a continuous coarse retraction $X \to A$.*

In Section 3.4, we use techniques of [Ro4], and Theorem 1.3.2, in order to prove Theorem 1.3.1(ii). In particular, as a subproduct of Theorem 1.3.1(ii), we obtain the following.

Theorem 1.3.4. *Let X and Y be Banach spaces, and let $\mathcal{E}$ be a 1-unconditional basic sequence. If X coarsely embeds into Y, then there exists a continuous coarse embedding $X \to (\bigoplus Y)_\mathcal{E}$ with uniformly continuous inverse. In particular, X simultaneously homeomorphically and coarsely embed into $(\bigoplus Y)_\mathcal{E}$.*

In Section 3.5, we look at N. Kalton's example of separable Banach spaces which are coarsely equivalent but are not uniformly equivalent, and show that we can actually get a stronger result. Precisely, we prove the following.

Theorem 1.3.5. *Let X and Y be Banach spaces, and $Q : Y \to X$ be a quotient map. If Q admits a coarse section, then Q admits a continuous coarse section. In particular, Y is simultaneously homeomorphically and coarsely equivalent to $Ker(Q) \oplus X$.*

Corollary 1.3.6. *There exist separable Banach spaces X and Y which are simultaneously homeomorphically and coarsely equivalent but not uniformly equivalent.*

The following problem lies in the core of the nonlinear geometry of Banach spaces and it remains open (see [Os2], Problem 11.17).

Problem 1.3.7. Does ℓ_2 coarsely (resp. uniformly) embed into every infinite dimensional Banach space?

In [Os1], Theorem 5.1, M. Ostrovskii has shown that ℓ_2 coarsely embeds into any Banach space containing a subspace with an unconditional basis and finite cotype. We prove the following stronger result in Section 3.2.

Theorem 1.3.8. *Let X be an infinite dimensional Banach space with an unconditional basis and finite cotype. Then ℓ_2 strongly embeds into X.*

At last, we dedicate Section 3.6 to study unconditional sums of coarsely equivalent (resp. uniformly equivalent) Banach spaces. In [Ka5], Theorem 4.6(ii), N. Kalton had shown that if X and Y are coarsely equivalent (resp. uniformly equivalent), then $\ell_p(X)$ and $\ell_p(Y)$ are coarsely equivalent (resp. uniformly equivalent). However, as N. Kalton himself noticed, his proof seems to be more complicated than necessary, and relies on the concepts of close (resp. uniformly close) Banach spaces. In Section 3.6, we present an easy argument which give us N. Kalton's result as a corollary.

Theorem 1.3.9. *Say X and Y are two coarsely equivalent (resp. uniformly equivalent, or simultaneously homeomorphically and coarsely equivalent) Banach spaces. Let $\mathcal{E}$ be a normalized 1-unconditional basic sequence. Then $(\oplus X)_{\mathcal{E}}$ and $(\oplus Y)_{\mathcal{E}}$ are coarsely equivalent (resp. uniformly equivalent, or simultaneously homeomorphically and coarsely equivalent).*

1.4 Weaker notions of nonlinear embeddings.

In Chapter 4, we study some different notions of nonlinear embeddings between Banach spaces which were introduced in [Ro4] and are weakenings of the notions of coarse and uniform embeddings. The main goal of this chapter is to provide the reader with evidence that the existence of those kinds of embeddings may represent a stronger restriction than one would think.

Given a map $f : (M, d) \to (N, \partial)$ between metric spaces, we say that f is *uncollapsed* if there exists some $t > 0$ such that $\rho_f(t) > 0$. The map f is called *solvent* if, for each $n \in \mathbb{N}$, there exists $R > 0$, such that

$$d(x, y) \in [R, R + n] \quad \text{implies} \quad \partial(f(x), f(y)) > n,$$

for all $x, y \in M$. For each $t \geqslant 0$, we define the *exact expansion modulus of f* as

$$\overline{\rho}_f(t) = \inf\{\partial(f(x), f(y)) \mid d(x, y) = t\}.$$

The map f is called *almost uncollapsed* if there exists some $t > 0$ such that $\overline{\rho}_f(t) > 0$.

It is clear from its definition, that expanding maps are both solvent and uncollapsed. Also, as $\rho_f(t) \leqslant \overline{\rho}_f(t)$, for all $t \in [0, \infty)$, uncollapsed maps are also almost uncollapsed. As a map $f : (M, d) \to (N, \partial)$ has uniformly continuous inverse if and only if $\rho_f(t) > 0$, for all $t > 0$, Diagram 1.4.1 holds.

$$\begin{tikzcd}
& \text{Expanding} \arrow[dl] \arrow[dr] & & \text{Uniformly continuous inverse} \arrow[dl] \\
\text{Solvent} \arrow[dr] & & \text{Uncollapsed} \arrow[dl] & \\
& \text{Almost uncollapsed} & &
\end{tikzcd} \tag{1.4.1}$$

None of the arrows in Diagram 1.4.1 reverse. Indeed, any bounded uniform embedding is uncollapsed (resp. almost uncollapsed), but it is not expanding (resp. solvent). Examples of uncollapsed maps which are not uniformly continuous are easy to be constructed, as you only need to make sure the map is not injective. At last, Proposition 4.2.3 below provides an example of a map $\mathbb{R} \to \ell_2(\mathbb{C})$ which is Lipschitz, solvent and collapsed (i.e., not uncollapsed), which covers the remaining arrows.

In [Ro4], Theorem 2, C. Rosendal showed that if there exists a uniformly continuous uncollapsed map $X \to Y$ between Banach spaces X and Y, then X strongly embeds into $\ell_p(Y)$, for any $p \in [1, \infty)$. C. Rosendal also showed that there exists no map $c_0 \to E$ which is both coarse and solvent (resp. uniformly continuous and almost uncollapsed), where E is any reflexive Banach space (see [Ro4], Proposition 63 and Theorem 64). This result is a strengthening of a result of N. Kalton that says that c_0 does not coarsely embed (resp. uniformly embed) into any reflexive space (see [Ka1], Theorem 3.6).

Those results naturally raise the following question.

Problem 1.4.1. Let X and Y be Banach spaces. Are the statements in Problem 1.2.6 equivalent to the following weaker statements?

(iv) X maps into Y by a map which is coarse and solvent.

(v) X maps into Y by a map which is uniformly continuous and almost uncollapsed.

Although we will not directly deal with Problem 1.2.6 and Problem 1.4.1 for an arbitrary Y, we intend to provide the reader with evidence that those problems either have a positive answer or that any possible differences between the aforementioned embeddings are often negligible.

For a Banach space X, let $q_X = \inf\{q \in [2, \infty) \mid X \text{ has cotype } q\}$ (see Section 2.7 for definitions regarding type and cotype). As mentioned above, M. Mendel and A. Naor proved that if a Banach space X either coarsely or uniformly embeds into a Banach space Y with nontrivial type, then $q_X \leqslant q_Y$ (see [MeN2], Theorem 1.9 and Theorem 1.11). In Section 4.3, we prove the following strengthening of this result.

Theorem 1.4.2. *Let X and Y be Banach spaces, and assume that Y has nontrivial type. If either*

(i) there exists a coarse solvent map $X \to Y$, or

(ii) there exists a uniformly continuous almost uncollapsed map $X \to Y$,

then, $q_X \leqslant q_Y$.

Theorem 1.4.2 gives us the following corollary.

Corollary 1.4.3. *Let $p, q \in [1, \infty)$ be such that $q > \max\{2, p\}$. Any uniformly continuous map $f : \ell_q \to \ell_p$ (resp. $f : L_q \to L_p$) must satisfy*

$$\sup_t \inf_{\|x-y\|=t} \|f(x) - f(y)\| = 0.$$

While the unit balls of the ℓ_p's are all uniformly equivalent to each other (see [OSc1], Theorem 2.1), Corollary 1.4.3 says that those uniform equivalences cannot be extended in any reasonable way.

In Section 4.4, we look at N. Kalton's Property $\mathcal{Q}$. This property was introduced by in [Ka1], Section 4, in order to study coarse and uniform embeddability into reflexive spaces. Let us recall the definition of Property $\mathcal{Q}$. Let $k \in \mathbb{N}$ and let $\mathbb{M} \subset \mathbb{N}$ be an infinite subset. Define $\mathcal{P}_k(\mathbb{M})$ as the set of all subset of $\mathbb{M}$ with exactly k elements. If $\bar{n} \in \mathcal{P}_k(\mathbb{M})$, we always write $\bar{n} = \{n_1, \ldots, n_k\}$ in increasing order, i.e., $n_1 < \ldots < n_k$. We make $\mathcal{P}_k(\mathbb{M})$ into a graph by saying that two distinct elements $\bar{n} = \{n_1, \ldots, n_k\}, \bar{m} = \{m_1, \ldots, m_k\} \in \mathcal{P}_k(\mathbb{M})$ are connected if they interlace, i.e., if either

$$n_1 \leqslant m_1 \leqslant n_2 \leqslant \ldots \leqslant n_k \leqslant m_k \quad \text{or} \quad m_1 \leqslant n_1 \leqslant m_2 \leqslant \ldots m_k \leqslant n_k.$$

We write $\bar{n} < \bar{m}$ if $n_k < m_1$. We endow $\mathcal{P}_k(\mathbb{M})$ with the shortest path metric. A Banach space X has *Property* $\mathcal{Q}$ if there exists a constant $Q_X > 0$ such that for all $k \in \mathbb{N}$, all L-

Lipschitz map $f : \mathcal{P}_k(\mathbb{N}) \to X$, and all $\lambda > 1$, there exists an infinite subset $\mathbb{M} \subset \mathbb{N}$ such that $\mathrm{diam}(f(\mathcal{P}_k(\mathbb{M}))) \leqslant \lambda Q_X^{-1} L$.

The following were proved in [Ka1], Corollary 4.3 and Corollary 4.6.

Theorem 1.4.4. ***(N. Kalton, 2007)*** *Let X be a Banach space. If either*

(i) X coarsely embed into a reflexive space, or

(ii) the unit ball of X uniformly embeds into a reflexive space,

then X has Property $\mathcal{Q}$

Theorem 1.4.5. ***(N. Kalton, 2007)*** *Let X be a Banach space with Property $\mathcal{Q}$ and nontrivial type. Then X is reflexive.*

Kalton proposed the following problem in [Ka1], Problem 6.5.

Problem 1.4.6. Let X be a separable Banach space. Does X have Property $\mathcal{Q}$ if and only if X coarsely embeds into a reflexive Banach space? Does X have Property $\mathcal{Q}$ if and only if the unit ball of X uniformly embeds into a reflexive Banach space?

In Section 4.4, we prove that Property $\mathcal{Q}$ is stable under those weaker kinds of embeddings (see Theorem 4.4.2). Although the stability of Property $\mathcal{Q}$ under coarse and uniform embeddings is implicit in [Ka1], to the best of our knowledge, this is not explicitly written in the literature. Theorem 4.4.2 allows us to obtain the following result (see Theorem 4.4.3 below for a stronger result).

Theorem 1.4.7. *Let X and Y be Banach spaces, and assume that Y is reflexive (resp. super-reflexive). If either*

(i) there exists a coarse solvent map $X \to Y$, or

(ii) there exists a uniformly continuous almost uncollapsed map $X \to Y$,

then, X is either reflexive (resp. super-reflexive) or X has a spreading model equivalent to the ℓ_1-basis (resp. trivial type).

Theorem 1.4.7 was proven in [Ka1], Theorem 5.1, for uniform and coarse embeddings into super-reflexive spaces. Although the result above for uniform and coarse embeddings into reflexive spaces is implicit in [Ka1], we could not find this result explicitly written anywhere in the literature.

It is worth noticing that Theorem 1.4.7 cannot be improved for embeddings of X into super-reflexive spaces in order to guarantee that X either is super-reflexive or has a spreading model equivalent to the ℓ_1-basis (see Remark 4.4.4 below).

As mentioned above, Problem 1.2.6 has a positive answer for $Y = \ell_p$, for all $p \in [1, 2]$ (see [No2], Theorem 5, and [Ran], page 1315). In Section 4.5, we show that Problem 1.4.1 also has a positive answer in the same settings. Precisely, we show the following.

Theorem 1.4.8. *Let X be a Banach space, and $Y = \ell_p$, for any $p \in [1, 2]$. Then Problem 1.4.1 has a positive answer.*

In Section 4.6, we give a positive answer to Problem 1.4.1 for $Y = \ell_\infty$. This is a strengthening of Theorem 5.3 of [Ka3], where N. Kalton shows that Problem 1.2.6 has a positive answer for $Y = \ell_\infty$. Moreover, N. Kalton showed that uniform (resp. coarse) embeddability into ℓ_∞ is equivalent to Lipschitz embeddability.

Theorem 1.4.9. *Let X be a Banach space, and $Y = \ell_\infty$. Then Problem 1.4.1 has a positive answer. Moreover, for $Y = \ell_\infty$, items (iv) and (v) of Problem 1.4.1 are also equivalent to Lipschitz embeddability into ℓ_∞.*

Even though we do not give a positive answer to Problem 1.2.6 and Problem 1.4.1, we believe that the aforementioned results provide considerable suggestive evidence that all the five different kinds of embeddings $X \hookrightarrow Y$ above preserve the geometric properties of X in a similar manner.

1.5 Coarse embeddings into superstable spaces.

D. Aldous showed in [Ald], Theorem 1.1, that every subspace of L_1 contains an isomorphic copy of ℓ_p, for some $p \geqslant 1$. In order to generalize Aldous' result, J. Krivine and B. Maurey introduced the notion of stable Banach spaces in [KrMau]. A metric space (M, d) is called *stable* if

$$\lim_{i,\mathcal{U}} \lim_{j,\mathcal{V}} d(x_i, y_j) = \lim_{j,\mathcal{V}} \lim_{i,\mathcal{U}} d(x_i, y_j),$$

for all bounded sequences $(x_i)_{i=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ in M, and all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ over $\mathbb{N}$. A Banach space X is called *stable* if $(X, \|\cdot - \cdot\|)$ is stable as a metric space. As L_p is stable for all $p \geqslant 1$ (see [KrMau], Theorem II.2), the following is a generalization of Aldous's result (see [KrMau], Theorem IV.1).

Theorem 1.5.1. ***(J. Krivine and B. Maurey, 1981)*** *Let X be a stable Banach space. There exists $p \in [1, \infty)$ such that X contains an $(1 + \varepsilon)$-isomorphic copy of ℓ_p, for all $\epsilon > 0$.*

In order to prove the theorem above, J. Krivine and B. Maurey looked at *types* on a stable Banach space X, i.e., functions $\sigma : X \to \mathbb{R}$ given by $\sigma(x) = \|x + a\|$, where a is an element of some ultrapower of X. In [KrMau], the authors showed that every stable Banach space must contain what was called an *ℓ_p-type*, which results in the existence of almost isometric copies of ℓ_p inside X, for some $p \geqslant 1$.

As shown in [Ray], J. Krivine and B. Maurey's result can be extendded to the nonlinear setting as follows. We say that a Banach space X is *superstable* if every Banach space which is finitely representable in X is also stable. Raynaud proved the following in [Ray] (see the corollary in page 34 of [Ray]).

Theorem 1.5.2. ***(Y. Raynaud, 1983)*** *If a Banach space X uniformly embeds into a superstable Banach space, then X contains an isomorphic copy of ℓ_p, for some $p \in [1, \infty)$.*

Raynaud's proof is also based on analyzing a space of types over the Banach space X. Precisely, the author shows that if X uniformly embeds into a superstable Banach space, then there exists an invariant stable metric d on X uniformly equivalent to the metric induced by the norm. Once one has an invariant stable metric, it is possible to define the space of types $\sigma : \mathbb{R} \times X \to \mathbb{R}$ as the closure of the family of maps $\big((\lambda, y) \in \mathbb{R} \times X \mapsto d(\lambda x, y)\big)_{x \in X}$ in the product space $\mathbb{R}^{\mathbb{R}\times X}$. Studying this new space of types, Raynaud shows that the type space of X must contain a so called ℓ_p-*type*, for some $p \in [1, \infty)$, which results in $\ell_p \hookrightarrow X$, for some $p \in [1, \infty)$. For more on stability and types on Banach spaces see [G-D], [HayMau] and [I].

N. Kalton asked the following in [Ka1], Problem 6.6.

Problem 1.5.3. Assume that a Banach space X coarsely embeds into a superstable Banach space. Does it follow that X contain an isomorphic copy of ℓ_p, for some $p \in [1, \infty)$?

In a joint work with Andrew Swift, although we were not able to obtain an answer to N. Kalton's problem, we obtained the following result.

Theorem 1.5.4. *If a Banach space X coarsely embeds into a superstable Banach space, then X has a basic sequence with an associated spreading model isomorphic to ℓ_p, for some $p \in [1, \infty)$.*

N. Kalton proved in [Ka1], Theorem 2.1, that any stable metric space embeds into some reflexive Banach space by a map which is both a uniform and a coarse embedding. In the same paper, N. Kalton asked if the converse of this result also holds. Precisely, the following is open (see [Ka1], Problem 6.1)

Problem 1.5.5. Does every (separable) reflexive Banach space embed coarsely (resp. uniformly) into a stable space?

By Raynaud's result, it is clear that there are separable reflexive spaces which do not embed into superstable spaces. However, to the best of our knowledge, it was unknown whether every reflexive Banach space coarsely embeds into a superstable Banach space. As a corollary of Theorem 1.7.3, we obtain the following.

Corollary 1.5.6. *There are separable reflexive Banach spaces which do not coarsely embed into any superstable Banach space.*

1.6 Coarse Lipschitz geometry and asymptotic structure.

In Chapter 6, we will be mainly interested in coarse Lipschitz embeddings and equivalences, and in what kind of stability properties these notions of nonlinear embeddings and nonlinear equivalences may have. Furthermore, we will mainly work with Banach spaces having some kind of asymptotic property. More specifically, we are concerned with asymptotically uniformly smooth Banach spaces, asymptotically uniformly convex Banach spaces, and Banach spaces having several different Banach-Saks-like properties. In order not to make this introduction too extensive, we will postpone some technical definitions from Banach space theory for later as well as our more technical results. The reader will find all the remaining background and notation in Section 6.1.

Along the lines of Problem 1.2.1, we prove the following in Section 6.2.

Theorem 1.6.1. *Let Y be a reflexive asymptotically uniformly smooth Banach space, and assume that a Banach space X coarse Lipschitz embeds into Y. Then X has the Banach-Saks property.*

As the Banach-Saks property implies reflexivity, Theorem 1.6.1 above is a strengthening of Theorem 4.1 of [BKaL], where the authors showed that if a separable Banach space X coarse Lipschitz embeds into a reflexive asymptotically uniformly smooth Banach space, then X must be reflexive. As the Tsirelson space T is a reflexive Banach space without the Banach-Saks property, Theorem 1.6.1 gives us the following new corollary.

Corollary 1.6.2. *The Tsirelson space does not coarse Lipschitz embed into any reflexive asymptotically uniformly smooth Banach space.*

In Section 6.2, we also prove some results on the linear theory of Banach spaces. Precisely, we show that an asymptotically uniformly smooth Banach space X must have the alternating Banach-Saks property (see Corollary 6.2.2). Using descriptive set theoretical arguments, we also show that the converse does not hold, i.e., that there are Banach spaces with the alternating Banach-Saks property which do not admit an asymptotically uniformly smooth renorming (see Proposition 6.2.8).

In Section 6.3, we study coarse embeddings $f : X \to Y$ between Banach spaces X and Y with specific asymptotic properties, and obtain a general result on how close to an affine map the expansion modulus ρ_f can be (see Theorem 6.3.1). Precisely, E. Guentner and J. Kaminker introduced the following quantity in [GuKa]: for Banach spaces X and Y, define $\alpha_Y(X)$ as the supremum of all $\alpha > 0$ for which there exists a coarse embedding $f : X \to Y$ and $L > 0$ such that $\rho_f(t) \geqslant L^{-1}t^\alpha - L$, for all $t \geqslant 0$. We call $\alpha_Y(X)$ the *compression exponent of X in Y*. As a simple consequence of Theorem 6.3.1, we obtain Theorem 1.6.3 below.

We denote by S the Schlumprecht space introduced in [Sc1], and, for each $p \in [1, \infty)$, we let S^p be the p-convexification of S and T^p be the p-convexification of the Tsirelson space T (see Subsection 6.1.5 for definitions).

Theorem 1.6.3. *Let $1 \leqslant p < q$. Then*

(i) $\alpha_{T^q}(T^p) \leqslant p/q$, *and*

(ii) $\alpha_{S^q}(S^p) \leqslant p/q$.

In particular, T^p (resp. S^p) does not coarse Lipschitz embed into T^q (resp. S^q).

The proof of Theorem 1.6.3 is asymptotic in nature, hence we obtain equivalent estimates for the compression exponent $\alpha_Y(X)$, where X and Y are Banach spaces satisfying some special asymptotic properties. In particular, the spaces T^q and S^q can be replaced in Theorem 1.6.3 by $(\oplus_n E_n)_{T^q}$ and $(\oplus_n E_n)_{S^q}$, where $(E_n)_{n=1}^\infty$ is any sequence of finite dimensional Banach spaces. See Theorem 6.3.3, Theorem 6.3.5 and Corollary 6.3.7 for precise statements.

We also apply our results to the hereditarily indecomposable Banach spaces $\mathfrak{X}^p$ defined by N. Dew in [D], and obtain that $\alpha_{\mathfrak{X}^q}(\mathfrak{X}^p) \leqslant p/q$, for $1 < p < q$ (see Corollary 6.3.8).

In Section 6.4, we prove a general theorem regarding the non existence of coarse Lipschitz embeddings $X \to Y_1 \oplus Y_2$, for Banach spaces X, Y_1, Y_2 with specific asymptotic properties (see Theorem 6.4.6). With that result in hands, we prove the following.

Theorem 1.6.4. *Let* $1 \leqslant p_1 < \ldots < p_n < \infty$, *and* $p \in [1, \infty)\backslash\{p_1, \ldots, p_n\}$. *Then neither* T^p *nor* ℓ_p *coarse Lipschitz embed into* $T^{p_1} \oplus \ldots \oplus T^{p_n}$. *In particular,* T^p *does not coarse Lipschitz embed into* T^q, *for all* $p, q \in [1, \infty)$ *with* $p \neq q$.

At last, we use Theorem 1.6.4 in order to obtain the following characterization.

Theorem 1.6.5. *Let* $1 < p_1 < \ldots < p_n < \infty$ *with* $2 \notin \{p_1, \ldots, p_n\}$. *A Banach space* Y *is coarsely equivalent (resp. uniformly equivalent) to* $X = T^{p_1} \oplus \ldots \oplus T^{p_n}$ *if and only if* Y *is linearly isomorphic to* $X \oplus \bigoplus_{j \in F} \ell_{p_j}$, *for some* $F \subset \{1, \ldots, n\}$.

Clearly, Theorem 1.6.5 is a strengthening of Theorem 5.8 of [JoLiS] mentioned above. However, just as in the case for $\ell_{p_1} \oplus \ldots \oplus \ell_{p_n}$, we still do not know whether the theorem above holds if $2 \in \{p_1, \ldots, p_n\}$.

1.7 The isomorphism group of the Gurarij space.

Chapter 7 differs slightly from the previous chapters of this dissertation, as we will not restrict ourselves only to Banach spaces. Precisely, in this chapter, we deal with embeddability of Polish groups into the isometry group of a Banach space. Recall, a *Polish space* is a separable topological space which is completely metrizable, i.e., there exists a complete metric compatible with its topology. A *Polish group* is a Polish space which is also a topological group.

A separable Banach space $\mathbb{G}$ is said to be a *Gurarij space* if, for all $\varepsilon > 0$, and all finite dimensional Banach spaces $E \subset F$, any isometry from E into $\mathbb{G}$ can be extended to an $(1 + \varepsilon)$-isomorphism from F into $\mathbb{G}$. In [Lu], W. Lusky proved that every two separable Banach spaces with this extension property are linearly isometric to each other. Therefore, the Gurarij spaces are unique up to isometry, and we refer to any such space as *the* Gurarij space $\mathbb{G}$.

Let X be a Banach space, and $\mathrm{Aff}(X)$ be the group of affine isometries of X endowed with the pointwise convergence topology. So, $\mathrm{Aff}(X)$ is a Polish group. By Mazur-Ulam's theorem, every surjective isometry $f : X \to X$ is affine. So $\mathrm{Aff}(X)$ is the group of surjective isometries of X. I. Yaacov showed (see [Y], Theorem 3.10) that the isometry group of the Gurarij space is universal for all Polish groups, i.e., every Polish group can be simultaneously homomorphically and homeomorphically embedded into $\mathrm{Aff}(\mathbb{G})$. However, I. Yaacov's result does not say anything regarding whether the large scale geometry of the Polish spaces can be preserved by those embeddings. Precisely, under which conditions can a Polish space be simultaneously homomorphically and homeomorphically embedded into $\mathrm{Aff}(\mathbb{G})$ by a map which is also a coarse or a coarse Lipschitz embedding?

Given a Polish group H, one can find a left-invariant metric d on H which is compatible with H's topology (see [Ke], Theorem 9.1). However, the metric d is by no means intrinsically defined, and different such metrics give us a different geometry on H. Therefore, the question above may sound vague and imprecise. To address this issue, we follow the approach of [Ro3]. Precisely, in [Ro3], C. Rosendal studied the problem of when a given Polish group H has a well-defined coarse type (resp. coarse Lipschitz type). For this, we need to introduce some terminology.

Let H be a metrizable topological group. A subset $A \subset H$ is said to have *property (OB) with respect to* H if A has finite diameter with respect to every compatible left-invariant metric on H. The Polish group H is said to have *property (OB)* if H has property (OB) with respect to itself, and H is said to be *locally (OB)* if there exists an open neighborhood of the identity with property (OB) with respect to H. Also, we say that H is *(OB) generated* if H is generated by an open set with property (OB) with respect to H.

A metric d on H is said to be *metrically proper* if all subsets of H with finite d-diameter have finite diameter with respect to any other compatible left-invariant metric on H, and

d is said to be *maximal* if, for any compatible left-invariant metric ∂ on H, there exists $K > 0$ such that $\partial \leqslant K \cdot d + K$. Clearly, if d is maximal, then d is metrically proper. Also, any two metrically proper compatible left-invariant metrics on a Polish space H are coarsely equivalent, and every two maximal compatible left-invariant metrics on a Polish space H are coarse Lipschitz equivalent (see [Ro3]).

Rosendal proved the following in [Ro3], Theorem 1 and Theorem 3.

Theorem 1.7.1. ***(C. Rosendal, 2014)*** *Let H be a Polish group. Then*

(i) H has a metricaly proper compatible left-invariant metric if and only if H is locally (OB), and

(ii) H has a maximal compatible left-invariant metric if and only if H is (OB) generated.

Therefore, a locally (OB) Polish group H has a well-defined coarse type, i.e., one can unambiguously talk about coarse maps between locally (OB) Polish groups without specifying the metrics on the respective spaces. Precisely, let H be a locally (OB) Polish group and (G, ∂) be a metric space, then a map $f : (H, d) \to (G, \partial)$ (resp. $f : (G, \partial) \to (H, d)$) is a coarse embedding, where d is *some* compatible left-invariant metrically proper metric on H, if and only if $f : (H, d) \to (G, \partial)$ (resp. $f : (G, \partial) \to (H, d)$) is a coarse embedding, for *all* compatible left-invariant metrically proper metric d on H. Hence, for locally (OB) Polish spaces H and G, we say that a map $f : H \to G$ is a *coarse embedding* if $f : (H, d) \to (G, \partial)$ is a coarse embedding, where d and ∂ are compatible left-invariant metrically proper metrics on H and G, respectively.

Similarly, a (OB) generated Polish group H has a well-defined coarse Lipschitz type, and we say that a map $f : H \to G$ is a *coarse Lipschitz embedding* if $f : (H, d) \to (G, \partial)$ is a coarse Lipschitz embedding, where d and ∂ are compatible left-invariant maximal metrics on H and G, respectively.

For a Banach space X, let $\mathrm{Iso_L}(X)$ be the closed subgroup of $\mathrm{Aff}(X)$ consisting of all the linear isometries of X. Along these lines, we prove the following theorems.

Theorem 1.7.2. *The group of linear isometries* $Iso_L(\mathbb{G})$ *has property (OB). In particular,* $Aff(\mathbb{G})$ *is (OB) generated and the map* $g \in Aff(\mathbb{G}) \mapsto g(x) \in \mathbb{G}$ *is a coarse Lipschitz equivalence, for all* $x \in \mathbb{G}$.

The next theorem is a strengthening of Theorem 3.10 of [Y] on the point of view of large scale geometry.

Theorem 1.7.3. *Let* $\mathbb{G}$ *be the Gurarij space, and* H *be a Polish group.*

(i) If H *is locally (OB), then there exists a simultaneously homomorphic and homeomorphic embedding* $\varphi : H \to Aff(\mathbb{G})$ *which is also a coarse embedding.*

(ii) If H *is (OB) generated, then there exists a simultaneously homomorphic and homeomorphic embedding* $\varphi : H \to Aff(\mathbb{G})$ *which is also a coarse Lipschitz embedding.*

Theorem 1.7.3 can be reformulated in the language of affine isometric actions. An affine isometric action $\alpha : H \curvearrowright X$ can be written as $\alpha(h)(x) = \pi(h)(x) + b(h)$, for all $h \in H$, and all $x \in X$, where $\pi : H \curvearrowright X$ is a linear isometric action and $b : H \to X$ a cocycle of π (see Subsection 7.2). Precisely, Theorem 1.7.3 is a corollary of the following result.

Theorem 1.7.4. *Let* $\mathbb{G}$ *be the Gurarij space, and* (H, d) *be a separable metric topological group. There exists an affine isometric action* $\alpha : H \curvearrowright \mathbb{G}$ *with a linear part* $\pi : H \curvearrowright \mathbb{G}$ *which induces a simultaneously homomorphic and homeomorphic embedding* $H \to Iso_L(\mathbb{G})$*, and a cocycle* $b : H \to \mathbb{G}$ *which is an isometric embedding.*

The theorem above can be seen as a strengthening of Theorem 45 of [Ro3]. Indeed, Theorem 45 of [Ro3] says that given a metric topological group (H, d), there exists a Banach space X for which the conclusion of Theorem 1.7.4 holds. Theorem 1.7.4 says that, if H is separable, then X can always be taken to be the Gurarij space.

At last, Theorem 1.7.2 and Proposition 79 of [Ro4] allow us to obtain the following.

Corollary 1.7.5. *Let* M *be a metric space and assume that there exists an isometric action* $Aff(\mathbb{G}) \curvearrowright M$ *with an unbounded orbit. Then* $\mathbb{G}$ *maps into* M *by a coarse solvent map.*

Chapter 2

Background and notation.

In this chapter, we give the basic backgroud needed for this dissertation regarding classic Banach space theory. Some of the most technical definitions will be introduced as needed during the chapters of this dissertation. For more on Banach space theory, we refer to *Topics in Banach Space Theory*, by F. Albiac and N. Kalton ([AlKa]), *Classical Banach spaces, Vol. I and II*, by J. Lindenstrauss and L. Tzafriri ([LiTz] and [LiTz]), and *Sequences and series in Banach spaces*, by J. Diestel ([Di]).

2.1 Banach space theory.

Throughout this dissertation, $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Recall that $(X, \|\cdot\|_X)$ is called a *Banach space* if X is a vector space over $\mathbb{K}$ and $\|\cdot\|_X$ is a norm on X generating a complete topology. In this dissertation, all Banach spaces are over the reals, unless explicitly noted. We usually omit the index X in $\|\cdot\|_X$, and simply write $\|\cdot\|$, as long as this does not cause any confusion. Also, we usually omit the norm of $(X, \|\cdot\|)$ when referring to it and simply refer to this space as X. We denote by B_X the closed unit ball of X, i.e., $B_X = \{x \in X \mid \|x\| \leqslant 1\}$, and by ∂B_X the unit sphere of X, i.e., $\partial B_X = \{x \in X \mid \|x\| = 1\}$. A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called *normalized* if $\|x_n\| = 1$, for all $n \in \mathbb{N}$, and *semi-normalized* if it is bounded and bounded away from zero, i.e., $\inf_n \|x_n\| > 0$.

Let X and Y be Banach spaces. Recall that a linear map $f : X \to Y$ is continuous if and only if it is bounded, i.e., if its norm $\|f\| \coloneqq \sup_{x\in B_X} \|f(x)\|$ is finite. The map f is called an *isomorphism* if f is a bijection and both f and f^{-1} are bounded. If f is an isomorphism with its image, i.e., $f : X \to f(X)$ is an isomorphism, we call f an *isomorphic embedding*. If $\|f\| = \|f^{-1}\| = 1$, f is called a *linear isometry*. If $Y = \mathbb{R}$, we denote the space of continuous linear functionals $f : X \to \mathbb{R}$ by X^*. The space X^* with the norm defined above is a Banach space, and it is called the *dual* of X.

Say X and Y are Banach spaces. We write $X \equiv Y$ to denote that X is linearly isometric to Y, and we write $X \cong Y$ to denote that X is (linearly) isomorphic to Y. A linear map $Q : Y \to X$ is called a *quotient map* if it is bounded and surjective. By the open mapping theorem, quotient maps are always open. A map $\varphi : X \to Y$ is called a *section for Q* if $Q \circ \varphi = \mathrm{Id}_X$.

Given a Banach space X, we say that a sequence $(x_n)_{n=1}^\infty$ in X is a *Schauder basis* for X if every element of X can be uniquely written as an infinite linear combination of $(x_n)_{n=1}^\infty$, i.e., for all $x \in X$ there exists a unique $(a_n)_{n=1}^\infty \in \mathbb{R}^\mathbb{N}$ such that $x = \sum_{n=1}^\infty a_n x_n$. If X has a Schauder basis $(x_n)_{n=1}^\infty$ we can define, for all $n \in \mathbb{N}$, natural projections $P_n(\sum_{i=1}^\infty a_i x_i) = \sum_{i=1}^n a_i x_i$. The uniform boundedness principle gives us that the norm of those projections are uniformly bounded. If $K = \sup_n \|P_n\|$, we say that the Schauder basis $(x_n)_{n=1}^\infty$ has *basis constant* K.

Say $(x_n)_{n=1}^\infty$ is a basis for the Banach space X. For $x = \sum_{n=1}^\infty a_n x_n \in X$, we write $\mathrm{supp}(x) = \{n \in \mathbb{N} \mid a_n \neq 0\}$. For all finite subsets $E, F \subset \mathbb{N}$, we write $E < F$ (resp. $E \leqslant F$) if $\max E < \min F$ (resp. $\max E \leqslant \min F$). We call a sequence $(y_n)_{n=1}^\infty$ in X a *block sequence of* $(x_n)_{n=1}^\infty$ if $\mathrm{supp}(y_n) < \mathrm{supp}(y_{n+1})$, for all $n \in \mathbb{N}$.

A sequence $(x_n)_{n=1}^\infty$ is called a *basic sequence* if it is a Schauder basis for its closed linear span. Equivalently, $(x_n)_{n=1}^\infty$ is a basic sequence if its elements are not zero and there exists $K > 0$ such that

$$\Big\|\sum_{i=1}^k a_i x_i\Big\| \leqslant K \Big\|\sum_{i=1}^n a_i x_i\Big\|,$$

for all $k, n \in \mathbb{N}$ with $k \leqslant n$, and for all $a_1, ..., a_n \in \mathbb{R}$. The infimum of the constants K for which the inequality above holds is called the *basic constant of* $(x_n)_{n=1}^{\infty}$. Given two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$, we say that they are *equivalent* if there exists $C \geqslant 1$ such that

$$\frac{1}{C}\Big\|\sum_{i=1}^{k} a_i x_i\Big\| \leqslant \Big\|\sum_{i=1}^{k} a_i y_i\Big\| \leqslant C\Big\|\sum_{i=1}^{k} a_i x_i\Big\|,$$

for all $k \in \mathbb{N}$, and all $a_1, ..., a_k \in \mathbb{R}$.

If a basis (resp. basic sequence) has the property that it remains a basis (resp. basic sequence) no matter how one reorders it, then the basis (resp. basic sequence) is called an *unconditional basis* (resp. *unconditional basic sequence*). Equivalently, a sequence $(x_n)_{n=1}^{\infty}$ is unconditional if its elements are not zero and there exists $K > 0$ such that

$$\Big\|\sum_{i=1}^{n} a_i x_i\Big\| \leqslant K\Big\|\sum_{i=1}^{n} b_i x_i\Big\|,$$

for all $n \in \mathbb{N}$, and all $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}$ such that $|a_i| \leqslant |b_i|$, for all $i \in \{1, ..., n\}$. The infimum of this constants is called the *unconditional constant of* $(x_n)_{n=1}^{\infty}$.

2.2 Examples.

2.2.1 $C(K)$ spaces.

An important class of Banach spaces are the $C(K)$ spaces. Let K be a compact metric space. Let $C(K) = \{f : K \to \mathbb{R} \mid f \text{ is continuous}\}$, and we endow $C(K)$ with the norm $\|f\| = \sup_{t \in K} |f(t)|$. This makes $C(K)$ into a Banach space. If $K = [0, 1]$, the space $C[0, 1]$ is universal for the class of separable Banach spaces, i.e., every separable Banach space X is linearly isometric to a subspace of $C[0, 1]$.

2.2.2 c_0 and ℓ_p spaces.

For $p \in [1, \infty)$, we define $\ell_p = \{(x_i)_{i=1}^\infty \in \mathbb{R}^\mathbb{N} \mid (\sum_{i=1}^\infty |x_i|^p)^{1/p} < \infty\}$, and endow ℓ_p with the norm

$$\|(x_i)_{i=1}^\infty\|_p = \left(\sum_{i=1}^\infty |x_i|^p\right)^{1/p}.$$

Similarly, we let $c_0 = \{(x_i)_{i=1}^\infty \in \mathbb{R}^\mathbb{N} \mid \lim_i |x_i| = 0\}$ and $\ell_\infty = \{(x_i)_{i=1}^\infty \in \mathbb{R}^\mathbb{N} \mid \sup_i |x_i| \leqslant \infty\}$, and endow both c_0 and ℓ_∞ with the norm

$$\|(x_i)_{i=1}^\infty\|_\infty = \sup_i |x_i|.$$

The spaces $\ell_p(\mathbb{C})$ and $c_0(\mathbb{C})$ are defined analogously as above, but with its elements having coordinates in $\mathbb{C}$. Also, for each $n \in \mathbb{N}$ and $p \in [1, \infty]$, we define the spaces ℓ_p^n (resp. $\ell_p^n(\mathbb{C})$) as being $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) endowed with the restriction of $\|\cdot\|_p$ to $\mathbb{R}^n$ (resp. $\mathbb{C}^n$).

An infinite dimensional Banach space X is called *minimal* if X isomorphically embeds into all of its infinite dimensional subspaces. The spaces c_0 and ℓ_p are all minimal.

2.2.3 Tsirelson and Schlumprecht spaces.

In 1974, B. Tsirelson constructed the first example of a (reflexive) Banach space which does not contain isomorphic copies of neither c_0 nor ℓ_p, for all $p \in [1, \infty)$ (see the theorem in page 57 of [Ts]). In the same year, T. Figiel and W. Johnson gave an implicit definition for the norm of the dual of Tsirelson's original space, and showed that this dual space shared the same property of not containing isomorphic copies of neither c_0 nor ℓ_p, for all $p \in [1, \infty)$ (see [FiJo]). Nowadays, T. Figiel and W. Johnson's space is the space which is usually referred to as being the Tsirelson space. We now describe this space. Let c_{00} denote the set of sequences of real numbers which are eventually zero, and let $\|\cdot\|_0$ be the max norm on c_{00}. We denote by T the *Tsirelson space* defined in [FiJo], i.e., T is the completion of c_{00} under the unique norm $\|\cdot\|$ satisfying

$$\|x\| = \max\Big\{\|x\|_0, \frac{1}{2}\cdot\sup\Big(\sum_{j=1}^{k}\|E_jx\|\Big)\Big\},$$

where the inner supremum above is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $k \leqslant E_1 < \ldots < E_k$. Tsirelson's original space is the dual of T.

Another Banach space which will be important for applications in this dissertation is the *Schlumprecht space* S. This space was constructed in [Sc1] and provided the first example of an arbitrarily distortable Banach space. We say that a Banach space $(X, \|\cdot\|)$ is *arbitrarily distortable* if for all $\lambda > 1$ there exists an equivalent norm $|||\cdot|||$ on X such that

$$\sup\Big\{\frac{|||y|||}{|||x|||} \mid x, y \in S_{(X,\|\cdot\|)}\Big\} \geqslant \lambda.$$

We define S as the completion of c_{00} under the unique norm $\|\cdot\|$ satisfying

$$\|x\| = \max\Big\{\|x\|_0, \sup\Big(\frac{1}{\log_2(k+1)}\sum_{j=1}^{k}\|E_jx\|\Big)\Big\},$$

where the inner supremum above is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $E_1 < \ldots < E_k$.

2.3 Unconditional sums.

Let $(X_n, \|\cdot\|_n)_{n=1}^\infty$ be a sequence of Banach spaces. Let $\mathcal{E} = (e_n)_{n=1}^\infty$ be a 1-unconditional basic sequence generating a space $(E, \|\cdot\|_E)$. We define the sum $(\oplus_n X_n)_{\mathcal{E}}$ to be the space of sequences $(x_n)_{n=1}^\infty$, where $x_n \in X_n$, for all $n \in \mathbb{N}$, such that

$$\|(x_n)_{n=1}^\infty\| := \Big\|\sum_{n\in\mathbb{N}}\|x_n\|_n e_n\Big\|_E < \infty.$$

One can check that $(\oplus_n X_n)_{\mathcal{E}}$ endowed with the norm $\|\cdot\|$ defined above is a Banach space. If the X_n's are all the same, say $X_n = X$, for all $n \in \mathbb{N}$, we write $(\oplus X)_{\mathcal{E}}$. Whenever $\mathcal{E}$ is the standard basis of either c_0 or ℓ_p, for some $p \in [1, \infty)$, we write $(\oplus_n X_n)_{c_0}$ or $(\oplus_n X_n)_{\ell_p}$, respectively. Moreover, if $X_n = X$, for all $n \in \mathbb{N}$, we write $c_0(X)$ and $\ell_p(X)$ instead.

2.4 Spreading models.

Let X be a Banach space and $(x_n)_{n=1}^{\infty}$ be a bounded sequence without Cauchy subsequences, and let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Then there exists a Banach space $(S, |||\cdot|||)$ containing X and a sequence $(\zeta_n)_{n=1}^{\infty}$ in S which is linearly independent over X such that, for all $y \in X$, and all $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, we have

$$\Big|\Big|\Big| y + \sum_{j=1}^{k} \alpha_j \zeta_j \Big|\Big|\Big| = \lim_{n_k, \mathcal{U}} \dots \lim_{n_1, \mathcal{U}} \Big\| y + \sum_{j=1}^{k} \alpha_j x_{n_j} \Big\|.$$

Without loss of generality, $S = X \oplus \overline{\operatorname{span}}\{\zeta_n \mid n \in \mathbb{N}\}$ (see [G-D], Chapter 2, Section 2, for a proof of this fact). The space S is called a *spreading model of* $(x_n)_{n=1}^{\infty}$ and the sequence $(\zeta_n)_{n=1}^{\infty}$ is called the *fundamental sequence* of the spreading model S. Notice that, if X is separable, by going to a subsequence of $(x_n)_{n=1}^{\infty}$ if necessary, we can assume that

$$\begin{aligned} \Big|\Big|\Big| y + \sum_{j=1}^{k} \alpha_j \zeta_j \Big|\Big|\Big| &= \lim_{n_k} \dots \lim_{n_1} \Big\| y + \sum_{j=1}^{k} \alpha_j x_{n_j} \Big\| \\ &= \lim_{(n_1, \dots, n_k) \to \infty} \Big\| y + \sum_{j=1}^{k} \alpha_j x_{n_j} \Big\|. \end{aligned}$$

A fundamental sequence $(\zeta_n)_{n=1}^{\infty}$ of a spreading model is 1-*spreading*, i.e., $(\zeta_n)_{n=1}^{\infty}$ is 1-equivalent to all of its subsequences. Also, the sequence $(\xi_n)_{n=1}^{\infty}$ is 1-sign unconditional, where $\xi_n = \zeta_{2n-1} - \zeta_{2n}$, for all $n \in \mathbb{N}$ (see [G-D], Proposition II.3.3). We refer to [ArT] and [G-D] for the theory of spreading models.

Remark 2.4.1. Spreading models are more usually defined in a slightly different manner. Precisely, we say that $(\zeta_n)_{n=1}^{\infty}$ is a spreading model of a sequence $(x_n)_{n=1}^{\infty}$ if, for all $\varepsilon > 0$, there exists $\ell \in \mathbb{N}$ such that

$$\left| \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_i \zeta_i \right\| \right| \leqslant \varepsilon,$$

for all $\ell \leqslant n_1 < \ldots < n_k$, and all $a_1, \ldots, a_k \in [-1, 1]$.

2.5 Finite representability.

Let X and Y be Banach spaces. We say that Y is *finitely representable* in X if for every finite dimensional subspace F and every $\varepsilon > 0$ there exists an isomorphism $f : F \to X$ such that $\|x\| \leqslant \|f(x)\| \leqslant (1 + \varepsilon)\|x\|$, for all $x \in F$.

If $\mathcal{P}$ stand for a class of Banach spaces (e.g., reflexive, stable, etc), we say that a Banach space X is *super*-$\mathcal{P}$ if every Banach space which is finitely representable in X has property $\mathcal{P}$. Notice that, as a Banach space X is always finitely representable into itself, then if X is super-$\mathcal{P}$, then X is $\mathcal{P}$.

2.6 Ultrapowers.

Let X be a Banach space, I be an index set, and $\mathcal{U}$ be a nonprincipal ultrafilter on I. We define

$$X^I/\mathcal{U} = \left\{ (x_i)_{i \in I} \in X^I \mid \sup_{i \in I} \|x_i\| < \infty \right\} / \sim,$$

where $(x_i)_{i \in I} \sim (y_i)_{i \in I}$ if $\lim_{i,\mathcal{U}} \|x_i - y_i\| = 0$. $X^I/\mathcal{U}$ is a Banach space with norm $\|x\| = \lim_{i,\mathcal{U}} \|x_i\|$, where $(x_i)_{i=1}^{\infty}$ is a representative of the class $x \in X^I/\mathcal{U}$. By abuse of notation, we will not distinguish between $(x_i)_{i=1}^{\infty}$ and its equivalence class. The space $X^I/\mathcal{U}$ is called an *ultrapower* of X.

Notice that every ultrapower $X^I/\mathcal{U}$ of a Banach space X is finitely representable in X (see [AlKa], Proposition 11.1.12(i)). On the other hand, if a separable Banach space Y is finitely representable in X, then Y is linearly isometrically embeddable into some ultrapower of X (see [AlKa], Proposition 11.1.12(ii)). In particular, as a Banach space X is reflexive (resp. stable) if and only if every separable subspace of itself is reflexive (resp. stable), it follows that a Banach space X is super-reflexive (resp. superstable) if and only if all of its ultrapowers are reflexive (resp. stable).

2.7 Type and cotype.

Let X be a Banach space and $p \in (1, 2]$. We say that X has *type* p if there exists $T > 0$ such that, for all $x_1, \ldots, x_n \in X$,

$$\mathbb{E}_\varepsilon \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|^p \leqslant T^p \sum_{j=1}^n \|x_j\|^p,$$

where the expectation above is taken with respect to a uniform choice of signs $\varepsilon = (\varepsilon_j)_{j=1}^n \in \{-1, 1\}^n$. The smallest T for which this holds is denoted $T_p(X)$. We say that X has *nontrivial type* if X has type p, for some $p \in (1, 2]$.

Let $q \in [2, \infty)$. We say that X has *cotype* q if there exists $C > 0$ such that, for all $x_1, \ldots, x_n \in X$,

$$\mathbb{E}_\varepsilon \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\|^q \geqslant \frac{1}{C^q} \sum_{j=1}^n \|x_j\|^q,$$

where the expectation above is taken with respect to a uniform choice of signs $\varepsilon = (\varepsilon_j)_{j=1}^n \in \{-1, 1\}^n$. The smallest C for which this holds is denoted $C_q(X)$. We say that X has *nontrivial cotype* if X has cotype q, for some $q \in [2, \infty)$.

Chapter 3

Coarse and uniform embeddings.

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In this chapter, we study the diference between coarse and uniform embeddings between Banach spaces. For that, we will go over the results in Section 1.3, which are contained in the paper *Coarse and uniform embeddings* (see [Br2]).

3.1 Space of positively homogeneous maps.

Let X and Y be Banach spaces. We denote by $\mathcal{H}(X,Y)$ the set consisting of all maps $f : X \to Y$ which are bounded on B_X and positively homogeneous, i.e.,

$$f(\alpha x) = \alpha f(x), \quad \text{for all} \quad \alpha \geqslant 0.$$

We define a norm on $\mathcal{H}(X,Y)$ by setting $\|f\| = \sup\{\|f(x)\| \mid x \in B_X\}$. The space $\mathcal{H}(X,Y)$ endowed with the norm $\|\cdot\|$ above is a Banach space. Clearly, $\|f(x)\| \leqslant \|f\| \cdot \|x\|$, for all $x \in X$. Denote by $\mathcal{HC}(X,Y)$ the subset of $\mathcal{H}(X,Y)$ consisting of continuous maps.

For $\varepsilon > 0$, we define $\|f\|_\varepsilon$ as the infimum of all $L > 0$ such that $L \geqslant \|f\|$ and

$$\|f(x) - f(y)\| \leqslant L \max\{\|x - y\|, \varepsilon\|x\|, \varepsilon\|y\|\},$$

for all $x, y \in X$. Clearly, we have

$$\|f\| \leqslant \|f\|_\varepsilon \leqslant \max\{1, 2\varepsilon^{-1}\}\|f\|,$$

for all $f \in \mathcal{H}(X, Y)$.

Let $f : (N, d) \to (M, \partial)$ be a map and fix $L, \varepsilon > 0$. We say that f is *of cL-type* (L, ε) if $\omega_f(t) \leqslant Lt + \varepsilon$, for all $t \geqslant 0$. The next proposition is a simple computation, and it can be found in [Ka4], Proposition 7.3.

Proposition 3.1.1. *Let X and Y be Banach spaces, and $\varphi : \partial B_X \to Y$ be a bounded map. Let $f : X \to Y$ be given by*

$$f(x) = \begin{cases} 0, & x = 0, \\ \|x\|\varphi\left(\frac{x}{\|x\|}\right), & x \neq 0. \end{cases}$$

Then $f \in \mathcal{H}(X, Y)$. If φ is also continuous, then $f \in \mathcal{HC}(X, Y)$. Moreover, let $L \geqslant 1$, $\varepsilon > 0$, and $K \geqslant 0$. If φ is of cL-type (L, ε), and $\|\varphi(x)\| \leqslant K$, for all $x \in \partial B_X$, then $\|f\|_\varepsilon \leqslant 2K + 4L$.

3.2 Strong embeddings into Banach spaces.

In this section, we show that if X uniformly embeds into a minimal Banach space Y, then X simultaneously coarsely and uniformly embeds into Y. For that, we will need Lemma 16 of [Ro4].

Lemma 3.2.1. ***(C. Rosendal, 2016)*** *Suppose X and E are Banach spaces and $P_n : E \to E$ is a sequence of bounded projections onto subspaces $E_n \subset E$ so that $E_m \subset Ker(P_n)$, for all*

$m \neq n$. Assume also that, for all $n \in \mathbb{N}$, there exists a uniform embedding $\sigma_n : X \to E_n$. Then X admits a strong embedding into E.

Proof Theorem 1.3.1(i). Let $\varphi : X \to Y$ be a uniform embedding. By W. Gowers' dichotomy, Y must contain either a hereditarily indecomposable Banach space or an unconditional basic sequence (see [Gow], Theorem 2). As Y is minimal, and a hereditarily indecomposable Banach space is not isomorphic to any of its proper subspaces (see [Gow], Theorem 4), Y must contain an unconditional basic sequence, say $(e_n)_{n=1}^{\infty}$. Let $(A_n)_{n=1}^{\infty}$ be a partition of $\mathbb{N}$ into infinite subsets, and set $E = \overline{\text{span}}\{e_j \mid j \in \mathbb{N}\}$ and $E_n = \overline{\text{span}}\{e_j \mid j \in A_n\}$, for all $n \in \mathbb{N}$. As Y is minimal, there exists a sequence of isomorphic embeddings $T_n : Y \to E_n$. So, $T_n \circ \varphi$ is a uniform embedding of X into E_n, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $P_n : E \to E_n$ denote the natural projection. We can now apply Lemma 3.2.1, so, X strongly embeds into Y. $\square$

Theorem 1.3.1(i) allows us to obtain some new examples. Let T denote the Tsirelson space introduced by T. Figiel and W. Johnson, and let S denote the Schlumprecht space. It is well known that both T^* and S are minimal Banach spaces (see [CSh], Theorem VI.a.1, and [AnS], Theorem 2.1, respectively). The following corollary is a trivial consequence of Theorem 1.3.1(i).

Corollary 3.2.2. *If a Banach space X uniformly embeds into T^* (resp. S), then X strongly embeds into T^* (resp. S).*

Proof of Theorem 1.3.8. By Corollary 3.3 of [AMauMi], there exists a uniform embedding $f : \ell_2 \to B_{\ell_2}$. Let $(e_n)_{n=1}^{\infty}$ be an unconditional basis for X. Let $(A_n)_{n=1}^{\infty}$ be a partition of $\mathbb{N}$ into infinite subsets. For each $n \in \mathbb{N}$, let $X_n = \overline{\text{span}}\{e_j \mid j \in A_n\}$. By Theorem 2.1 of [OSc1], there exists a uniform equivalence $\sigma_n : B_{\ell_2} \to B_{X_n}$, for each $n \in \mathbb{N}$. By Lemma 3.2.1, we are done. $\square$

Corollary 3.2.3. *Let X be an infinite dimensional space with an unconditional basis and finite cotype. Then L_p strongly embeds into X, for all $p \in [1, 2]$. In particular, ℓ_p strongly embeds into X, for all $p \in [1, 2]$.*

Proof. This is a simple consequence of the fact that L_p strongly embeds into $L_2 \equiv \ell_2$, for all $p \in [1,2]$ (see Remark 5.10 of [MeN1]). $\square$

We finish this section with the following natural question.

Problem 3.2.4. Does ℓ_2 strongly embed into every infinite dimensional Banach space?

3.3 Approximating coarse maps by continuous coarse maps.

In this section, we study when a coarse map can be assumed to be also continuous. Our goal is to prove a general theorem (Theorem 1.3.2) and then use it to obtain applications to the Banach space setting. Precisely, we end this section showing that the existence of a coarse retraction $X \to Y$, where X and Y and Banach spaces and $Y \subset X$, implies the existence of a continuous coarse retraction $X \to Y$ (Corollary 1.3.3). In Section 3.4, we use Theorem 1.3.2 in order to show that if a Banach space X coarsely embeds into a minimal Banach space Y, then X simultaneously coarsely and homeomorphically embeds into Y by a map with uniformly continuous inverse (Theorem 1.3.1(ii)). Finally, in Section 3.5, we use Theorem 1.3.2 to prove that the existence of a coarse section for a quotient map implies the existence of a continuous coarse section.

J. Dugundji proved (see [Du], Theorem 4.1) the following: let M be a metric space, $A \subset M$ be a closed subspace, E be a normed space (or, more generally, a locally convex topological vector space), and $f : A \to E$ be a continuous map, then f can be extended to a continuous map $\varphi : M \to E$. However, J. Dugundji was only interested in continuous maps and did not care about having any control over the value of $\|\varphi(x) - \varphi(a)\|$, for $x \in M$, and $a \in A$. Proposition 3.3.2 below is the modification of Theorem 4.1 of [Du] that we will need for our settings.

Lemma 3.3.1. *Let M be a metric space, $A \subset M$ be a closed subspace, and $\alpha > 0$. There exists a locally finite open cover $\mathcal{U}$ of $M \backslash A$ such that*

(i) diam$(U) < \alpha$, for all $U \in \mathcal{U}$, and

(ii) for all $a \in A$, and all neighborhoods V of a, there exists a neighborhood $V' \subset V$ of a such that, for all $U \in \mathcal{U}$, $U \cap V' \neq \varnothing$ implies $U \subset V$.

The lemma above is Lemma 2.1 of [Du]. Although, item (i) above does not explicitly appear in Lemma 2.1 of [Du], it is clear from its proof that the diameters of the elements of $\mathcal{U}$ can be taken to be arbitrarily small.

Proposition 3.3.2. *Let (M, d) be a metric space, and $A \subset M$ be a closed subspace. Let E be a normed space, and let $f : A \to E$ be a continuous coarse map. Then, for all $\lambda > 1$, and all $\gamma > 0$, there exists a continuous map $\varphi : M \to conv(f(A))$ extending f such that*

$$\|\varphi(x) - \varphi(a)\| \leqslant \omega_f(\lambda \cdot d(x, A) + d(x, a) + \gamma),$$

for all $x \in M$, and all $a \in A$.

Proof. Without loss of generality, assume $\lambda < 2$. Let $\mathcal{U} = \{U_j\}_{j \in J}$ be a locally finite open cover for the metric space $M \backslash A$ given by Lemma 3.3.1 for $\alpha = \gamma/(1+\lambda)$. For each $j \in J$, pick $x_j \in U_j$, and $a_j \in A$ such that $d(x_j, a_j) \leqslant \lambda \cdot d(x_j, A)$. For each $j \in J$, let $\psi_j(x) = d(x, U_j^c)$, for all $x \in M$.

Let $\Psi = \sum_{j \in J} \psi_j$, and define $\varphi : M \to \text{conv}(f(A))$ by

$$\varphi(x) = \begin{cases} f(x), & \text{if} \quad x \in A, \\ \sum_{j \in J} \frac{\psi_j(x)}{\Psi(x)} f(a_j), & \text{if} \quad x \notin A. \end{cases}$$

Clearly, φ extends f, and, as $\mathcal{U}$ is locally finite, φ is continuous on $M \backslash A$. Let us observe that φ is also continuous on A. Pick $a \in A$, and let $\varepsilon > 0$. By the continuity of f, there exists

$\delta > 0$ such that $d(a, a') < \delta$ implies $\|f(a) - f(a')\| < \varepsilon$, for all $a' \in A$. Pick $\delta' \in (0, \delta/6)$ such that, for all $j \in J$, $U_j \cap B(a, \delta') \neq \varnothing$ implies $U_j \subset B(a, \delta/6)$. Say $x \in B(a, \delta')\backslash A$, so x belongs to only finitely many elements of $\mathcal{U}$, say $U_{i_1}, \ldots, U_{i_k}$. By our choice of δ', $d(x, x_{i_j}) < \delta/3$ and $d(a_{i_j}, x_{i_j}) < \lambda\delta/6 < \delta/3$, for all $j \in \{1, \ldots, k\}$. Hence,

$$d(a_{i_j}, a) \leqslant d(a_{i_j}, x_{i_j}) + d(x_{i_j}, x) + d(x, a) < \frac{\delta}{3} + \frac{\delta}{3} + \delta' < \delta,$$

for all $j \in \{1, \ldots, n\}$. By our choice of δ, this gives us that

$$\|\varphi(x) - \varphi(a)\| \leqslant \sum_{j=1}^{k} \frac{\psi_{i_j}(x)}{\Psi(x)} \cdot \|f(a_{i_j}) - f(a)\| < \varepsilon.$$

So φ is continuous.

Let $x \in M$, and $a \in A$. If $x \in A$, it follows that $\|\varphi(x) - \varphi(a)\| \leqslant \omega_f(d(x, a))$, so assume $x \notin A$. Let $U_{i_1}, \ldots, U_{i_k}$ be the only elements of $\mathcal{U}$ containing x. As $\operatorname{diam}(U_j) < \gamma/(1 + \lambda)$, for all $j \in J$, it follows that $d(x_{i_j}, x) < \gamma/(1 + \lambda)$, for all $j \in \{1, \ldots, k\}$. Hence, we must have

$$\begin{aligned}
\|\varphi(x) - \varphi(a)\| &\leqslant \sum_{j=1}^{k} \frac{\psi_{i_j}(x)}{\Psi(x)} \cdot \|f(a_{i_j}) - f(a)\| \\
&\leqslant \sum_{j=1}^{k} \frac{\psi_{i_j}(x)}{\Psi(x)} \cdot \omega_f\big(d(a_{i_j}, x_{i_j}) + d(x_{i_j}, x) + d(x, a)\big) \\
&\leqslant \sum_{j=1}^{k} \frac{\psi_{i_j}(x)}{\Psi(x)} \cdot \omega_f\big(\lambda \cdot d(x, A) + (1 + \lambda) \cdot d(x_{i_j}, x) + d(x, a)\big) \\
&\leqslant \omega_f(\lambda \cdot d(x, A) + \gamma + d(x, a)),
\end{aligned}$$

and we are done. $\square$

We can now prove the main theorem of this section.

Proof of Theorem 1.3.2. Let $\theta : M \to E$ be the continuous extension of $\varphi\restriction A$ given by Proposition 3.3.2 for $\lambda = 2$, and some $\gamma > 0$ such that

$$\omega_\varphi(\gamma) + \omega_{\varphi\restriction A}(4\gamma) \leqslant \inf_{s>0} \omega_\varphi(s) + \inf_{s>0} \omega_{\varphi\restriction A}(s) + \delta.$$

Let $U = \{x \in M \mid d(x, A) < \gamma\}$, and let $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover for the metric space $M \backslash A$ such that $\mathrm{diam}(U_j) < \gamma$, for all $j \in J$. So, $\mathcal{U}' = \{U, U_j\}_{j \in J}$ is an open cover for M, and, as M is paracompact, $\mathcal{U}'$ has a locally finite refinement (see [Mu], Theorem 41.4). Hence, there exists a family of open sets $\mathcal{V} = \{V_i\}_{i \in I}$ refining $\mathcal{U}$ such that $\{U, V_i\}_{i \in I}$ is a locally finite open cover of M. For each $i \in I$, pick $x_i \in V_i$, let $\psi_i(x) = d(x_i, V_i^c)$, and let $\psi_U(x) = \max\{0, 1 - d(x, A)/\gamma\}$, for all $x \in M$. So $\psi_U(x) = 1$, if $x \in A$, and $\psi_U(x) = 0$, if $x \notin U$.

Let $\Psi = \psi_U + \sum_{i \in I} \psi_i$, and define $\Phi : M \to \mathrm{conv}(\varphi(M))$ by

$$\Phi(x) = \frac{\psi_U(x)}{\Psi(x)} \theta(x) + \sum_{i \in I} \frac{\psi_i(x)}{\Psi(x)} \varphi(x_i).$$

As $\{U, V_i\}_{i \in I}$ is locally finite, Φ is continuous. Also, as $\psi_i(x) = 0$, for all $x \in A$, and all $i \in I$, it is clear that $\Phi\restriction A = \varphi\restriction A$.

Let $x \in M \backslash A$, and let $V_{i_1}, \ldots, V_{i_k}$ be the only elements of $\mathcal{V}$ containing x. As $\mathrm{diam}(V_i) < \gamma$, for all $i \in I$, we have that $d(x, x_{i_j}) < \gamma$, for all $j \in \{1, \ldots, k\}$. Hence,

$$\begin{aligned} \|\varphi(x) - \Phi(x)\| &\leqslant \frac{\psi_U(x)}{\Psi(x)} \cdot \|\varphi(x) - \theta(x)\| + \sum_{j=1}^k \frac{\psi_{i_j}(x)}{\Psi(x)} \cdot \|\varphi(x) - \varphi(x_{i_j})\| \\ &\leqslant \frac{\psi_U(x)}{\Psi(x)} \cdot \|\varphi(x) - \theta(x)\| + \omega_\varphi(\gamma) \cdot \sum_{j=1}^k \frac{\psi_{i_j}(x)}{\Psi(x)}. \end{aligned}$$

If $x \notin U$, this shows that $\|\varphi(x) - \Phi(x)\| \leqslant \omega_\varphi(\gamma)$. If $x \in U$, pick $a \in A$ such that $d(x, a) < \gamma$.

Then, as $\theta(a) = \varphi(a)$, we have that

$$\begin{aligned}
\|\varphi(x) - \theta(x)\| &\leqslant \|\varphi(x) - \varphi(a)\| + \|\theta(a) - \theta(x)\| \\
&\leqslant \omega_\varphi(\gamma) + \omega_{\varphi\restriction A}(\lambda \cdot d(x, A) + d(x, a) + \gamma) \\
&\leqslant \omega_\varphi(\gamma) + \omega_{\varphi\restriction A}(4\gamma).
\end{aligned}$$

So, we are done. □

Corollary 3.3.3. *Let Y be a Banach space and $A \subset Y$ be a closed subset. Let $\varphi : Y \to A$ be a retraction. Then, for all $\delta > 0$, there exists a continuous retraction $\Phi : Y \to conv(A)$ such that*

$$\sup_{x \in Y} \|\varphi(x) - \Phi(x)\| \leqslant \inf_{s>0} \omega_\varphi(s) + \delta.$$

In particular, if φ is coarse, so is Φ.

Proof. As $\varphi\restriction A = \mathrm{Id}\restriction A$, we have that $\omega_{\varphi\restriction X}(t) = t$, for all t. A straightforward application of Theorem 1.3.2 finishes the proof. □

Proof of Corollary 1.3.3. This is a particular case of Corollary 3.3.3 above. □

In the case where $A = \varnothing$, the Φ given by Theorem 1.3.2 is not only continuous, but even locally Lipschitz. Let (M, d) and (N, ∂) be metric spaces. We call a map $f : M \to N$ *locally Lipschitz* if for each $x \in M$, there exists a neighborhood of x in which f is Lipschitz.

Proposition 3.3.4. *Let (M, d) be a metric space, and let E be a normed space. Let $\varphi : M \to E$ be a map. Then, for all $\delta > 0$, there exists a locally Lipschitz map $\Phi : M \to conv(\varphi(M))$ such that*

$$\sup_{x \in M} \|\varphi(x) - \Phi(x)\| \leqslant \inf_{s>0} \omega_\varphi(s) + \delta.$$

In particular, if M coarsely embeds into E, then M coarsely embeds into E by a locally Lipschitz map.

Proof. Let γ, $\mathcal{V} = \{V_i\}_{i\in I}$, $\{x_i\}_{i\in I}$, $(\psi_i)_{i\in\mathbb{N}}$, Ψ and Φ be as in the proof of Theorem 1.3.2 (with $A = \emptyset$, and $U = \emptyset$). We only need to notice that Φ is locally Lipschitz. Let $x \in M$. Then, there exists $\varepsilon > 0$ such that $B(x,\varepsilon)$ intersects only finitely many elements of $\mathcal{V}$, say $V_{i_1}, \dots, V_{i_k}$. Without loss of generality, we can assume that $x \in V_{i_1}$, and that $B(x, 2\varepsilon) \subset V_{i_1}$. So $\Psi(y) \geqslant \varepsilon$, for all $y \in B(x,\varepsilon)$. Therefore, as $\psi_i(y)/\Psi(y) \leqslant 1$, for all $y \in M$, and all $i \in I$, we have that

$$\begin{aligned}\Big|\frac{\psi_i(z)}{\Psi(z)} - \frac{\psi_i(y)}{\Psi(y)}\Big| &\leqslant \frac{|\psi_i(z) - \psi_i(y)|}{\Psi(z)} + \frac{|\Psi(z) - \Psi(y)|}{\Psi(z)} \cdot \frac{\psi_i(y)}{\Psi(y)} \\ &\leqslant \Big(\frac{1+k}{\varepsilon}\Big) d(z,y),\end{aligned}$$

for all $z, y \in B(x,\varepsilon)$. Hence, letting $L = \max\{\|\varphi(x_{i_l})\| \mid 1 \leqslant l \leqslant k\}$, we have

$$\|\Phi(z) - \Phi(y)\| \leqslant L\Big(\frac{k+k^2}{\varepsilon}\Big) d(z,y),$$

for all $z, y \in B(x,\varepsilon)$. □

We had just shown that if (M,d) coarsely embeds into a Banach space E, then it coarsely embeds by a continuous map. We would like to obtain that the existence of coarse embeddings actually guarantee us the existence of simultaneously coarse and homeomorphic embeddings. In the next proposition, we show that injectivity of the embedding is not a problem.

Proposition 3.3.5. *Let (M,d) be a separable metric space and let E be an infinite dimensional Banach space. Let $\varphi : M \to E$ be a map. Then, for all $\delta > 0$, there exists an injective continuous map $\Phi : M \to E$ such that*

$$\sup_{x\in M} \|\varphi(x) - \Phi(x)\| \leqslant \inf_{s>0} \omega_\varphi(s) + \delta.$$

In particular, if a separable Banach space X coarsely embeds into a Banach space Y, then X coarsely embeds into Y by an injective continuous map.

Proof. Let $\varphi : M \to E$ be a coarse map, and $\delta > 0$. Pick $\gamma > 0$ such that $\omega_\varphi(\gamma) + 2\gamma < \inf_{s>0} \omega_\varphi(s) + \delta$. Let $Z \subset E$ be a closed infinite dimensional separable subspace such that the quotient space E/Z is infinite dimensional. As M is separable, M isometrically embeds into the space of continuous function on $[0,1]$ with the supremum norm, $C[0,1]$ (see [FHHaMoZ], Corollary 5.9). Therefore, as $C[0,1]$ is homeomorphic to B_Z (see [K]), it follows that M homeomorphically embeds into $\gamma \cdot B_Z$. Say $\theta : M \to \gamma \cdot B_Z$ is such embedding.

Let $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ be a countable locally finite cover of M such that $\operatorname{diam}(U_n) < \gamma$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, pick $x_n \in U_n$, and let $\psi_n(x) = d(x, U_n^c)$, for all $x \in M$.

Define a sequence $(y_n)_{n=1}^\infty$ in E as follows. Pick $y_1 \in B(\varphi(x_1), \gamma)\backslash Z$. Say $y_1, \ldots, y_k$ had been chosen. Then pick $y_{k+1} \in B(\varphi(x_{k+1}), \gamma)\backslash(Z \oplus \operatorname{span}\{y_1, \ldots, y_k\})$. Let $\Psi = \sum_{n\in\mathbb{N}} \psi_n$, and define $\Phi : M \to E$ by

$$\Phi(x) = \theta(x) + \sum_{n\in\mathbb{N}} \frac{\psi_n(x)}{\Psi(x)} y_n.$$

for all $x \in M$. Clearly, Φ is continuous, and satisfies the required inequality. To notice that Φ is injective, notice that, by our choice of $(y_n)_{n=1}^\infty$, if $\Phi(x) = \Phi(y)$, then $\psi_n(x)/\Psi(x) = \psi_n(y)/\Psi(y)$, for all $n \in \mathbb{N}$. So, $\theta(x) = \theta(y)$, which implies $x = y$.

The last claim follows from the facts that (i) if $\dim(X) < \infty$, then $\dim(Y) \geqslant \dim(X)$ (see [NoYu], Theorem 2.2.5 and Example 2.2.6), and (ii) if an infinite dimensional Banach space X coarsely embeds into Y, then Y is also infinite dimensional. □

3.4 Simultaneously homeomorphic and coarse embeddings.

In this section, we show that if a Banach space X coarsely embeds into a minimal Banach space Y, then X simultaneously homeomorphically and coarsely embeds into Y. In order to show that, we show that there exists a map $X \to (\oplus Y)_\mathcal{E}$, where $\mathcal{E}$ is any 1-unconditional basic sequence, which is simultaneously a homeomorphic and coarse embedding.

The following lemma is an application of the methods of [Ro4] to our specific setting (see

[Ro4], Lemma 16).

Lemma 3.4.1. *Suppose X and E are Banach spaces and $P_n : E \to E$ is a sequence of bounded projections onto subspaces $E_n \subset E$ so that $E_m \subset Ker(P_n)$, for all $m \neq n$. Assume also that, for all $n \in \mathbb{N}$, there exists a coarse embedding $\sigma_n : X \to E_n$ which is also continuous. Then X homeomorphically coarsely embeds into E by a map with uniformly continuous inverse.*

Proof. Let us define a continuous coarse map $\psi : X \to \mathrm{Ker}(P_1)$ such that ψ^{-1} exists and is uniformly continuous. Then, by setting $\Psi : X \to E$ as $\Psi(x) = \sigma_1(x) + \psi(x)$, for all $x \in X$, we have that Ψ is a continuous coarse embedding with uniformly continuous inverse. Indeed, Ψ is clearly coarse and continuous. As $\|\sigma_1(x) - \sigma_1(y)\| = \|P_1(\Psi(x) - \Psi(y))\| \leqslant \|P_1\| \cdot \|\Psi(x) - \Psi(y)\|$, for all $x, y \in X$, it follows that Ψ is expanding. As $\|\psi(x) - \psi(y)\| = \|(\mathrm{Id} - P_1)(\Psi(x) - \Psi(y))\| \leqslant \|\mathrm{Id} - P_1\| \cdot \|\Psi(x) - \Psi(y)\|$, for all $x, y \in X$, it follows that Ψ has uniformly continuous inverse.

Without loss of generality, we can assume that $\sigma_n(0) = 0$, for all $n \in \mathbb{N}$. As each σ_n is a coarse embedding, there exist sequences $(L_n)_{n\in\mathbb{N}}$ and $(\Delta_n)_{n\in\mathbb{N}}$ of positive numbers such that $\omega_{\sigma_n}(t) \leqslant L_n t + L_n$ (see Proposition 1.1.1) and $\rho_{\sigma_n}(\Delta_n) > 1$, for all $n \in \mathbb{N}$, and all $t \in [0, \infty)$. We can assume that $\Delta_n \geqslant 1$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\psi_n : X \to E_n$ be given by

$$\psi_n(x) = \frac{\sigma_n(n\Delta_n x)}{n\Delta_n L_n 2^n},$$

and let $\psi(x) = \sum_{n>1} \psi_n(x)$, for all $x \in X$. Clearly, $\psi_n(0) = 0$, for all $n \in \mathbb{N}$, and $\psi(0) = 0$.

Claim: ψ is well defined, coarse, continuous, and ψ^{-1} is uniformly continuous.

For all $x, y \in X$, and all $n \in \mathbb{N}$, there are $x_0, \ldots, x_n \in X$, such that $x_0 = n\Delta_n x$, $x_n = n\Delta_n y$, and $\|x_{j-1} - x_j\| = \Delta_n \|x - y\|$, for all $1 \leqslant j \leqslant n$. So, by the triangle inequality,

$$\|\sigma_n(n\Delta_n x) - \sigma_n(n\Delta_n y)\| \leqslant \sum_{j=1}^{n} \|\sigma_n(x_{j-1}) - \sigma_n(x_j)\| \leqslant n \cdot \omega_{\sigma_n}(\Delta_n \|x - y\|).$$

Hence, as $\Delta_n \geqslant 1$, for all $n \in \mathbb{N}$, we have that

$$\begin{aligned}\Big\| \sum_{n=l}^{m} \psi_n(x) - \sum_{n=l}^{m} \psi_n(y) \Big\| &\leqslant \sum_{n=l}^{m} \frac{\|\sigma_n(n\Delta_n x) - \sigma_n(n\Delta_n y)\|}{n\Delta_n L_n 2^n} \\ &\leqslant \sum_{n=l}^{m} \frac{\omega_{\sigma_n}(\Delta_n\|x-y\|)}{\Delta_n L_n 2^n} \leqslant \frac{\|x-y\|+1}{2^{l-1}}.\end{aligned}$$

In particular, as $\psi_n(0) = 0$, for all $n \in \mathbb{N}$, we have that $\| \sum_{n=l}^{m} \psi_n(x)\| \leqslant (\|x\|+1)/2^{l-1}$, for all $x \in X$, and all $l, m \in \mathbb{N}$, with $l \leqslant m$. Hence, ψ is well defined. Similarly, the argument above gives us that $\omega_\psi(t) \leqslant t+1$, for all $t > 0$, so ψ is coarse.

Let $x \in X$, and $\varepsilon > 0$. Choosing $N \in \mathbb{N}$ such that $1/2^N < \varepsilon/4$, we have that, for all $y \in X$, with $\|x-y\| \leqslant 1$,

$$\begin{aligned}\|\psi(x) - \psi(y)\| &\leqslant \sum_{n\leqslant N} \|\psi_n(x) - \psi_n(y)\| + \sum_{n>N} \frac{\|x-y\|+1}{2^n} \\ &\leqslant \sum_{n\leqslant N} \|\psi_n(x) - \psi_n(y)\| + \frac{\varepsilon}{2}.\end{aligned}$$

By the continuity of each ψ_n at x, there exists $\delta \in (0,1)$ such that $\sum_{n\leqslant N} \|\psi_n(x) - \psi_n(y)\| < \varepsilon/2$, whenever $\|x-y\| < \delta$. Then, $\|\psi(x)-\psi(y)\| < \varepsilon$, if $\|x-y\| < \delta$. So ψ is continuous.

Let us show that ψ^{-1} exists and it is uniformly continuous. For this, we only need to show that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$,

$$\|x-y\| > \varepsilon \;\Rightarrow\; \|\psi(x)-\psi(y)\| > \delta.$$

As $\rho_{\sigma_n}(\Delta_n) > 1$, for all $n \in \mathbb{N}$, if $x, y \in X$ and $\|x-y\| > 1$, then $\|\sigma_n(\Delta_n x) - \sigma_n(\Delta_n y)\| > 1$. Fix $\varepsilon > 0$, and pick $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Then, if $\|x-y\| > \varepsilon$, we have that

$$\begin{aligned}\|\psi(x)-\psi(y)\| &\geqslant \frac{\|\sigma_n(n\Delta_n x) - \sigma_n(n\Delta_n y)\|}{\|P_n\| n\Delta_n L_n 2^n} \\ &\geqslant \frac{1}{\|P_n\| n\Delta_n L_n 2^n},\end{aligned}$$

Hence, ψ^{-1} is uniformly continuous, and we are done. $\square$

Proof of Theorem 1.3.1(ii). Let $\varphi : X \to Y$ be a coarse embedding. By Theorem 1.3.2, we can assume that φ is also continuous. As in the proof of item (i) of Theorem 1.3.1, Y contains an unconditional basic sequence $(e_n)_{n=1}^{\infty}$. Let $(A_n)_{n=1}^{\infty}$ be a partition of $\mathbb{N}$ into infinite subsets, and set $E = \overline{\text{span}}\{e_j \mid j \in \mathbb{N}\}$ and $E_n = \overline{\text{span}}\{e_j \mid j \in A_n\}$, for all $n \in \mathbb{N}$. As Y is minimal, there exists a sequence of isomorphic embeddings $T_n : Y \to E_n$. So, $T_n \circ \varphi$ is a continuous coarse embedding of X into E_n, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $P_n : E \to E_n$ denote the natural projection. We can now apply Lemma 3.4.1, so, X simultaneously homeomorphically and coarsely embeds into Y by a map with uniformly continuous inverse. $\square$

The following corollary is a trivial consequence of Theorem 1.3.1(ii).

Corollary 3.4.2. *If a Banach space X coarsely embeds into T^* (resp. S), then X simultaneously homeomorphically and coarsely embeds into T^* (resp. S) by a map with uniformly continuous inverse.*

Proof of Theorem 1.3.4. If X coarsely embeds into Y, by Theorem 1.3.2, X coarsely embeds into Y by a continuous map. Let $E = (\bigoplus Y)_{\mathcal{E}}$, and $E_n = \{(x_n)_{n=1}^{\infty} \in E \mid \forall j \neq n,\ x_j = 0\}$, for all $n \in \mathbb{N}$. Then, by Lemma 3.4.1, X homeomorphically coarsely embeds into E by a map with uniformly continuous inverse. $\square$

The following simpler version of Problem 1.2.6 could be slightly easier to prove, and it would be a significant advance on this problem.

Problem 3.4.3. Let X and Y be Banach spaces, and assume that X coarsely embeds into Y. Does B_X uniformly embed into Y? What if Y is minimal?

It is worth noticing that one cannot hope that X coarsely embeds into Y if and only if B_X uniformly embeds into Y (even if we restrict ourselves to minimal spaces Y). Indeed, it is well known that all the ℓ_p's have uniformly equivalent balls (see [OSc1], Theorem 2.1), but ℓ_p does not coarsely embed into ℓ_2 for any $p > 2$ (see [JoRan], Theorem 1, or [MeN2], Theorem 1.11).

3.5 Continuous coarse sections.

In [Ka4], N. Kalton proved (Theorem 8.9) that the concepts of coarse and uniform equivalences are actually distinct concepts, i.e., N. Kalton presented two Banach spaces X and Y which are coarsely equivalent but not uniformly equivalent. However, the coarse equivalence presented in [Ka4] only preserves the large scale geometries of X and Y and does not need to be a homeomorphism. In this section, we show that N. Kalton's example is actually an example of Banach spaces which are simultaneously homeomorphically and coarsely equivalent, but not uniformly equivalent.

Let X and Y be Banach spaces, and let $Q : Y \to X$ be a quotient map. If $A \subset X$, we say that $f : A \to Y$ is a *section of* Q if $Q \circ f = \mathrm{Id}_A$. N. Kalton's argument is based on the construction of a quotient map $Q : Y \to X$ for which a coarse section $X \to Y$ exists, but X does not coarse Lipschitz embed into Y by map which is also uniformly continuous (see [Ka4], Theorem 8.8). In particular, Q has no uniformly continuous section $X \to Y$. In this section, we show that if a quotient map $Q : Y \to X$ admits a coarse section, then it admits a continuous coarse section. As a corollary, we get the strengthening of N. Kalton's result mentioned above.

The proof of the following lemma uses ideas in the proof of Proposition 6.5 of [Ka4].

Lemma 3.5.1. *Let X and Y be Banach spaces, and let $Q : Y \to X$ be a quotient map. Assume that there exists a coarse section $\varphi : X \to Y$. Then, there exists $L > 1$ such that, for every $\varepsilon > 0$, there exists a continuous section $\psi : \partial B_X \to Y$ of cL-type (L, ε).*

Proof. Let $\varphi : X \to Y$ be a coarse section. So, there exists $L > 1$ such that $\omega_\varphi(t) \leqslant Lt + L$, for all $t > 0$ (see Proposition 1.1.1). Fix $\varepsilon \in (0, 1)$, and let us show that the required continuous section ψ of cL-type (L, ε) exists.

For each $n \in \mathbb{N}$, let $\varphi_n(x) = \varphi(nx)/n$. So, each φ_n is a coarse section, and $\omega_{\varphi_n}(t) \leqslant Lt + Ln^{-1}$, for all $t > 0$. For each $n \in \mathbb{N}$, let $\Phi_n : X \to Y$ be the continuous map given by

Theorem 1.3.2 applied to φ_n, L/n, and $A = \varnothing$. Hence, we have that

$$\sup_{x\in X} \|\varphi_n(x) - \Phi_n(x)\| \leqslant \frac{2L}{n},$$

for all $n \in \mathbb{N}$. In particular, $\omega_{\Phi_n}(t) \leqslant Lt + 5L/n$, and

$$\|x - Q(\Phi_n(x))\| \leqslant \|x - Q(\varphi_n(x))\| + \|Q(\varphi_n(x)) - Q(\Phi_n(x))\| \leqslant \frac{2L\|Q\|}{n}$$

for all $n \in \mathbb{N}$, and all $x \in X$. Let $\lambda \in (0,1)$ be such that $\sum_{n\in\mathbb{N}} \lambda^n < \frac{\varepsilon}{8L}$. Fix $n_0 \in \mathbb{N}$ large enough so that $2L\|Q\|/n_0 < \lambda$, and $5L/n_0 < \varepsilon/2$.

Let $h : X \to Y$ be given by

$$h(x) = \begin{cases} 0, & \text{if} \quad x = 0, \\ \frac{\|x\|}{2}\Big(\Phi_{n_0}\Big(\frac{x}{\|x\|}\Big) - \Phi_{n_0}\Big(-\frac{x}{\|x\|}\Big)\Big), & \text{if} \quad x \neq 0. \end{cases}$$

Then h is continuous, positively homogeneous, and bounded on bounded sets. Also, it is clear that $\|x - Q(h(x))\| \leqslant \lambda\|x\|$, for all $x \in X$. Let $g(x) = x - Q(h(x))$. Then, as g is positively homogeneous, we have that $\|g^n(x)\| \leqslant \lambda^n\|x\|$, for all $n \in \mathbb{N}$, and all $x \in X$. Set $g^0(x) = x$ and let

$$\psi(x) = \sum_{n=0}^{\infty} h(g^n(x)),$$

for all $x \in \partial B_X$.

As h is positively homogeneous, the series above converges uniformly on bounded sets, so ψ is well defined and continuous. Also, as $g^n(x) - Q(h(g^n(x))) = g^{n+1}(x)$, we have that

$$Q(\psi(x)) = \sum_{n=0}^{\infty} (g^n(x) - g^{n+1}(x)) = x,$$

for all $x \in X$. So $\psi : \partial B_X \to Y$ is a continuous section of Q.

It remains to notice that ψ is of cL-type (L, ε). Notice that, as $5L/n_0 < \varepsilon/2$, we have that

$\sup_{x\in B_X}\|h(x)\| < 2L$ and $\omega_{h_{|\partial B_X}}(t) \leqslant Lt + \varepsilon/2$, for all $t > 0$. Then

$$\begin{aligned}\|\psi(x)-\psi(y)\| &\leqslant \|h(x)-h(y)\| + \sum_{n=1}^{\infty}\|h(g^n(x))\| + \sum_{n=1}^{\infty}\|h(g^n(y))\| \\ &\leqslant L\|x-y\| + \frac{\varepsilon}{2} + 2\cdot\sup_{x\in B_X}\|h(x)\|\cdot\sum_{n=1}^{\infty}\lambda^n \\ &\leqslant L\|x-y\| + \varepsilon,\end{aligned}$$

for all $x, y \in \partial B_X$. So, ψ is of cL-type (L,ε), and we are done. □

The next technical lemma is the continuous version of Lemma 7.4 of [Ka4], and it will play a fundamental role in the proof of Theorem 1.3.5.

Lemma 3.5.2. *Let X and Y be Banach spaces and consider a map $t \in [0,\infty) \mapsto f_t \in \mathcal{H}(X,Y)$ with the property that, for some $K > 0$,*

$$\|f_t\|_{e^{-2t}} \leqslant K, \quad \textit{and} \quad \|f_t - f_s\| \leqslant K|t-s|, \quad \forall\ t,s \geqslant 0.$$

Define $F : X \to Y$ as

$$F(x) = \begin{cases} f_0(x), & \|x\| \leqslant 1, \\ f_{\ln\|x\|}(x), & \|x\| > 1. \end{cases}$$

Then F is coarse. Moreover, if $f_t \in \mathcal{HC}(X,Y)$, for all $t \geqslant 0$, then F is continuous.

In Lemma 7.4 of [Ka4], the author shows that the map F above is coarse, and, under the assumption that f_t is *uniformly* continuous, for all $t \geqslant 0$, the author shows that F is also uniformly continuous. Therefore, we only present the proof that F is continuous if each f_t is so.

Sketch of the proof. For convenience, let $f_t = f_0$, if $t < 0$. In the proof of Lemma 7.4 of

[Ka4], N. Kalton shows that

$$\|F(x) - F(z)\| \leqslant 3K\|x - z\| + 2K\min\{\|x\|, \|x\|^{-1}\} + 2K\min\{\|z\|, \|z\|^{-1}\}, \tag{3.5.1}$$

for all $x, z \in X$. In particular, $\omega_F(t) \leqslant 3Kt + 4K$, so F is coarse.

Let us show that F is continuous if each $f_t \in \mathcal{HC}(X, Y)$, and the map $t \mapsto f_t$ is continuous. Note that, as $F(x) = f_0(x)$ if $\|x\| \in [0, 1)$, F is continuous at x if $\|x\| \in [0, 1)$. Therefore, we only need to show that F is continuous at x if $\|x\| \geqslant 1$.

Let $x \in X$, with $\|x\| \geqslant 1$, and fix $\varepsilon > 0$. Pick $\delta_0 \in (0, 1)$ such that $K\delta_0 < \varepsilon/6$, and $a > 1$ such that $4K/a < \varepsilon/2$. If $\|x\| > a$, pick $\delta_1 \in (0, \min\{\delta_0, \|x\| - a\})$. By Equation 3.5.1, if $\|x - z\| < \delta_1$, we have

$$\|F(x) - F(z)\| \leqslant 3K\delta_1 + 4Ka^{-1} < \varepsilon.$$

Say $\|x\| \leqslant a$. Let $b = \ln(a+1)$. Pick $N > b$ such that $Kb/N < \varepsilon/(3e^b)$. Then $|s - t| \leqslant b/N$ implies $\|f_s - f_t\| < \varepsilon/(3e^b)$.

By the continuity of each f_t, there exists $\delta_2 \in (0, \min\{b/N, 1\})$ such that $\|x - z\| < \delta_2$ implies

$$\|f_{kb/N}(x) - f_{bk/N}(z)\| < \varepsilon/3, \quad \text{for all} \quad 0 \leqslant k \leqslant N.$$

Making δ_2 smaller if necessary, we can also assume that $\|x - z\| < \delta_2$ implies $|\ln\|x\| - \ln\|z\|| < b/(2N)$.

Fiz $z \in X$ with $\|x - z\| < \delta_2$. As $\|x\| \in [1, a]$, we have that $\ln\|x\|, \max\{\ln\|z\|, 0\} \in [0, b]$. Therefore, as $|\ln\|x\| - \ln\|z\|| < b/(2N)$, there exists $k \in \{0, \ldots, N\}$ such that

$$\left|\ln\|x\| - \frac{kb}{N}\right|, \left|\max\{\ln\|z\|, 0\} - \frac{kb}{N}\right| \leqslant \frac{b}{N}.$$

As $\|x - z\| < \delta_2$, we have that $\|x\|/e^b, \|z\|/e^b \leqslant 1$. Therefore, we conclude that

$$\begin{aligned}\|F(x) - F(z)\| &\leqslant \|F(x) - f_{kb/N}(x)\| + \|f_{kb/N}(x) - f_{kb/N}(z)\| \\ &\quad + \|f_{kb/N}(z) - F(z)\| \\ &\leqslant \frac{\varepsilon\|x\|}{3e^b} + \frac{\varepsilon}{3} + \frac{\varepsilon\|z\|}{3e^b} \leqslant \varepsilon.\end{aligned}$$

So, F is continuous at x. $\square$

Proposition 3.5.3. *Let $Q : Y \to X$ be a quotient map. Assume that there exist a constant $L > 1$, a sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive real numbers converging to zero, and a sequence of continuous sections $\varphi_n : \partial B_X \to Y$ such that φ_n is of cL-type (L, ε_n), for all $n \in \mathbb{N}$. Then Q has a continuous coarse section.*

Proof. Without loss of generality, we may assume that $\varepsilon_n < e^{-2n}$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $\psi_n(x) = 1/2(\varphi_n(x) - \varphi_n(-x))$, for all $x \in \partial B_X$. So, each ψ_n is a continuous section of Q of cL-type (L, ε_n) and $\|\psi_n(x)\| \leqslant 2L$, for all $n \in \mathbb{N}$, and all $x \in \partial B_X$.

By Proposition 3.1.1, we can extend each ψ_n to an $f_{n-1} \in \mathcal{HC}(X, Y)$ so that f_{n-1} is a section of Q, and $\|f_{n-1}\|_{e^{-2n}} \leqslant 8L$, for all $n \in \mathbb{N}$. For each $t \geqslant 0$, we define $f_t : X \to Y$ as follows. If $t \in [n-1, n]$, let

$$f_t(x) = (n - t)f_{n-1}(x) + (t - n + 1)f_n(x).$$

Clearly $t \mapsto f_t$ is continuous. Indeed, $\|f_t - f_s\| \leqslant 4L|t - s|$, for all $t, s \in [n-1, n]$.

Notice that $\|f_t\|_{e^{-2t}} \leqslant 8L$, for all $t \geqslant 0$. Let F be the map obtained by Lemma 3.5.2 for the maps $(f_t)_{t \geqslant 0}$. Then F is a continuous coarse section of Q. $\square$

Proof of Theorem 1.3.5. If the quotient map $Q : Y \to X$ has a coarse section $X \to Y$, it follows from Lemma 3.5.1 and Proposition 3.5.3 that Q has a continuous coarse section. Let $\varphi : X \to Y$ be such section. Then, the map $y \mapsto (y - \varphi(Q(y)), Q(y))$ is both a homeomorphism and a coarse equivalence between Y and $\mathrm{Ker}(Q) \oplus X$ with inverse $(x, z) \mapsto x + \varphi(z)$. $\square$

Proof of Corollary 1.3.6. By Proposition 8.4 and Theorem 8.8 of [Ka4], there exist separable Banach spaces X and Y, and a quotient map $Q : Y \to X$ which admits a coarse section $\varphi : X \to Y$, but X does not coarse Lipschitz embed into Y by a uniformly continuous map. Hence, Y and $\mathrm{Ker}(Q) \oplus X$ are not uniformly equivalent. By Theorem 1.3.5, Y and $\mathrm{Ker}(Q) \oplus X$ are simultaneously homeomorphically and coarsely equivalent. □

This raises the question of when two Banach spaces are simultaneously homeomophically and coarsely equivalent. It is well known that any two Banach spaces with the same density character are homeomorphic (see [K], and [To]). But what about if X and Y are coarsely equivalent? Can we get both coarse equivalence and topological equivalence at the same time?

Problem 3.5.4. Let X and Y be Banach spaces, and assume that X and Y are coarsely equivalent. Are X and Y simultaneously coarsely and homeomorphically equivalent?

It is worth noticing that, for separable spaces, the existence of a coarse equivalence easily implies the existence of a measurable coarse equivalence.

Proposition 3.5.5. *Let X and Y be separable Banach spaces, and assume that X is coarsely equivalent to Y. Then, there exists a coarse equivalence $X \to Y$ which is also a Borel bijection.*

Proof. Without loss of generality, we can assume that X and Y are infinite dimensional (see Proposition 2.2.4, and Theorem 2.2.5 of [NoYu]). Let $\{x_n\}_n$ and $\{y_n\}_n$ be $(1,1)$-nets in X and Y such that $x_n \mapsto y_n$ defines a Lipschitz isomorphism. Let $A_1 = B(x_1, 1) \setminus \cup_{i>1} B(x_i, 1/2)$, and

$$A_n = B(x_n, 1) \setminus \Big(\bigcup_{i<n} A_i \cup \bigcup_{i>n} B(x_i, 1/2) \Big),$$

for all $n > 1$. We define a sequence of subsets $(C_n)_{n=1}^{\infty}$ of Y analogously. It is clear that $X = \sqcup_n A_n$, $Y = \sqcup_n C_n$, that A_n and C_n are Borel, and that A_n and C_n are Borel isomorphic (see [Ke], Theorem 15.6), for all $n \in \mathbb{N}$. Let $f_n : A_n \to C_n$ be Borel isomorphisms. Define a map $\varphi : X \to Y$ by setting $\varphi(x) = f_n(x)$, if $x \in A_n$. It should be clear that φ is both a coarse equivalence and a Borel bijection. □

3.6 Unconditional sums of coarse and uniform equivalences.

In [Ka5], N. Kalton proved (Theorem 4.6(ii)) that if X and Y are coarsely equivalent (resp. uniformly equivalent), then $\ell_p(X)$ and $\ell_p(Y)$ are coarsely equivalent (resp. uniformly equivalent). However, as N. Kalton pointed out, his proof seems to be much more complicated than necessary, and it relies on results about close (resp. uniformly close) Banach spaces. In this section, we give a direct proof for a general theorem (see Theorem 3.6.1 below) which gives us N. Kalton's result as a corollary.

Proof of Theorem 1.3.9. Let $\varphi : X \to Y$ be a coarse equivalence (resp. uniform equivalence). Assume $\varphi(0) = 0$. For each $n \in \mathbb{N}$, let $\varphi_n(\cdot) = 2^{-n}\varphi(2^n\cdot)$. Define $\Phi : (\oplus X)_\mathcal{E} \to (\oplus Y)_\mathcal{E}$ by letting $\Phi(x) = (\varphi_n(x_n))_{n=1}^\infty$, for all $x = (x_n)_{n=1}^\infty \in (\oplus X)_\mathcal{E}$.

Claim: Φ is well defined and coarse (resp. uniformly continuous).

Let $L > 0$, be such that $\omega_\varphi(t) < Lt + L$, for all $t > 0$. So, $\omega_{\varphi_n}(t) < Lt + L2^{-n}$, for all $t > 0$. Let us first notice that Φ is well defined. Let $x = (x_n)_{n=1}^\infty \in (\oplus X)_\mathcal{E}$. For $\varepsilon > 0$, pick $N \in \mathbb{N}$ so that $\|\sum_{n>N} \|x_n\| e_n\| < \varepsilon/2L$, and $\sum_{n>N} 2^{-n} < \varepsilon/2L$. Then, for $k > l > N$, we have

$$\begin{aligned}\Big\|\sum_{n=l}^{k} \|\varphi_n(x_n)\| e_n\Big\| &\leqslant \Big\|\sum_{n=l}^{k}\Big(L\|x_n\| + \frac{L}{2^n}\Big)e_n\Big\| \\ &\leqslant \Big\|\sum_{n=l}^{k} L\|x_n\| e_n\Big\| + \Big\|\sum_{n=l}^{k}\frac{L}{2^n}e_n\Big\| < \varepsilon.\end{aligned}$$

Hence, the sum $\sum_{n\in\mathbb{N}} \varphi_n(x)$ converges for every x, so Φ is well defined.

Say $x, y \in (\oplus X)_{\mathcal{E}}$. Then

$$\|\Phi(x) - \Phi(y)\| = \Big\| \sum_{n\in\mathbb{N}} \|\varphi_n(x_n) - \varphi_n(y_n)\| e_n \Big\|$$
$$\leqslant \Big\| \sum_{n\in\mathbb{N}} L\|x_n - y_n\| e_n \Big\| + \Big\| \sum_{n\in\mathbb{N}} \frac{L}{2^n} e_n \Big\| \leqslant L\|x - y\| + L.$$

So Φ is coarse.

Assume φ is uniformly continuous, let us show that Φ is also uniformly continuous. Fix $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $\sum_{n>N} 2^{-n} < \varepsilon/3L$. Choose $\delta > 0$ such that $\delta < \varepsilon/3L$, and $\|\varphi_n(x_n) - \varphi_n(y_n)\| < \varepsilon/3N$, for all $n \leqslant N$, and all $x_n, y_n \in X$ such that $\|x_n - y_n\| < \delta$. Then, if $\|x - y\| < \delta$, we have

$$\|\Phi(x) - \Phi(y)\|$$
$$\leqslant \Big\| \sum_{n\leqslant N} \|\varphi_n(x_n) - \varphi_n(y_n)\| e_n \Big\| + \Big\| \sum_{n>N} L\|x_n - y_n\| e_n \Big\| + \Big\| \sum_{n>N} \frac{L}{2^n} e_n \Big\|$$
$$\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

This shows that Φ is uniformly continuous.

Say φ is a uniform equivalence. Notice that $\varphi_n^{-1}(\cdot) = 2^{-n}\varphi^{-1}(2^n\cdot)$, therefore, $\Phi^{-1}(\cdot) = (\varphi_n^{-1}(\cdot))_{n=1}^{\infty}$, and, by the same arguments as above, Φ^{-1} is uniformly continuous. Hence, $(\oplus X)_{\mathcal{E}}$ and $(\oplus Y)_{\mathcal{E}}$ are uniformly equivalent.

If φ is a coarse equivalence, let $\psi : Y \to X$ be a coarse inverse for φ. Let $\psi_n(\cdot) = 2^{-n}\psi(2^n\cdot)$, and $\Psi = (\psi_n)_{n=1}^{\infty}$. Then, by the same arguments above, Ψ is coarse. One can easily see that Φ and Ψ are coarse inverses of each other, so we are done.

The case of simultaneous homeomorphic and coarse equivalences follows analogously. □

The proof above actually gives us the following slightly stronger result.

Theorem 3.6.1. *Let $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ be sequences of Banach spaces, and let $\varphi_n : X_n \to Y_n$ be a coarse equivalence (resp. uniform equivalence, or simultaneously homeomorphic and* *coarse equivalence), for each $n \in \mathbb{N}$. Let $\mathcal{E}$ be a normalized* 1*-unconditional basic sequence. Assume that*

$$\sup_n \lim_{t\to\infty} \frac{\omega_{\varphi_n}(t)}{t} < \infty \quad \textit{and} \quad \inf_n \lim_{t\to\infty} \frac{\rho_{\varphi_n}(t)}{t} > 0.$$

Then $(\oplus_n X_n)_{\mathcal{E}}$ and $(\oplus_n Y_n)_{\mathcal{E}}$ are coarsely equivalent (resp. uniformly equivalent, or simultaneously homeomorphically and coarsely equivalent).

Proof. Let us work with the uniform equivalence case. Without loss of generality, we assume that $\varphi_n(0) = 0$, for all $n \in \mathbb{N}$. Let $L > 0$ be large enough so that $\lim_{t\to\infty} \omega_{\varphi_n}(t)/t < L$, and $\lim_{t\to\infty} \rho_{\varphi_n}(t)/t > 1/L$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, pick $t_n > 0$ such that $\omega_{\varphi_n}(t) < Lt$, and $\rho_{\varphi_n}(t) > t/L$, for all $n \in \mathbb{N}$, and all $t \geqslant t_n$. Then,

$$\omega_{\varphi_n}(t) < Lt + Lt_n \quad \text{and} \quad \rho_{\varphi_n}(t) > \frac{1}{L}t - \frac{1}{L}t_n,$$

for all $n \in \mathbb{N}$, and all $t > 0$. Hence, it is easy to check that $\omega_{\varphi_n^{-1}}(t) < Lt + t_n$, for all n, and all $t > 0$. Setting

$$\tilde{\varphi}_n(x) = \frac{1}{2^n t_n}\varphi_n(2^n t_n x),$$

we have that each $\tilde{\varphi}_n$ is a uniform equivalence between X_n and Y_n, and that

$$\omega_{\tilde{\varphi}_n}(t) < Lt + \frac{L}{2^n} \quad \text{and} \quad \omega_{\tilde{\varphi}_n^{-1}}(t) < Lt + \frac{1}{2^n},$$

for all $n \in \mathbb{N}$, and all $t > 0$. The proof now follows analogously the proof of Theorem 1.3.9. For the coarse equivalence case we only need to work with the coarse inverses of φ_n's instead of its inverses, and proceed similarly. □

Corollary 3.6.2. *Let X and Y be coarsely equivalent (resp. uniformly equivalent, or simultaneously homeomorphically and coarsely equivalent) Banach spaces, then $\ell_p(X)$ and $\ell_p(Y)$ are coarsely equivalent (resp. uniformly equivalent, or simultaneously homeomorphically and coarsely equivalent).*

Remark 3.6.3. The conditions on ω_{φ_n} and ρ_{ρ_n} in Theorem 3.6.1 cannot be omitted. Indeed, let $q_X = \inf\{q \in [2,\infty) \mid X \text{ has cotype } q\}$, for any Banach space X. Then, by Theorem 1.11 (resp. Theorem 1.9) of [MeN2], if a Banach space X coarsely (resp. uniformly) embeds into a Banach space Y with nontrivial type, then $q_X \leqslant q_Y$. Therefore, $(\oplus_n \ell_\infty^n)_2$ does not coarsely (resp. uniformly) embed into $(\oplus_n \ell_2^n)_2 \cong \ell_2$, as $q_{(\oplus_n \ell_\infty^n)_2} = \infty$ and $q_{\ell_2} = 2$.

Clearly, the method above gives us that, if X coarse Lipschitz embeds into Y, then $(\oplus X)_\mathcal{E}$ coarse Lipschitz embeds into $(\oplus Y)_\mathcal{E}$. However, the same does not work for coarse and uniform embeddings. Indeed, we know that ℓ_1 strongly embeds into ℓ_2. However, $\ell_2(\ell_1)$ neither coarsely nor uniformly embeds into $\ell_2 \equiv \ell_2(\ell_2)$ (see page 1108 of [NS]). On the other hand, if $\mathcal{E}$ is the standard basis of c_0, we do have an analogous result. Indeed, if $\varphi : X \to Y$ is a uniform embedding and $\varphi(0) = 0$, then $\Phi = (\varphi)_{n=1}^\infty$ is a uniform embedding of $(\oplus X)_{c_0}$ into $(\oplus Y)_{c_0}$. If φ is a coarse embedding, then $\Phi = (\varphi)_{n=1}^\infty$ does not need to be well defined, so the same argument does not work. However, without loss of generality, we can assume that $\varphi(x) = 0$, for all $x \in B_X$. Then, the map $\Phi = (\varphi)_{n=1}^\infty$ is well defined, and it is a coarse embedding.

Remark 3.6.4. We should notice that, in [Ka5], N. Kalton only deals with what he calls "coarse homeomorphisms", i.e., a coarse equivalence which is also a bijection. However it is easy to show that X and Y are coarsely homeomorphic if and only if X and Y are coarsely equivalent, for all Banach spaces X and Y. This follows from the easy fact that if X and Y are coarsely equivalent, then X and Y have the same density character, which equals the cardinality of any net in X and Y (for separable Banach spaces this follows from Proposition 3.5.5).

Chapter 4

Weaker notions of nonlinear embeddings.

In this chapter, we study some notions of nonlinear embeddings which are weakenings of the notions of coarse and uniform embeddings. More precisely, we study what we can say when a Banach space X maps into another Banach space Y by a map which is both solvent and coarse (resp. almost uncollapsed and uniformly continuous). The main goal is to provide the reader with evidence that those notions may not be as weaker as one would think. For that, we will go over the results contained in Section 1.4, which are in the paper *Weaker notions of nonlinear embeddings between Banach spaces* (see [Br4]).

4.1 Preliminaries.

The following proposition, proved in [Ro4], Lemma 60, gives us a useful equivalent definition of solvent maps.

Proposition 4.1.1. *Let X be a Banach space and M be a metric space. Then a coarse map $f : X \to M$ is solvent if and only if $\sup_{t>0} \overline{\rho}_f(t) = \infty$.*

Although the statement of the next proposition is different from Proposition 63 of [Ro4], its proof is the same. However, as its proof is very simple and as this result will play an important role in this chapter, for the convenience of the reader, we include its proof here.

Proposition 4.1.2. *Let X and Y be a Banach space, and let $\mathcal{E}$ be an 1-unconditional basic sequence. Assume that there exists a uniformly continuous almost uncollapsed map $\varphi : X \to Y$. Then, there exists a uniformly continuous solvent map $\Phi : X \to (\oplus Y)_{\mathcal{E}}$.*

Proof. Let $\varphi : X \to Y$ be a uniformly continuous almost uncollapsed map. As φ is almost uncollapsed, pick $t > 0$ such that $\overline{\rho}_\varphi(t) > 0$. As φ is uniformly continuous, pick a sequence of positive reals $(\varepsilon_n)_{n=1}^\infty$ such that

$$\|x - y\| < \varepsilon_n \quad \Rightarrow \quad \|\varphi(x) - \varphi(y)\| < \frac{1}{n2^n},$$

for all $x, y \in X$.

For each $n \in \mathbb{N}$, let $\Phi_n(x) = n \cdot \varphi\big(\frac{\varepsilon_n}{n} x\big)$, for all $x \in X$. Then, for $n_0 \in \mathbb{N}$, and $x, y \in X$, with $\|x - y\| \leqslant n_0$, we have that

$$\|\Phi_n(x) - \Phi_n(y)\| = n \cdot \left\|\varphi\Big(\frac{\varepsilon_n}{n} x\Big) - \varphi\Big(\frac{\varepsilon_n}{n} y\Big)\right\| \leqslant \frac{1}{2^n},$$

for all $n \geqslant n_0$. Define $\Phi : X \to (\oplus Y)_{\mathcal{E}}$ by letting $\Phi(x) = (\Phi_n(x))_{n=1}^\infty$, for all $x \in X$. By the above, Φ is well-define and it is uniformly continuous. Now notice that, if $\|x - y\| = tn/\varepsilon_n$, then $\|\frac{\varepsilon_n}{n} x - \frac{\varepsilon_n}{n} y\| = t$. Hence, if $\|x - y\| = tn/\varepsilon_n$, we have that

$$\|\Phi(x) - \Phi(y)\| \geqslant \|\Phi_n(x) - \Phi_n(y)\| = n \cdot \left\|\varphi\Big(\frac{\varepsilon_n}{n} x\Big) - \varphi\Big(\frac{\varepsilon_n}{n} y\Big)\right\| \geqslant n \cdot \overline{\rho}_\varphi(t).$$

So, as $\overline{\rho}_\varphi(t) > 0$, we have that $\lim_n \overline{\rho}_\Phi(tn/\varepsilon_n) = \infty$. By Proposition 4.1.1, Φ is solvent. $\square$

4.2 Cocycles.

By the Mazur-Ulam Theorem (see [MazU]), any surjective isometry $A : Y \to Y$ of a Banach space Y is affine, i.e., there exists a surjective linear isometry $T : Y \to Y$, and some $y_0 \in Y$, such that $A(y) = T(y) + y_0$, for all $y \in Y$. Therefore, if G is a group, every isometric action $\alpha : G \curvearrowright Y$ of G on the Banach space Y is an affine isometric action, i.e., there exists an isometric linear action $\pi : G \curvearrowright Y$, and a map $b : G \to Y$ such that

$$\alpha_g(y) = \pi_g(y) + b(g),$$

for all $g \in G$, and all $y \in Y$. The map $b : G \to Y$ is called the *cocycle of* α, and it is given by $b(g) = \alpha_g(0)$, for all $g \in G$. As α is an action by isometries, we have that

$$\|b(g) - b(h)\| = \|\alpha_g(0) - \alpha_h(0)\| = \|\alpha_{h^{-1}g}(0)\| = \|b(h^{-1}g)\|$$

for all $g, h \in G$. Hence, if G is a metric group, a continuous cocycle $b : G \to Y$ is automatically uniformly continuous.

Remark 4.2.1. If $(X, \|\cdot\|)$ is a Banach space, we look at $(X, +)$ as an additive group with a metric given by the norm $\|\cdot\|$. So, we can work with affine isometric actions $\alpha : X \curvearrowright Y$ of the additive group $(X, +)$ on a Banach space Y.

Let $\alpha : G \curvearrowright Y$ be an action by affine isometries. Its cocycle b is called a *coboundary* if there exists $\xi \in Y$ such that $b(g) = \xi - \pi_g(\xi)$, for all $g \in G$. Clearly, b is a coboundary if and only if α has a fixed point. Also, if Y is reflexive, then $\mathrm{Im}(b)$ is bounded if and only if b is a coboundary. Indeed, if b is a coboundary, it is clear that $\mathrm{Im}(b)$ is bounded. Say $\mathrm{Im}(b)$ is bounded and let $\mathcal{O}$ be an orbit of the action α. Then the closed convex hull $\overline{\mathrm{conv}}(\mathcal{O})$ must be bounded, hence weakly compact (as Y is reflexive). Therefore, by Ryll-Nardzewski fixed-point theorem (see [R-N]), there exists $\xi \in Y$ such that $\alpha_g(\xi) = \xi$, for all $g \in G$. So, $b(g) = \xi - \pi_g(\xi)$, for all $g \in G$.

The discussion above is well-known, and we isolate it in the proposition below.

Proposition 4.2.2. *Let G be a group and Y be a Banach space. Let $\alpha : G \curvearrowright Y$ be an action by affine isometries with cocycle b. Then b is a coboundary if and only if α has a fixed point. Moreover, if Y is reflexive, then b is a coboundary if and only if b is bounded.*

As we are interested in studying the relations between maps which are expanding, solvent, uncollapsed, and almost uncollapsed, it is important to know that those are actually different classes of maps. The next proposition shows that there are maps which are both solvent and collapsed (see [Ed], Theorem 2.1, for a similar example). In particular, such maps are not expanding.

Proposition 4.2.3. *There exists an affine isometric action $\mathbb{R} \curvearrowright \ell_2(\mathbb{C})$ whose cocycle is Lipschitz, solvent, and collapsed.*

Proof. Define an action $U : \mathbb{R} \curvearrowright \mathbb{C}^{\mathbb{N}}$ by letting

$$U_t(x) = \Big(\exp\Big(\frac{2\pi i t}{2^{2^n}}\Big) x_n \Big)_{n=1}^{\infty},$$

for all $t \in \mathbb{R}$, and all $x = (x_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}$. Let $w = (1, 1, \dots) \in \mathbb{C}^{\mathbb{N}}$ and define an action $\alpha : \mathbb{R} \curvearrowright \mathbb{C}^{\mathbb{N}}$ as $\alpha_t(x) = w + U_t(x - w)$, for all $t \in \mathbb{R}$, and all $x \in \mathbb{C}^{\mathbb{N}}$. So,

$$(\alpha_t(x))_m = \exp\Big(\frac{2\pi i t}{2^{2^m}}\Big) x_m + 1 - \exp\Big(\frac{2\pi i t}{2^{2^m}}\Big), \tag{4.2.1}$$

for all $t \in \mathbb{R}$, all $x = (x_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}$, and all $m \in \mathbb{N}$. As $|1 - \exp(\theta i)| \leqslant |\theta|$, for all $\theta \in \mathbb{R}$, it follows that $(1 - \exp(2\pi i t/2^{2^n}))_{n=1}^{\infty} \in \ell_2(\mathbb{C})$, for all $t \in \mathbb{R}$. Hence, $\alpha_t(x) \in \ell_2(\mathbb{C})$, for all $t \in \mathbb{R}$, and all $x \in \ell_2(\mathbb{C})$. So, α restricts to an action $\alpha : \mathbb{R} \curvearrowright \ell_2(\mathbb{C})$. By Equation 4.2.1, it follows that $\alpha : \mathbb{R} \curvearrowright \ell_2(\mathbb{C})$ is an affine isometric action.

Let $b : \mathbb{R} \to \ell_2(\mathbb{C})$ be the cocycle of $\alpha : \mathbb{R} \curvearrowright \ell_2(\mathbb{C})$, i.e., $b(t) = \alpha_t(0)$, for all $t \in \mathbb{R}$. Then, an easy induction gives us that $b(t) = w - U_t(w)$, for all $t \in \mathbb{R}$. Let $C = \sum_{n\in\mathbb{N}} \left(\frac{2\pi}{2^{2^n}}\right)^2$, then

$$\|b(t)\|^2 = \sum_{n\in\mathbb{N}} \left|1 - \exp\left(\frac{2\pi i t}{2^{2^n}}\right)\right|^2 \leqslant \sum_{n\in\mathbb{N}} \left(\frac{2\pi t}{2^{2^n}}\right)^2 = C|t|^2,$$

for all $t \in \mathbb{R}$. So, b is Lipschitz.

For $t \neq 0$, $0 \in \mathbb{C}^{\mathbb{R}}$ is the only fixed point of U_t. Hence, w is the only fixed point of α_t. So, as $w \notin \ell_2(\mathbb{C})$, $\alpha : \mathbb{R} \rightharpoonup \ell_2(\mathbb{C})$ has no fixed points. Therefore, b is unbounded (see Proposition 4.2.2). By Proposition 4.1.1, b is solvent.

Pick $L > 0$ such that $Ls \leqslant 2^s - 1$, for all $s \in \mathbb{N}$. If $k \in \mathbb{N}$ is large enough, say $2\pi/2^{2^k L} < 1$, we have that

$$\begin{aligned}
\|b(2^{2^k})\|^2 &= \sum_{n>k} \left|1 - \exp\left(\frac{2\pi i 2^{2^k}}{2^{2^n}}\right)\right|^2 \leqslant \sum_{n>k} \left(\frac{2\pi 2^{2^k}}{2^{2^n}}\right)^2 \\
&= \sum_{s\in\mathbb{N}} \left(\frac{2\pi}{2^{2^k(2^s-1)}}\right)^2 \leqslant \sum_{s\in\mathbb{N}} \left(\frac{2\pi}{2^{2^k Ls}}\right)^2 \\
&\leqslant \frac{2\pi}{2^{2^k L} - 1}.
\end{aligned}$$

Hence, $\|b(2^{2^k})\| \to 0$, as $k \to \infty$. So, b is collapsed. □

Problem 4.2.4. Is there a map $X \to Y$ which is collapsed, almost uncollapsed and bounded (in particular not solvent), for some Banach spaces X and Y?

4.3 Preservation of cotype.

M. Mendel and A. Naor solved in [MeN2] the long standing problem in Banach space theory of giving a completely metric definition for the cotype of a Banach space. As a subproduct of this, they have shown that if a Banach space X coarsely (resp. uniformly) embeds into a Banach space Y with nontrivial type, then $q_X \leqslant q_Y$ (see [MeN2], Theorem 1.9 and Theorem 1.11). In this section we prove Theorem 1.4.2, which shows that the hypothesis on the embedding $X \hookrightarrow Y$ can be weakened.

For every $m \in \mathbb{N}$, we denote by $\mathbb{Z}_m$ the set of integers modulo m. For every $n, m \in \mathbb{N}$, we denote the normalized counting measure on $\mathbb{Z}_m^n$ by $\mu = \mu_{n,m}$, and the normalized counting measure on $\{-1,0,1\}^n$ by $\sigma = \sigma_n$.

Definition 4.3.1. ***(Metric cotype)*** *Let (M,d) be a metric space and $q, \Gamma > 0$. We say that (M,d) has* metric cotype q with constant Γ *if, for all $n \in \mathbb{N}$, there exists an even integer m, such that, for all $f : \mathbb{Z}_m^n \to M$,*

$$\sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d\Big(f\Big(x + \frac{m}{2}e_j\Big), f(x)\Big)^q d\mu(x) \tag{4.3.1}$$
$$\leqslant \Gamma^q m^q \int_{\{-1,0,1\}^n} \int_{\mathbb{Z}_m^n} d\big(f(x+\varepsilon), f(x)\big)^q d\mu(x) d\sigma(\varepsilon).$$

The infimum of the constants Γ for which (M,d) has metric cotype q with constant Γ is denoted by $\Gamma_q(M)$. Given $n \in \mathbb{N}$ and $\Gamma > 0$, we define $m_q(M,n,\Gamma)$ as the smallest even integer m such that Inequality 4.3.1 holds, for all $f : \mathbb{Z}_m^n \to M$. If no such m exists we set $m_q(M,n,\Gamma) = \infty$.

The following is the main theorem of [MeN2]. Although we will not use this result in this dissertation, we believe it is worth mentioning.

Theorem 4.3.2. ***(M. Mendel and A. Naor, 2008)*** *Let X be a Banach space and $q \in [2,\infty)$. Then X has metric cotype q if and only if X has cotype q. Moreover,*

$$\frac{1}{2\pi} C_q(X) \leqslant \Gamma_q(X) \leqslant 90 C_q(X),$$

where $C_q(X)$ is the q-cotype constant of X.

We start by proving a simple property of solvent maps.

Proposition 4.3.3. *Let (M,d) and (N,∂) be metric spaces, $\varphi : M \to N$ be a solvent map, and $S > 0$. If $[a_n,b_n]_{n=1}^{\infty}$ is a sequence of intervals of the real line such that $\lim_n a_n = \infty$,*

$b_n - a_n < S$ and $a_{n+1} - a_n < S$, for all $n \in \mathbb{N}$, then, we must have

$$\sup_n \inf\{\overline{\rho}_\varphi(t) \mid t \in [a_n, b_n]\} = \infty.$$

Proof. Let $k > 0$. Pick $N \in \mathbb{N}$ so that $N \geqslant \{a_1 + S, k, 2S\}$, and let $R \geqslant 0$ be such that $d(x, y) \in [R, R + N]$ implies $\partial(f(x), f(y)) > N$, for all $x, y \in M$. Then there exists $n \in \mathbb{N}$ such that $[a_n, b_n] \subset [R, R + N]$. Indeed, if $a_1 < R$ let $n = \max\{j \in \mathbb{N} \mid a_j < R\} + 1$, and if $R \leqslant a_1$ let $n = 1$. Hence,

$$\inf\{\overline{\rho}_\varphi(t) \mid t \in [a_n, b_n]\} \geqslant \inf\{\overline{\rho}_\varphi(t) \mid t \in [R, R + N]\} \geqslant N \geqslant k.$$

As k was chosen arbitrarily, we are done. □

The following lemma is a version of Lemma 7.1 of [MeN2] in the context of the modulus $\overline{\rho}$ instead of ρ. It's proof is analogous to the proof of Lemma 7.1 of [MeN2] but we include it here for completeness. Let $n \in \mathbb{N}$ and $r \in [1, \infty]$. In what follows, $\ell_r^n(\mathbb{C})$ denotes the complex Banach space $(\mathbb{C}^n, \|\cdot\|_r)$, where $\|\cdot\|_r$ denotes the ℓ_r-norm in $\mathbb{C}^n$.

Lemma 4.3.4. *Let (M, d) be a metric space, $n \in \mathbb{N}$, $q, \Gamma > 0$, and $r \in [1, \infty]$. Then, for every map $f : \ell_r^n(\mathbb{C}) \to M$, and every $s > 0$, we have*

$$n^{1/q}\overline{\rho}_f(2s) \leqslant \Gamma \cdot m_q(M, n, \Gamma) \cdot \omega_f\left(\frac{2\pi s n^{1/r}}{m_q(M, n, \Gamma)}\right)$$

(if $r = \infty$, we use the notation $1/r = 0$).

Proof. In order to simplify notation, let $m = m_q(M, n, \Gamma)$ and assume $r < \infty$ (if $r = \infty$, the same proof holds with the ℓ_r-norm substituted by the max-norm below). Let $e_1, \ldots, e_n$ be the standard basis of $\ell_r^n(\mathbb{C})$. Let $h : \mathbb{Z}_m^n \to \ell_r^n(\mathbb{C})$ be given by

$$h(x) = s \cdot \sum_{j=1}^n e^{\frac{2\pi i x_j}{m}} e_j,$$

for all $x = (x_j)_j \in \mathbb{Z}_m^n$, and define $g : \mathbb{Z}_m^n \to M$ by letting $g(x) = f(h(x))$, for all $x = (x_j)_j \in \mathbb{Z}_m^n$. Then, as

$$d(g(x+\varepsilon), g(x)) \leqslant \omega_f\Big(s\Big(\sum_{j=1}^n |e^{\frac{2\pi i\varepsilon_j}{m}} - 1|^r\Big)^{1/r}\Big) \leqslant \omega_f\Big(\frac{2\pi s n^{1/r}}{m}\Big),$$

for all $\varepsilon = (\varepsilon_j)_{j=1}^n \in \{-1,0,1\}^n$ and all $x = (x_j)_{j=1}^n \in \mathbb{Z}_m^n$, we must have

$$\int_{\{-1,0,1\}^n}\int_{\mathbb{Z}_m^n} d\big(g(x+\varepsilon), g(x)\big)^q d\mu(x) d\sigma(\varepsilon) \leqslant \omega_f\Big(\frac{2\pi s n^{1/r}}{m}\Big)^q.$$

Also, as $\|h(x+\frac{m}{2}e_j) - h(x)\| = 2s$, for all $x \in \mathbb{Z}_m^n$, and all $j \in \{1,\dots,n\}$, we have that $d(g(x+\frac{m}{2}e_j), g(x)) \geqslant \overline{\rho}_f(2s)$, for all $x \in \mathbb{Z}_m^n$, and all $j \in \{1,\dots,n\}$. Hence,

$$\sum_{j=1}^n \int_{\mathbb{Z}_m^n} d\Big(g\Big(x + \frac{m}{2}e_j\Big), g(x)\Big)^q d\mu(x) \geqslant n\overline{\rho}_f(2s)^q.$$

Therefore, by the definition of $m_q(M,n,\Gamma)$, we conclude that

$$n\overline{\rho}_f(2s)^q \leqslant \Gamma^q m^q \omega_f\Big(\frac{2\pi s n^{1/r}}{m}\Big)^q.$$

Raising both sides to the $(1/q)$-th power, we are done. □

We can now prove the main result of this section.

Proof of Theorem 1.4.2. First, let us notice that we only need to prove the case in which φ is coarse and solvent. Indeed, let $\varphi : X \to Y$ be a uniformly continuous almost uncollapsed map, then X maps into $\ell_2(Y)$ by a uniformly continuous solvent map (see Proposition 4.1.2). As $p_{\ell_2(Y)} = p_Y$ and $q_{\ell_2(Y)} = q_Y$, there is no loss of generality if we assume that φ is solvent.

If $q_Y = \infty$ we are done, so assume $q_Y < \infty$. Suppose $q_X > q_Y$. Pick $q \in (q_Y, q_X)$ such that $1/q - 1/q_X < 1$, and let $\alpha = 1/q - 1/q_X$ (if $q_X = \infty$, we use the notation $1/q_X = 0$).

Let $(\varepsilon_n)_{n=1}^\infty$ be a sequence in $(0,1)$ such that $(1+\varepsilon_n)n^\alpha \leqslant n^\alpha + 1$, for all $n \in \mathbb{N}$. By Maurey-Pisier Theorem (see [MauPi]), ℓ_{q_X} is finitely representable in X. Considering $\ell_p(\mathbb{C})$

as a real Banach space, we have that $\ell_p(\mathbb{C})$ is finitely representable in ℓ_p, so $\ell_p(\mathbb{C})$ is finitely representable in X. Therefore, looking at $\ell_p^n(\mathbb{C})$ as real Banach spaces, we can pick a (real) isomorphic embedding $f_n : \ell_{q_X}^n(\mathbb{C}) \to X$ such that $\|x\| \leqslant \|f_n(x)\| \leqslant (1+\varepsilon_n)\|x\|$, for all $x \in \ell_{q_X}^n(\mathbb{C})$. For each $n \in \mathbb{N}$, let $g_n = \varphi \circ f_n$. Hence,

$$\begin{aligned}\overline{\rho}_{g_n}(t) &= \inf\{\|\varphi(f_n(x)) - \varphi(f_n(y))\| \mid \|x-y\| = t\}\\ &\geqslant \inf\{\overline{\rho}_\varphi(\|f_n(x) - f_n(y)\|) \mid \|x-y\| = t\}\\ &\geqslant \inf\{\overline{\rho}_\varphi(a) \mid a \in [t, (1+\varepsilon_n)t]\},\end{aligned}$$

for all $n \in \mathbb{N}$, and all $t \in [0,\infty)$. Also, as $\varepsilon_n \in (0,1)$, we have that $\omega_{g_n}(t) \leqslant \omega_\varphi(2t)$, for all $n \in \mathbb{N}$, and all $t \in [0,\infty)$.

As Y has nontrivial type and as $q > q_Y$, Theorem 4.1 of [MeN2] gives us that, for some $\Gamma > 0$, $m_q(Y,n,\Gamma) = O(n^{1/q})$. Therefore, there exists $A > 0$ and $n_0 \in \mathbb{N}$ such that $m_q(Y,n,\Gamma) \leqslant An^{1/q}$, for all $n > n_0$. On the other hand, by Lemma 2.3 of [MeN2], $m_q(Y,n,\Gamma) \geqslant n^{1/q}/\Gamma$, for all $n \in \mathbb{N}$. Hence, applying Lemma 4.3.4 with $s = n^\alpha$ and $r = q_X$, we get that, for all $n > n_0$,

$$\inf\{\overline{\rho}_\varphi(2a) \mid a \in [2n^\alpha, 2n^\alpha + 2]\} \leqslant \Gamma A \omega_\varphi(4\pi\Gamma).$$

As $\alpha < 1$, we have that $\sup_n |(n+1)^\alpha - n^\alpha| < \infty$. Therefore, by Proposition 4.3.3, the supremum over n of the left hand side above is infinite. As φ is coarse, this gives us a contradiction. □

Proof of Corollary 1.4.3. If $p > 1$, this follows straight from Theorem 1.4.2, the fact that $q_{\ell_p} = \max\{2,p\}$ and that ℓ_p has nontrivial type. If $p = 1$, let $g : \ell_1 \to \ell_2$ be a uniform embedding (see [No2], Theorem 5). Then the conclusion of the corollary must hold for the map $g \circ f : \ell_q \to \ell_2$, which implies that it holds for f as well. □

4.4 Property $\mathcal{Q}$.

For each $k \in \mathbb{N}$, let $\mathcal{P}_k(\mathbb{N})$ denote the set of all subset of $\mathbb{N}$ with exactly k elements endowed $\mathcal{P}_k(\mathbb{N})$ with the metric given in Section 1.4. For $\varepsilon, \delta > 0$, a metric space (M, d) is said to have *Property* $\mathcal{Q}(\varepsilon, \delta)$ if for all $k \in \mathbb{N}$, and all $f : \mathcal{P}_k(\mathbb{N}) \to M$ with $\omega_f(1) \leqslant \delta$, there exists an infinite subset $\mathbb{M} \subset \mathbb{N}$ such that

$$d(f(\bar{n}), f(\bar{m})) \leqslant \varepsilon, \quad \text{for all} \quad \bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}).$$

For each $\varepsilon > 0$, we define $\Delta_M(\varepsilon)$ as the supremum of all $\delta > 0$ so that (M, d) has Property $\mathcal{Q}(\varepsilon, \delta)$. For a Banach space X, it is clear that there exists $Q_X \geqslant 0$ such that $\Delta_X(\varepsilon) = Q_X\varepsilon$, for all $\varepsilon > 0$. The Banach space X is said to have *Property* $\mathcal{Q}$ if $Q_X > 0$.

Remark 4.4.1. Notice that this definition of Property $\mathcal{Q}$ is slightly different from the definition given in Section 1.4. However, the definition above is N. Kalton's original definition and it is easy to see that they are equivalent to each other. The reason why we introduce this equivalent definition here will be clear in the proof of Theorem 4.4.2 below.

Theorem 4.4.2. *Let X and Y be Banach spaces, and assume that Y has Property $\mathcal{Q}$. If either*

(i) there exists a coarse solvent map $X \to Y$, or

(ii) there exists a uniformly continuous map $\varphi : B_X \to Y$ such that $\overline{\rho}_\varphi(t) > 0$, for some $t \in (0, 1)$,

then, X has Property $\mathcal{Q}$. In particular, if there exists a uniformly continuous almost uncollapsed map $X \to Y$, then, X has Property $\mathcal{Q}$.

Proof. (i) Assume $\varphi : X \to Y$ is a coarse solvent map. In particular, $\omega_\varphi(1) > 0$. Fix $j \in \mathbb{N}$, and pick $R > 0$ such that

$$\|x - y\| \in [R, R + j] \quad \text{implies} \quad \|\varphi(x) - \varphi(y)\| > j, \tag{4.4.1}$$

for all $x, y \in X$. Assume that X does not have Property $\mathcal{Q}$. So, $\Delta_X(R) = 0$, and there exists $k \in \mathbb{N}$, and $f : \mathcal{P}_k(\mathbb{N}) \to X$ with $\omega_f(1) \leqslant 1$, such that, for all infinite $\mathbb{M} \subset \mathbb{N}$, there exists $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M})$ such that $\|f(\bar{n}) - f(\bar{m})\| > R$. By standard Ramsey theory (see [T], Theorem 1.3), we can assume that $\|f(\bar{n}) - f(\bar{m})\| > R$, for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M})$.

Pick a positive $\theta < j$. As $\omega_f(1) \leqslant 1$, we have that $\|f(\bar{n}) - f(\bar{m})\| \in [R, k]$, for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M})$. Therefore, applying Ramsey theory again, we can get an infinite subset $\mathbb{M}' \subset \mathbb{M}$, and $a \in [R, k]$ such that $\|f(\bar{n}) - f(\bar{m})\| \in [a, a + \theta]$, for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}')$. By our choice of θ, it follows that

$$\left\| \frac{R}{a} f(\bar{n}) - \frac{R}{a} f(\bar{m}) \right\| \in [R, R + j], \quad \text{for all} \quad \bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}'). \tag{4.4.2}$$

Let $Q_Y > 0$ be the constant given by the fact that Y has Property $\mathcal{Q}$. Let $g = (R/a)f$. As $R/a \leqslant 1$, we have that $\omega_{\varphi \circ g}(1) \leqslant \omega_\varphi(1)$. As $\Delta_Y(2\omega_\varphi(1)Q_Y^{-1}) = 2\omega_\varphi(1)$, we get that there exists $\mathbb{M}'' \subset \mathbb{M}'$ such that

$$\|\varphi(g(\bar{n})) - \varphi(g(\bar{m}))\| \leqslant 2\omega_\varphi(1)Q_Y^{-1}, \tag{4.4.3}$$

for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}'')$. As j was chosen arbitrarily, (4.4.1), (4.4.2) and (4.4.3) above gives us that $j < 2\omega_\varphi(1)Q_Y^{-1}$, for all $j \in \mathbb{N}$. As φ is coarse, this gives us a contradiction.

(ii) Assume $\varphi : B_X \to Y$ is a uniformly continuous map, and let $t \in (0, 1)$ be such that $\overline{\rho}_\varphi(t) > 0$. As φ is uniformly continuous, we can pick $\rho \in (t, 1)$, $s, r \in (0, \rho)$ with $s < t < r$, and $\gamma > 0$, such that

$$\|x - y\| \in [s, r] \quad \text{implies} \quad \|\varphi(x) - \varphi(y)\| > \gamma, \tag{4.4.4}$$

for all $x, y \in \rho B_X$. Assume that X does not have Property $\mathcal{Q}$. So, $\Delta_X(s) = 0$. Fix $j \in \mathbb{N}$. Then, there exists $k \in \mathbb{N}$, and $f : \mathcal{P}_k(\mathbb{N}) \to X$ with $\omega_f(1) \leqslant j^{-1}$, such that, for all infinite $\mathbb{M} \subset \mathbb{N}$, there exists $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M})$ such that $\|f(\bar{n}) - f(\bar{m})\| > s$. Without loss of generality, we can assume that $\|f(\bar{n}) - f(\bar{m})\| > s$, for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M})$.

Pick a positive $\theta < (r - s)$. As $\|f(\bar{n}) - f(\bar{m})\| \in [s, k]$, we can use Ramsey theory once again to pick an infinite $\mathbb{M}' \subset \mathbb{M}$, and $a \in [s, k]$ such that $\|f(\bar{n}) - f(\bar{m})\| \in [a, a + \theta]$, for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}')$. By our choice of θ, it follows that

$$\left\| \frac{s}{a} f(\bar{n}) - \frac{s}{a} f(\bar{m}) \right\| \in [s, r], \quad \text{for all} \quad \bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}'). \tag{4.4.5}$$

Let $\bar{m}_0$ be the first k elements of $\mathbb{M}'$, and $\mathbb{M}'' = \mathbb{M}' \backslash \bar{m}_0$. For each $\bar{n} \in \mathcal{P}_k(\mathbb{M}'')$, let $h(\bar{n}) = (s/a)(f(\bar{n}) - f(\bar{m}_0))$. Then, $h(\bar{n}) \in \rho B_X$, and $\|h(\bar{n}) - h(\bar{m})\| \in [s, r]$, for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}'')$. As $s/a \leqslant 1$, we have $\omega_h(1) \leqslant \omega_f(1)$. Hence, $\omega_{\varphi \circ h}(1) \leqslant \omega_\varphi(j^{-1})$.

Let $Q_Y > 0$ be the constant given by the fact that Y has Property $\mathcal{Q}$. Hence, as $\Delta_Y(2\omega_\varphi(j^{-1})Q_Y^{-1}) = 2\omega_\varphi(j^{-1})$, there exists $\mathbb{M}''' \subset \mathbb{M}''$ such that

$$\|\varphi(h(\bar{n})) - \varphi(h(\bar{m}))\| \leqslant 2\omega_\varphi(j^{-1})Q_Y^{-1}, \tag{4.4.6}$$

for all $\bar{n} < \bar{m} \in \mathcal{P}_k(\mathbb{M}''')$. As j was chosen arbitrarily, (4.4.4), (4.4.5) and (4.4.6) gives us that $\gamma < 2\omega_\varphi(j^{-1})Q_Y^{-1}$, for all $j \in \mathbb{N}$. As φ is uniformly continuous, this gives us a contradiction. $\square$

We can now prove the following generalization of Theorem 5.1 of [Ka1].

Theorem 4.4.3. *Let X and Y be Banach spaces, and assume that Y is reflexive (resp. super-reflexive). If either*

(i) there exists a coarse solvent map $X \to Y$, or

(ii) there exists a uniformly continuous map $\varphi : B_X \to Y$ such that $\overline{\rho}_\varphi(t) > 0$, for some $t \in (0, 1)$,

then, X is either reflexive (resp. super-reflexive) or X has a spreading model equivalent to the ℓ_1-basis (resp. trivial type).

Proof. By Corollary 4.3 of [Ka1], any reflexive Banach space has Property $\mathcal{Q}$. By Theorem 4.5 of [Ka1], a Banach space with Property $\mathcal{Q}$ must be either reflexive or have a spreading model equivalent to the ℓ_1-basis (in particular, have nontrivial type). Therefore, if Y is reflexive, the result now follows from Theorem 4.4.2.

For an index set I and an ultrafilter $\mathcal{U}$ on I, denote by $X^I/\mathcal{U}$ the ultrapower of X with respect to $\mathcal{U}$. Say Y is super-reflexive. In particular, by Corollary 4.3 of [Ka1], every ultrapower of Y has Property $\mathcal{Q}$. If X maps into Y by a coarse and solvent map, then $X^I/\mathcal{U}$ maps into $Y^I/\mathcal{U}$ by a coarse and solvent map. Therefore, it follows from Theorem 4.4.2 that every ultrapower of X has Property Q. Suppose X has nontrivial type. Then, all ultrapowers of X have nontrivial type. Therefore, by Theorem 4.5 of [Ka1], we conclude that all ultrapowers of X are reflexive. Hence, item (i) follows.

Similarly, if there exists $\varphi : B_X \to Y$ as in item (ii), then the unit balls of ultrapowers of X are mapped into ultrapowers of Y by maps with the same properties as φ, and item (ii) follows. □

Proof of Theorem 1.4.7. Item (ii) of Theorem 1.4.7 follows directly from item (ii) of Theorem 4.4.3. □

Remark 4.4.4. The statement in Theorem 1.4.7 cannot be improved so that if X embeds into a super-reflexive space, then X is either super-reflexive or it has an ℓ_1-spreading model. Indeed, it was proven in Proposition 3.1 of [NS] that $\ell_2(\ell_1)$ strongly embeds into L_p, for all $p \geqslant 4$. As $(\oplus_n \ell_1^n)_{\ell_2} \subset \ell_2(\ell_1)$, it follows that $(\oplus_n \ell_1^n)_{\ell_2}$ strongly embeds into L_4. However $(\oplus_n \ell_1^n)_{\ell_2}$ is neither super-reflexive nor contains an ℓ_1-spreading model.

4.5 Embeddings into Hilbert spaces.

In [Ran], N. Randrianarivony showed that a Banach space X coarsely embeds into a Hilbert space if and only if it uniformly embeds into a Hilbert space. This result together with Theorem 5 of [No2], gives a positive answer to Problem 1.2.6 for $Y = \ell_p$, for $p \in [1, 2]$.

In this section, we show that Problem 1.4.1 also has a positive answer if Y is ℓ_p, for any $p \in [1,2]$.

First, let us prove a simple lemma. For $\delta > 0$, a subset S of a metric space (M,d) is called *δ-dense* if $d(x,S) < \delta$, for all $x \in M$.

Lemma 4.5.1. *Let (M,d) and (N,∂) be Banach spaces and $S \subset M$ be a δ-dense set, for some $\delta > 0$. Let $f : M \to N$ be a coarse map such that $f\restriction S$ is solvent. Then f is solvent.*

Proof. Let $n \in \mathbb{N}$. As $f\restriction S$ is solvent and $\omega_f(\delta) < \infty$, we can pick $R > 0$ such that

$$d(x,y) \in [R-2\delta, R+n+2\delta] \quad \text{implies} \quad \partial(f(x),f(y)) > n + 2\omega_f(\delta),$$

for all $x,y \in S$. Pick $x,y \in X$, with $d(x,y) \in [R,R+n]$. As S is δ-dense, we can pick $x',y' \in S$ such that $d(x,x') \leqslant \delta$ and $d(y,y') \leqslant \delta$. Hence, $d(x',y') \in [R-2\delta, R+n+2\delta]$, which gives us that $\partial(f(x'),f(y')) > n + 2\omega_f(\delta)$. Therefore, we conclude that $\partial(f(x),f(y)) > n$. □

The next lemma is an adaptation of Proposition 2 of [Ran], and its proof is analogous to the proof of Theorem 1 of [JoRan]. Before stating the lemma, we need the following definition: a map $K : X \times X \to \mathbb{R}$ is called a *negative definite kernel* (resp. *positive definite kernel*) if

(i) $K(x,y) = K(y,x)$, for all $x,y \in X$, and

(ii) $\sum_{i,j} K(x_i,x_j)c_ic_j \leqslant 0$ (resp. $\sum_{i,j} K(x_i,x_j)c_ic_j \geqslant 0$), for all $n \in \mathbb{N}$, all $x_1,\ldots,x_n \in X$, and all $c_1,\ldots,c_n \in \mathbb{R}$, with $\sum_i c_i = 0$.

A function $f : X \to R$ is called *negative definite* (resp. *positive definite*) if $K(x,y) = f(x-y)$ is a negative definite kernel (resp. positive definite kernel).

Lemma 4.5.2. *Let X be a Banach space and assume that X maps into a Hilbert space by a map which is coarse and solvent. Then there exist $\alpha > 0$, a map $\overline{\rho} : [0,\infty) \to [0,\infty)$, with $\limsup_{t\to\infty} \overline{\rho}(t) = \infty$, and a continuous negative definite function $g : X \to \mathbb{R}$ such that* *(i)* $g(0) = 0$, *and*

(ii) $\overline{\rho}(\|x\|) \leqslant g(x) \leqslant \|x\|^{2\alpha}$, *for all* $x \in X$.

Sketch of the Proof of Lemma 4.5.2. Let H be a Hilbert space and consider a coarse solvent map $f : X \to H$. Without loss of generality, we may assume that $\|f(x) - f(y)\| \leqslant \|x - y\|$, for all $x, y \in X$, with $\|x - y\| \geqslant 1$.

Claim 1: Let $\alpha \in (0, 1/2)$. Then X maps into a Hilbert space by a map which is α-Hölder and solvent.

As H is Hilbert, the assignment $(x, y) \mapsto \|f(x) - f(y)\|^2$ is a negative definite kernel on X (this is a simple computation and it is contained in the proof of Proposition 3.1 of [No1]). Hence, for all $\alpha \in (0, 1)$, the kernel $N(x, y) = \|f(x) - f(y)\|^{2\alpha}$ is also negative definite (see [No1], Lemma 4.2). So, there exists a Hilbert space H_α and a map $f_\alpha : X \to H_\alpha$ such that $N(x, y) = \|f_\alpha(x) - f_\alpha(y)\|^2$, for all $x, y \in X$ (see [No1], Theorem 2.3(2)). This gives us that

$$\big(\overline{\rho}_f(\|x - y\|)\big)^\alpha \leqslant \|f_\alpha(x) - f_\alpha(y)\| \leqslant \|x - y\|^\alpha,$$

for all $x, y \in X$, with $\|x - y\| \geqslant 1$. In particular, f_α is solvent. Hence, if $N \subset X$ is a 1-net (i.e., a maximal 1-separated set), the restriction $f_{\alpha|N} : N \to H_\alpha$ is α-Hölder and solvent. Using that $\alpha \in (0, 1/2)$, Theorem 19.1 of [WWi] gives us that there exists an α-Hölder map $F_\alpha : X \to H_\alpha$ extending $f_{\alpha|N}$. By Lemma 4.5.1, F_α is also solvent. This finishes the proof of Claim 1.

By Claim 1 above, we can assume that $f : X \to H$ is an α-Hölder solvent map, with $\alpha \in (0, 1/2)$. Set $N(x, y) = \|f(x) - f(y)\|^2$, for all $x, y \in X$. So, N satisfies

$$\big(\overline{\rho}_f(\|x - y\|)\big)^2 \leqslant N(x, y) \leqslant \|x - y\|^{2\alpha}, \tag{4.5.1}$$

for all $x, y \in X$. Let μ be an invariant mean on the bounded functions $X \to \mathbb{R}$ (see [BenLi], Appendix C, for the definition of an invariant mean, and [BenLi], Theorem C.1, for the existence of such invariant mean), and define

$$g(x) = \int_X N(y + x, y) d\mu(y), \quad \text{for all} \quad x \in X.$$

Let $\overline{\rho}(t) = (\overline{\rho}_f(t))^2$, for all $t \geqslant 0$. As $\int_X 1 d\mu = 1$, Inequality 4.5.1 gives us that items (i) and (ii) are satisfied. As f is solvent, we also have that $\limsup_{t\to\infty} \overline{\rho}(t) = \infty$. The proof that g is a negative definite kernel is contained in Step 2 of [JoRan] and the proof that g is continuous is contained in Step 3 of [JoRan]. As both proofs are simple computations, we omit them here. □

We can now prove the main theorem of this section. For that, given a probability space $(\Omega, \mathcal{A}, \mu)$, we denote by $L_0(\mu)$ the space of all measurable functions $\Omega \to \mathbb{C}$ with metric determined by convergence in probability.

Theorem 4.5.3. *Let X be a Banach space. Then the following are equivalent.*

(i) X coarsely embeds into a Hilbert space.

(ii) X uniformly embeds into a Hilbert space.

(iii) X strongly embeds into a Hilbert space.

(iv) X maps into a Hilbert space by a map which is coarse and solvent.

(v) X maps into a Hilbert space by a map which is uniformly continuous and almost uncollapsed.

(vi) There is a probability space $(\Omega, \mathcal{A}, \mu)$ such that X is linearly isomorphic to a subspace of $L_0(\mu)$.

Proof. We only need to show that (iv) implies (vi). Indeed, the equivalence between (i), (ii), and (vi) were established in [Ran], Theorem 1 (see the paragraph preceeding Theorem 1 of [Ran] as well). By [Ro4], Theorem 2, if X uniformly embeds into a Hilbert space H then X

strongly embeds into $\ell_2(H)$. Hence, (ii) and (iii) are also equivalent. Using Proposition 4.1.2 with $\mathcal{E}$ being the standard basis of ℓ_2, we get that (v) implies (iv). Hence, once we show that (iv) implies (vi), all the equivalences will be established.

Let H be a Hilbert space and $f : X \to H$ be a coarse solvent map. Let $\alpha > 0$, $\overline{\rho}$ and $g : X \to \mathbb{R}$ be given by Lemma 4.5.2. Define $F(x) = e^{-g(x)}$, for all $x \in X$. So, F is a positive definite function (see [No1], Theorem 2.2). As F is also continuous, by Lemma 4.2 of [AMauMi] applied to F, there exist a probability space $(\Omega, \mathcal{A}, \mu)$ and a continuous linear operator $U : X \to L_0(\mu)$ such that

$$F(tx) = \int_\Omega e^{itU(x)(w)} d\mu(w), \quad \text{for all} \quad t \in \mathbb{R}, \quad \text{and all} \quad x \in X.$$

As U is continuous, we only need to show that U is injective and its inverse is continuous. Suppose false. Then there exists a sequence $(x_n)_{n=1}^\infty$ in the unit sphere of X such that $\lim_n U(x_n) = 0$. By the definition of convergence in $L_0(\mu)$, this gives us that $\lim_n F(tx_n) = 1$, for all $t \in \mathbb{R}$. As $\limsup_{t\to\infty} \overline{\rho}(t) = \infty$, we can pick $t_0 > 0$ such that $e^{-\overline{\rho}(t_0)} < 1/2$. Hence, we have that

$$F(t_0 x_n) = e^{-g(t_0 x_n)} \leqslant e^{-\overline{\rho}(\|t_0 x_n\|)} = e^{-\overline{\rho}(t_0)} < \frac{1}{2}, \quad \text{for all} \quad n \in \mathbb{N}.$$

As $\lim_n F(t_0 x_n) = 1$, this gives us a contradiction. □

Proof of Theorem 1.4.8. This is a trivial consequence of Theorem 4.5.3 and the equivalence between coarse and uniform embeddability into ℓ_p, for $p \in [1, 2]$ (see [No2], Theorem 5). □

4.6 Embeddings into ℓ_∞.

Kalton proved in [Ka3], Theorem 5.3, that uniform embeddability into ℓ_∞, coarse embeddability into ℓ_∞ and Lipschitz embeddability into ℓ_∞ are all equivalent. In this section, we show that Problem 1.4.1 also has a positive answer if $Y = \ell_\infty$.

The following lemma is Lemma 5.2 of [Ka3]. Although in [Ka3] the hypothesis on the map are stronger, this is not used in their proof.

Lemma 4.6.1. *Let X be a Banach space and assume that there exists a Lipschitz map $X \to \ell_\infty$ that is also almost uncollapsed. Then X Lipschitz embeds into ℓ_∞.*

Proof. Let $f : X \to \ell_\infty$ be a Lipschitz almost uncollapsed map. Pick $t > 0$ such that $\overline{\rho}_f(t) > 0$. Define a map $F : X \to \ell_\infty(\mathbb{Q}_+ \times \mathbb{N})$ by setting $F(x)(q,n) = q^{-1} f(qx)_n$, for all $x \in X$, and all $(q,n) \in \mathbb{Q}_+ \times \mathbb{N}$. Then

$$\|F(x) - F(y)\| = \sup_{(q,n)\in\mathbb{Q}_+\times\mathbb{N}} q^{-1}\big|f(qx)_n - f(qy)_n\big| \leqslant \mathrm{Lip}(f)\cdot\|x-y\|.$$

So, F is also Lipschitz. Now notice that, as f is continuous, we have that

$$\|F(x) - F(y)\| = \sup_{q>0} q^{-1}\|f(qx) - f(qy)\|.$$

Hence, if $x \neq y$, by letting $q = t\|x-y\|^{-1}$, we obtain that

$$\|F(x) - F(y)\| \geqslant \frac{\|x-y\|}{t}\cdot\Big\|f\Big(\frac{tx}{\|x-y\|}\Big) - f\Big(\frac{ty}{\|x-y\|}\Big)\Big\| \geqslant \frac{\overline{\rho}_f(t)}{t}\cdot\|x-y\|.$$

So, F is a Lipschitz embedding. □

Proof of Theorem 1.4.9. By Theorem 5.3 of [Ka3], items (i), (ii) and (iii) of Problem 1.2.6 are all equivalent. Using Proposition 4.1.2 with $\mathcal{E}$ being the standard basis of c_0, we have that item (v) of Problem 1.4.1 implies item (iv) of Problem 1.4.1. Hence, we only need to show that item (iv) of Problem 1.4.1 implies that X Lipschitz embeds into ℓ_∞. For that, let $f : X \to \ell_\infty$ be a coarse solvent map. Without loss of generality, we may assume that $\|f(x) - f(y)\| \leqslant \|x-y\|$, for all $x, y \in X$, with $\|x-y\| \geqslant 1$. Let $N \subset X$ be a 1-net. Then $f{\restriction}N$ is 1-Lipschitz and solvent. Recall that ℓ_∞ is a 1-*absolute Lipschitz retract*, i.e., every Lipschitz map $g : A \to \ell_\infty$, where M is a metric space and $A \subset M$, has a $\mathrm{Lip}(g)$-Lipschitz extension (see [Ka2], Subsection 3.3). Let F be a Lipschitz extension of $f{\restriction}N$. By Lemma 4.5.1, F is solvent. Hence, by Lemma 4.6.1, it follows that X Lipschitz embeds into ℓ_∞. □

4.7 Open questions.

Besides Problem 1.2.6 and Problem 1.4.1, there are many other interesting questions regarding those weaker kinds of embeddings. We mention a couple of them in this section.

Raynaud proved in [Ray] (see the corollary in page 34 of [Ray]) that if a Banach space X uniformly embeds into a superstable space (see [Ray] for definitions), then X must contain an ℓ_p, for some $p \in [1,\infty)$. Hence, in the context of those weaker embeddings, it is natural to ask the following.

Problem 4.7.1. Say an infinite dimensional Banach space X maps into a superstable space by a map which is both uniformly continuous and almost uncollapsed. Does it follow that X must contain ℓ_p, for some $p \in [1,\infty)$.

We refer to Problem 5.8.6 below for a similar question.

The properties of a map being solvent (resp. almost uncollapsed) are not necessarily stable under Lipschitz isomorphisms. Hence, the following question seems to be really important for the theory of solvent (resp. almost uncollapsed) maps between Banach spaces.

Problem 4.7.2. Assume that there is no coarse solvent (resp. uniformly continuous almost uncollapsed) map $X \to Y$. Is this also true for any renorming of X?

At last, we would like to notice that we have no results for maps $X \to Y$ which are coarse and almost uncollapsed. Hence, we ask the following.

Problem 4.7.3. What can we say if X maps into Y by a map which is coarse and almost uncollapsed map? Is this enough to obtain any restriction in the geometries of X and Y?

Chapter 5

Coarse embeddings into superstable Banach spaces.

In this chapter, we study nonlinear embeddability into superstable spaces. The goal of this chapter is to show that if a Banach space X coarsely embeds into a superstable Banach space, then X must contain an ℓ_p-spreading model, for some $p \in [1,\infty)$. For that, we will go over the results contained in Section 1.5, which are in the paper *Coarse embedings into superstable spaces* (see [BrSw]).

5.1 Preliminaries.

Given a Banach space X, we define stability and superstability as in Section 1.5. By Theorem II.1 of [KrMau] and Theorem 0.1 of [Ray], both stability and superstability are closed under taking ℓ_p-sums, for $p \in [1,\infty)$. Precisely, given $p \in [1,\infty)$, if X is stable (resp. superstable), then $\ell_p(X)$ is also stable (resp. superstable). We will be using this property without mentioning throughout this chapter.

We say that (M,d) is a *pseudometric space* if $d : X \times X \to \mathbb{R}_+$ is a *pseudometric*, i.e., if d is symmetric map satisfying the triangular inequality. Given pseudometric spaces (M,d), (N,∂) and a map $f : M \to N$, we define ω_f and ρ_f by the formulas given in Equation

1.1.1 and Equation 1.1.2, and define uniform and coarse embeddings, and solvent and almost uncollapsed maps analogously as in Chapter 1.

5.2 Baire class 1 functions.

Let X and Y be metrizable topological spaces. A function $f\colon X \to Y$ is called *Baire class 1* if the inverse image of any open subset of Y under f is an F_σ subset of X. If Y is separable, then the set of continuity for f is a comeager G_δ subset of X. If Y is separable and $(f_n\colon X \to Y)_{n=1}^\infty$ is a sequence of Baire class 1 functions, then $(f_n)_{n=1}^\infty\colon X \to Y^{\mathbb{N}}$ is a Baire class 1 function. The pointwise limit of a sequence of continuous functions from X to Y is a Baire class 1 function. The restriction of a Baire class 1 function is a Baire class 1 function. For proofs of these facts and more information about Baire class 1 functions, see [Ke] and [Kur].

Lemma 5.2.1. *Let X be a metrizable σ-compact topological space, Y a topological space, and let $f\colon X \times Y \to \mathbb{R}$ be separately continuous. Given a metric d inducing the topology of X and a countable family $\mathcal{K}$ of compact subsets of X such that $X = \bigcup_{K\in\mathcal{K}} K$; if there is $\delta > 0$ such that for each $x \in X$, $B_\delta(x) \cap K \neq \varnothing$ for only finitely many $K \in \mathcal{K}$, then f is the pointwise limit of a sequence of continuous functions.*

Proof. For each $n \in \mathbb{N}$, let $\{x_{n,i}\}_{i=1}^\infty$ be a $\frac{\delta}{2(n+1)}$-dense set in (X, d) such that $\left|\{x_{n,i}\}_{i=1}^\infty \cap K\right| < \infty$ for every $K \in \mathcal{K}$. For each $n, i \in \mathbb{N}$, define $g_{n,i}\colon X \to \mathbb{R}_+$ by

$$g_{n,i}(x) = \max\left\{\frac{\delta}{n+1} - d\left(x_{n,i}, x\right), 0\right\}$$

for every $x \in X$. Note that $g_{n,i}$ is continuous and given $x \in X$, $g_{n,i}\restriction_{B_{\delta/2}(x)}$ is a nonzero function for some but only finitely many $i \in \mathbb{N}$. Thus the function $h_{n,i} := \frac{g_{n,i}}{\sum_{j=1}^\infty g_{n,j}}$ is well-defined and continuous. For each $n \in \mathbb{N}$, define $f_n\colon X \times Y \to \mathbb{R}$ by

$$f_n(x,y) = \sum_{i=1}^{\infty} f\left(x_{n,i}, y\right) h_{n,i}(x)$$

for every $(x,y) \in X \times Y$ and note that f_n is itself continuous by the separate continuity of f and the observation on $g_{n,i}\restriction_{B_{\delta/2}(x)}$. The sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to f. Indeed, take any $(x,y) \in X \times Y$ and any $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that $|f(x,y) - f(x',y)| < \varepsilon$ when $d(x,x') < \frac{\delta}{N}$. Then, for $n \geqslant N$,

$$\begin{aligned}
|f(x,y) - f_n(x,y)| &= \left|\sum_{i=1}^{\infty} \left(f(x,y) - f\left(x_{n,i}, y\right)\right) h_{n,i}(x)\right| \\
&\leqslant \sum_{i=1}^{\infty} \left|f(x,y) - f\left(x_{n,i}, y\right)\right| h_{n,i}(x) \\
&< \varepsilon \cdot \sum_{i=1}^{\infty} h_{n,i}(x) \\
&= \varepsilon.
\end{aligned}$$

□

Given a set X and a family of functions $\mathcal{F}$ from $X \times X$ to X, define the sequence of subsets $\left(\mathcal{F}^{[k]}\right)_{k=1}^{\infty}$ of X^X recursively by

$$\begin{aligned}
\mathcal{F}^{[0]} &= \{x \mapsto x\} \\
\mathcal{F}^{[k+1]} &= \left\{x \mapsto f(x, g(x)) \mid f \in \mathcal{F}, g \in \mathcal{F}^{[k]}\right\}.
\end{aligned}$$

The following lemma will give us Lemma 5.5.5 below, which is essential for the proof of Theorem 1.7.3.

Lemma 5.2.2. *Let X be a separable metric space and $\mathcal{F}$ a countable family of Baire class 1 functions from $X \times X$ to X. There is a comeager G_δ subset E of X such that g is continuous on E for all $g \in \bigcup_{k=1}^{\infty} \mathcal{F}^{[k]}$.*

Proof. Certainly, g is continuous on $E_0 = X$ for $g \in \mathcal{F}^{[0]}$. Suppose $k \in \mathbb{N}_0$ is such that there is a comeager G_δ subset E_k of X such that g is continuous on E_k for all $g \in \mathcal{F}^{[k]}$. For each $g \in \mathcal{F}^{[k]}$, let $\Gamma_g = \{(x, g(x)) \mid x \in E_k\}$. Since $\mathcal{F}$ is a countable family of Baire class 1 functions with separable codomain X, there is a comeager G_δ subset F_g of Γ_g such that $f\restriction_{\Gamma_g}$ is continuous on F_g for all $f \in \mathcal{F}$. Let $\pi\colon X \times X \to X$ be the first coordinate projection. Consider $U = \Gamma_g \cap V \times W$, where V, W are open subsets of X; and suppose $x \in \pi(U)$, so that $(x, g(x)) \in U$. As W is open and $g(x) \in W$, there is $r_1 > 0$ such that $B_{r_1}(g(x)) \subseteq W$. Since g is continuous on E_k, there is $r_2 > 0$ such that $g(B_{r_2}(x) \cap E_k) \subseteq B_{r_1}(g(x))$. Thus $(V \cap B_{r_2}(x)) \cap E_k$ is an open neighborhood of x in E_k contained in $\pi(U)$. Since $x \in \pi(U)$ was arbitrary, $\pi(U)$ is open in E_k. And U was an arbitrary element in a basis for the topology on Γ_g, so $\pi(U)$ is open in E_k whenever U is open in Γ_g. It follows easily that $\pi(F_g)$ is a comeager G_δ subset of E_k since F_g is a comeager G_δ subset of Γ_g. Let $E_{k+1} = \bigcap_{g \in \mathcal{F}^{[k]}} \pi(F_g)$. Since $\mathcal{F}^{[k]}$ is countable, E_{k+1} is a comeager G_δ subset of E_k, and therefore also of X, since E_k is a comeager G_δ subset of X. Now take any $g \in \mathcal{F}^{[k+1]}$. Then there is $f \in \mathcal{F}$ and $g' \in \mathcal{F}^{[k]}$ such that $g(x) = f(x, g'(x))$ for all $x \in X$. And if $x \in E_{k+1}$, then by construction x is a point of continuity for g' and $(x, g'(x))$ is a point of continuity for $f\restriction_{\Gamma_{g'}}$. Therefore x is a point of continuity for g. Thus, we have constructed a comeager G_δ subset E_{k+1} of E_k such that g is continuous on E_{k+1} for all $g \in \mathcal{F}^{[k+1]}$. And so we can recursively define such E_k for all $k \in \mathbb{N}$. The result follows by taking $E = \bigcap_{k=0}^{\infty} E_k$. $\square$

5.3 Making coarse maps "invariant".

In this section, we use Markov-Kakutani's fixed-point theorem in order to show that coarse embeddings may be modified and made more "tamed" if we allow ourselves to substitute its codomain by an ultrapower of the ℓ_1-sum of the original space. Precisely, we have the following.

Theorem 5.3.1. *Let X and Y be Banach spaces and $f : X \to Y$ a coarse map. Then there* *exists an ultrafilter* $\mathcal{U}$ *on an index set* I*, and a map* $F : X \to \ell_1(Y)^I/\mathcal{U}$*, such that*

$$\rho_f(\|x-y\|) \leqslant \|F(x)-F(y)\| \leqslant \omega_f(\|x-y\|),$$

and

$$\|F(x)-F(y)\| = \|F(x-y)\|, \quad \textit{for all} \quad x, y \in X.$$

Proof. Define $C \subseteq \mathbb{R}^{X\times X}$ by letting $D \in C$ if and only if

$$\rho_f(\|x-y\|) \leqslant D(x,y) \leqslant \omega_f(\|x-y\|),$$

for all $x, y \in X$. So, C is relatively compact. Indeed, one only needs to notice that

$$C \subseteq \prod_{(x,y)\in X\times X} [0, \omega_f(\|x-y\|)].$$

Hence, as f is coarse, C is relatively compact. Let $d : X \times X \to \mathbb{R}$ be given by $d(x,y) = \|f(x)-f(y)\|$, for all $x, y \in X$. So, $d \in C$.

For each $z \in X$, define $\hat{z}\colon \mathbb{R}^{X\times X} \to \mathbb{R}^{X\times X}$ by letting $\hat{z}(g)(x,y) = g(x+z, y+z)$ for all $g \in \mathbb{R}^{X\times X}$ and all $x, y \in X$. Let $A = \overline{\mathrm{conv}}\{\hat{z}(d) \mid z \in X\} \subseteq \mathbb{R}^{X\times X}$. By the definition of the pointwise convergence topology on $\mathbb{R}^{X\times X}$, we have that $A \subseteq C$. The family $\{\hat{z}\restriction_A\}_{z\in X}$ is easily seen to be a commuting family of continuous, affine self-mappings of the compact convex subset A of $\mathbb{R}^{X\times X}$. Hence, by Markov-Kakutani's fixed-point theorem, there exists $D \in A$ such that $\hat{z}(D) = D$ for all $z \in X$. That is, $D(x+z, y+z) = D(x,y)$ for all $x, y, z \in X$. Say $D = \lim_{i\in\mathcal{U}} D_i$, where I is an index set, $\mathcal{U}$ is some nonprincipal ultrafilter on I, and $D_i \in \mathrm{conv}\{\hat{z}(d) \mid z \in X\}$, for all $i \in I$. For each $i \in I$, we have that $D_i = \sum_{j=1}^{s(i)} \alpha_{i,j}\hat{z}_{i,j}(d)$, for some finite sequence $(\alpha_{i,j})_{j=1}^{s(i)}$ of non negative real numbers such that $\sum_{j=1}^{s(i)} \alpha_{i,j} = 1$, and some finite sequence $(z_{i,j})_{j=1}^{s(i)}$ in X.

For $y_1, \ldots, y_n \in Y$, we denote $(y_1, \ldots, y_n, 0, 0, \ldots) \in \ell_1(Y)$ by $\oplus_{j=1}^n y_j$. Consider the map

$$\begin{aligned} F = (F_i)_{i \in I} : X &\longrightarrow \ell_1(Y)^I/\mathcal{U} \\ x &\longmapsto \Big(\bigoplus_{j=1}^{s(i)} \alpha_{i,j} \big(f(x + z_{i,j}) - f(z_{i,j}) \big) \Big)_{i \in I}. \end{aligned}$$

As $\sup_{i \in I} \|F_i(x)\|_{\ell_1(Y)} \leqslant \omega_f(\|x\|)$, for all $x \in X$, the map F is well-defined. By the definition of the norm on $\ell_1(Y)^I/\mathcal{U}$, we have that

$$\|F(x) - F(y)\|_{\ell_1(Y)^I/\mathcal{U}} = D(x, y),$$

for all $x, y \in X$. Therefore, as $D(x, y) = D(x - y, 0)$, for all $x, y \in X$, and $F(0) = 0$, we are done. $\square$

Corollary 5.3.2. *Let $(X, \|\cdot\|)$ be a Banach space. If X is coarsely embeddable into a superstable Banach space, then there exists an invariant stable pseudometric d on X such that the identity map* $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ *is a coarse equivalence.*

Proof. Suppose Y is a superstable Banach space and $f\colon X \to Y$ is a coarse embedding. Let $F\colon X \to \ell_1(Y)^{\mathcal{I}}/\mathcal{U}$ be obtain from Theorem 5.3.1 applied to f. The map $d\colon X \times X \to \mathbb{R}_+$ defined by $d(x, y) = \|F(x) - F(y)\|$ for all $x, y \in X$ can easily be seen to be an invariant pseudometric on X, and the stability of d follows from the stability of $\ell_1(Y)^{\mathcal{I}}/\mathcal{U}$. Finally, $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is a coarse equivalence since $\rho_{\mathrm{Id}} = \rho_f$ and $\omega_{\mathrm{Id}} = \omega_f$, by the definitions of F and d. $\square$

Remark. Although this will not be needed for the main result in these notes, Corollary 5.3.2 can actually be improved to show the existence of a coarsely equivalent invariant stable *metric* on X. Indeed, by Theorem 1.3.4 (see [Br2], Theorem 1.6) if X and Y are Banach spaces and $f\colon X \to Y$ is a coarse embedding, then there is a coarse embedding $\hat{f}\colon X \to \ell_1(Y)$ with uniformly continuous inverse (meaning $\rho_{\hat{f}}(t) > 0$ whenever $t > 0$). Raynaud has shown (see [Ray], Theorem 0.1) that the ℓ_p-sum of a superstable space is again superstable, and so the same proof as in Corollary 5.3.2 with $\ell_1(Y)$ replacing Y and $\hat{f}$ replacing f will yield that $\mathrm{Id} : (X, \|\cdot\|) \to (X, d)$ is a coarse embedding with uniformly continuous inverse. In particular, d is a metric.

In the remaining of this section, we use Theorem 5.3.1 to prove a result on the uniform embeddability of the ball of a given Banach space into a superstable space (Theorem 5.3.3).

Theorem 5.3.3. *If a Banach space X maps into a superstable space by a map which is both uniformly continuous and almost uncollapsed, then B_X uniformly embeds into a superstable space.*

Before proving Theorem 5.3.3, we need the following proposition.

Proposition 5.3.4. *Let X and Y be Banach spaces and $f : X \to Y$ be a solvent map such that $\|f(x)-f(y)\| = \|f(x-y)\|$ for all $x, y \in X$. Then, for every norm bounded subset $B \subseteq X$, $f\restriction_B$ has a Lipschitz inverse.*

Proof. First notice that,

$$\|f(x)\| = \|f(x) - f(0)\| = \|f(0) - f(x)\| = \|f(-x)\|,$$

for all $x \in X$. Therefore,

$$\|f(mx)\| = \|f((m-1)x) - f(-x)\| \leqslant \|f((m-1)x)\| + \|f(x)\|,$$

for all $x \in X$, and all $m \in \mathbb{N}$. So, $\|f(mx)\| \leqslant m \cdot \|f(x)\|$, for all $x \in X$, and all $m \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $B \subseteq N \cdot B_X$. As f is solvent, we can find $n, R > 2N$ such that $\|x\| \in [R, R+n]$ implies $\|f(x)\| > n$. By our choice of n and R, for each $x \in 2N \cdot B_X$ we can pick $m_x \in \mathbb{N}$ such that $\|m_x x\| \in [R, R+n]$. Hence,

$$\|f(x)\| > \frac{n}{m_x} \geqslant \frac{n}{R+n}\|x\|,$$

for all $x \in 2N \cdot B_X$. This gives us that $\|f(x) - f(y)\| \geqslant \frac{n}{R+n}\|x - y\|$, for all $x, y \in B$. $\square$

Proof of Theorem 5.3.3. If X maps into a superstable space by a uniformly continuous almost uncollapsed map, then, by Proposition 63 of [Ro4], X maps into a superstable space by a map which is both uniformly continuous and solvent. Notice that Theorem 5.3.2 remains valid replacing ρ_f by $\overline{\rho}_f$. Indeed, the exact same proof remains valid replacing ρ_f by $\overline{\rho}_f$. Therefore, X maps into a superstable space Y by a uniformly continuous solvent map F such that $\|F(x) - F(y)\| = \|F(x - y)\|$, for all $x, y \in X$. By Proposition 5.3.4, $F\restriction_{B_X}$ has a Lipschitz inverse. In particular, B_X uniformly embeds into a superstable space. $\square$

5.4 Type space.

From now on, we consider a separable infinite dimensional Banach space $(X, \|\cdot\|)$ which admits an invariant stable pseudometric d coarsely equivalent to $\|\cdot\|$, and the corresponding identity map $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$. By Corollary 5.3.2, such d exists as long as X coarsely embeds into a superstable space.

Remark 5.4.1. Notice that, by Remark 5.3, we can actually assume that d is a metric. However, in order to obtain the isomorphism constant in Remark 5.8 below, we need to work with d being the pseudometric given by Corollary 5.3.2.

Let Δ be a countable $\|\cdot\|$-dense $\mathbb{Q}$-vector subspace of X. Given $x \in \Delta$, define the function $\overline{x} \in \mathbb{R}_+^{\mathbb{Q}\times\Delta}$ by $\overline{x}(\lambda, y) = d(\lambda x, y)$ for all $(\lambda, y) \in \mathbb{Q} \times \Delta$. The *space of types* on $(\Delta, d\restriction_{\Delta\times\Delta})$, which we denote by $\mathcal{T}$, is defined to be the closure of $\{\overline{x}\}_{x\in\Delta}$ in $\mathbb{R}^{\mathbb{Q}\times\Delta}$ (with the topology of pointwise convergence). An element σ of $\mathcal{T}$ is called a *type*, and is called a *realized type* if $\sigma = \overline{x}$ for some $x \in \Delta$, in which case σ is also called the *type realized by x*. The type $\overline{0}$ is called the *null* or *trivial* type.

Note that the countability of $\mathbb{Q} \times \Delta$ implies that $\mathcal{T}$ is metrizable, and so every $\sigma \in \mathcal{T}$ can be expressed as $\lim_{n\to\infty} \overline{x}_n$ for some sequence $(x_n)_{n=1}^{\infty}$ in Δ. Such a sequence is called a *defining sequence* for σ. Note also that in this case $\sigma(\lambda, x) = \lim_{n,\mathcal{U}} d(\lambda x_n, x)$ for every

$(\lambda, x) \in \mathbb{Q} \times \Delta$ and every nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{N}$. In particular, $\lim_{n\to\infty} d(x_n, 0)$ exists, and so $(x_n)_{n=1}^{\infty}$ is a d-bounded (and therefore also $\|\cdot\|$-bounded) sequence in Δ.

For every $M \in \mathbb{R}_+$, we let $\mathcal{T}_M = \{\sigma \in \mathcal{T} \mid \sigma(1,0) \leqslant M\}$. We will need the following lemma.

Lemma 5.4.2. *Say $M \in \mathbb{R}_+$. Then $\mathcal{T}_M$ is compact.*

Proof. Say $\sigma \in \mathcal{T}_M$, and $(x_n)_{n=1}^{\infty}$ is a defining sequence for σ. As $\lim_{n\to\infty} d(x_n, 0) = \sigma(1, 0) \leqslant M$, we may suppose that the defining sequence for σ is contained in the d-ball of radius $M+1$ around 0. As $\mathrm{Id}\colon (X, \|\cdot\|) \to (X, d)$ is expanding, there exists $R < \infty$ such that $t \leqslant R$ whenever $\rho_{\mathrm{Id}}(t) \leqslant M+1$. Then, since $\rho_{\mathrm{Id}}(\|x_n\|) \leqslant d(x_n, 0) \leqslant M+1$ for every $n \in \mathbb{N}$, we have

$$\sigma(\lambda, x) = \lim_n d(\lambda x_n, x) \leqslant \lim_n (d(\lambda x_n, 0) + d(0, x)) \leqslant \omega_{\mathrm{Id}}(|\lambda| R) + d(0, x)$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. That is, we have

$$\mathcal{T}_M \subseteq \prod_{(\lambda,x)\in\mathbb{Q}\times\Delta} [0, \omega(|\lambda|R) + d(x, 0)],$$

since $\sigma \in \mathcal{T}_M$ was arbitrary. By Tychonoff's theorem and the fact that $\mathcal{T}_M$ is closed, we are finished. $\square$

Corollary 5.4.3. *The metric space $\mathcal{T}$ is σ-locally compact.*

Lemma 5.4.4. *Suppose $\sigma, \tau \in \mathcal{T}$. Then if $(w_n)_{n=1}^{\infty}, (x_n)_{n=1}^{\infty}$ are defining sequences for σ and $(y_n)_{n=1}^{\infty}, (z_n)_{n=1}^{\infty}$ are defining sequences for τ, then*

(i) The limits $\lim_n \overline{\alpha w_n}$ and $\lim_n \overline{\alpha x_n}$ exist and are equal for every $\alpha \in \mathbb{Q}$.

(ii) The limits $\lim_n \lim_m \overline{w_n + y_m}$ and $\lim_n \lim_m \overline{x_n + z_m}$ exist and are equal.

Proof. Item (i) follows immediately from the definitions. By a straightforward argument using the invariance and stability of d, item (ii) also follows. $\square$

Definition 5.4.5. *Let $\sigma, \tau \in \mathcal{T}$ and let $(x_n)_{n=1}^\infty, (y_m)_{m=1}^\infty$ be any defining sequences for σ and τ, respectively. We define the* dilation *operation on $\mathcal{T}$ by $(\alpha, \sigma) \in \mathbb{Q} \times \mathcal{T} \mapsto \alpha \cdot \sigma \in \mathcal{T}$, where $\alpha \cdot \sigma := \lim_n \overline{\alpha x_n}$. We define the* convolution *operation on $\mathcal{T}$ by $(\sigma, \tau) \in \mathcal{T} \times \mathcal{T} \mapsto \sigma * \tau \in \mathcal{T}$, where $\sigma * \tau := \lim_n \lim_m \overline{x_n + y_m}$. By Lemma 5.4.4, both dilation and the convolution are well defined. For $(\sigma_j)_{j=1}^k \subseteq \mathcal{T}$, we define $\circledast_{j=1}^k \sigma_j$ in the obvious way, and we allow dilation to bind more strongly than convolution in our notation, i.e., we write $\alpha \cdot \sigma * \tau$ meaning $(\alpha \cdot \sigma) * \tau$.*

It follows easily from the definition above that, given $\sigma \in \mathcal{T}$ and a defining sequence $(x_n)_{n=1}^\infty$ for σ, we have $\alpha \cdot \sigma(\lambda, x) = \sigma(\lambda\alpha, x)$ for every $(\lambda, x) \in \mathbb{Q} \times \Delta$ and $\sigma * \tau = \lim_n \overline{x}_n * \tau$ for every $\tau \in \mathcal{T}$. Furthermore, using the invariance and stability of d, it is easily shown that the convolution is associative and commutative, and that dilation distributes over convolution.

Lemma 5.4.6. *Dilation is a right-continuous map from $\mathbb{Q} \times \mathcal{T}$ to $\mathcal{T}$.*

Proof. Fix $\alpha \in \mathbb{Q}$ and suppose $(\sigma_n)_{n=1}^\infty$ is a sequence in $\mathcal{T}$ converging to $\sigma \in \mathcal{T}$. Then $\alpha \cdot \sigma(\lambda, x) = \sigma(\lambda\alpha, x) = \lim_{n\to\infty} \sigma_n(\lambda\alpha, x) = \lim_{n\to\infty} \alpha \cdot \sigma_n(\lambda, x)$ for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. Thus $\alpha \cdot \sigma = \lim_{n\to\infty} \alpha \cdot \sigma_n$. This was for an arbitrary converging sequence in $\mathcal{T}$, so dilation is right continuous. ☐

Lemma 5.4.7. *Convolution is a separately continuous map from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$.*

Proof. Let D be a metric compatible with the topology on $\mathcal{T}$. Fix $\tau \in \mathcal{T}$ and suppose $(\sigma_n)_{n=1}^\infty$ is a sequence in $\mathcal{T}$ converging to $\sigma \in \mathcal{T}$. For each $n \in \mathbb{N}$, let $(x_{n,m})_{m=1}^\infty$ be a defining sequence for σ_n, and let $m_n \in \mathbb{N}$ be such that $D(\sigma_n, \overline{x}_{n,m_n}) < \frac{1}{n}$ and $D(\overline{x}_{n,m_n} * \tau, \sigma_n * \tau) < \frac{1}{n}$. Then $(x_{n,m_n})_{n=1}^\infty$ is a defining sequence for σ by the triangle inequality; and so, again by triangle inequality, $\sigma * \tau = \lim_n \sigma_n * \tau$. This was for an arbitrary converging sequence in $\mathcal{T}$, so convolution (which is commutative) is separately continuous. ☐

Corollary 5.4.8. *Convolution is a Baire class 1 map from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$.*

Proof. Given $(\lambda, x) \in \mathbb{Q} \times \Delta$, let $\Phi_{\lambda,x} \colon \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ be defined by $\Phi_{\lambda,x}(\sigma, \tau) = \sigma * \tau(\lambda, x)$ for all $\sigma, \tau \in \mathcal{T}$. Choose a compatible metric D for the topology on $\mathcal{T}$ and note that there is $\delta > 0$ such that $D(\sigma, \tau) \geqslant \delta$ whenever $|\sigma(1,0) - \tau(1,0)|$ is large enough. Now, by Lemma 5.4.7 and the topology on $\mathcal{T}$, $\Phi_{\lambda,x}$ is separately continuous; and by Lemma 5.4.2, $\mathcal{T}_M$ is compact for every $M \in \mathbb{R}_+$. Thus; applying Lemma 5.2.1 with $\mathcal{T}$ replacing both X and Y, $\Phi_{\lambda,x}$ replacing f, D replacing d, $\{\mathcal{T}_{M+1} \backslash \mathrm{int}\mathcal{T}_M\}_{M=0}^{\infty}$ replacing $\mathcal{K}$, and with δ as above in the statement of Lemma 5.2.1; we have that $\Phi_{\lambda,x}$ is the pointwise limit of a sequence of continuous functions, and is therefore Baire class 1. As this is true for any $(\lambda, x) \in \mathbb{Q} \times \Delta$, the convolution is itself Baire class 1. □

The sequence in the statement of our main theorem will be a defining sequence for one of the types in $\mathcal{T}$. We already know that a defining sequence $(x_n)_{n=1}^{\infty}$ for a type σ is bounded in norm, but we want to put a condition on σ that guarantees $(x_n)_{n=1}^{\infty}$ is eventually bounded away from zero in norm. This motivates our next definition.

Definition 5.4.9. *A type $\sigma \in \mathcal{T}$ is called* admissible *if $\sigma(1,0) > \inf_{t>0} \omega_{\mathrm{Id}}(t)$.*

Note that if σ is an admissible type and $(x_n)_{n=1}^{\infty}$ is a defining sequence for σ, then $\liminf_n \omega_{\mathrm{Id}}(\|x_n\|) \geqslant \lim_n d(x_n, 0) = \sigma(1,0) > \inf_{t>0} \omega_{\mathrm{Id}}(t)$. Thus, since ω_{Id} is an increasing function, we can find $\delta > 0$ such that $(x_n)_{n=1}^{\infty}$ is eventually δ-bounded in norm away from zero. From this point forward, we will let the Greek letter γ stand for the value $\inf_{t>0} \omega_{\mathrm{Id}}(t)$.

Remark. If $\mathrm{Id} : (X, \|\cdot\|) \to (X, d)$ is uniformly continuous, then $\gamma = 0$. If, in addition, d is a metric, then the inequality in our definition is trivial, and every nontrivial type will be admissible. Given our assumption that d is coarsely equivalent to $\|\cdot\|$, we do not need to place any additional conditions on a type to guarantee its defining sequences to be norm bounded. Had this not been the case, we would have had to include such a condition in our definition of admissibility. One condition we could use would be to require a type σ to also satisfy the inequality $\sigma(1,0) < \sup_{t<\infty} \rho_{\mathrm{Id}}(t)$ (a trivial inequality in our case). In [Ray], where the author is concerned with an invariant stable metric d *uniformly* equivalent to $\|\cdot\|$, the author does exactly this.

We have established a condition to put on a type to guarantee its defining sequences are bounded in norm and eventually bounded away from zero in norm. In our goal to obtain a basic sequence, we now need an extra condition which will guarantee that a type's defining sequences contain no norm Cauchy subsequences.

Definition 5.4.10. *We say that a type σ is* symmetric *if $\sigma = (-1) \cdot \sigma$, i.e., if $\sigma(\lambda, x) = \sigma(-\lambda, x)$, for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. Let $\mathcal{S} = \{\sigma \in \mathcal{T} \mid \sigma \text{ is symmetric}\}$ and let $\mathcal{S}_M = \mathcal{S} \cap \mathcal{T}_M$.*

Note that by Lemma 5.4.6, $\mathcal{S}$ is closed, and therefore $\mathcal{S}_M$ is compact for all $M \in \mathbb{R}_+$.

Proposition 5.4.11. *Say $\sigma \in \mathcal{T}$ is an admissible symmetric type and $(x_n)_{n=1}^\infty$ is a defining sequence for σ. Then $(x_n)_n^\infty$ has no $\|\cdot\|$-Cauchy subsequence.*

Proof. Suppose false. By taking a subsequence, we can assume that $(x_n)_{n=1}^\infty$ converges in norm to some $x \in X$. Then, as σ is symmetric, we have that

$$\begin{aligned}
&\liminf_{n\to\infty} d(\lambda x_n, -\lambda x_n) \\
&\qquad = \liminf_{n\to\infty} \Big(d(\lambda x_n, -\lambda x_n) - \sigma(\lambda, -\lambda x_n) + \sigma(-\lambda, -\lambda x_n) \Big) \\
&\qquad = \liminf_{n\to\infty} \lim_{m\to\infty} \Big(d(\lambda x_n, -\lambda x_n) - d(\lambda x_m, -\lambda x_n) + d(-\lambda x_m, -\lambda x_n) \Big) \\
&\qquad \leqslant \liminf_{n\to\infty} \lim_{m\to\infty} \Big(d(\lambda x_n, \lambda x_m) + d(-\lambda x_m, -\lambda x_n) \Big) \\
&\qquad \leqslant 2 \cdot \liminf_{n\to\infty} \liminf_{m\to\infty} \omega_{\mathrm{Id}}(|\lambda| \cdot \|x_n - x_m\|) \\
&\qquad = 2\gamma,
\end{aligned}$$

for all $\lambda \in \mathbb{Q}$. This gives us that $\rho_{\mathrm{Id}}(\|\lambda x\|) \leqslant \liminf_n \rho_{\mathrm{Id}}(2\|\lambda x_n\|) \leqslant 2\gamma$, for all $\lambda \in \mathbb{Q}$. As d is coarsely equivalent to the norm of X, this can only happen if $x = 0$. But then the admissibility of σ yields

$$\gamma < \sigma(1, 0) = \lim_{n\to\infty} d(x_n, 0) \leqslant \liminf_{n\to\infty} \omega_{\mathrm{Id}}(\|x_n\|) = \gamma,$$

a contradiction. □

5.5 Conic classes.

We will need the following definition for a minimality argument later on.

Definition 5.5.1. *A nonempty subset $\mathcal{C}$ of $\mathcal{S}$ is called a* conic class *if*

(i) $\mathcal{C} \neq \{\overline{0}\}$,

(ii) $\lambda \cdot \sigma \in \mathcal{C}$ *for all* $\lambda \in \mathbb{Q}$ *and* $\sigma \in \mathcal{C}$, *and*

(iii) $\sigma * \tau \in \mathcal{C}$ *for all* $\sigma, \tau \in \mathcal{C}$.

Moreover, $\mathcal{C}$ *is called* admissible *if* $\mathcal{C}$ *contains an admissible type, i.e., if there exists* $\sigma \in \mathcal{C}$ *such that* $\sigma(1,0) > \gamma$.

Lemma 5.5.2. *The set* $\mathcal{S}$ *is a closed admissible conic class.*

Proof. That $\mathcal{S}$ is closed follows from Lemma 5.4.6. The properties (ii) and (iii) follow easily from the definitions of dilation and convolution and from the invariance of d. All that remains is to show that there is an admissible (and therefore nontrivial) type σ in $\mathcal{S}$. Let $R < \infty$ be such that $\rho_{\mathrm{Id}}(t) > \gamma$ whenever $t \geqslant R$. By the infinite-dimensionality of X, there is a bounded R-separated sequence $(x_n)_{n=1}^{\infty}$ in $(X, \|\cdot\|)$. After possibly taking a subsequence, we may suppose that $(x_n)_{n=1}^{\infty}$ is a defining sequence for some $\sigma \in \mathcal{T}$. In this case,

$$\begin{aligned}(\sigma * (-1) \cdot \sigma)(1,0) &= \lim_n \lim_m d(x_n - x_m, 0) \\ &\geqslant \inf_{n \neq m} d(x_n - x_m, 0) \\ &\geqslant \rho_{\mathrm{Id}}(R) \\ &> \gamma\end{aligned}$$

That is, the symmetric type $\sigma * (-1) \cdot \sigma$ is admissible. Therefore $\mathcal{S}$ is a closed admissible conic class. $\square$

Lemma 5.5.3. *Let σ be an admissible type. Given any $0 \leqslant r_1 < r_2$, there is $\alpha \in \mathbb{Q}_+$ such that $\rho_{\mathrm{Id}}(r_1) \leqslant \alpha \cdot \sigma(1,0) \leqslant \omega_{\mathrm{Id}}(r_2)$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a defining sequence for σ. The admissibility of σ implies that $(x_n)_{n=1}^{\infty}$ is a $\|\cdot\|$-bounded sequence which is eventually $\|\cdot\|$-bounded away from 0. Thus, we may suppose after possibly taking a subsequence that $\lim_n \|x_n\|$ exists and is nonzero. Let $\alpha \in \mathbb{Q}_+$ be such that $\lim_n \|\alpha x_n\| \in [r_1, r_2]$. As $\alpha \cdot \sigma(1,0) = \lim_n d(\alpha x_n, 0)$, we then have

$$\rho_{\mathrm{Id}}(r_1) \leqslant \alpha \cdot \sigma(1,0) \leqslant \omega_{\mathrm{Id}}(r_2).$$

$\square$

Proposition 5.5.4. *Every closed admissible conic class contains a minimal closed admissible conic class.*

Proof. Fix a closed admissible conic class $\mathcal{C}$. Let $\mathcal{F}$ be the family of closed admissible conic classes contained in $\mathcal{C}$ ordered by reverse set inclusion and let $\{\mathcal{C}_i\}_{i \in I}$ be some chain in $\mathcal{F}$.

Claim: $\bigcap_{i \in I} \mathcal{C}_i$ is a closed admissible conic class.

Certainly, $\bigcap_{i \in I} \mathcal{C}_i \subseteq \mathcal{S}$ is closed and satisfies conditions (ii) and (iii) in the definition of conic class. So we only need to show that $\bigcap_{i \in I} \mathcal{C}_i$ contains an admissible type. For that, fix $R < \infty$ such that $\rho_{\mathrm{Id}}(t) > \gamma$ whenever $t \geqslant R$ and let $\mathcal{B}_i = C_i \cap (\mathcal{T}_{\omega_{\mathrm{Id}}(R+1)} \setminus \mathrm{int} \mathcal{T}_{\rho_{\mathrm{Id}}(R)})$ for all $i \in I$. By Lemma 5.4.2, $\mathcal{B}_i$ is compact. Given $i \in I$, let $\sigma_i \in \mathcal{C}_i$ be admissible. By the previous lemma, there is $\alpha_i \in \mathbb{Q}_+$ such that $\alpha_i \cdot \sigma_i \in \mathcal{B}_i$, so $\mathcal{B}_i$ is nonempty. Hence, $\{\mathcal{B}_i\}_{i \in I}$ is a family of compact sets with the finite intersection property, which gives us that $\bigcap_{i \in I} \mathcal{B}_i \subseteq \bigcap_{i \in I} \mathcal{C}_i$ is nonempty. By our choice of R, $\bigcap_{i \in I} \mathcal{B}_i$ can only contain admissible types, hence $\bigcap_{i \in I} \mathcal{C}_i$ contains an admissible type, and the claim is proved.

As $\bigcap_{i \in I} \mathcal{C}_i$ is a closed admissible conic class, it is an upper bound for the chain $\{\mathcal{C}_i\}_{i \in I}$ in $\mathcal{F}$.

By Zorn's lemma, $\mathcal{F}$ has a maximal element. That is, $\mathcal{C}$ contains a minimal closed admissible conic class. $\square$

Lemma 5.5.5. *Let $\mathcal{C}$ be a closed admissible conic class. Then there is an admissible $\sigma \in \mathcal{C}$ such that σ is a common point of continuity for the family of functions $\{\sigma \mapsto \ast_{j=1}^{n} \alpha_j \cdot \sigma \mid n \in \mathbb{N}, \alpha \in \mathbb{Q}^n\} \subseteq \mathcal{C}^{\mathcal{C}}$.*

Proof. By Lemma 5.2.2 and Corollary 5.4.8 (with $\mathcal{C}$ replacing X and $\{\sigma \mapsto \alpha \cdot \sigma * \beta \cdot \sigma \mid \alpha, \beta \in \mathbb{Q}\}$ replacing $\mathcal{F}$), there is a comeager G_δ subset E of $\mathcal{C}$ such that g is continuous on E for all $g \in \{\sigma \mapsto \ast_{j=1}^{n} \alpha_j \cdot \sigma \mid n \in \mathbb{N}, \alpha \in \mathbb{Q}^n\} \subseteq \mathcal{C}^{\mathcal{C}}$. But $\mathcal{C}$ is closed, and so is locally compact, by Corollary 5.4.3. Therefore E is dense in $\mathcal{C}$, by the Baire category theorem, and the statement follows by the admissibility of $\mathcal{C}$. $\square$

5.6 Model associated to an admissible symmetric type.

Let σ be an admissible symmetric type and $(x_n)_{n=1}^{\infty}$ be a defining sequence for σ. Then the sequence $(x_n)_{n=1}^{\infty}$ is bounded, and by Proposition 5.4.11, has no $\|\cdot\|$-Cauchy subsequence. Thus, given a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we may define a spreading sequence $(\zeta_n)_{n=1}^{\infty}$ and a spreading model $S = X \oplus \overline{\text{span}}\{\zeta_n \mid n \in \mathbb{N}\}$ associated to $(x_n)_{n=1}^{\infty}$ and $\mathcal{U}$ as in Section 2.4. As in Section 2.4, we let $(\xi_n)_{n=1}^{\infty}$ be given by $\xi_n = \zeta_{2n-1} - \zeta_{2n}$, for all $n \in \mathbb{N}$.

Let $\tau = \sigma * (-1) \cdot \sigma$. As $\sigma = \lim_n \overline{x}_n$, we may assume after taking a subsequence that $\tau = \lim_n \overline{y}_n$ where $y_n = x_{2n-1} - x_{2n}$. As $(x_n)_{n=1}^{\infty}$ has no $\|\cdot\|$-Cauchy subsequence, we may further assume after taking another subsequence that $\inf_{n \neq m} \|x_n - x_m\| > 0$. As $\tau(1, 0) = \lim_n d(y_n, 0) \geqslant \rho_{\mathrm{Id}}(\inf_{n \neq m} \|x_n - x_m\|)$, by dilating σ, we can also assume that τ is an admissible type. It is clear that $(\xi_n)_{n=1}^{\infty}$ is the spreading model of $(y_n)_{n=1}^{\infty}$ for the ultrafilter $\mathcal{U}$.

From this point forward, we fix a minimal closed admissible conic class $\mathcal{C}$ and an admissible $\phi \in \mathcal{C}$ that is a common point of continuity for the family of functions $\mathcal{F} = \{\sigma \mapsto \ast_{j=1}^{n} \alpha_j \cdot \sigma \mid n \in \mathbb{N}, \alpha \in \mathbb{Q}^n\} \subseteq \mathcal{C}^{\mathcal{C}}$ such that $\psi = \phi * (-1) \cdot \phi$ is admissible (this is possible by Lemma 5.5.5). We also fix a defining sequence $(x_n)_{n=1}^{\infty}$ for ϕ with unique (see Section 2.4) spreading model $(S, \|\cdot\|)$ such that $y_n = x_{2n-1} - x_{2n}$ is a defining sequence for ψ. We will let $(\zeta_n)_{n=1}^{\infty}$ be the spreading sequence associated with S and $\xi_n = \zeta_{2n-1} - \zeta_{2n}$ for every $n \in \mathbb{N}$.

Definition 5.6.1. *Given $k \in \mathbb{N}$, $\overline{\alpha} = (\alpha_i)_{i=1}^{k} \in \mathbb{Q}^k$, we say that $\sum_{j=1}^{k} \alpha_j \zeta_j$* realizes *the type $*_{j=1}^{k} \alpha_j \cdot \phi$.*

Remark 5.6.2. Notice that, if $u = \sum_{j=1}^{k_1} \alpha_j \zeta_j$ realizes σ, and $v = \sum_{j=k_1+1}^{k_2} \beta_j \zeta_j$ realizes τ, it follows that $u + v$ realizes $\sigma * \tau$.

5.6.1 Basic properties of $\|\cdot\|$.

We will now prove some technical lemmas which will be important in the proof of the main theorem of these notes.

Lemma 5.6.3. *Say $u \neq v \in \operatorname{span}_{\mathbb{Q}}\{\zeta_n \mid n \in \mathbb{N}\}$ realize σ and τ, respectively. Then for every $(\lambda, y) \in \mathbb{Q} \times \Delta$,*

$$\sup_{0<\varepsilon\leqslant|\lambda|\|u-v\|} \rho_{\mathrm{Id}}(|\lambda|\|u - v\| - \varepsilon) \leqslant \sigma(\lambda, y) + \tau(\lambda, y)$$

and

$$|\sigma(\lambda, y) - \tau(\lambda, y)| \leqslant \inf_{\varepsilon>0} \omega_{\mathrm{Id}}(|\lambda|\|u - v\| + \varepsilon).$$

In particular, we have for each $\delta > 0$ that

(i) $\|u\| > \delta$ implies $\sigma(1, 0) \geqslant \rho_{\mathrm{Id}}(\delta)$, and

(ii) $\sigma(1, 0) > \omega_{\mathrm{Id}}(\delta)$ implies $\|u\| \geqslant \delta$.

Proof. Say $u = \sum_{j=1}^k \alpha_j \zeta_j$ and $v = \sum_{j=1}^k \beta_j \zeta_j$, for some $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{Q}$. Then

$$\begin{aligned}
\rho_{\mathrm{Id}}(|\lambda|\|\!|u - v|\!\| - \varepsilon) &\leqslant \limsup_{n_k} \ldots \limsup_{n_1} \rho_{\mathrm{Id}}\Big(\Big\|\lambda \sum_{j=1}^k (\alpha_j - \beta_j) x_{n_j}\Big\|\Big) \\
&\leqslant \lim_{n_k} \ldots \lim_{n_1} d\Big(\lambda \sum_{j=1}^k (\alpha_j - \beta_j) x_{n_j}, 0\Big) \\
&\leqslant \lim_{n_k} \ldots \lim_{n_1} \Big(d\Big(\lambda \sum_{j=1}^k \alpha_j x_{n_j}, y\Big) + d\Big(\lambda \sum_{j=1}^k \beta_j x_{n_j}, y\Big)\Big) \\
&= \sigma(\lambda, y) + \tau(\lambda, y)
\end{aligned}$$

for every $0 < \varepsilon < |\lambda|\|\!|u - v|\!\|$. Similarly,

$$\begin{aligned}
|\sigma(\lambda, y) - \tau(\lambda, y)| &= \lim_{n_k} \ldots \lim_{n_1} \Big| d\Big(\lambda \sum_{j=1}^k \alpha_j x_{n_j}, y\Big) - d\Big(\lambda \sum_{j=1}^k \beta_j x_{n_j}, y\Big)\Big| \\
&\leqslant \lim_{n_k} \ldots \lim_{n_1} d\Big(\lambda \sum_{j=1}^k (\alpha_j - \beta_j) x_{n_j}, 0\Big) \\
&\leqslant \liminf_{n_k} \ldots \liminf_{n_1} \omega_{\mathrm{Id}}\Big(\Big\|\lambda \sum_{j=1}^k (\alpha_j - \beta_j) x_{n_j}\Big\|\Big) \\
&\leqslant \omega_{\mathrm{Id}}(|\lambda|\|\!|u - v|\!\| + \varepsilon)
\end{aligned}$$

for all $\varepsilon > 0$. The particular case follows by letting $v = 0$ and $\lambda = 1$. $\square$

Let $H = \mathrm{span}_{\mathbb{Q}}\{\xi_i \mid i \in \mathbb{N}\} \subseteq S$. Given $\overline{\alpha} = (\alpha_j)_{j=1}^m \in \mathbb{Q}^{<\mathbb{N}}$, we define a bounded linear map $T_{\overline{\alpha}} \colon \overline{H} \to \overline{H}$ as follows. For each $n \in \mathbb{N}$ let

$$T_{\overline{\alpha}}(\xi_n) = \sum_{j=1}^m \alpha_j \xi_{mn+j-1}$$

and extend $T_{\overline{\alpha}}$ linearly to H. As $(\xi_n)_{n=1}^\infty$ is 1-spreading, we have that $\|\!|T_{\overline{\alpha}}(u)|\!\| \leqslant \|\alpha\|_1 \|\!|u|\!\|$, for all $u \in H$. Hence, we can extend $T_{\overline{\alpha}}$ to a bounded operator $T_{\overline{\alpha}} : \overline{H} \to \overline{H}$. If $\overline{\alpha} = (\alpha_1)$ is a sequence of length 1, then $T_{\overline{\alpha}} u$ is just the scaling of u by α_1, and we write $T_{\alpha_1} u := T_{\overline{\alpha}} u = \alpha_1 u$.

We also define the function $\widehat{T}_{\overline{\alpha}}\colon \mathcal{C} \to \mathcal{C}$ by letting

$$\widehat{T}_{\overline{\alpha}}(\sigma) = \mathop{\ast}_{j=1}^{m} \alpha_j \cdot \sigma$$

for all $\sigma \in \mathcal{C}$.

Lemma 5.6.4. *Let $\overline{\alpha} = (\alpha_i)_{i=1}^n, \overline{\beta} = (\beta_j)_{j=1}^m \in \mathbb{Q}^{<\mathbb{N}}$. Let $\overline{\gamma} = (\gamma_k)_{k=1}^{nm} \in \mathbb{Q}^{<\mathbb{N}}$, where $\gamma_k = \alpha_i\beta_j$ whenever $k = n(j-1)+i$. Then $T_\alpha \circ T_\beta = T_\gamma$ and $\widehat{T}_\alpha \circ \widehat{T}_\beta = \widehat{T}_\gamma$.*

Proof. For any $k \in \mathbb{N}$,

$$\begin{aligned}
(T_{\overline{\alpha}} \circ T_{\overline{\beta}})(\xi_k) &= T_{\overline{\alpha}}\Big(\sum_{j=1}^{m} \beta_j \xi_{mk+j-1}\Big) \\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_i\beta_j\xi_{n(mk+j-1)+i-1} \\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} \alpha_i\beta_j\xi_{nmk+n(j-1)+i-1} \\
&= \sum_{\ell=1}^{nm} \gamma_\ell \xi_{nmk+\ell-1} \\
&= T_{\overline{\gamma}}(\xi_k)
\end{aligned}$$

Therefore $T_{\overline{\alpha}} \circ T_{\overline{\beta}} = T_{\overline{\gamma}}$, by linearity and continuity. Similarly,

$$(\widehat{T}_{\overline{\alpha}} \circ \widehat{T}_{\overline{\beta}})(\sigma) = \widehat{T}_{\alpha}\Big(\mathop{\ast}_{j=1}^{m} \beta_j\sigma\Big) = \mathop{\ast}_{j=1}^{m}\mathop{\ast}_{i=1}^{n} \alpha_i\beta_j\sigma = \mathop{\ast}_{\ell=1}^{nm} \gamma_\ell\sigma = \widehat{T}_{\overline{\gamma}}(\sigma).$$

for all $\sigma \in \mathcal{C}$, and so $\widehat{T}_{\overline{\alpha}} \circ \widehat{T}_{\overline{\beta}} = \widehat{T}_{\overline{\gamma}}$. $\square$

The previous lemma justifies the following definition.

Definition 5.6.5. *Let $\overline{\alpha} = (\alpha_i)_{i=1}^n, \overline{\beta} = (\beta_j)_{j=1}^m \in \mathbb{Q}^{<\mathbb{N}}$. We define $\overline{\alpha} \circ \overline{\beta} = (\gamma_k)_{k=1}^{nm} \in \mathbb{Q}^{<\mathbb{N}}$ by $\gamma_k = \alpha_i\beta_j$ whenever $k = n(j-1)+i$. We define $\overline{\alpha}^{\circ k}$ recursively by letting $\overline{\alpha}^{\circ 1} = \overline{\alpha}$ and $\overline{\alpha}^{\circ k+1} = \overline{\alpha} \circ \overline{\alpha}^{\circ k}$ for every $k \in \mathbb{N}$.*

Remark 5.6.6. Notice that, $\widehat{T}_{\overline{\alpha}}^k = \widehat{T}_{\overline{\alpha}^{\circ k}}$ for all $\overline{\alpha} \in \mathbb{Q}^{<\mathbb{N}}$ and all $k \in \mathbb{N}$.

Lemma 5.6.7. *Let $\overline{\alpha} = (\alpha_j)_{j=1}^m \in \mathbb{Q}^{<\mathbb{N}}$. Say $u \in H$ realizes the type τ. Then, $T_{\overline{\alpha}}(u)$ realizes $\widehat{T}_{\overline{\alpha}}(\tau)$.*

Proof. Suppose $u = \sum_{i=1}^n \lambda_i \xi_i$, so $\tau = \ast_{i=1}^n \lambda_i \cdot \psi$. Then

$$T_{\overline{\alpha}}(u) = \sum_{i=1}^n \lambda_i \sum_{j=1}^m \alpha_j \xi_{mi+j-1} = \sum_{j=1}^m \sum_{i=1}^n \alpha_j \lambda_i \xi_{mi+j-1}$$

which realizes the type

$$\mathop{\ast}_{i=1}^{m} \mathop{\ast}_{j=1}^{n} \alpha_j \lambda_i \cdot \psi = \mathop{\ast}_{j=1}^{m} \alpha_j \cdot \mathop{\ast}_{i=1}^{n} \lambda_i \cdot \psi = \widehat{T}_{\overline{\alpha}}(\tau).$$

$\square$

Lemma 5.6.8. *Say $u, v \in H$ realize σ and τ, respectively. Let $(\overline{\alpha}_i)_{i=1}^N, (\overline{\beta}_i)_{i=1}^N \subseteq \mathbb{Q}^{<\mathbb{N}}$ and $(b_i)_{i=1}^N \in \mathbb{Q}^N$. Then for every $(\lambda, y) \in \mathbb{Q} \times \Delta$, we have that*

$$\Big| \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i} \sigma(\lambda, y) - \mathop{\ast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\beta}_i} \tau(\lambda, y) \Big| \leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}}\Big(|\lambda| \sum_{i=1}^N |b_i| \cdot \| T_{\overline{\alpha}_i} u - T_{\overline{\beta}_i} v \| + \varepsilon \Big).$$

Proof. For each $m \in \mathbb{N}$, let $s_m \colon H \to H$ be the linear map given by $s_m(\xi_n) = \xi_{n+m}$ for each $n \in \mathbb{N}$. We construct sequences $(u_i)_{i=1}^N, (v_i)_{i=1}^N \subseteq H$ recursively as follows. Let $u_1 = b_1 T_{\overline{\alpha}_1} u$ and $v_1 = b_1 T_{\overline{\beta}_1} v$. Given u_i, v_i for some $1 \leqslant i < N$, let $m_i = \max\{\mathrm{supp}(u_i) \cup \mathrm{supp}(v_i)\}$ and then let $u_{i+1} = b_{i+1} s_{m_i}(T_{\overline{\alpha}_{i+1}} u)$ and $v_{i+1} = b_{i+1} s_{m_i}(T_{\overline{\beta}_{i+1}} v)$. Clearly, both sequences $(u_i)_{i=1}^N$ and $(v_i)_{i=1}^N$ have disjoint supports. Hence, by Lemma 5.6.7 and Remark 5.6.2, $\sum_{i=1}^N u_i$ and $\sum_{i=1}^N v_i$ realize $\ast_{i=1}^N b_i \cdot \widehat{T}_{\overline{\alpha}_i} \sigma$ and $\ast_{i=1}^N b_i \cdot \widehat{T}_{\overline{\beta}_i} \tau$, respectively. Thus, by Lemma 5.6.3 and the fact that $(\xi_n)_{n=1}^\infty$ is 1-spreading, we have that

$$\begin{aligned}\Big|\mathop{\circledast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i}\sigma(\lambda, y) - \mathop{\circledast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\beta}_i}\tau(\lambda, y)\Big| &\leqslant \inf_{\varepsilon>0} \omega_{\mathrm{Id}}\Big(|\lambda| \cdot \Big\|\Big| \sum_{i=1}^{N}(u_i - v_i)\Big\|\Big| + \varepsilon\Big) \\ &\leqslant \inf_{\varepsilon>0} \omega_{\mathrm{Id}}\Big(|\lambda| \sum_{i=1}^{N} \|| u_i - v_i \|| + \varepsilon\Big) \\ &= \inf_{\varepsilon>0} \omega_{\mathrm{Id}}\Big(|\lambda| \sum_{i=1}^{N} |b_i| \cdot \|| T_{\overline{\alpha}_i} u - T_{\overline{\beta}_i} v \|| + \varepsilon\Big).\end{aligned}$$

□

5.7 Coarse approximating sequences.

The goal of this section is to show that the type ψ satisfies the conclusion of Proposition 5.7.7 below. For that, we introduce the notion of coarse approximating sequences.

Definition 5.7.1. *Let $u = \sum_{i=1}^{k} \alpha_i \xi_i \in span\{\xi_n \mid n \in \mathbb{N}\}$. We say that a vector $v \in span\{\xi_n \mid n \in \mathbb{N}\}$ is a* spreading *of u if $v = \sum_{i=1}^{k} \alpha_i \xi_{n_i}$ for some $n_1 < \ldots < n_k \in \mathbb{N}$.*

Definition 5.7.2. *Let $(\overline{\alpha}_i)_{i=1}^N \subseteq \mathbb{Q}^{<\mathbb{N}}$ and $(\beta_i)_{i=1}^N \in \mathbb{R}_+^N$. A sequence of types $(\sigma_n)_{n=1}^\infty \subseteq \mathcal{C}$ is called a* coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence *if there exists a sequence $(u_n)_{n=1}^\infty \subseteq H$ and sequences $(u_{i,n})_{n=1}^\infty \subseteq H$ for each $1 \leqslant i \leqslant N$ such that*

(i) u_n realizes σ_n for all $n \in \mathbb{N}$,

(ii) $u_{i,n}$ is a spreading of u_n for each $n \in \mathbb{N}$ and $1 \leqslant i \leqslant N$, and

(iii) $\lim_n \|| T_{\overline{\alpha}_i}(u_n) - \beta_i u_{i,n} \|| = 0$ for all $1 \leqslant i \leqslant N$.

Lemma 5.7.3. *Suppose $\overline{\alpha} \in \mathbb{Q}^{<\mathbb{N}}$, $\beta \geqslant 0$, and $(u_n)_{n=1}^\infty \subseteq H$. If there is a spreading $(u'_n)_{n=1}^\infty$ of $(u_n)_{n=1}^\infty$ such that $\lim_n \|| T_{\overline{\alpha}}(u_n) - \beta u'_n \|| = 0$, then for every $k \in \mathbb{N}$ there is a spreading $(u''_n)_{n=1}^\infty$ of $(u_n)_{n=1}^\infty$ such that $\lim_n \|| T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n \|| = 0$.*

Proof. For $k = 1$ the result is trivial. Suppose the result holds for some $k \in \mathbb{N}$. Let $(u''_n)_{n=1}^{\infty}$ be a spreading of $(u_n)_{n=1}^{\infty}$ such that $\lim_n |||T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n||| = 0$. By the definition of $T_{\overline{\alpha}}$, it follows that $(T_{\overline{\alpha}}(u''_n))_{n=1}^{\infty}$ is a spreading of $(T_{\overline{\alpha}}(u_n))_{n=1}^{\infty}$, so there exists a spreading $(u'''_n)_{n=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$ such that also $(T_{\overline{\alpha}}(u''_n) - \beta u'''_n)_{n=1}^{\infty}$ is a spreading of $(T_{\overline{\alpha}}(u_n) - \beta u'_n)_{n=1}^{\infty}$. Thus, by the 1-equivalence of $(\xi_n)_{n=1}^{\infty}$ with all its subsequences,

$$\begin{aligned}
|||T_{\overline{\alpha}}^{k+1}(u_n) - \beta^{k+1} u'''_n||| &\leqslant |||T_{\overline{\alpha}}^{k+1}(u_n) - T_{\overline{\alpha}}(\beta^k u''_n)||| + |||T_{\overline{\alpha}}(\beta^k u''_n) - \beta^{k+1} u'''_n||| \\
&= |||T_{\overline{\alpha}}(T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n)||| + \beta^k |||T_{\overline{\alpha}}(u''_n) - \beta u'''_n||| \\
&\leqslant |||T_{\overline{\alpha}}||| \cdot |||T_{\overline{\alpha}}^k(u_n) - \beta^k u''_n||| + \beta^k |||T_{\overline{\alpha}}(u_n) - \beta u'_n|||.
\end{aligned}$$

Therefore $\lim_n |||T_{\overline{\alpha}}^{k+1}(u_n) - \beta^{k+1} u'''_n||| = 0$, so the result holds for $k+1$. By induction, we are finished. $\square$

With the above lemma and Lemma 5.6.4, we have the following corollary.

Corollary 5.7.4. *If $(\sigma_n)_{n=1}^{\infty}$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence, then it is also a coarse $(\overline{\alpha}_i^{\circ k}, \beta_i^k)_{i=1}^N$-approximating sequence for every $k \in \mathbb{N}$.*

Lemma 5.7.5. *Suppose $(\overline{\alpha}_i)_{i=1}^N \subseteq \mathbb{Q}^{<\mathbb{N}}$ is such that $\overline{\alpha}_i \circ \overline{\alpha}_j = \overline{\alpha}_j \circ \overline{\alpha}_i$ for all $1 \leqslant i, j \leqslant N$. Then there are $(\beta_i)_{i=1}^N \in \mathbb{R}^N$ and $(\sigma_n)_{n=1}^{\infty} \subseteq \mathcal{C}$ such that $(\sigma_n)_{n=1}^{\infty}$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence and $\beta_i \in [\|\overline{\alpha}_i\|_\infty, \|\overline{\alpha}_i\|_1]$ for each $1 \leqslant i \leqslant N$. Moreover, we may choose $(\sigma_n)_{n=1}^{\infty}$ so that for all $n \in \mathbb{N}$, $b_1 \leqslant \sigma_n(1, 0) \leqslant b_2$ for some $\gamma < b_1 \leqslant b_2$ not depending on n.*

Proof. For those $\overline{\alpha}_i$'s that are length 1 sequences, the proposition is clear with $\{\beta_i\} = \overline{\alpha}_i$. So suppose for each $1 \leqslant i \leqslant N$ that $\overline{\alpha}_i$ is a sequence of length at least 2. As the basis $(\xi_n)_{n=1}^{\infty}$ of $\overline{H}$ is 1-unconditional and 1-spreading, we have that $\|\overline{\alpha}_i\|_\infty |||u||| \leqslant |||T_{\overline{\alpha}_i}(u)||| \leqslant \|\overline{\alpha}_i\|_1 |||u|||$, for all $u \in \overline{H}$ and all $1 \leqslant i \leqslant N$. Also, for each $1 \leqslant i \leqslant N$, it is clear from the definition of $T_{\overline{\alpha}_i}$ that $|||T_{\overline{\alpha}_i}(u) - \xi_1||| > 0$ for all $u \in \overline{H}$, and so $T_{\overline{\alpha}_i}$ is not invertible. Hence, the spectrum of $T_{\overline{\alpha}_i}$ has a real non-negative boundary point, and so $T_{\overline{\alpha}_i}$ has a real non-negative approximate eigenvalue for each $1 \leqslant i \leqslant N$ by Proposition IV.1 of [KrMau]. By Lemma 5.6.4, $T_{\overline{\alpha}_i}$ commutes with

$T_{\overline{\alpha}_j}$ for all $1 \leqslant i, j \leqslant N$. Thus, by Proposition 12.18 of [BenLi], there exists $(\beta_i)_{i=1}^N \in \mathbb{R}_+^N$ and a single normalized sequence $(u_n)_{n=1}^\infty \subseteq \overline{H}$ such that $\lim_n |||T_{\overline{\alpha}_i} u_n - \beta_i u_n||| = 0$ for every $1 \leqslant i \leqslant N$. As $|||u_n||| = 1$ for each $n \in \mathbb{N}$, the bounds above for $|||T_{\overline{\alpha}_i}(u)|||$ yield that $\beta_i \in [\|\overline{\alpha}_i\|_\infty, \|\overline{\alpha}_i\|_1]$ for each $1 \leqslant i \leqslant N$. By density, one may assume that $(u_n)_{n=1}^\infty \subseteq H$ and $1 \leqslant |||u_n||| \leqslant 2$ for all $n \in \mathbb{N}$. Finally, let $\delta > 0$ be such that $\rho_{\mathrm{Id}}(\delta/2) > \gamma$ and let σ_n be the type realized by δu_n for each $n \in \mathbb{N}$. The result now follows by letting $b_1 = \rho_{\mathrm{Id}}(\delta)$ and $b_2 = \omega_{\mathrm{Id}}(3\delta)$ (see Lemma 5.6.3). $\square$

Lemma 5.7.6. *Suppose $(\overline{\alpha}_i)_{i=1}^N \subseteq \mathbb{Q}^{<\mathbb{N}}$ is such that $\overline{\alpha}_i \circ \overline{\alpha}_j = \overline{\alpha}_j \circ \overline{\alpha}_i$ for all $1 \leqslant i, j \leqslant N$. Then there is $(\beta_i)_{i=1}^N \in \mathbb{R}^N$ such that every $\sigma \in \mathcal{C}$ is the limit of a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence and $\beta_i \in [\|\overline{\alpha}_i\|_\infty, \|\overline{\alpha}_i\|_1]$ for all $1 \leqslant i \leqslant N$.*

Proof. Let $\gamma < b_1 \leqslant b_2$, $(\beta_i)_{i=1}^N \in \mathbb{R}^N$ and $(\sigma_n)_{n=1}^\infty$ be given by Lemma 5.7.5, so that $(\sigma_n)_{n=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence and $b_1 \leqslant \sigma_n(1, 0) \leqslant b_2$ for every $n \in \mathbb{N}$. Let $\tilde{\mathcal{C}}$ be the subset of $\mathcal{C}$ consisting of all types of $\mathcal{C}$ which are the limit of a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence. As $\mathcal{T}_{b_1,b_2} := \{\sigma \in \mathcal{T} \mid b_1 \leqslant \sigma(1, 0) \leqslant b_2\}$ is compact and metrizable, $(\sigma_n)_{n=1}^\infty$ has a converging subsequence which converges to an element $\sigma \in \mathcal{C} \cap \mathcal{T}_{b_1,b_2}$. A subsequence of a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence is still a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence, so we have that $\tilde{\mathcal{C}} \neq \{0\}$, and in particular $\tilde{\mathcal{C}}$ contains an admissible type.

By the minimality of $\mathcal{C}$, it is enough to show that $\tilde{\mathcal{C}}$ is a closed conic class. Suppose $\sigma \in \tilde{\mathcal{C}}$ and $(\sigma_n)_{n=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence converging to σ. Then, by Lemma 5.4.6, $\lambda \cdot \sigma$ is the limit of $(\lambda \cdot \sigma_n)_{n=1}^\infty$, which is easily seen to be a coarse $(\overline{\alpha}_i, \beta_i)$-approximating sequence for every $\lambda \in \mathbb{Q}$. Thus $\tilde{C}$ is closed under dilation by any $\lambda \in \mathbb{Q}$.

Let D be a metric compatible with the topology of $\mathcal{T}$. Say $\sigma, \tau \in \tilde{\mathcal{C}}$ and let us show that $\sigma * \tau \in \tilde{\mathcal{C}}$. Let $(\sigma_n)_{n=1}^\infty$ and $(\tau_n)_{n=1}^\infty$ be coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequences in $\mathcal{C}$ converging to σ and τ, respectively. As the convolution is separately continuous, we have $\lim_k \sigma_k * \tau = \sigma * \tau$ and, for a fixed $k \in \mathbb{N}$, $\lim_n \sigma_k * \tau_n = \sigma_k * \tau$. For each $k \in \mathbb{N}$, let $n(k) \geqslant k$

be such that

$$D(\sigma_k * \tau_{n(k)}, \sigma_k * \tau) \leqslant 2^{-k}.$$

If we set $\sigma'_k = \sigma_k * \tau_{n(k)}$, then $\lim_k \sigma'_k = \sigma * \tau$.

For each $1 \leqslant i \leqslant N$, let $(u_n)_{n=1}^\infty$, $(u_{i,n})_{n=1}^\infty$, $(v_n)_{n=1}^\infty$ and $(v_{i,n})_{n=1}^\infty$ be sequences realizing $(\sigma_n)_{n=1}^\infty$ and $(\tau_n)_{n=1}^\infty$ respectively, as given by the definition of coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequences. By translating the supports of $v_{n(k)}$ and $v_{i,n(k)}$ if necessary, we may assume that $\mathrm{supp}(u_k) < \mathrm{supp}(v_{n(k)})$ and $\mathrm{supp}(u_{i,k}) < \mathrm{supp}(v_{i,n(k)})$ for all $1 \leqslant i \leqslant N$ and $k \in \mathbb{N}$. Let $(z_k)_{k=1}^\infty = (u_k + v_{n(k)})_{k=1}^\infty$, so z_k realizes σ'_k for each $k \in \mathbb{N}$. Set $(z_{i,k})_{k=1}^\infty = (u_{i,k} + v_{i,n(k)})_{k=1}^\infty$ for all $1 \leqslant i \leqslant N$, so $z_{i,k}$ is a spreading of z_k. This gives us that $(\sigma'_k)_{k=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence. Thus $\sigma * \tau \in \tilde{\mathcal{C}}$, and so $\tilde{\mathcal{C}}$ is closed under convolution.

Finally, let us show that $\tilde{\mathcal{C}}$ is closed. Say $(\sigma_k)_{k=1}^\infty$ is a sequence in $\tilde{\mathcal{C}}$ converging to $\sigma \in \mathcal{C}$. For each $k \in \mathbb{N}$, there exists a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence $(\sigma_{k,n})_{n=1}^\infty$ in $\mathcal{C}$ converging to σ_k. For each $k \in \mathbb{N}$, let $(u_{k,n})_{n=1}^\infty$ be a sequence realizing $(\sigma_{k,n})_{n=1}^\infty$ and let $(u_{k,i,n})_{n=1}^\infty$ be a spreading of $(u_{k,n})_{n=1}^\infty$ for each $1 \leqslant i \leqslant N$ as given by Definition 5.7.2. For each $k \in \mathbb{N}$, choose an integer $n(k) \geqslant k$ such that $D(\sigma_{k,n(k)}, \sigma_k) \leqslant 1/k$ and $|||T_{\overline{\alpha}_i}(u_{k,n(k)}) - \beta_i u_{k,i,n(k)}||| < 1/k$ for each $1 \leqslant i \leqslant N$. Set $\tau_k = \sigma_{k,n(k)}$ for each $k \in \mathbb{N}$. Then $(\tau_k)_{k=1}^\infty$ is a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-approximating sequence converging to σ. That is, $\sigma \in \tilde{\mathcal{C}}$. Thus $\tilde{\mathcal{C}}$ is closed since σ was an arbitrary limit point. By what was shown, $\tilde{\mathcal{C}}$ is a closed admissible conic class contained in $\mathcal{C}$ and by the minimality of $\mathcal{C}$, we are finished. □

Proposition 5.7.7. *Suppose $(\overline{\alpha}_i)_{i=1}^N \subseteq \mathbb{Q}^{<\mathbb{N}}$ is such that $\overline{\alpha}_i \circ \overline{\alpha}_j = \overline{\alpha}_j \circ \overline{\alpha}_i$ for all $1 \leqslant i, j \leqslant N$. There exists $\overline{\beta} = (\beta_i)_{i=1}^N \in \mathbb{R}^N$ such that $\beta_i \in [\|\overline{\alpha}_i\|_\infty, \|\overline{\alpha}_i\|_1]$ for all $1 \leqslant i \leqslant N$ and*

$$\limsup_m \left| \mathop{\ast}_{i=1}^N b_i \cdot \widehat{T}_{\overline{\alpha}_i}^{k_i} \psi(\lambda, x) - \mathop{\ast}_{i=1}^N b_i \beta_{i,m}^{k_i} \cdot \psi(\lambda, x) \right| \leqslant \gamma$$

for every $(b_i)_{i=1}^N \in \mathbb{Q}^N$, every $(k_i)_{i=1}^N \in \mathbb{N}^N$, every $(\lambda, x) \in \mathbb{Q} \times \Delta$, and every sequence $(\overline{\beta}_m)_{m=1}^\infty \subseteq \mathbb{Q}_+^N$ converging to $\overline{\beta}$, where $\overline{\beta}_m = (\beta_{i,m})_{i=1}^N$ for all $m \in \mathbb{N}$.

Proof. Let $(\beta_i)_{i=1}^N \in \mathbb{R}^N$ be given by Lemma 5.7.6 and let $(\phi_n)_{n=1}^\infty$ be a coarse $(\overline{\alpha}_i, \beta_i)_{i=1}^N$-

approximating sequence converging to ϕ, also given by Lemma 5.7.6. For each $n \in \mathbb{N}$ let $\psi_n = \phi_n * (-1) \cdot \phi_n$. Then, by our choice of ϕ (see Lemma 5.5.5) we have that

$$\lim_n \mathop{\circledast}_{i=1}^{N} b_i \cdot \widehat{T}^{k_i}_{\overline{\alpha}_i} \psi_n(\lambda, x) = \mathop{\circledast}_{i=1}^{N} b_i \cdot \widehat{T}^{k_i}_{\overline{\alpha}_i} \psi(\lambda, x)$$

and

$$\lim_n \mathop{\circledast}_{i=1}^{N} b_i \beta^{k_i}_{i,m} \cdot \psi_n(\lambda, x) = \mathop{\circledast}_{i=1}^{N} b_i \beta^{k_i}_{i,m} \cdot \psi(\lambda, x)$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$ and all $m \in \mathbb{N}$.

By Corollary 5.7.4, $(\phi_n)_{n=1}^\infty$ is a coarse $(\overline{\alpha}_i^{\circ k_i}, \beta_i^{k_i})_{i=1}^N$-approximating sequence and we can pick a sequence $(u_n)_{n=1}^\infty$ realizing $(\phi_n)_{n=1}^\infty$ and sequences $(u_{i,n})_{n=1}^\infty$ which are spreadings of $(u_n)_{n=1}^\infty$ and satisfy $\lim_n |||T_{\overline{\alpha}^{\circ k_i}} u_n - \beta_i^{k_i} u_{i,n}||| = 0$ for every $1 \leqslant i \leqslant N$. For each $n \in \mathbb{N}$, let $u'_n \in H$ have the same basis coordinates as u_n except shifted over so that the supports of u_n and u'_n are disjoint. It is easy to see that, for each $1 \leqslant i \leqslant N$ and $n \in \mathbb{N}$, we can pick a spreading of u_n, say $u'_{i,n}$, so that $T_{\overline{\alpha}_i^{\circ k_i}} u'_n - \beta^{k_i} u'_{i,n}$ is a spreading of $T_{\overline{\alpha}_i^{\circ k_i}}(u_n) - \beta^{k_1} u_{i,n}$ and such that $u_{i,n}$ and $u'_{i,n}$ have disjoint supports.

Notice that both $u_n - u'_n$ and $u_{i,n} - u'_{i,n}$ realize ψ_n. Therefore, by Lemma 5.6.8, we have that

$$\begin{aligned}
&\Big| \mathop{\circledast}_{i=1}^{N} b_i \cdot \widehat{T}^{k_i}_{\overline{\alpha}_i} \psi_n(\lambda, x) - \mathop{\circledast}_{i=1}^{N} b_i \beta^{k_i}_{i,m} \cdot \psi_n(\lambda, x) \Big| \\
&\quad = \Big| \mathop{\circledast}_{i=1}^{N} b_i \cdot \widehat{T}_{\overline{\alpha}_i^{\circ k_i}} \psi_n(\lambda, x) - \mathop{\circledast}_{i=1}^{N} b_i \beta^{k_i}_{i,m} \cdot \psi_n(\lambda, x) \Big| \\
&\quad \leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \Big(|\lambda| \sum_{i=1}^{N} |b_i| \cdot |||T_{\overline{\alpha}_i^{\circ k_i}}(u_n - u'_n) - \beta^{k_i}_{i,m}(u_{i,n} - u'_{i,n})||| + \varepsilon \Big) \\
&\quad \leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \Big(2|\lambda| \sum_{i=1}^{N} |b_i| \cdot |||T_{\overline{\alpha}_i^{\circ k_i}} u_n - \beta^{k_i}_{i,m} u_{i,n}||| + \varepsilon \Big) \\
&\quad \leqslant \inf_{\varepsilon > 0} \omega_{\mathrm{Id}} \Big(2|\lambda| \Big(\sum_{i=1}^{N} |b_i| \cdot \Big(|||T_{\overline{\alpha}_i^{\circ k_i}} u_n - \beta^{k_i}_i u_n||| + |\beta^{k_i}_i - \beta^{k_i}_{i,m}| \cdot |||u_n||| \Big) \Big) + \varepsilon \Big)
\end{aligned}$$

for all $(\lambda, x) \in \mathbb{Q} \times \Delta$. As the sequence $(u_n)_{n=1}^\infty$ is bounded (see Lemma 5.6.3), taking the limit superiors over n and m in the inequality above yields the result. □

5.8 Coarse ℓ_p-types and coarse c_0-types.

In this section, we will define a notion of ℓ_p-type and c_0-type and use Proposition 5.7.7 in order to show that ψ satisfies this property. Finally, we will show that $\overline{H}$ is isomorphic to some ℓ_p.

Definition 5.8.1. *Let $p \in [1, \infty)$. We say that ψ is a* coarse ℓ_p-type *if there exists $L > 0$ such that, for all $(\lambda, y) \in \mathbb{Q} \times \Delta$, and all $\overline{\alpha} = (\alpha_i)_{i=1}^N \in \mathbb{Q}^{<\mathbb{N}}$, we have*

$$\limsup_m \Big| \mathop{\ast}_{i=1}^{N} \alpha_i \cdot \psi(\lambda, y) - t_m \cdot \psi(\lambda, y) \Big| \leqslant L.$$

for all $(t_m)_{m=1}^\infty \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$. The type ψ is called a coarse c_0-type *if, for all $(\lambda, y) \in \mathbb{Q} \times \Delta$, and all $(\alpha_i)_{i=1}^N \in \mathbb{Q}^{<\mathbb{N}}$, we have*

$$\Big| \mathop{\ast}_{i=1}^{N} \alpha_i \cdot \psi(\lambda, y) - \max_{1 \leqslant i \leqslant N} |\alpha_i| \cdot \psi(\lambda, y) \Big| \leqslant L.$$

Proposition 5.8.2. *The type ψ is either a coarse c_0-type or a coarse ℓ_p-type for some $p \in [1, \infty)$.*

Proof. Let $\overline{\alpha}_2 = (1, 1)$ and $\overline{\alpha}_3 = (1, 1, 1)$, and notice that $\overline{\alpha}_2 \circ \overline{\alpha}_3 = \overline{\alpha}_3 \circ \overline{\alpha}_2$. Let $\beta_2, \beta_3 \in \mathbb{R}$ be given by Proposition 5.7.7 for $\overline{\alpha}_2 = (1)$ and $\overline{\alpha}_3 = (1, 1)$, respectively. Let $(\beta_{2,m})_{m=1}^\infty, (\beta_{3,m})_{m=1}^\infty \subseteq \mathbb{Q}$ be nonzero increasing sequences converging to β_2, β_3 respectively. By our choice of β_2 and β_3, we have that

$$\limsup_m \Big| b \cdot \mathop{\ast}_{i=1}^{j^k} \psi(\lambda, x) - b\beta_{j,m}^k \cdot \psi(\lambda, x) \Big| \leqslant \gamma$$

for all $j \in \{2, 3\}$, all $b \in \mathbb{Q}$, all $k \in \mathbb{N}$, and all $(\lambda, x) \in \mathbb{Q} \times \Delta$.

Let $\ell, k \in \mathbb{N}$ be such that $3^k \leqslant 2^\ell < 3^{k+1}$. As $(\xi_n)_{n=1}^\infty$ is 1-unconditional, we have

$$\Big\|\Big|\sum_{i=1}^{3^k}\xi_i\Big|\Big\| \leqslant \Big\|\Big|\sum_{i=1}^{2^\ell}\xi_i\Big|\Big\| \leqslant \Big\|\Big|\sum_{i=1}^{3^{k+1}}\xi_i\Big|\Big\|.$$

Let $a_\ell \in \mathbb{Q}$ be between $\frac{1}{2}|\!|\!| \sum_{i=1}^{2^\ell}\xi_i |\!|\!|$ and $|\!|\!| \sum_{i=1}^{2^\ell}\xi_i |\!|\!|$. Then, for any $\mu > 0$,

$$\mu \leqslant \Big\|\Big|\mu \cdot \frac{\sum_{i=1}^{3^{k+1}}\xi_i}{a_\ell}\Big|\Big\|.$$

As $\mathrm{Id} : (X, \|\cdot\|) \to (X, d)$ is expanding, we can pick $\mu \in \mathbb{Q}$ such that $\rho_{\mathrm{Id}}(\mu/2) > 2\omega_{\mathrm{Id}}(1) + \gamma$ and $\eta \in \mathbb{Q}$ such that $\rho_{\mathrm{Id}}(\eta\|\xi_1\|/2) > 2\omega_{\mathrm{Id}}(1) + \gamma$. Let $M \in \mathbb{N}$ be such that

$$\Big|\frac{\mu}{a_\ell} \cdot \mathop{\ast}_{i=1}^{3^{k+1}} \psi(1,0) - \frac{\mu\beta_{3,M}^\ell}{a_\ell} \cdot \psi(1,0)\Big| \leqslant \gamma + \omega_{\mathrm{Id}}(1)$$

and let $N \geqslant M$ be such that

$$\Big|\frac{\eta}{\beta_{2,M}^\ell} \cdot \mathop{\ast}_{i=1}^{2^\ell} \psi(1,0) - \frac{\eta\beta_{2,N}^\ell}{\beta_{2,M}^\ell} \cdot \psi(1,0)\Big| \leqslant \gamma + \omega_{\mathrm{Id}}(1).$$

Then as $(\mu/a_\ell) \cdot (\sum_{i=1}^{3^{k+1}} \xi_i)$ realizes $(\mu/a_l) \cdot \ast_{i=1}^{3^{k+1}} \psi$, by Lemma 5.6.3(i), we have that

$$2\omega_{\mathrm{Id}}(1) + \gamma < \frac{\mu}{a_\ell} \cdot \mathop{\ast}_{i=1}^{3^{k+1}} \psi(1,0) \leqslant \frac{\mu\beta_{3,M}^{k+1}}{a_\ell} \cdot \psi(1,0) + \gamma + \omega_{\mathrm{Id}}(1).$$

Therefore, as $(\mu\beta_{3,M}^{k+1}/a_\ell) \cdot \xi_1$ realizes $(\mu\beta_{3,M}^{k+1}/a_\ell) \cdot \psi$, by Lemma 5.6.3(ii), we have

$$1 \leqslant \frac{\beta_{3,M}^{k+1}\mu}{a_\ell} \cdot |\!|\!|\xi_1|\!|\!|. \tag{5.8.1}$$

Similarly, by Lemma 5.6.3(i) and the fact that $(\eta\beta_{2,N}^\ell/\beta_{2,M}^\ell) \cdot \xi_1$ realizes $(\eta\beta_{2,N}^\ell/\beta_{2,M}^\ell) \cdot \psi$, we have

$$\frac{\eta}{\beta_{2,M}^\ell} \cdot \mathop{\ast}_{i=1}^{2^\ell} \psi(1,0) \geqslant \frac{\eta\beta_{2,N}^\ell}{\beta_{2,M}^\ell} \cdot \psi(1,0) - \gamma - \omega_{\mathrm{Id}}(1) > \omega_{\mathrm{Id}}(1).$$

Hence, as $(\eta/\beta^{\ell}_{2,M})\cdot(\sum_{i=1}^{2^\ell}\xi_i)$ realizes $(\eta/\beta^{\ell}_{2,M})\cdot *_{i=1}^{2^\ell}\psi$, Lemma 5.6.3(ii) gives us

$$\frac{2\eta a_\ell}{\beta^{\ell}_{2,M}} \geqslant \left|\!\left|\!\left|\frac{\eta\sum_{i=1}^{2^\ell}\xi_i}{\beta^{\ell}_{2,M}}\right|\!\right|\!\right| \geqslant 1. \tag{5.8.2}$$

Combining Inequalities (5.8.1) and (5.8.2), we obtain

$$\frac{\beta_3^k}{\beta_2^\ell} = \lim_M \frac{\beta^k_{3,M}}{\beta^\ell_{2,M}} \geqslant \frac{1}{2\eta\mu\beta_3|\!|\!|\xi_1|\!|\!|}.$$

The lower bound for β_3^k/β_2^ℓ above does not depend on k and ℓ, as long as $2^\ell < 3^{k+1}$. Similarly, we get a lower bound for β_2^ℓ/β_3^k, which also does not depend on k and ℓ, as long as $3^k \leqslant 2^\ell$. We conclude that there exist $a, b > 0$ such that for all k and ℓ, with $3^k \leqslant 2^\ell < 3^{k+1}$, we have

$$a \leqslant \frac{\beta_3^k}{\beta_2^\ell} \leqslant b.$$

Therefore, there exists $L \geqslant 0$ such that $\beta_2 = 2^L$, and $\beta_3 = 3^L$. Also, as $\beta_2 \leqslant 2$, we must have $L \in [0,1]$. The same argument works for arbitrary $n, m \in \mathbb{N}$ instead of 2 and 3. Hence, we have $\beta_n = n^L$, for all $n \in \mathbb{N}$, where β_n is given by Proposition 5.7.7 for

$$\overline{\alpha} = \underbrace{(1,\dots,1)}_{n}.$$

Case 1: Say $L \neq 0$. Then ψ is a coarse ℓ_p-type, for $p = 1/L$.

Fix $\overline{\alpha} = (\alpha_i)_{i=1}^N \in \mathbb{Q}^N$ and a sequence $(t_m)_{m=1}^\infty \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$. Let $\epsilon > 0$ and, for each $1 \leqslant j \leqslant N$, let $r_j \in \mathbb{Q}_+$ be such that $||\alpha_j| - r_j^{1/p}| < \varepsilon$. Find a common denominator $m \in \mathbb{N}$ so that for each $1 \leqslant j \leqslant N$ there is $n_j \in \mathbb{N}_0$ such that $r_j = n_j/m$. Let $s > 0$ be a rational number such that $|s - (1/m)^{1/p}| < \varepsilon$. For each $1 \leqslant j \leqslant N$, let $(\beta_{j,m})_{m=1}^\infty \subseteq \mathbb{Q}$ be a sequence converging to $n_j^{1/p}$and let $(\beta_m)_{m=1}^\infty \subseteq \mathbb{Q}$ be a sequence converging to $(\sum_{j=1}^N n_j)^{1/p}$.

By Lemma 5.6.8,

$$\Big|\mathop{\ast}_{j=1}^{N}\alpha_j\cdot\psi(\lambda,x)-\mathop{\ast}_{j=1}^{N}s\beta_{j,k}\cdot\psi(\lambda,x)\Big|\leqslant\omega_{\mathrm{Id}}\Big(|\lambda|\sum_{j=1}^{N}|\alpha_j-s\beta_{j,k}|\|\|\xi_j\|\|+\varepsilon\Big)$$

and

$$|s\beta_m\cdot\psi(\lambda,x)-t_m\cdot\psi(\lambda,x)|\leqslant\omega_{\mathrm{Id}}(|\lambda||s\beta_m-t_m|\|\|\xi_1\|\|+\varepsilon).$$

for all $(\lambda,x)\in\mathbb{Q}\times\Delta$. By Proposition 5.7.7 and what was shown above with $L=1/p$, we have that

$$\limsup_m\Big|\mathop{\ast}_{j=1}^{N}s\beta_{j,m}\cdot\psi(\lambda,x)-\mathop{\ast}_{j=1}^{N}s\cdot\mathop{\ast}_{i=1}^{n_j}\psi(\lambda,x)\Big|\leqslant\gamma$$

and

$$\limsup_m\Big|s\cdot\mathop{\ast}_{j=1}^{N}\mathop{\ast}_{i=1}^{n_j}\psi(\lambda,x)-s\beta_m\cdot\psi(\lambda,x)\Big|\leqslant\gamma$$

for all $(\lambda,x)\in\mathbb{Q}\times\Delta$.

Combining the four inequalities above with the triangle inequality, taking a limit superior over m, and letting $\epsilon\to 0$, one obtains

$$\limsup_m\Big|\mathop{\ast}_{j=1}^{N}\alpha_j\cdot\psi(\lambda,x)-t_m\cdot\psi(\lambda,x)\Big|\leqslant 4\gamma$$

for all $(\lambda,x)\in\mathbb{Q}\times\Delta$. Therefore ψ is a coarse ℓ_p-type.

Case 2: Say $L=0$. Then ψ is a coarse c_0-type.

Fix $\overline{\alpha}=(\alpha_i)_{i=1}^N\in\mathbb{Q}^N$ such that $\alpha_1=1$ and $\alpha_j\leqslant 1$ for $2\leqslant j\leqslant N$ (the general case will follow by dilation). Using Proposition 5.7.7, find $\beta\geqslant 1$ and a nonzero increasing sequence $(\beta_m)_{m=1}^{\infty}\subseteq\mathbb{Q}$ converging to β such that

$$\limsup_m|b\cdot\widehat{T}^k_{\overline{\alpha}}\psi(\lambda,x)-b\beta_m^k\cdot\psi(\lambda,x)|\leqslant\gamma$$

for all $b \in \mathbb{Q}$, $k \in \mathbb{N}$ and $(\lambda, x) \in \mathbb{Q} \times \Delta$. We will show $\beta \leqslant 1$. Fix $k \in \mathbb{N}$ and note that $\widehat{T}_{\overline{\alpha}}^k \psi = \ast_{i_k=1}^N \cdots \ast_{i_1=1}^N (\prod_{\ell=1}^k \alpha_{i_\ell}) \cdot \psi$ (using the definition of $\widehat{T}_{\overline{\alpha}}$ and the distributivity of dilation over convolution). After combining like terms using the commutativity of convolution, by Proposition 5.7.7 and what was shown above with $L = 0$, we have

$$\Big| b \cdot \widehat{T}_{\overline{\alpha}}^k \psi(\lambda, x) - b \cdot \mathop{\ast}_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \psi(\lambda, x) \Big| \leqslant \gamma$$

where $F = \{\overline{n} = (n_j)_{j=1}^k \in \mathbb{N}_0^k \mid \sum_{j=1}^k n_j = k\}$ for every $b \in \mathbb{Q}$ and $(\lambda, x) \in \mathbb{Q} \times \Delta$. Now, take any $\mu \in \mathbb{Q}$ such that $\rho_{\mathrm{Id}}(\mu \|\|\xi_1\|\|/2) > 2\omega_{\mathrm{Id}}(1) + 2\gamma$. Fix $M \in \mathbb{N}$, and let $N \geqslant M$ be such that $|\frac{\mu}{\beta_M^k} \widehat{T}_{\overline{\alpha}}^k \psi(1, 0) - \frac{\mu \beta_N^k}{\beta_M^k} \cdot \psi(1, 0)| \leqslant \gamma + \omega_{\mathrm{Id}}(1)$. Then combining the two inequalities above yields

$$\Big| \frac{\mu \beta_N^k}{\beta_M^k} \cdot \psi(1, 0) - \frac{\mu}{\beta_M^k} \cdot \mathop{\ast}_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \psi(1, 0) \Big| \leqslant 2\gamma + \omega_{\mathrm{Id}}(1).$$

As $(\mu \beta_N^k / \beta_M^k) \xi_1$ realizes $(\mu \beta_N^k / \beta_M^k) \cdot \psi$, we have, by Lemma 5.6.3(i), $\frac{\mu}{\beta_M^k} \cdot \ast_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \psi(1, 0) \geqslant \omega_{\mathrm{Id}}(1)$. So, as $\frac{\mu}{\beta_M^k} \sum_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \xi_{I(\overline{n})}$ realizes $\frac{\mu}{\beta_M^k} \cdot \ast_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \psi$ for any injective map $I \colon F \to \mathbb{N}$, we have, by Lemma 5.6.3(ii),

$$1 \leqslant \Big\|\Big| \frac{\mu}{\beta_M^k} \sum_{\overline{n} \in F} (\prod_{j=1}^k \alpha_j^{n_j}) \cdot \xi_{I(\overline{n})} \Big\|\Big| \leqslant \frac{\mu \|\|\xi_1\|\|}{\beta_M^k} \prod_{\alpha_j < 1} \frac{1}{1 - \alpha_j}.$$

But this was for any $k, M \in \mathbb{N}$, and so we must have $\beta \leqslant 1$. That is, $\beta = 1$. Therefore ψ is a coarse c_0-type. □

We can now prove the following.

Proposition 5.8.3. *If ψ is a coarse ℓ_p-type, for some $p \in [1, \infty)$, then $(\xi_n)_{n=1}^\infty$ is equivalent to the ℓ_p-basis. If ψ is a coarse c_0-type, then (ξ_n) is equivalent to the c_0-basis.*

Proof. Say $\psi \in \mathcal{T}$ is a coarse ℓ_p-type for some $p \in [1, \infty)$ (the c_0 case will be analogous). Say $L > 0$ is such that, for all $(\alpha_j)_{j=1}^N \in \mathbb{Q}^{<\mathbb{N}}$, all $(t_m)_{m=1}^\infty \subseteq \mathbb{Q}$ converging to $\|\overline{\alpha}\|_p$, and all

$(\lambda, y) \in \mathbb{Q} \times \Delta$, we have

$$\limsup_m \Big| \mathop{\circledast}_{j=1}^{N} \alpha_j \cdot \psi(\lambda, y) - t_m \cdot \psi(\lambda, y)\Big| \leqslant L. \tag{5.8.3}$$

Let $(e_n)_{n=1}^\infty$ be the standard basis of ℓ_p, and let $Y = \mathrm{span}\{e_n \mid n \in \mathbb{N}\}$. Let us show that the map $T\colon Y \to \mathrm{span}\{\xi_n \mid n \in \mathbb{N}\}$ defined by sending e_n to $\xi_n/|||\xi_1|||$ for each $n \in \mathbb{N}$ and extending linearly is an isomorphism. Hence, T extends to an isomorphism between ℓ_p and $\overline{\mathrm{span}}\{\xi_n \mid n \in \mathbb{N}\}$, and we are done.

We first show that T bounded. Fix $\varepsilon > 0$ and let $b \in \mathbb{Q}$ be such that $1/|||\xi_1||| < b < 1/|||\xi_1||| + \varepsilon$. For each $\overline{\alpha} = (\alpha_i)_{i=1}^N \in \mathbb{Q}^{<\mathbb{N}}$, let $t_{\overline{\alpha}} \in \mathbb{Q}$ be such that $|t_{\overline{\alpha}} - \|\overline{\alpha}\|_p| < \varepsilon$ and $|\circledast_{j=1}^N \alpha_j \cdot \psi(b, 0) - t_{\overline{\alpha}} \cdot \psi(b,0)| \leqslant L + \varepsilon$. By Lemma 5.6.3 and Inequality 5.8.3, we have that

$$\begin{aligned}
\rho_{\mathrm{Id}}\Big(\Big|\Big|\Big|\sum_{i=1}^N \alpha_i \frac{\xi_j}{|||\xi_1|||}\Big|\Big|\Big| - \varepsilon\Big) &\leqslant \rho_{\mathrm{Id}}\Big(\Big|\Big|\Big|\sum_{i=1}^N \alpha_i b \xi_j\Big|\Big|\Big| - \varepsilon\Big) \\
&\leqslant \mathop{\circledast}_{i=1}^{N} \alpha_i \cdot \psi(b, 0) \\
&\leqslant t_{\overline{\alpha}} \cdot \psi(b,0) + L + \varepsilon \\
&\leqslant \omega_{\mathrm{Id}}\Big(b|||\xi_1|||t_{\overline{\alpha}} + \varepsilon\Big) + L + \varepsilon \\
&\leqslant \omega_{\mathrm{Id}}\Big(\|\overline{\alpha}\|_p + 2\varepsilon + \varepsilon\|\xi_1\|\|\overline{\alpha}\|_p + \varepsilon^2\|\xi_1\|\Big) + L + \varepsilon,
\end{aligned}$$

for all $\overline{\alpha} = (\alpha_i)_{i=1}^N \in \mathbb{Q}^{<\mathbb{N}}$. Hence, as $\mathrm{Id} : (X, \|\cdot\|) \to (X, d)$ is expanding, there exists $K > 0$ such that $\|\overline{\alpha}\|_p \leqslant 1$ implies $|||\sum_{i=1}^N \alpha_i \frac{\xi_i}{|||\xi_1|||}||| \leqslant K$. Therefore T is bounded.

Clearly, T is a bijection. Let us show that T^{-1} is bounded. By Lemma 5.6.3 and Inequality

5.8.3, we have that

$$\begin{aligned}\rho_{\mathrm{Id}}\Big(\|\overline{\alpha}\|_p - 2\varepsilon\Big) - L - \varepsilon &\leqslant \rho_{\mathrm{Id}}\Big(bt_{\overline{\alpha}}\|\!|\xi_1\|\!| - \varepsilon\Big) - L - \varepsilon \\ &\leqslant t_{\overline{\alpha}} \cdot \psi(b,0) - L - \varepsilon \\ &\leqslant \mathop{\ast}_{i=1}^{N} \alpha_i \cdot \psi(b,0) \\ &\leqslant \omega_{\mathrm{Id}}\Big(b\Big\|\!\Big|\sum_{i=1}^{N}\alpha_i\xi_i\Big\|\!\Big|\Big) \\ &\leqslant \omega_{\mathrm{Id}}\Big(\Big\|\!\Big|\sum_{i=1}^{N}\alpha_i\frac{\xi_i}{\|\!|\xi_1\|\!|}\Big\|\!\Big| + \varepsilon\|\!|\xi_1\|\!|\Big\|\!\Big|\sum_{i=1}^{N}\alpha_i\frac{\xi_i}{\|\!|\xi_1\|\!|}\Big\|\!\Big|\Big).\end{aligned}$$

for all $\overline{\alpha} = (\alpha_i)_{i=1}^N \in \mathbb{Q}^{<\mathbb{N}}$. Hence, as $\mathrm{Id} : (X, \|\cdot\|) \to (X, d)$ is expanding, there exists some $R > 0$ such that $\|\!|\sum_{i=1}^N \alpha_i \frac{\xi_i}{\|\!|\xi_1\|\!|}\|\!| \leqslant 1$ implies $\|\overline{\alpha}\|_p < R$. So T^{-1} is bounded. □

Proof of Theorem 1.7.3. By Corollary 5.3.2, if X coarsely embeds into a superstable space Y, there exists an invariant stable pseudometric d on X which is coarsely equivalent to the norm of X. Hence, we can define the type space $\mathcal{T}$ as in Section 5.4. By Proposition 5.5.4, there exists a minimal closed admissible conic class $\mathcal{C}$. Let $\phi \in \mathcal{C}$ be given by Lemma 5.5.5. Without loss of generality, $\psi = \phi * (-1) \cdot \phi$ is admissible. By Proposition 5.8.2, ψ is either a c_0-type or an ℓ_p-type, for some $p \in [1, \infty)$. Hence, by Proposition 5.8.3, X has either an ℓ_p-spreading model or a c_0-spreading model.

Assume that X has a c_0-spreading model. In particular, c_0 is finitely represented in X. Hence, c_0 isomorphically embeds into an ultrapower of X. As ultrapowers of X coarsely embed into ultrapowers of Y, this gives us that c_0 coarsely embeds into an ultrapower of Y, which is a stable space. By Theorem 2.1 of [Ka1], stable spaces coarsely embed into reflexive spaces. Therefore, c_0 coarsely embeds into a reflexive space. By Theorem 3.6 of [Ka1], this cannot happen, so we have a contradiction. Therefore, X contains an ℓ_p-spreading model, for some $p \in [1, \infty)$.

Let $(x_n)_{n=1}^\infty$ be a bounded sequence in X without Cauchy subsequences whose spreading model is isomorphic to ℓ_p. Let us observe now that $(x_n)_{n=1}^\infty$ can be assumed to be a basic sequence. By Rosenthal's ℓ_1-Theorem, either $(x_n)_{n=1}^\infty$ has a subsequence which is isomorphic to ℓ_1, or it has a weakly Cauchy subsequence. Assume that $(x_n)_{n=1}^\infty$ is weakly Cauchy. Then $(y_n)_{n=1}^\infty$ is weakly null and it has an ℓ_p-spreading model, where $y_n = x_n - x_{n+1}$, for all $n \in \mathbb{N}$. Hence, by taking a subsequence, we can assume that $(y_n)_{n=1}^\infty$ is basic. $\square$

Remark. By the last inequality of Case 1 in Proposition 5.8.2, and by following the proof of Proposition 5.8.3, we find an upper bound of

$$\Big(\inf_{\varepsilon>0}\sup \rho_{\mathrm{Id}}^{-1}([0,\omega_{\mathrm{Id}}(1)+5\gamma+\varepsilon])\Big)^2$$

for the Banach-Mazur distance between ℓ_p and the spreading model associated to $(y_n)_{n=1}^\infty$.

Proof of Corollary 1.5.6. This follows from the fact that the original Tsirelson space (see [Ts]) does not have an ℓ_p-spreading model. $\square$

Remark 5.8.4. Another example of a reflexive Banach space that does not coarsely embed into any superstable space is the space constructed by E. Odell and Th. Schlumprecht in [OSc2]. Indeed, this follows from Theorem 1.7.3 and the fact that every spreading model of their space contains neither a subspace isomorphic to c_0 nor to ℓ_p (see [OSc2], Theorem 1.4).

As mentioned in the introduction, our work is not enough to solve Problem 1.5.3. The following is a natural approach to give a negative answer to Problem 1.5.3, given Theorem 1.7.3.

Problem 5.8.5. Let T be the Tsirelson space. Does T or T^p (i.e., the p-convexification of T) for some $p \in [1,\infty)$ coarsely embed into a superstable Banach space?

At last, in the spirit of Chapter 4, we ask the following.

Problem 5.8.6. Say an infinite dimensional Banach space X maps into a superstable space by a map which is both coarse and solvent. Does it follow that X must contain an ℓ_p-spreading model, for some $p \in [1,\infty)$.

Chapter 6

Coarse Lipschitz geometry and asymptotic structure

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In this chapter, we study coarse Lipschitz embeddings and equivalences between Banach spaces and what kind of stability properties this notions of nonlinear embeddings and nonlinear equivalences may have. Furthermore, we will mainly work with Banach spaces having some kind of asymptotic property. For that, we will go over the results contained in Section 1.6, which are in the paper *Asymptotic structure and coarse Lipschitz geometry of Banach spaces* (see [Br3]).

6.1 Preliminaries.

In this section, we will introduce some notation and terminology which will be essential for this chapter.

6.1.1 p-convex and p-concave Banach spaces.

Let X be a Banach space with 1-unconditional basis $(e_n)_{n=1}^{\infty}$, and let $p \in (1, \infty)$. We say that the basis $(e_n)_{n=1}^{\infty}$ is *p-convex with convexity constant C* (resp. *p-concave with concavity constant C*), if

$$\Big\| \sum_{j \in \mathbb{N}} (|x_j^1|^p + \ldots + |x_j^k|^p)^{1/p} e_j \Big\|^p \leqslant C^p \sum_{n=1}^{k} \|x^n\|^p,$$

$$\Big(\text{resp.} \quad C^p \Big\| \sum_{j \in \mathbb{N}} (|x_j^1|^p + \ldots + |x_j^k|^p)^{1/p} e_j \Big\|^p \geqslant \sum_{n=1}^{k} \|x^n\|^p \Big),$$

for all $x^1 = \sum_{j=1}^{\infty} x_j^1 e_j, \ldots, x^k = \sum_{j=1}^{\infty} x_j^k e_j \in X$. We say that the basis $(e_n)_{n=1}^{\infty}$ satisfies an *upper ℓ_p-estimate with constant C* (resp. *lower ℓ_p-estimate with constant C*), if

$$\|x_1 + \ldots + x_k\|^p \leqslant C^p \sum_{n=1}^{k} \|x_n\|^p \quad \Big(\text{resp. } C^p \|x_1 + \ldots + x_k\|^p \geqslant \sum_{n=1}^{k} \|x_n\|^p \Big),$$

for all $x_1, \ldots, x_k \in X$ with disjoint supports. Clearly, a p-convex (resp. p-concave) basis with constant C satisfies an upper (resp. lower) ℓ_p-estimate with constant C.

6.1.2 p-convexification.

Let X be a Banach space with a 1-unconditional basis $(e_n)_{n=1}^{\infty}$. For any $p \in [1, \infty)$, we define the *p-convexification of X* as follows. Let

$$X^p = \Big\{ (x_n)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} \mid x^p := \sum_{n \in \mathbb{N}} |x_n|^p e_n \in X \Big\},$$

and endow X^p with the norm $\|x\|_p = \|x^p\|^{1/p}$, for all $x \in X^p$. By abuse of notation, we denote by $(e_n)_{n=1}^{\infty}$ the sequence of coordinate vectors in X^p. It is clear that $(e_n)_{n=1}^{\infty}$ is a 1-unconditional basis for X^p and that $X^1 = X$. Also, the triangle inequality gives us that X^p is p-convex with constant 1.

6.1.3 Asymptotically p-uniformly smooth and convex spaces.

Let X be a Banach space. We define the *modulus of asymptotic uniform smoothness of* X as

$$\overline{\rho}_X(t) = \sup_{x\in\partial B_X} \inf_{\dim(X/E)<\infty} \sup_{h\in\partial B_E} \|x+th\| - 1.$$

We say that X is *asymptotically uniformly smooth* if $\lim_{t\to 0_+} \overline{\rho}_X(t)/t = 0$. If there exists $p \in (1,\infty)$ and $C > 0$ such that $\overline{\rho}_X(t) \leqslant Ct^p$, for all $t \in [0,1]$, we say that X is *asymptotically p-uniformly smooth*. Every asymptotically uniformly smooth Banach space is asymptotically p-uniformly smooth for some $p \in (1,\infty)$ (this was first proved in [KnOSc] for separable Banach spaces, and later generalized for any Banach space in [Ra], Theorem 1.2).

Let X be a Banach space. We define the *modulus of asymptotic uniform convexity of* X as

$$\overline{\delta}_X(t) = \inf_{x\in\partial B_X} \sup_{\dim(X/E)<\infty} \inf_{h\in\partial B_E} \|x+th\| - 1.$$

We say that X is *asymptotically uniformly convex* if $\overline{\delta}_X(t) > 0$, for all $t > 0$. If there exists $p \in (1,\infty)$ and $C > 0$ such that $\overline{\delta}_X(t) \geqslant Ct^p$, for all $t \in [0,1]$, we say that X is *asymptotically p-uniformly convex*.

The following proposition is straight forward.

Proposition 6.1.1. *Let $p \in (1,\infty)$ and let X be a Banach space with a 1-unconditional basis satisfying an upper ℓ_p-estimate (resp. lower ℓ_p-estimate) with constant 1. Then X is asymptotically p-uniformly smooth (resp. asymptotically p-uniformly convex).*

6.1.4 Banach-Saks properties.

A Banach space X is said to have the *Banach-Saks property* if every bounded sequence $(x_n)_{n=1}^\infty$ in X has a subsequence $(x_{n_j})_{j=1}^\infty$ such that $(\frac{1}{k}\sum_{j=1}^k x_{n_j})_{k=1}^\infty$ converges. A Banach space X is said to have the *alternating Banach-Saks property* if every bounded sequence $(x_n)_{n=1}^\infty$ in X has a subsequence $(x_{n_j})_{j=1}^\infty$ such that $(\frac{1}{k}\sum_{j=1}^k \varepsilon_j x_{n_j})_{k=1}^\infty$ converges, for some $(\varepsilon_j)_{j=1}^\infty \in \{-1,1\}^{\mathbb{N}}$. For a detailed study of this properties, we refer to [Be].

Let $p \in (1,\infty)$. A Banach space X is said to have the *p-Banach-Saks property* (resp. *p-co-Banach-Saks property*), if for every semi-normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X, there exists a subsequence $(x_{n_j})_{j=1}^\infty$ and $c > 0$ such that

$$\|x_{n_1} + \ldots + x_{n_k}\| \leqslant ck^{1/p} \quad (\text{resp.} \quad \|x_{n_1} + \ldots + x_{n_k}\| \geqslant ck^{1/p}),$$

for all $k \in \mathbb{N}$, and all $k \leqslant n_1 < \ldots < n_k$.

The following is a combination of Proposition 1.2, Proposition 1.3, and Proposition 1.6 of [DimGoJ] (Proposition 1.6 of [DimGoJ] only mentions the p-Banach-Saks property, but a straight forward modification of their proof gives us the result for the p-co-Banach-Saks property).

Proposition 6.1.2. *Let $p \in (1,\infty)$ and let X be a Banach space. If X asymptotically p-uniformly smooth (resp. asymptotically p-uniformly convex), then X has the p-Banach-Saks property (resp. p-co-Banach-Saks property)*

6.1.5 Convexifications of the Tsirelson and Schlumprecht spaces.

As in Subsection 2.2.3, we define the Tsirelson space T as the completion of c_{00} under the unique norm $\|\cdot\|$ satisfying

$$\|x\| = \max\Big\{\|x\|_0, \frac{1}{2}\cdot\sup\Big(\sum_{j=1}^k \|E_j x\|\Big)\Big\},$$

where the inner supremum above is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $k \leqslant E_1 < \ldots < E_k$. Therefore, for each $p \in (1,\infty)$, the norm $\|\cdot\|_p$ of the p-convexified Tsirelson space T^p satisfies

$$\|x\|_p = \max\Big\{\|x\|_0, \frac{1}{2^{1/p}}\cdot\sup\Big(\sum_{j=1}^k \|E_j x\|_p^p\Big)^{1/p}\Big\},$$

where the inner supremum above is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $k \leqslant E_1 < \ldots < E_k$ (see [CSh], Chapter X, Section E).

As T^p satisfies an upper ℓ_p-estimate with constant 1, it follows that T^p is asymptotically p-uniformly smooth and it has the p-Banach-Saks property. Also, T^p has the p-co-Banach-Saks property. Indeed, let $(e_n)_{n=1}^\infty$ be the standard basis for T^p. If $(x_n)_{n=1}^\infty$ is a normalized block subsequence of $(e_n)_{n=1}^\infty$, then

$$2^{-1/p}k^{1/p} = 2^{-1/p}\Big(\sum_{n=k}^{2k-1}\|x_n\|_p^p\Big)^{1/p} \leqslant \Big\|\sum_{n=k}^{2k-1}x_n\Big\|_p,$$

for all $k \in \mathbb{N}$. Therefore, as for any normalized weakly null sequence $(x_n)_{n=1}^\infty$ in T^p, one can find a block sequence $(y_n)_{n=1}^\infty$ which is equivalent to a subsequence of $(x_n)_{n=1}^\infty$, we conclude that T^p has the p-co-Banach-Saks.

Remark 6.1.3. Let $p \in (1,\infty)$. Then T^p does not contain ℓ_r for any $r \in [1,\infty)$ (this is shown in [Jo2] for T, and the result for T^p follows analogously). Similarly, by duality arguments, T^{p*} does not contain ℓ_r for any $r \in [1,\infty)$ (the reader can find more on T^p and similar duality arguments in [CSh]).

As in Subsection 2.2.3, we define the Schlumprecht space S as the completion of c_{00} under the unique norm $\|\cdot\|$ satisfying

$$\|x\| = \max\Big\{\|x\|_0, \sup\Big(\frac{1}{\log_2(k+1)}\sum_{j=1}^k\|E_jx\|\Big)\Big\},$$

where the inner supremum above is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $E_1 < \ldots < E_k$. Similarly as with the p-convexified Tsirelson space, the norm $\|\cdot\|_p$ of the p-convexified Schlumprecht space S^p is given by

$$\|x\|_p = \max\Big\{\|x\|_0, \sup\Big(\frac{1}{\log_2(k+1)}\sum_{j=1}^k\|E_jx\|_p^p\Big)^{1/p}\Big\},$$

where the inner supremum above is taken over all finite sequences $(E_j)_{j=1}^k$ of finite subsets of $\mathbb{N}$ such that $E_1 < \ldots < E_k$ (see [D], page 59).

Similarly to T^p, S^p is asymptotically p-uniformly smooth and has the p-Banach-Saks property, for $p \in (1, \infty)$.

6.1.6 Almost p-co-Banach-Saks property.

Although T^p has the p-co-Banach-Saks property, S^p does not. However, S^p satisfies a weaker property that will be enough for our goals. Let $p \in (1, \infty)$. We say that a Banach space X has the *almost p-co-Banach-Saks property* if for every semi-normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X there exists a subsequence $(x_{n_j})_{j=1}^\infty$, and a sequence of positive numbers $(\theta_j)_{j=1}^\infty$ in $[1, \infty)$ such that $\lim_{j\to\infty} j^\alpha \theta_j^{-1} = \infty$, for all $\alpha > 0$, and

$$\|x_{n_1} + \ldots + x_{n_k}\| \geqslant k^{1/p}\theta_k^{-1},$$

for all $k \in \mathbb{N}$, and all $k \leqslant n_1 < \ldots < n_k$. Clearly, S^p has the almost p-co-Banach-Saks property, with $\theta_k = \log_2(k+1)^{1/p}$, for all $k \in \mathbb{N}$.

6.2 Asymptotic uniform smoothness and the alternating Banach-Saks property.

In this section, we are going to show that asymptotically uniformly smooth Banach spaces must have the alternating Banach-Saks property (Corollary 6.2.2), but the converse does not hold (see Proposition 6.2.8). Also, we show that if a Banach space X coarse Lipschitz embeds into a reflexive space Y which is also asymptotically uniformly smooth, then X must have the Banach-Saks property (Theorem 1.6.1). As any space with the Banach-Saks property is reflexive, this is a strengthening of Theorem 4.1 of [BKaL], which says that, under the same hypothesis, X must be reflexive.

Proposition 6.2.1. *Let X be a Banach space with the p-Banach-Saks property, for some $p \in (1, \infty)$, and assume that X does not contain ℓ_1. Then X has the alternating Banach-Saks property. In particular, if X is also reflexive, then X has the Banach-Saks property.*

Proof. Assume X does not have the alternating Banach-Saks property. Then, there exist $\delta > 0$ and a bounded sequence $(x_n)_{n=1}^{\infty}$ in X such that, for all $k \in \mathbb{N}$, all $\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}$, and all $n_1 < \ldots < n_k \in \mathbb{N}$, we have

$$\Big\|\frac{1}{k}\sum_{j=1}^{k} \varepsilon_j x_{n_j}\Big\| > \delta \tag{6.2.1}$$

(see [Be], Theorem 1, page 369). As X does not contain ℓ_1, by Rosenthal's ℓ_1-theorem (see [Ros]), we can assume that $(x_n)_{n=1}^{\infty}$ is weakly Cauchy. Hence, the sequence $(x_{2n-1} - x_{2n})_{n=1}^{\infty}$ is weakly null. By Equation (6.2.1), it is also semi-normalized. Therefore, as X has the p-Banach-Saks property, by taking a subsequence if necessary, we have that

$$\Big\|\sum_{j=1}^{k}(x_{n_{2j-1}} - x_{n_{2j}})\Big\| \leqslant ck^{1/p},$$

for all $k \in \mathbb{N}$, and some constant $c > 0$ independent of k. By Equation (6.2.1), we get that

$$\delta < \Big\|\frac{1}{2k}\sum_{j=1}^{2k}(-1)^{j+1} x_{n_j}\Big\| \leqslant \frac{c}{2}k^{1/p-1}.$$

As this holds for all $k \in \mathbb{N}$, and $p > 1$, if we let $k \to \infty$, we get that $\delta = 0$, which is a contradiction.

For reflexive spaces, the alternating Banach-Saks property and the Banach-Saks property are equivalent (see [Be], Proposition 2), so the last statement of the proposition follows. □

Corollary 6.2.2. *Let X be an asymptotically uniformly smooth Banach space. Then X has the alternating Banach-Saks property. In particular, if X is also reflexive, then X has the Banach-Saks property.*

Proof. As X is asymptotically uniformly smooth, X cannot contain ℓ_1. Therefore, we only need to notice that X has the p-Banach-Saks property, for some $p \in (1, \infty)$, and apply Proposition 6.2.1. By Theorem 1.2 of [Ra], X is asymptotically p-uniformly smooth, for some $p \in (1, \infty)$. Therefore, by Proposition 6.1.2 above, we have that X has the p-Banach-Saks property, so we are done. □

For each $k \in \mathbb{N}$, we want to define a new metric on $\mathcal{P}_k(\mathbb{N})$ (see Section 1.4). In order to avoid confusion, we use a diferent notation for $\mathcal{P}_k(\mathbb{N})$ in this chapter. Let $k \in \mathbb{N}$, and $\mathbb{M} \subset \mathbb{N}$, be an infinite subset. Define $G_k(\mathbb{M})$ as the set of all subsets of $\mathbb{M}$ with k elements. We write $\bar{n} = (n_1, \dots, n_k) \in G_k(\mathbb{M})$ always in an increasing order, i.e., $n_1 < \dots < n_k$. We define a metric $d = d_k$ on $G_k(\mathbb{M})$ by letting

$$d(\overline{n}, \overline{m}) = |\{j \mid n_j \neq m_j\}|,$$

for all $\overline{n} = (n_1, \dots, n_k), \overline{m} = (m_1, \dots, m_k) \in G_k(\mathbb{M})$.

The following will play an important role in many of the results in this chapter. This result was proved in [KaRa], Theorem 4.2 (see also Theorem 6.1 of [KaRa]).

Theorem 6.2.3. *Let $p \in (1, \infty)$, and let Y be a reflexive asymptotically p-uniformly smooth Banach space. There exists $K > 0$ such that, for all infinite subset $\mathbb{M} \subset \mathbb{N}$, all $k \in \mathbb{N}$, and all bounded map $f : G_k(\mathbb{M}) \to Y$, there exists an infinite subset $\mathbb{M}' \subset \mathbb{M}$ such that*

$$diam(f(G_k(\mathbb{M}'))) \leqslant K\, Lip(f) k^{1/p}.$$

Proof of Theorem 1.6.1. Let $f : X \to Y$ be a coarse Lipschitz embedding. Pick $C > 0$ so that $\omega_f(t) \leqslant Ct + C$, $\rho_f(t) \geqslant C^{-1}t - C$, for all $t \geqslant 0$. Assume that X does not have the Banach-Saks property. By [Be], page 373, there exists $\delta > 0$ and a sequence $(x_n)_{n=1}^{\infty}$ in B_X

such that, for all $k \in \mathbb{N}$, and all $n_1 < \ldots < n_{2k} \in \mathbb{N}$, we have that

$$\Big\| \frac{1}{2k} \sum_{j=1}^{k} (x_{n_j} - x_{n_{k+j}}) \Big\| \geqslant \delta.$$

For each $k \in \mathbb{N}$, define $\varphi_k : G_k(\mathbb{N}) \to X$ by setting $\varphi_k(n_1, \ldots, n_k) = x_{n_1} + \ldots + x_{n_k}$, for all $(n_1, \ldots, n_k) \in G_k(\mathbb{N})$. Therefore, $\mathrm{diam}(\varphi_k(G_k(\mathbb{M}))) \geqslant 2k\delta$, and we have that $\mathrm{diam}(f \circ \varphi_k(G_k(\mathbb{M}))) \geqslant 2k\delta C^{-1} - C$, for all $k \in \mathbb{N}$, and all infinite $\mathbb{M} \subset \mathbb{N}$.

As, $\mathrm{Lip}(\varphi_k) \leqslant 2$, we have that $\mathrm{Lip}(f \circ \varphi_k) \leqslant 3C$. As Y is asymptotically uniformly smooth, there exists $p \in (1, \infty)$ for which Y is asymptotically p-uniformly smooth (see [Ra], Theorem 1.2). By Theorem 6.2.3, there exists $K = K(Y) > 0$ and $\mathbb{M} \subset \mathbb{N}$ such that $\mathrm{diam}(f \circ \varphi_k(G_k(\mathbb{M}))) \leqslant 3KCk^{1/p}$, for all $k \in \mathbb{N}$. We conclude that

$$2k\delta C^{-1} - C \leqslant 3KCk^{1/p},$$

for all $k \in \mathbb{N}$. As $p > 1$, this gives us a contradiction if we let $k \to \infty$. $\square$

The following was asked in [GLZi], Problem 2, and it remains open.

Problem 6.2.4. If a Banach space X coarse Lipschitz embeds into a reflexive asymptotically uniformly smooth Banach space Y, does it follow that X has an asymptotically uniformly smooth renorming?

Problem 6.2.5. Let N be a metric space. We say that a family of metric spaces $(M_k)_{k=1}^\infty$ *uniformly Lipschitz embeds into* N if there exists $C > 0$ and Lipschitz embeddings $f_k : M_k \to N$ such that $\mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1}) < C$, for all $k \in \mathbb{N}$. Does the family $(G_k(\mathbb{N}), d)_{k=1}^\infty$ uniformly Lipschitz embed into any Banach space without an asymptotically uniformly smooth renorming?

As noticed in [GLZi], Problem 6, a positive answer to Problem 6.2.5 together with Theorem 6.2.3 would give us a positive answer to Problem 6.2.4.

It is worth noticing that the Banach-Saks property is not stable under uniform equivalences, hence, it is not stable under coarse Lipschitz isomorphisms either. Indeed, if $(p_n)_{n=1}^\infty$ is a sequence in $(1,\infty)$ converging to 1, then $(\oplus_n \ell_{p_n})_{\ell_2}$ is uniformly equivalent to $(\oplus_n \ell_{p_n})_{\ell_2} \oplus \ell_1$ (see [BenLi], page 244). The space $(\oplus_n \ell_{p_n})_{\ell_2}$ has the Banach-Saks property, while $(\oplus_n \ell_{p_n})_{\ell_2} \oplus \ell_1$ does not.

Let $\mathcal{G}(\mathbb{N})$ denote the set of finite subsets of $\mathbb{N}$. We endow $\mathcal{G}(\mathbb{N})$ with the metric D given by

$$D(\overline{n}, \overline{m}) = |\overline{n} \Delta \overline{m}|,$$

for all $\overline{n} = (n_1, \ldots, n_k), \overline{m} = (m_1, \ldots, m_l) \in \mathcal{G}(\mathbb{N})$, where $\overline{n} \Delta \overline{m}$ denotes the symmetric difference between the sets $\overline{n}$ and $\overline{m}$.

Proposition 6.2.6. *$\mathcal{G}(\mathbb{N})$ Lipschitz embeds into any Banach space X without the alternating Banach-Saks property. Moreover, for any $\varepsilon > 0$, the Lipschitz embedding $f : \mathcal{G}(\mathbb{N}) \to X$ can be chosen so that $Lip(f) \cdot Lip(f^{-1}) < 1 + \varepsilon$.*

Proof. By Theorem 1 of [Be], page 369, for all $\eta > 0$, there exists a bounded sequence $(x_n)_{n=1}^{\infty}$ in X such that, for all $k \in \mathbb{N}$, all $\varepsilon_1, \ldots, \varepsilon_k \in \{-1, 1\}$, and all $n_1 < \ldots < n_k$, we have

$$1 - \eta \leqslant \left\| \frac{1}{k} \sum_{j=1}^{k} \varepsilon_j x_{n_j} \right\| \leqslant 1 + \eta.$$

Define $\varphi : \mathcal{G}(\mathbb{N}) \to X$ by setting $\varphi(n_1, \ldots, n_k) = x_{n_1} + \ldots + x_{n_k}$, for all $(n_1, \ldots, n_k) \in \mathcal{G}(\mathbb{N}) \backslash \{\varnothing\}$, and $\varphi(\varnothing) = 0$. Then, we have that

$$(1 - \eta) \cdot D(\overline{n}, \overline{m}) \leqslant \|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \leqslant (1 + \eta) \cdot D(\overline{n}, \overline{m})$$

for all $\overline{n}, \overline{m} \in \mathcal{G}(\mathbb{N})$. □

Problem 6.2.7. If X has the Banach-Saks property, does it follow that $\mathcal{G}(\mathbb{N})$ does not Lipschitz embed into X? In other words, if X is a reflexive Banach space, do we have that $\mathcal{G}(\mathbb{N})$ Lipschitz embed into X if and only if X does not have the Banach-Saks property?

By Corollary 6.2.2 above, any Banach space with an asymptotically uniformly smooth renorming has the alternating Banach-Saks property. To the best of our knowledge, there is no known example of a Banach space which has the alternating Banach-Saks property but does not admit an asymptotically uniformly smooth renorming. However, using descriptive set theoretical arguments, one can show the existence of such spaces. Recall, (X, Ω) is called a *standard Borel space* if X is a set and Ω is a σ-algebra on X which is the Borel σ-algebra associated to a Polish topology on X (i.e., a topology generated by a complete separable metric). A subset $A \subset X$ is called *analytic* if it is the image of a standard Borel space under a Borel map. We refer to [Do] and [Br1], Section 2, for more details on the descriptive set theory of separable Banach spaces.

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$ endowed with the supremum norm. Let

$$\mathrm{SB} = \{X \in C[0,1] \mid X \text{ is a closed linear subspace}\},$$

and endow SB with the Effros-Borel structure, i.e., the σ-algebra generated by

$$\{X \in \mathrm{SB} \mid X \cap U \neq \varnothing\}, \quad \text{for} \quad U \subset C[0,1] \quad \text{open.}$$

This makes SB into a standard Borel space and, as $C[0,1]$ contains isometric copies of every separable Banach space, SB can be seen as a coding set for the class of all separable Banach spaces. Therefore, we can talk about Borel and analytic classes of separable Banach spaces.

By [Br1], Theorem 17, the subset ABS $\subset$ SB of Banach spaces with the alternating Banach-Saks is not analytic. On the other hand, letting AUS $= \{X \in \mathrm{SB} \mid X$ is asymptotically uniformly smooth$\}$, we have

$$X \in \mathrm{AUS} \Leftrightarrow \forall \varepsilon \in \mathbb{Q}_+ \exists \delta \in \mathbb{Q}_+ \forall t \in \mathbb{Q}_+ \Big(t < \delta \Rightarrow \overline{\rho}_X(t) < \varepsilon t\Big).$$

As $\{X \in \mathrm{SB} \mid \dim(C[0,1]/X) < \infty\}$ is Borel, it is easy to check that the condition $A(t, \varepsilon) \subset$

SB given by

$$X \in A(t, \varepsilon) \Leftrightarrow \overline{\rho}_X(t) < \varepsilon t$$

defines an analytic subset of SB (for similar arguments, we refer to [Do], Chapter 2, Section 2.1). So, AUS must be analytic. Hence, letting AUSable $\subset$ SB be the subset of Banach spaces with an asymptotically uniformly smooth renorming, we have that

$$X \in \text{AUSable} \Leftrightarrow \exists Y \in \text{AUS} \;\text{ such that }\; X \cong Y.$$

As the isomorphism relation in SB $\times$ SB forms an analytic set (see [Do], page 11), it follows that AUSable is analytic. This discussion together with Corollary 6.2.2 gives us the following.

Proposition 6.2.8. *AUSable $\subsetneq$ ABS. In particular, there exist separable Banach spaces with the alternating Banach-Saks property which do not admit an asymptotically uniformly smooth renorming.*

6.3 Asymptotically p-uniformly convex/smooth spaces.

In this section, we will use results from [KaRa] in order to obtain some restrictions on coarse embeddings $X \to Y$, where the spaces X and Y are assumed to have some asymptotic properties (see Theorem 6.3.1). We obtain restrictions on the existence of coarse embeddings between the convexified Tsirelson spaces (Theorem 1.6.3(i)), convexified Schlumprecht spaces (Theorem 1.6.3(ii)), and some specific hereditarily indecomposable spaces introduced in [D] (Corollary 6.3.8).

Theorem 6.3.1. *Let $p, q \in (1, \infty)$. Let X be an infinite dimensional Banach space with the p-co-Banach-Saks property and not containing ℓ_1. Let Y be a reflexive asymptotically q-uniformly smooth Banach space. Then, there exists no coarse embedding $f : X \to Y$ such that*

$$\limsup_{k\to\infty} \frac{\rho_f(k^{1/p})}{k^{1/q}} = \infty.$$

Proof. Let $f : X \to Y$ be a coarse embedding. So, there exists $C > 0$, such that $\omega_f(t) \leqslant Ct + C$, for all $t > 0$. As X does not contain ℓ_1, by Rosenthal's ℓ_1-theorem, we can pick a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X, with $\inf_{n \neq m} \|x_n - x_m\| > 0$. For each $k \in \mathbb{N}$, define a map $\varphi_k : G_k(\mathbb{N}) \to X$ by letting

$$\varphi_k(n_1, \ldots, n_k) = x_{n_1} + \ldots + x_{n_k},$$

for all $(n_1, \ldots, n_k) \in G_k(\mathbb{N})$. So, φ_k is a bounded map.

If $d((n_1, \ldots, n_k), (m_1, \ldots, m_k)) \leqslant 1$, then $\| \sum_{j=1}^k x_{n_j} - \sum_{j=1}^k x_{m_j} \| \leqslant 2$. So, $\mathrm{Lip}(f \circ \varphi_k) \leqslant 3C$. By Theorem 6.2.3, there exists $K = K(Y) > 0$ and an infinite subset $\mathbb{M}_k \subset \mathbb{N}$ such that

$$\mathrm{diam}(f \circ \varphi_k(G_k(\mathbb{M}_k))) \leqslant 3KCk^{1/q}.$$

Without loss of generality, we may assume that $\mathbb{M}_{k+1} \subset \mathbb{M}_k$, for all $k \in \mathbb{N}$. Let $\mathbb{M} \subset \mathbb{N}$ diagonalize the sequence $(\mathbb{M}_k)_{k=1}^\infty$, say $\mathbb{M} = (n_j)_{j=1}^\infty$. If a sequence $(y_n)_{n=1}^\infty$ is weakly null, so is $(y_{2n-1} - y_{2n})_{n=1}^\infty$. Therefore, using the fact that X has the p-co-Banach-Saks property to the weakly null sequence $(x_{n_{2j-1}} - x_{n_{2j}})_{j=1}^\infty$, we get that there exists $c > 0$ such that, for all $k \in \mathbb{N}$, there exists $m_1 < \ldots < m_{2k} \in \mathbb{M}_k$, such that

$$\Big\| \sum_{j=1}^k (x_{m_{2j-1}} - x_{m_{2j}}) \Big\| \geqslant ck^{1/p}.$$

Therefore, we have that $\mathrm{diam}(\varphi_k(G_k(\mathbb{M}_k))) \geqslant ck^{1/p}$, which implies that $\mathrm{diam}(f \circ \varphi_k(G_k(\mathbb{M}_k))) \geqslant \rho_f(ck^{1/p})$, for all $k \in \mathbb{N}$. So,

$$\rho_f(ck^{1/p}) \leqslant 3KCk^{1/q},$$

for all $k \in \mathbb{N}$. Therefore, if $\limsup_{k \to \infty} \rho_f(k^{1/p}) k^{-1/q} = \infty$, we get a contradiction. □

Remark 6.3.2. Let X be any Banach space containing a sequence $(x_n)_{n=1}^\infty$ which is asymptotically ℓ_1, i.e., there exists $L > 0$ such that, for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$(x_{n_j})_{j=1}^m$ is L-equivalent to $(e_j)_{j=1}^m$, for all $k \leqslant n_1 < \ldots < n_m \in \mathbb{N}$, where $(e_j)_{j=1}^\infty$ is the standard ℓ_1-basis. Then, proceeding exactly as above, we can show that there exists no coarse embedding $f : X \to Y$ such that

$$\limsup_{k\to\infty} \frac{\rho_f(k)}{k^{1/q}} = \infty,$$

where $q \in (1, \infty)$ and Y is a reflexive asymptotically q-uniformly smooth Banach space.

Let X and Y be Banach spaces. We define $\alpha_Y(X)$ as the supremum of all $\alpha > 0$ for which there exists a coarse embedding $f : X \to Y$ and $L > 0$ such that

$$L^{-1}\|x - y\|^\alpha - L \leqslant \|f(x) - f(y)\|,$$

for all $x, y \in X$. We call $\alpha_Y(X)$ the *compression exponent of X in Y*, or the *Y-compression of X*. If, for all $\alpha > 0$, no such f and L exist, we set $\alpha_Y(X) = 0$. As ω_f is always bounded by an affine map (as X is a Banach space), it follows that $\alpha_Y(X) \in [0, 1]$. Also, $\alpha_Y(X) = 0$ if X does not coarsely embed into Y.

The quantity $\alpha_Y(X)$ was first introduced by E. Guentner and J. Kaminker in [GuKa]. For a detailed study of $\alpha_{\ell_q}(\ell_p)$, $\alpha_{L_q}(\ell_p)$, $\alpha_{\ell_q}(L_p)$, and $\alpha_{L_q}(L_p)$, where $p, q \in (0, \infty)$, we refer to [B].

Using this terminology, let us reinterpret Theorem 6.3.1.

Theorem 6.3.3. *Let $1 < p < q$. Let Y be a reflexive asymptotically q-uniformly smooth Banach space. The following holds.*

(i) If X contains a sequence which is asymptotically ℓ_1, then $\alpha_Y(X) \leqslant 1/q$.

(ii) If X is an infinite dimensional Banach space with the p-co-Banach-Saks property and not containing ℓ_1, then $\alpha_Y(X) \leqslant p/q$.

In particular, X does not coarse Lipschitz embed into Y.

Proof. (ii) Let $L > 0$ and $f : X \to Y$ be a coarse embedding such that $\rho_f(t) \geqslant L^{-1}t^\alpha - L$, for all $t > 0$. By Theorem 6.3.1, we must have

$$\limsup_{k\to\infty} k^{\alpha/p-1/q}L^{-1} - Lk^{-1/q} < \infty.$$

Therefore, $\alpha/p - 1/q \leqslant 0$, and the result follows.

(i) This follows from Remark 6.3.2 and the same reasoning as item (ii) above. □

Notice that Y being reflexive in Theorem 6.3.3 cannot be removed. Indeed, c_0 contains a Lipschitz copy of any separable metric space (see [A]), and it is also asymptotically q-uniformly smooth, for any $q \in (1, \infty)$.

Corollary 6.3.4. *Let $1 < p < q$. Let X be asymptotically p-uniformly convex, and Y be reflexive and asymptotically q-uniformly smooth. Then $\alpha_Y(X) \leqslant p/q$.*

Asking the Banach space X to have the p-co-Banach-Saks property in Theorem 6.3.3 is actually too much, and we can weaken this condition by only requiring X to have the almost p-co-Banach-Saks property. Precisely, we have the following.

Theorem 6.3.5. *Let $1 < p < q$. Let X be an infinite dimensional Banach space with the almost p-co-Banach-Saks property. Let Y be a reflexive asymptotically q-uniformly smooth Banach space. Then $\alpha_Y(X) \leqslant p/q$. In particular, X does not coarse Lipschitz embed into Y.*

Proof. Let $f : X \to Y$ be a coarse embedding and pick $C > 0$ such that $\omega_f(t) \leqslant Ct + C$, for all $t \geqslant 0$. If X contains ℓ_1, the result follows from Theorem 6.3.3(i). If X does not contain ℓ_1, we can pick a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X, with $\inf_{n\neq m} \|x_n - x_m\| > 0$. By taking a subsequence of $(x_n)_{n=1}^\infty$ if necessary, pick $(\theta_k)_{k=1}^\infty$ as in the definition of the almost p-co-Banach-Saks property. Define $\varphi_k : G_k(\mathbb{N}) \to X$ by letting $\varphi_k(n_1, \ldots, n_k) = x_{n_1} + \ldots + x_{n_k}$, for all $(n_1, \ldots, n_k) \in G_k(\mathbb{N})$.

Following the proof of Theorem 6.3.1, we get that

$$\rho_f(k^{1/p}\theta_k^{-1}) \leqslant 3KCk^{1/q},$$

for all $k \in \mathbb{N}$. Let $L > 0$ and $\alpha > 0$ be such that $\rho_f(t) \geqslant L^{-1}t^\alpha - L$, for all $t > 0$. Then,

$$k^{\alpha/p-1/q}\theta_k^{-\alpha}L^{-1} \leqslant 4KC,$$

for big enough $k \in \mathbb{N}$. As $\lim_{k\to\infty} k^\beta\theta_k^{-\alpha} = \infty$, for all $\beta > 0$, we must have that $\alpha/p - 1/q \leqslant 0$. □

Remark 6.3.6. Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space X with the following property: there exists a sequence of positive reals $(\theta_j)_{j=1}^\infty$ in $[1,\infty)$ such that $\lim_{j\to\infty} j^\alpha\theta_j^{-1} = \infty$, for all $\alpha > 0$, and

$$k\theta_k^{-1} \leqslant \| \pm x_{n_1} + \ldots + \pm x_{n_k}\|, \tag{$*$}$$

for all $n_1 < \ldots < n_k \in \mathbb{N}$. The proof of Theorem 6.3.5 gives us that $\alpha_Y(X) \leqslant 1/q$, for any reflexive asymptotically q-uniformly smooth Banach space Y, with $q > 1$.

Let $q > 1$, and let $(E_n)_{n=1}^\infty$ be a sequence of finite dimensional Banach spaces. Let $\mathcal{E}$ be a 1-unconditional basic sequence. Notice that, if $\mathcal{E}$ generates a reflexive asymptotically q-uniformly smooth Banach space, then $(\oplus_n E_n)_\mathcal{E}$ is also reflexive and asymptotically q-uniformly smooth. Hence, Theorem 6.3.3 and Theorem 6.3.5 gives us the following corollary.

Corollary 6.3.7. *Let $1 < p < q$, and let $(E_n)_{n=1}^\infty$ be a sequence of finite dimensional Banach spaces. Let $\mathcal{E}$ be a 1-unconditional basic sequence generating a reflexive asymptotically q-uniformly smooth Banach space. The following holds.*

(i) If X contains a sequence with Property $(*)$, then $\alpha_{(\oplus_n E_n)_\mathcal{E}}(X) \leqslant 1/q$.

(ii) If X is an infinite dimensional Banach space with the almost p-co-Banach-Saks property, then $\alpha_{(\oplus_n E_n)_\mathcal{E}}(X) \leqslant p/q$.

In particular, X does not coarse Lipschitz embed into $(\oplus_n E_n)_\mathcal{E}$.

Proof of Theorem 1.6.3. (i) As noticed in Subsection 6.1.5, T^p has the p-co-Banach-Saks property, and is asymptotically p-uniformly smooth, for all $p \in (1, \infty)$. Therefore, as T^p is reflexive (see [OScZs], Proposition 5.3(b)), for all $p \in [1, \infty)$, the result follows from Theorem 6.3.3 (or Corollary 6.3.7).

(ii) For any $p \in (1, \infty)$, S^p has the almost p-co-Banach-Saks property and is asymptotically p-uniformly smooth. By Theorem 8 and Proposition 2(2) of [CKaKutMa], S^p is reflexive, for all $p \in [1, \infty)$. So, the result follows from Corollary 6.3.7. □

A Banach space X is called *hereditarily indecomposable* if none of its subspaces can be decomposed as a sum of two infinite dimensional Banach spaces. In Chapter 5 of [D], for each $p \in (1, \infty)$, Dew constructed a hereditarily indecomposable space $\mathfrak{X}_p$ with a basis $(e_n)_{n=1}^\infty$ satisfying the following properties: (i) $\mathfrak{X}_p$ is reflexive, (ii) the base $(e_n)_{n=1}^\infty$ satisfies an upper ℓ_p-estimate with constant 1, and (iii) if $(x_n)_{n=1}^\infty$ is a block sequence of $(e_n)_{n=1}^\infty$, then, for all $n \in \mathbb{N}$,

$$\Big\|\sum_{j=1}^n x_j\Big\| \geqslant f(n)^{-1/p}\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p},$$

where $f : \mathbb{N} \to [0, \infty)$ is a function such that, among other properties, $\lim_{n\to\infty} n^\alpha f(n)^{-1} = \infty$, for all $\alpha > 0$. In particular, $\mathfrak{X}_p$ has the almost p-co-Banach-Saks property, and it is asymptotically p-uniformly smooth. This, together with Theorem 6.3.5, gives us the following.

Corollary 6.3.8. *Let $1 < p < q$. Then $\alpha_{\mathfrak{X}^q}(\mathfrak{X}^p) \leqslant p/q$. In particular, $\mathfrak{X}_p$ does not coarse Lipschitz embeds into $\mathfrak{X}_q$.*

Problem 6.3.9. Let $1 \leqslant p < q$. Does $\alpha_{T^q}(T^p) = \alpha_{S^q}(S^p) = p/q$? If $p > 1$, does $\alpha_{\mathfrak{X}^q}(\mathfrak{X}^p) = p/q$ hold?

Remark 6.3.10. It is worth noticing that, if $p > \max\{q, 2\}$, then $\alpha_{T^q}(T^p) = 0$. Indeed, for all $r \geqslant 2$, T^r has cotype $r + \varepsilon$ for all $\varepsilon > 0$ (see [DiJT], page 305). On the other hand, if $r < 2$, then T^r has cotype 2. This follows from the fact that, for any $\varepsilon > 0$, T^r has an equivalent norm satisfying a lower $\ell_{(r+\varepsilon)}$-estimate (we explain this in the proof of Corollary 1.6.4 below), then, by Theorem 1.f.7 and Proposition 1.f.3(i) of [LiTz], T^r has cotype 2. Similarly, by Theorem 1.f.7 and Proposition 1.f.3(ii), T^r has nontrivial type, for all $r \in (1,\infty)$. By Theorem 1.11 of [MeN2], if a Banach space X coarsely embeds into a Banach space Y with nontrivial type, then

$$\inf\{q \in [2,\infty) \mid X \text{ has cotype } q\} \leqslant \inf\{q \in [2,\infty) \mid Y \text{ has cotype } q\}.$$

Therefore, we conclude that T^p does not coarsely embed into T^q, if $p > \max\{q,2\}$. So, $\alpha_{T^q}(T^p) = 0$.

Problem 6.3.11. Let $1 \leqslant q < p \leqslant 2$. What can we say about $\alpha_{T^q}(T^p)$?

We finish this section with an application of Theorem 6.3.3, Theorem 6.3.5, and Theorem 3.4 of [AlB]. By looking at the proof of Theorem 3.4 of [AlB], one can easily see that the authors proved a stronger result than the one stated in their paper. Precisely, the authors proved the following.

Theorem 6.3.12. *Let $0 < p < q$. There exist maps $(\psi_j : \mathbb{R} \to \mathbb{R})_{j=1}^\infty$ such that, for all $x, y \in \mathbb{R}$,*

$$A_{p,q}|x-y|^p \leqslant \max\{|\psi_j(x) - \psi_j(y)|^q \mid j \in \mathbb{N}\}$$

and

$$\sum_{j\in\mathbb{N}} |\psi_j(x) - \psi_j(y)|^q \leqslant B_{p,q}|x-y|^p,$$

where $A_{p,q}, B_{p,q}$ are positive constants.

Proposition 6.3.13. *Let $1 \leqslant p < q$. There exists a map $f : T^p \to (\oplus T^q)_{T^q}$ which is simultaneously a coarse and a uniform embedding such that $\rho_f(t) \geqslant Ct^{p/q}$, for some $C > 0$. In particular, $\alpha_{(\oplus T^q)_{T^q}}(T^p) = p/q$.*

Proof. Let $(\psi_j)_{j=1}^\infty$, $A_{p,q}$, and $B_{p,q}$ be given by Theorem 6.3.12. Define $f : T^p \to (\oplus T_q)_{T^q}$ by letting

$$f(x) = \Big((\psi_j(x_n) - \psi_j(0))_{j=1}^\infty\Big)_{n=1}^\infty,$$

for all $x = (x_n)_{n=1}^{\infty} \in T^p$. One can easily check that f satisfies

$$A_{p,q}^{1/q} \|x - y\|^{p/q} \leqslant \|f(x) - f(y)\| \leqslant B_{p,q}^{1/q} \|x - y\|^{p/q},$$

for all $x, y \in T^p$.

As T^q is q-convex, it is easy to see that $(\oplus T^q)_{T^q}$ is asymptotic q-uniformly smooth. Hence, as $(\oplus T^q)_{T^q}$ is reflexive, we conclude that $\alpha_{(\oplus T^q)_{T^q}}(T^p) = p/q$. □

Corollary 6.3.14. *T strongly embeds into a super-reflexive Banach space.*

Proof. It is easy to check that $(\oplus T^2)_{T^2}$ is super-reflexive. Indeed, super-reflexivity is equivalent to a uniformly convex renorming. Hence, if $\mathcal{E}$ is a 1-unconditional basis generating a super-reflexive space, and X is a super-reflexive space, then so is $(\oplus X)_{\mathcal{E}}$ (see [LiTz], page 100). □

Similarly as above, we get the following proposition.

Proposition 6.3.15. *Let $1 \leqslant p < q$. There exists a map $f : S^p \to (\oplus S^q)_{S^q}$ which is simultaneously a coarse and a uniform embedding such that $\rho_f(t) \geqslant Ct^{p/q}$, for some $C > 0$. In particular, $\alpha_{(\oplus S^q)_{S^q}}(S^p) = p/q$.*

6.4 Coarse Lipschitz embeddings into sums.

In this last section, we will be specially interested in the nonlinear geometry of the Tsirelson space and its convexifications. In order to obtain Theorem 1.6.4, we will prove a technical result on the coarse Lipschitz non embeddability of certain Banach spaces into the direct sum of Banach spaces with certain p-properties (Theorem 6.4.6). The main goal of this section is to characterize the Banach spaces which are coarsely (resp. uniformly) equivalent to $T^{p_1} \oplus \ldots \oplus T^{p_n}$, for $p_1, \ldots, p_n \in (1, \ldots, \infty)$, and $2 \notin \{p_1, \ldots, p_n\}$.

Given $x, y \in X$, and $\delta > 0$ the *approximate midpoint between x and y with error δ* is given by

$$\mathrm{Mid}(x,y,\delta) = \{z \in X \mid \max\{\|x-z\|, \|y-z\|\} \leqslant 2^{-1}(1+\delta)\|x-y\|\}.$$

The following lemma is an asymptotic version of Lemma 1.6(i) of [JoLiS] and Lemma 3.2 of [KaRa].

Lemma 6.4.1. *Let X be an asymptotically p-uniformly smooth Banach space, for some $p \in (1,\infty)$. There exists $c > 0$ such that, for all $x, y \in X$, all $\delta > 0$, and all weakly null sequence $(x_n)_{n=1}^\infty$ in B_X, there exists $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, we have*

$$u + \delta^{1/p}\|v\|x_n \in Mid(x,y,c\delta),$$

where $u = \frac{1}{2}(x+y)$, and $v = \frac{1}{2}(x-y)$.

Proof. By Proposition 1.3 of [DimGoJ], there exists $c > 0$ such that, for all weakly null sequence $(x_n)_{n=1}^\infty$ in B_X, we have

$$\limsup_n \|x + x_n\|^p \leqslant \|x\|^p + c \cdot \limsup_n \|x_n\|^p.$$

Fix such sequence. As $\|x - (u + \delta^{1/p}\|v\|x_n)\| = \|v - \delta^{1/p}\|v\|x_n\|$, we get

$$\limsup_n \left\|x - \left(u + \delta^{1/p}\|v\|x_n\right)\right\|^p \leqslant (1+c\delta)\|v\|^p.$$

Therefore, as $(1+c\delta)^{1/p} < 1 + c\delta$, there exists $n_0 \in \mathbb{N}$ such that $\|x - (u + \delta^{1/p}\|v\|x_n)\| \leqslant (1+c\delta)\|v\|$, for all $n > n_0$. Similarly, we can assume that $\|y - (u + \delta^{1/p}\|v\|x_n)\| \leqslant (1+c\delta)\|v\|$, for all $n > n_0$. $\square$

The following lemma is a simple modification of Lemma 3.3 of [KaRa], or Lemma 1.6(ii) of [JoLiS], so we omit its proof.

Lemma 6.4.2. *Suppose $1 \leqslant p < \infty$, and let X be Banach space with a 1-unconditional basis $(e_n)_{n=1}^{\infty}$ satisfying a lower ℓ_p-estimate with constant 1. For all $x, y \in X$, and all $\delta > 0$, there exists a compact subset $K \subset X$, such that*

$$Mid(x, y, \delta) \subset K + 2\delta^{1/p}\|v\|B_X,$$

where $u = \frac{1}{2}(x+y)$, and $v = \frac{1}{2}(x-y)$.

For each $s > 0$, let

$$\mathrm{Lip}_s(f) = \sup_{t \geqslant s} \frac{\omega_f(t)}{t} \quad \text{and} \quad \mathrm{Lip}_\infty(f) = \inf_{s>0} \mathrm{Lip}_s(f).$$

We will need the following proposition, which can be found in [KaRa] as Proposition 3.1.

Proposition 6.4.3. *Let X be a Banach space and M be a metric space. Let $f : X \to M$ be a coarse map with $Lip_\infty(f) > 0$. Then, for all $\varepsilon, t > 0$, and all $\delta \in (0,1)$, there exists $x, y \in X$ with $\|x - y\| > t$ such that*

$$f(Mid(x, y, \delta)) \subset Mid(f(x), f(y), (1+\varepsilon)\delta).$$

The following lemma will play the same role in our settings as Proposition 3.5 did in [KaRa].

Lemma 6.4.4. *Let $1 \leqslant q < p$. Let X be an asymptotically p-uniformly smooth Banach space, and Y be a Banach space with a 1-unconditional basis satisfying a lower ℓ_q-estimate with constant 1. Let $f : X \to Y$ be a coarse map. Then, for any $t > 0$, and any $\delta \in (0,1)$, there exists $x \in X$, $\tau > t$, and a compact subset $K \subset Y$ such that, for any weakly null sequence $(x_n)_{n=1}^{\infty}$ in B_X, there exists $n_0 \in \mathbb{N}$ such that*

$$f(x + \tau x_n) \in K + \delta\tau B_Y, \quad \textit{for all} \quad n > n_0.$$

Proof. If $\mathrm{Lip}_\infty(f) = 0$, then there exists $\tau > t$ such that $\mathrm{Lip}_\tau(f) < \delta$. Hence, $\omega_f(\tau) < \delta\tau$, and the result follows by letting $x = 0$ and $K = \{f(0)\}$. Indeed, if $z \in B_X$, we have

$$\|f(\tau z) - f(0)\| \leqslant \omega_f(\|\tau z\|) \leqslant \omega_f(\tau) \leqslant \delta\tau.$$

Assume $\mathrm{Lip}_\infty(f) > 0$. In particular, $C = \mathrm{Lip}_s(f) > 0$, for some $s > 0$. Let $c > 0$ be given by Lemma 6.4.1 applied to X and p. As $q < p$, we can pick $\nu \in (0,1)$ such that $2C(2c)^{1/q}\nu^{1/q-1/p} < \delta$. By Proposition 6.4.3, there exists $u, v \in X$ such that $\|u - v\| > \max\{s, 2t\nu^{-1/p}\}$ and

$$f(\mathrm{Mid}(u, v, c\nu)) \subset \mathrm{Mid}(f(u), f(v), 2c\nu).$$

Let $x = \frac{1}{2}(u + v)$, and $\tau = \nu^{1/p}\|\frac{1}{2}(u - v)\|$ (so $\tau > t$). Fix a weakly null sequence $(x_n)_{n=1}^\infty$ in B_X. Then, by Lemma 6.4.1, there exists $n_0 \in \mathbb{N}$ such that $x + \tau x_n \in \mathrm{Mid}(u, v, c\nu)$, for all $n > n_0$. So,

$$f(x + \tau x_n) \subset f(\mathrm{Mid}(u, v, c\nu)) \subset \mathrm{Mid}(f(u), f(v), 2c\nu),$$

for all $n > n_0$. Let $K \subset Y$ be given by Lemma 6.4.2 applied to Y, $f(u), f(v) \in Y$, and $2c\nu$. So,

$$\mathrm{Mid}(f(u), f(v), 2c\nu) \subset K + 2(2c)^{1/q}\nu^{1/q}\frac{\|f(u) - f(v)\|}{2}B_Y.$$

As $\mathrm{Lip}_s(f) = C$, and as $\|u - v\| > s$, we have $\|f(u) - f(v)\| \leqslant C\|u - v\| = 2C\tau\nu^{-1/p}$. Hence,

$$2(2c)^{1/q}\nu^{1/q}\frac{\|f(u) - f(v)\|}{2} \leqslant 2C(2c)^{1/q}\nu^{1/q-1/p}\tau < \delta\tau,$$

and we are done. $\square$

Remark 6.4.5. Lemma 6.4.4 remains valid if we only assume that X has an *equivalent* norm with which X becomes asymptotically p-uniformly smooth. Indeed, let $M \geqslant 1$ be such that $B_{(X,\|\cdot\|)} \subset M \cdot B_{(X,|||\cdot|||)}$. Fix $t > 0$, and $\delta \in (0,1)$. Applying Lemma 6.4.4 to $(X, |||\cdot|||)$ with $t' = M.t$ and $\delta' = \delta/M$, we obtain $x \in X$, $\tau' > t'$, and a compact set $K \subset Y$. The result now follows by letting $\tau = \tau'/M$.

Theorem 6.4.6. *Let $1 \leqslant q_1 < p < q_2$. Assume that*

(i) X is an asymptotically p-uniformly smooth Banach space with the p-co-Banach-Saks property, and it does not contain ℓ_1,

(ii) Y_1 is a Banach space with a 1-unconditional basis satisfying a lower ℓ_{q_1}-estimate with constant 1, and

(iii) Y_2 is a reflexive asymptotically q_2-uniformly smooth Banach space.

Then X does not coarse Lipschitz embed into $Y_1 \oplus Y_2$.

Proof. Let $Y_1 \oplus_1 Y_2$ denote the space $Y_1 \oplus Y_2$ endowed with the norm $\|(y_1, y_2)\| = \|y_1\| + \|y_2\|$, for all $(y_1, y_2) \in Y_1 \oplus Y_2$. Assume $f = (f_1, f_2) : X \to Y_1 \oplus_1 Y_2$ is a coarse Lipschitz embedding. As f is a coarse Lipschitz embedding, there exists $C > 0$ such that $\rho_f(t) \geqslant C^{-1}t - C$, and $\omega_{f_2}(t) \leqslant Ct + C$, for all $t > 0$.

Fix $k \in \mathbb{N}$, and $\delta \in (0,1)$. Then, by Lemma 6.4.4, there exists $\tau > k$, $x \in X$, and a compact subset $K \subset Y_1$, such that, for any weakly null sequence $(y_n)_{n=1}^{\infty}$ in B_X, there exists $n_0 \in \mathbb{N}$, such that

$$f_1(x + \tau y_n) \in K + \delta\tau B_{Y_1},$$

for all $n > n_0$.

As X does not contain ℓ_1, by Rosenthal's ℓ_1-theorem, we can pick a normalized weakly null sequence $(x_n)_{n=1}^{\infty}$ in X, with $\inf_{n \neq m} \|x_n - x_m\| > 0$. As X has the p-Banach-Saks property (Proposition 6.1.2), there exists $c > 0$ (independent of k) such that, by going to a subsequence if necessary, we have

$$\|x_{n_1} + \ldots + x_{n_k}\| \leqslant ck^{1/p},$$

for all $n_1 < \ldots < n_k \in \mathbb{N}$. Define a map $\varphi_{k,\delta} : G_k(\mathbb{N}) \to X$ by letting

$$\varphi_{k,\delta}(n_1, \ldots, n_k) = x + \frac{\tau}{c} k^{-1/p}(x_{n_1} + \ldots + x_{n_k}),$$

for all $(n_1,\dots,n_k)\in G_k(\mathbb{N})$.

As $d((n_1,\dots,n_k),(m_1,\dots,m_k))\leqslant 1$ implies $\|\sum_{j=1}^k x_{n_j}-\sum_{j=1}^k x_{m_j}\|\leqslant 2$, we have that $\mathrm{Lip}(f_2\circ\varphi_{k,\delta})\leqslant 2\tau Ck^{-1/p}c^{-1}+C$. Therefore, by Theorem 6.2.3, there exists $\mathbb{M}_{k,\delta}\subset\mathbb{N}$ such that

$$\mathrm{diam}(f_2\circ\varphi_{k,\delta}(G_k(\mathbb{M}_{k,\delta})))\leqslant 2K\tau Ck^{1/q_2-1/p}c^{-1}+KCk^{1/q_2},$$

for some $K>0$ independent of k and δ.

Notice that, if $(n_1^j,\dots,n_k^j)_{j=1}^\infty$ is a sequence in $G_k(\mathbb{M}_{k,\delta})$, with $n_k^j<n_1^{j+1}$, for all $j\in\mathbb{N}$, then $(x_{n_1^j}+\ldots+x_{n_k^j})_{j=1}^\infty$ is a weakly null sequence in $ck^{1/p}\cdot B_X$. Therefore,

$$f_1\circ\varphi_{k,\delta}(n_1^j,\dots,n_k^j)\in K+\delta\tau B_{Y_1},$$

for large enough j. This argument and standard Ramsey theory, gives us that, by passing to a subsequence of $\mathbb{M}_{k,\delta}$, we can assume that, for all $(n_1,\dots,n_k)\in G_k(\mathbb{M}_{k,\delta})$,

$$f_1\circ\varphi_{k,\delta}(n_1,\dots,n_k)\in K+\delta\tau B_{Y_1}.$$

Therefore, as K is compact, by passing to a further subsequence, we can assume that $\mathrm{diam}(f_1\circ\varphi_{k,\delta}(G_k(\mathbb{M}_{k,\delta})))\leqslant 3\delta\tau$ (see Lemma 4.1 of [KaRa]).

We have shown that, for all $k\in\mathbb{N}$, and all $\delta\in(0,1)$, there exists a subsequence $\mathbb{M}_{k,\delta}\subset\mathbb{N}$ such that

$$\mathrm{diam}(f\circ\varphi_{k,\delta}(G_k(\mathbb{M}_{k,\delta})))\leqslant 2K\tau Ck^{1/q_2-1/p}c^{-1}+KCk^{1/q_2}+3\delta\tau. \tag{6.4.1}$$

We may assume that $\mathbb{M}_{k+1,\delta}\subset\mathbb{M}_{k,\delta}$, for all $k\in\mathbb{N}$, and all $\delta\in(0,1)$. For each $\delta\in(0,1)$, let $\mathbb{M}_\delta\subset\mathbb{N}$ diagonalize the sequence $(\mathbb{M}_{k,\delta})_{k=1}^\infty$.

As X has the p-co-Banach-Saks property, arguing similarly as in the proof of Theorem 6.3.1, we get that there exists $d>0$ (independent of k) such that, for all $k\in\mathbb{N}$, there exists

$n_1 < \ldots < n_{2k} \in \mathbb{M}_{k,\delta}$, such that

$$\Big\| \sum_{j=1}^{k} (x_{n_{2j-1}} - x_{n_{2j}}) \Big\| \geqslant dk^{1/p}.$$

Therefore, $\mathrm{diam}(\varphi_{k,\delta}(G_k(\mathbb{M}_\delta))) \geqslant \tau d/c$, which implies that

$$\mathrm{diam}(f \circ \varphi_{k,\delta}(G_k(\mathbb{M}_\delta))) \geqslant \tau d(cC)^{-1} - C, \tag{6.4.2}$$

for all $k \in \mathbb{N}$, and all $\delta \in (0,1)$. So, Equation (6.4.1) and Equation (6.4.2) give us that

$$\tau d(cC)^{-1} - C \leqslant 2K\tau Ck^{1/q_2 - 1/p}c^{-1} + KCk^{1/q_2} + 3\delta\tau.$$

for all $k \in \mathbb{N}$, and all $\delta \in (0,1)$. As $\tau > k$, this gives us that

$$d(cC)^{-1} - Ck^{-1} \leqslant 2KCk^{1/q_2 - 1/p}c^{-1} + KCk^{1/q_2 - 1} + 3\delta$$

for all $k \in \mathbb{N}$, and all $\delta \in (0,1)$. As $q_2 > p > 1$, by letting $k \to \infty$ and $\delta \to 0$, we get a contradiction. □

If $T = (T_1, T_2) : X \to Y_1 \oplus Y_2$ is a linear isomorphic embedding, then either $T_1 : X \to Y_1$ or $T_2 : X \to Y_2$ is not strictly singular, i.e., $T_i : X_0 \to Y_i$ is a linear isomorphic embedding, for some infinite dimensional subspace $X_0 \subset X$, and some $i \in \{1,2\}$. Is there an analog of this result for coarse Lipschitz embeddings? Precisely, we ask the following.

Problem 6.4.7. Let X, Y_1 and Y_2 be Banach spaces and consider a coarse Lipschitz embedding $f = (f_1, f_2) : X \to Y_1 \oplus Y_2$. Is there an infinite dimensional subspace $X_0 \subset X$ such that either $f_1 : X_0 \to Y_1$ or $f_2 : X_0 \to Y_2$ is a coarse Lipschitz embedding?

We can now prove Theorem 1.6.4, which will be essential in the proof of Theorem 1.6.5.

Proof of Theorem 1.6.4. Say $m \in \{1, \ldots, n-1\}$ is such that $p \in (p_m, p_{m+1})$ (the other cases have analogous proofs). Then $(T^{p_{m+1}} \oplus \ldots \oplus T^{p_n})_{\ell_\infty}$ is reflexive (see [OScZs], Proposition

5.3(b)). Also, it is easy to see that $(T^{p_{m+1}} \oplus \ldots \oplus T^{p_n})_{\ell_\infty}$ is asymptotically p_{m+1}-uniformly smooth. By Theorem 6.4.6, it is enough to prove the following claim.

Claim: Fix $\varepsilon > 0$ such that $p_m + \varepsilon < p$. $(T^{p_1} \oplus \ldots \oplus T^{p_m})_{\ell_{p_m}}$ can be renormed so that it has a 1-unconditional basis satisfying a lower $\ell_{(p_m+\varepsilon)}$-estimate with constant 1.

For each $k \in \mathbb{N}$ and $p \in [1, \infty)$, denote by $P_k = P_k^p : T^p \to T^p$ the projection on the first k coordinates, and let $Q_k = \mathrm{Id} - P_k$. By Proposition 5.6 of [JoLiS], there exists $M \in [1, \infty)$ and $N \in \mathbb{N}$ such that $Q_N(T^{p_j})$ has an equivalent norm with $(p_j + \varepsilon)$-concavity constant M, for all $j \in \{1, \ldots, m\}$ (precisely, the modified Tsirelson norm has this property, see [CSh] for definition).

As the shift operator on the basis of T^p is an isomorphism onto $Q_1(T^p)$, we have that $T^p \cong Q_k(T^p)$, for all $k \in \mathbb{N}$, and all $p \in [1, \infty)$. Therefore, it follows that $(T^{p_1} \oplus \ldots \oplus T^{p_m})_{\ell_{p_m}}$ has an equivalent norm with $(p_m + \varepsilon)$-concavity constant M. By Proposition 1.d.8 of [LiTz], we can assume that $M = 1$. As a q-concave basis with constant 1 satisfies a lower ℓ_q-estimate with constant 1, we are done. $\square$

Before given the proof of Theorem 1.6.5, we need a lemma. For that, we must introduce some natation. Let $p \in (1, \infty)$. A Banach space X is said to be *as.* $\mathcal{L}_p$ if there exists $\lambda \geqslant 1$ so that for every $n \in \mathbb{N}$ there is a finite codimensional subspace $Y \subset X$ so that every n-dimensional subspace of Y is contained in a subspace of X which is λ-isomorphic to $L_p(\mu)$, for some μ. As noticed in [JoLiS], Proposition 2.4.a, an as. $\mathcal{L}_p$ space is super-reflexive. Also, the p-convexifications T^p are as. $\mathcal{L}_p$ (see [JoLiS], page 440).

The following lemma, although not explicitily written, is contained in the proof of Proposition 2.7 of [JoLiS]. For the convenience of the reader, we provide its proof here.

Lemma 6.4.8. *Say $1 < p_1 < \ldots < p_n < \infty$ and $X = X^{p_1} \oplus \ldots \oplus X^{p_n}$, where X^{p_j} is as. $\mathcal{L}_{p_j}$, for all $j \in \{1, \ldots, n\}$. Assume that Y is coarsely equivalent to X.*

(i) Then there exists a separable Banach space W such that $Y \oplus W$ is Lipschitz equivalent to $\bigoplus_{j=1}^n (X^{p_j} \oplus L_{p_j})$.

(ii) Moreover, if $Y = Y^{p_1} \oplus \ldots \oplus Y^{p_n}$*, where* Y^{p_j} *is as.* $\mathcal{L}_{p_j}$*, for all* $i \in \{1, \ldots, n\}$*, then* $\bigoplus_{j=1}^n (Y^{p_j} \oplus L_{p_j})$ *is Lipschitz equivalent to* $\bigoplus_{j=1}^n (X^{p_j} \oplus L_{p_j})$*.*

Proof. Let Z be a Banach space and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. In order to simplify notation, let $Z_{\mathcal{U}} = Z^{\mathbb{N}}/\mathcal{U}$, where $Z^{\mathbb{N}}/\mathcal{U}$ is the ultrapower of Z with respect to $\mathcal{U}$. Notice that $z \in Z \mapsto (z)_{n=1}^\infty \in Z_{\mathcal{U}}$ is a linear isometric embedding. If Z is reflexive, Z is 1-complemented in the ultrapower $Z_{\mathcal{U}}$ (where the projection is given by $(z_n)_{n=1}^\infty \in Z_{\mathcal{U}} \mapsto w\text{-}\lim_{n \in \mathcal{U}} z_n \in Z$), and we write $Z_{\mathcal{U}} = Z \oplus Z_{\mathcal{U},0}$. Also, we have that $(Z \oplus E)_{\mathcal{U}} = Z_{\mathcal{U}} \oplus E_{\mathcal{U}}$. We can now prove the lemma. For simplicity, let us assume that $n = 2$.

(i) Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. As Y is coarsely equivalent to X, $Y_{\mathcal{U}}$ is Lipschitz equivalent to $X_{\mathcal{U}} = X_{\mathcal{U}}^{p_1} \oplus X_{\mathcal{U}}^{p_2}$ (see [Ka2], proposition 1.6). As the spaces $X_{\mathcal{U}}^{p_j}$ are reflexive, using the separable complementation property for reflexive spaces (see [FiJoP], Section 3), we can pick complemented separable subspaces $W \subset Y_{\mathcal{U},0}$, and $X_{j,0} \subset X_{\mathcal{U},0}^{p_j}$, for $j \in \{1, 2\}$, such that $Y \oplus W$ is Lipschitz equivalent to $(X^{p_1} \oplus X_{1,0}) \oplus (X^{p_2} \oplus X_{2,0})$. By enlarging $X_{j,0}$ and W, if necessary, we can assume that $X_{j,0} = L_{p_j}$, for $j \in \{1, 2\}$ (this follows from Proposition 2.4.a of [JoLiS], Theorem I(ii) and Theorem III(b) of [LiRos]).

(ii) The same argument as why $X_{1,0} \oplus X_{2,0}$ can be enlarged so that $X_{1,0} \oplus X_{2,0} = L_{p_1} \oplus L_{p_2}$ gives us that W can also be assumed to be $L_{p_1} \oplus L_{p_2}$. □

We can now prove Theorem 1.6.5. As mentioned in Section 1.2, Theorem 1.6.5 was proved in [JoLiS] (Theorem 5.8) for the cases $1 < p_1 < \ldots < p_n < 2$ and $2 < p_1 < \ldots < p_n < \infty$. In our proof, Theorem 1.6.4 will play a similar role as Corollary 1.7 of [JoLiS] did in their proof. Also, we use ideas in the proof of Theorem 5.3 of [KaRa] in order to unify the cases $1 < p_1 < \ldots < p_n < 2$ and $2 < p_1 < \ldots < p_n < \infty$. In order to avoid an unnecessarily extensive proof, we will only present the parts of the proof that require Theorem 1.6.4 above, and therefore are different from what can be found in the present literature.

Sketch of the proof of Theorem 1.6.5. By Proposition 5.7 of [JoLiS], T^p is uniformly equivalent to $T^p \oplus \ell_p$, for all $p \in [1,\infty)$. So, the backwards direction follows. Let us prove the forward direction. As uniform equivalence implies coarse equivalence, it is enough to assume that Y is coarsely equivalent to X. By Theorem 1.6.4, Y does not contain ℓ_2. Let $m \in \{1,\dots,n-1\}$ be such that $2 \in (p_m, p_{m+1})$ (if such m does not exist, the result simply follows from Theorem 5.8 of [JoLiS]).

Claim 1: $X \oplus \bigoplus_{j=1}^n L_{p_j}$ and $Y \oplus \bigoplus_{j=1}^n L_{p_j}$ are Lipschitz equivalent.

By Lemma 6.4.8(i), there exists a separable Banach space W so that $Y \oplus W$ is Lipschitz equivalent to $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j})$. Hence, the image of Y through this Lipschitz equivalence is the range of a Lipschitz projection in $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j})$. Therefore, by Theorem 2.2 of [HeM], we have that Y is isomorphic to a complemented subspace of $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j})$. Let A be this isomorphic embedding. For each $i \in \{m+1,\dots,n\}$, let $\pi_i : Y \to L_{p_i}$ be the composition of A with the projection $\bigoplus_{j=1}^n (T^{p_j} \oplus L_{p_j}) \to L_{p_i}$. As Y does not contain ℓ_2, π_i factors through ℓ_{p_i} (see [Jo1]). Hence, Y is isomorphic to a complemented subspace of

$$\bigoplus_{j=1}^m (T^{p_j} \oplus L_{p_j}) \oplus \bigoplus_{j=m+1}^n (T^{p_j} \oplus \ell_{p_j}).$$

As $Z_1 := \bigoplus_{j=1}^m (T^{p_j} \oplus L_{p_j})$ and $Z_2 := \bigoplus_{j=m+1}^n (T^{p_j} \oplus \ell_{p_j})$ are totally incomparable (i.e., none of their infinite dimensional subspaces are isomorphic), $Y \cong Y_1 \oplus Y_2$, where Y_1 and Y_2 are complemented subspaces of Z_1 and Z_2, respectively (see [EWo], Theorem 3.5). Hence, Y_1^* is complemented in Z_1^*. Notice that, as Y is coarsely equivalent to the super-reflexive space X, Y is also super-reflexive (see [Ri], Theorem 1A). Hence, Y_1 is super-reflexive, and so is Y_1^*. As Y_1 has cotype 2 (see Remark 6.3.10) and Y_1^* has nontrivial type (as Y_1^* is super-reflexive), it follows that Y_1^* has type 2 (see the remark below Theorem 1 in [Pi]). So, Y_1^* does not contain a copy of ℓ_2. Indeed, otherwise Y_1^* would contain a complemented copy of ℓ_2 (see [Mau]), contradicting that Y_1 does not contain a copy of ℓ_2.

Proceeding similarly as above and using that Y_1^* does not contain ℓ_2, the main theorem of [Jo1] implies that Y_1^* is isomorphic to a complemented subspace of $\bigoplus_{j=1}^m (T^{p_j *} \oplus \ell_{\tilde{p}_j})$, where each $\tilde{p}_j$ is the conjugate of p_j (i.e., $1/p_j + 1/\tilde{p}_j = 1$). Therefore, Y_1 embeds into $\bigoplus_{j=1}^{m}(T^{p_j} \oplus \ell_{p_j})$ as a complemented subspace. This gives us that Y embeds into $\bigoplus_{j=1}^{n}(T^{p_j} \oplus \ell_{p_j})$ as a complemented subspace.

As the spaces $(T^{p_j} \oplus \ell_{p_j})_{j=1}^{n}$ are totally incomparable, we can write Y as $Y_{p_1} \oplus \ldots \oplus Y_{p_n}$, where each Y_{p_j} is a complemented subspace of $T^{p_j} \oplus \ell_{p_j}$ (see [EWo], Theorem 3.5) and it is an as. $\mathcal{L}_{p_j}$ (see [JoLiS], Lemma 2.5 and Proposition 2.7). By Lemma 6.4.8(ii), we have that $X \oplus \bigoplus_{j=1}^{n} L_{p_j}$ and $Y \oplus \bigoplus_{j=1}^{n} L_{p_j}$ are Lipschitz equivalent.

Claim 2: There exists a quotient W of $L_{p_1} \oplus \ldots \oplus L_{p_n}$ such that $Y \oplus W$ is isomorphic to $X \oplus \bigoplus_{j=1}^{n} L_{p_j}$.

The prove of Claim 2 is the same as the proof of the claim in Proposition 2.10 of [JoLiS], so we do not present it here. Let us assume the claim and finish the proof. As X does not contain any ℓ_s, every operator of X into $\oplus_{j=1}^{n} L_{p_j}$ is strictly singular (see [KrMau], Theorem II.2 and Theorem IV.1). Therefore, by [EWo] (or [LiTz], Theorem 2.c.13), $Y \cong Y_X \oplus Y_L$ and $W \cong W_X \oplus W_L$, where Y_X and W_X are complemented subspaces of X, Y_L and W_L are complemented subspaces of $\oplus_{j=1}^{n} L_{p_j}$, and $X \cong Y_X \oplus W_X$. Proceeding as in the proof of Claim 1 above, we get that Y_L is complemented in $\oplus_{j=1}^{n} \ell_{p_j}$. So, Y_L is either finite dimensional or isomorphic to $\oplus_{j \in F} \ell_{p_j}$, for some $F \subset \{1, \ldots, n\}$.

Let us show that W_X is finite dimensional. Suppose this is not the case. As W is a quotient of $\oplus_{j=1}^{n} L_{p_j}$, and W_X is complemented in W, we have that W_X^* embeds into $\oplus_{j=1}^{n} L_{\tilde{p}_j}$, where each $\tilde{p}_j$ is the conjugate of p_j. Therefore, it follows that W_X^* must contain some ℓ_s (see [KrMau], Theorem II.2 and Theorem IV.1). As W_X^* embeds into X^*, and X^* does not contain any ℓ_s, this gives us a contradiction.

As $X \cong Y_X \oplus W_X$, and $\dim(W_X) < \infty$, we have that $\dim(X/Y_X) < \infty$. Therefore, as X is isomorphic to its hyperplanes, we conlude that $Y_X \cong X$. So, we are done. □

Problem 6.4.9. Does Theorem 1.6.5 hold if $2 \in \{p_1, \ldots, p_n\}$?

Problem 6.4.10. What can we say if a Banach space X is either coarsely or uniformly equivalent to the Tsirelson space T?

Remark 6.4.11. It is worth noticing that, using Remark 6.4.5 and adapting the proofs of Theorem 5.5 and Theorem 5.7 of [KaRa] to our settings, one can show that $(\oplus T_p)_{T_q}$ does not coarse Lipschitz embed into $T_p \oplus T_q$, for all $p, q \in [1, \infty)$ with $p \neq q$.

Chapter 7

The isomorphism group of the Gurarij space.

In this chapter, we study homeomorphic embeddings of Polish groups into the isometry group of the Gurarij space, and how make sure those maps also preserve the large scale geometry of the Polish group.

7.1 The Gurarij space

Let X and Y be Banach spaces, and let $\varepsilon > 0$. We say that a linear map $f : X \to Y$ is an $(1+\varepsilon)$-*isomorphism* if

$$(1+\varepsilon)^{-1}\|x\| < \|f(x)\| < (1+\varepsilon)\|x\|,$$

for all $x \in X$. As we saw in Chapter 1, a Banach space X is said to be a *Gurarij space* if for all finite dimensional Banach spaces $E \subset F$, for all $\varepsilon > 0$, and all linear isometry $f : E \to X$, there exists an $(1+\varepsilon)$-isomorphism $g : F \to X$ extending f. W. Lusky was the first one to show that the Gurarij space is unique up to isomorphism (see [Lu]). In [KuS], W. Kubis, and S. Solecki gave an elementary prove of the uniqueness of the Gurarij space. In particular, they showed the following (see [KuS], Theorem 1.1).

Theorem 7.1.1. ***(W. Kubis, S. Solecki, 2013)*** *Let $\mathbb{G}$ be the Gurarij space, and $\varepsilon > 0$. Let $F \subset \mathbb{G}$ be a finite dimensional subspace and $f : F \to \mathbb{G}$ be an $(1+\varepsilon)$-isomorphism. Then, there exists a linear surjective isometry $g : \mathbb{G} \to \mathbb{G}$ such that $\|g \restriction F - f\| < \varepsilon$.*

7.2 Group of affine isometries.

If (X, d) is a metric space, we denote by $\mathrm{Iso}(X)$ the group of isometries of X endowed with the pointwise convergence topology. If X is a Banach space, we denote the group of affine isometries of X by $\mathrm{Aff}(X)$, and endow $\mathrm{Aff}(X)$ with the pointwise convergence topology. Denote by $\mathrm{Iso}_{\mathrm{L}}(X)$ the closed subgroup of $\mathrm{Aff}(X)$ consisting of the linear isometries of X. As we saw in Section 4.2, by Mazur-Ulam's theorem, every surjective isometry $f : X \to X$ is affine, i.e., there exists $g \in \mathrm{Iso}_{\mathrm{L}}(X)$, and $x \in X$, such that $f(y) = g(y) + x$, for all $y \in X$. So, for a Banach space X, we have that $\mathrm{Aff}(X) = \mathrm{Iso}(X)$. The group $\mathrm{Aff}(X)$ can be seen as the semi-direct product

$$\mathrm{Aff}(X) = \mathrm{Iso}_{\mathrm{L}}(X) \ltimes X,$$

where $\mathrm{Iso}_{\mathrm{L}}(X) \ltimes X$ is the topological product space $\mathrm{Iso}_{\mathrm{L}}(X) \times X$ endowed with the group operation $(f, x) * (g, y) = (f \circ g, x + f(y))$.

A homomorphism $\varphi : H \to \mathrm{Aff}(X)$ can be seen as an affine isometric action $\alpha : H \curvearrowright X$ with a linear part $\pi : H \curvearrowright X$ and a cocycle $b : H \to X$. That is, $\pi : H \curvearrowright X$ is a linear isometric action on X, b is a map satisfying the cocycle equation for the action π, i.e.,

$$b(hg) = \pi(h)(b(g)) + b(h),$$

for all $h, g \in H$, and $\alpha(h)(x) = \pi(h)(x) + b(h)$, for all $h \in H$, and all $x \in X$.

If $\mathrm{Iso}_{\mathrm{L}}(X)$ has property (OB), then $\mathrm{Aff}(X)$ is (OB) generated. Indeed, by Example 39 of [Ro3], B_X has property (OB) relative to $(X, +)$. Hence, as the product of two subsets $A, B \subset G$ with property (OB) relative to a Polish proup G still has property (OB) relative to G (see [Ro3], Lemma 7), we have that

$$\mathrm{Iso_L}(X) \times B_X = \Big(\{\mathrm{Id}\} \times B_X\Big) * \Big(\mathrm{Iso_L}(X) \times \{0\}\Big)$$

has property (OB) relative to $\mathrm{Aff}(X)$, if $\mathrm{Iso_L}(X)$ has property (OB). As $\mathrm{Iso_L}(X) \times B_X$ generates $\mathrm{Aff}(X)$, it follows that $\mathrm{Aff}(X)$ is (OB) generated. C. Rosendal proved the following in [Ro3], page 21.

Lemma 7.2.1. *If $Iso_L(X)$ has property (OB), then the map $g \in Aff(X) \mapsto g(x) \in X$ is a coarse Lipschitz equivalence, for all $x \in X$.*

7.3 Approximately oligomorphic subgroups.

In this section, we prove a lemma which will give us a more Banach space theoretical characterization of approximately oligomorphic subgroups $G \leqslant \mathrm{Iso_L}(X)$, where X is a Banach space. First, we need a couple of definitions.

Let $(X, \|\cdot\|)$ be a Banach space. For $n \in \mathbb{N}$, we view B_X^n as a metric space with the supremum metric $\|\cdot - \cdot\|_\infty$ induced by $\|\cdot\|$. The natural action of a subgroup $G \leqslant \mathrm{Iso_L}(X)$ on X extends to an action on B_X^n coordinatewise, i.e., if $\bar{x} = (x_1, ..., x_n) \in X^n$, and $g \in G$, we have $g \cdot \bar{x} = (g(x_1), ..., g(x_n))$. For a subset $A \subset B_X^n$, we write $G \cdot A = \{g \cdot \bar{a} \mid g \in G,\ \bar{a} \in A\}$.

Definition 7.3.1. *Let $(X, \|\cdot\|)$ be a normed space and let $G \leqslant Iso_L(X)$. We say that G is* approximately oligomorphic *if, for all $n \in \mathbb{N}$, and all $\varepsilon > 0$, there exist a finite set $A \subset B_X^n$ such that $G \cdot A$ is ε-dense in B_X^n.*

For a metric space (X, d), let $\mathcal{K}(X) = \{K \subset X \mid K \text{ is compact}\}$. If d is bounded, we can define a metric on $\mathcal{K}(X)$, called *Hausdorff metric*, by saying that, given $K, L \in \mathcal{K}(X)$,

$$d_H(K, L) < \varepsilon \Leftrightarrow K \subset B(L, \varepsilon) \ \text{ and } \ L \subset B(K, \varepsilon),$$

for all $\varepsilon \geqslant 0$. For more on Hausdorff metric see [Ke].

A normalized basis $\{x_i\}_{i=1}^n$ of an n-dimensional Banach space X is called a *Auerbach basis* if there exists a normalized sequence of biorthogonal vectors for $\{x_i\}_{i=1}^n$, i.e., there are $\{x_i^*\}_{i=1}^n$ in X^* such that $x_i^*(x_j) = \delta_{ij}$, and $\|x_i\| = \|x_i^*\| = 1$, for all $j, i \in \{1, ..., n\}$. By Lemma 2.22 of [Os2], every finite dimensional Banach space has an Auerbach basis. If $\{x_i\}_{i=1}^n$ is an Auerbach basis for X, its basic constant is at most $n-1$. Indeed, for all $x \in X$, we have $x = \sum_{i=1}^n x_i^*(x)x_i$. Hence, for $m < n$, we have

$$\Big\|\sum_{i=1}^m x_i^*(x)x_i\Big\| \leqslant \sum_{i=1}^m |x_i^*(x)| \leqslant (n-1)\cdot\|x\|.$$

In particular, if $\{x_i\}_{i=1}^n$ is an Auerbach basis for its span, $\varepsilon \in (0, 1/2)$, and $\{y_i\}_{i=1}^n$ is such that $\|x_i - y_i\| < \varepsilon/2n^2$, for all $i \in \{1, ..., n\}$, then $\{y_i\}_{i=1}^n$ is also a basis for its span, and its basic constant is at most $3n$ (see [AlKa], Theorem 1.3.9).

A topological group G is called *Roelcke precompact* if for any open neighborhood of the identity V there exists a finite set F such that $G = VFV$.

Lemma 7.3.2. *Consider a Banach space X and a subgroup $G \leqslant Iso_L(X)$. Then, the following are equivalent.*

(i) $\forall n \in \mathbb{N}$, $\forall \varepsilon > 0$, there exist finite dimensional subspaces $F_1, ..., F_k \subset X$ such that, for all subspace $E \subset X$ with dimension at most n, there exists $i \in \{1, ..., k\}$ and $g \in G$ such that $d_H(B_E, B_{g(F_i)}) < \varepsilon$.

(ii) G is approximately oligomorphic.

(iii) G is Roelcke precompact.

In particular, in the point of view of the model theory of metric structures, if $G = Iso_L(X)$, (i) holds for G if and only if the theory of B_X, i.e., Th(B_X), is ω-categorical.

A word or two on the last statement of the lemma above is neeeded. Informally speaking, the theory of B_X, i.e., Th(B_X), consists of all the "sentences" which are true in B_X, and the fact that $\mathrm{Th}(B_X)$ is ω-categorical means that any other bounded metric space M such that every "sentence" which is true in B_X is also true in M is isometric to B_X. As model theory is not the focus of this dissertation, we refer to [Sch] for more details and precise definitions regarding the model theory of metric structures, $\mathrm{Th}(B_X)$, and ω-categoricity.

Proof of Lemma 7.3.2. (i)$\Rightarrow$(ii) Fix $n \in \mathbb{N}$, and $\varepsilon > 0$. By (i), we can pick $F_1, ..., F_k$ such that for all subspace $E \subset X$ with dimension at most n, there exists $i \in \{1, ..., k\}$ and $g \in G$ such that $d_H(B_E, B_{g(F_i)}) < \varepsilon/2$. For each $i \in \{1, ..., k\}$, pick a finite $\varepsilon/2$-net A_j of $B^n_{F_j}$. Set $A = \cup_{i=1}^k A_j$, so A is finite and $A \subset B^n_X$.

Let $\bar{x} \in B^n_X$, and set $E = \mathrm{span}\{\bar{x}\}$. Then $\dim E \leqslant n$, so there exists $i \in \{1, ..., k\}$ and $g \in G$ such that $d_H(B_E, B_{g(F_i)}) < \varepsilon/2$. Pick $\bar{z} \in B^n_{g(F_i)}$ such that $\|\bar{x} - \bar{z}\|_\infty < \varepsilon/2$, and pick $\bar{y} \in A$ such that $\|g^{-1} \cdot \bar{z} - \bar{y}\|_\infty < \varepsilon/2$. Hence, $\|\bar{x} - g \cdot \bar{y}\| < \varepsilon$, so $G \cdot A$ is ε-dense in B^n_X.

(ii)$\Rightarrow$ (i) Fix $n \in \mathbb{N}$, and $\varepsilon \in (0, 1/2)$. By (ii), we can pick a finite subset $A \subset B^n_X$ such that, for all $\bar{x} \in B^n_X$ there exists $\bar{y} \in A$ and $g \in G$ such that $\|\bar{x} - g \cdot \bar{y}\|_\infty < \varepsilon/6n^2$. Let $F_1, ..., F_k \subset X$ be given by the linear spans of all the n-tuples of A.

Let $E \subset X$ be an m-dimensional subspace, with $m \leqslant n$. By Auerbach's theorem, E has an Auerbach basis, say $x_1, ..., x_m$. Let $\bar{x} = (x_1, ..., x_m, 0, ..., 0) \in B^n_X$, and pick $\bar{y} \in A$ and $g \in G$ such that $\|\bar{x} - g \cdot \bar{y}\|_\infty < \varepsilon/6n^2$. Without loss of generality, we can assume that $\bar{y} = (y_1, ..., y_m, 0, ..., 0)$. Pick $i \in \{1, ..., k\}$ such that $F_i = \mathrm{span}\{\bar{y}\}$. In particular, $y_1, ..., y_m$ is a basis for F_i, with basic constant at most $3n$.

Let us show that $d_H(B_E, B_{g(F_i)}) < \varepsilon$. Pick $x \in B_E$, so $x = \sum_{j=1}^m x_j^*(x) x_j$. Let $y = \sum_{j=1}^m x_j^*(x) g(y_j)$, and $y' = y/(1 + \varepsilon/2)$. Then

$$\|x - y\| \leqslant \sum_{j=1}^m |x_j^*(x)| \cdot \|x_j - g(y_j)\| < \frac{\varepsilon}{6n^2} \cdot m \cdot \|x\| < \frac{\varepsilon}{2},$$

so $\|y\| < 1 + \varepsilon/2$. Hence, $\|y'\| \in B_{g(F_i)}$, and, as $\|y - y'\| < \varepsilon/2$, we have that $\|x - y'\| < \varepsilon$. On the other hand, let $y \in B_{g(F_i)}$, say $y = \sum_{j=1}^m a_j g(y_j)$. Let $x = \sum_{j=1}^m a_j x_j$, and $x' = x/(1 + \varepsilon/2)$.

Then

$$\|y - x\| \leqslant \sum_{j=1}^{m} |a_j| \cdot \|g(y_j) - x_j\| < \frac{\varepsilon}{6n^2} \cdot m \cdot 3n \cdot \|y\| < \frac{\varepsilon}{2},$$

so $\|x\| < 1 + \varepsilon/2$. Hence $\|x'\| \in B_E$, and, as $\|x - x'\| < \varepsilon/2$, we have that $\|y - x'\| < \varepsilon$. This concludes the proof of the lemma.

The equivalence (ii)$\Leftrightarrow$(iii) is given by Proposition 1.22 of [Ro2]. The last statement of the Lemma follows from Theorem 4.25 of [Sch]. □

7.4 Proof of the Theorems.

A minor modification of Theorem 5.2 of [Ro1], gives us the following.

Theorem 7.4.1. ***(C. Rosendal, 2009)*** *Let $(X, \|\cdot\|)$ be a Banach space and G be a closed subgroup of $Iso_L(X)$. If G is approximately oligomorphic, then G has property (OB).*

Theorem 7.4.2. *$Iso_L(\mathbb{G})$ is approximately oligomorphic, Roeckle precompact, and $Th(B_{\mathbb{G}})$ is ω-categorical. In particular, $Iso_L(\mathbb{G})$ has property (OB).*

Proof. For this, we only need to show that (i) of Lemma 7.3.2 holds for $G = \mathrm{Iso}_{\mathrm{L}}(\mathbb{G})$. Fix $n \in \mathbb{N}$, and $\varepsilon > 0$. For each $m \in \mathbb{N}$, let $\mathcal{F}_m$ be the set of equivalence classes of m-dimensional Banach spaces with respect to the equivalence relation of isometry between Banach spaces. Let D be the Banach-Mazur distance on $\mathcal{F}_m$. Then $(\mathcal{F}_m, D)$ is a compact metric space, for all $m \in \mathbb{N}$. Let $\varepsilon_1 = \log(1 + \varepsilon^2)$. There exist finitely many finite dimensional Banach spaces $F_1, ..., F_k$ such that for any m-dimensional Banach space E, with $m \leqslant n$, there exists $i \in \{1, ..., k\}$ such that $D(E, F_i) < \varepsilon_1$. As the Gurarij space is isometrically universal for all separable Banach spaces, we can assume that $F_1, ..., F_k \subset \mathbb{G}$.

Let $E \subset \mathbb{G}$ be an m-dimensional subspace, with $m \leqslant n$. Pick $i \in \{1, ..., k\}$ such that $D(E, F_i) < \varepsilon$. Let $f : F_i \to E$ be an $(1 + \varepsilon_2)$-isomorphism, for some $\varepsilon_2 > 0$ such that $\log(1 + \varepsilon_2)^2 < \varepsilon_1$ (so $\varepsilon_2 < \varepsilon$). By Theorem 7.1.1, there exists $g \in \mathrm{Iso}_{\mathrm{L}}(\mathbb{G})$ such that $\|g \restriction F_j - f\| < \varepsilon_2$. Clearly, $d_H(B_E, B_{g(F_i)}) < \varepsilon_2$, and we are done. □

Proof of Theorem 1.7.2. This follows from Theorem 7.4.2, Lemma 7.2.1 and the discussion preceding Lemma 7.2.1. □

Definition 7.4.3. *Let X and Y be Banach spaces. We say that a linear isometric embedding $i : X \hookrightarrow Y$ is a g-*embedding *if there exists a continuous homomorphism $\Theta : Iso_L(X) \to Iso_L(Y)$ such that $\Theta(g)(i(x)) = i(g(x))$, for all $g \in Iso_L(X)$, and all $x \in X$.*

Considering the notation of Definition 7.4.3, let $i^{-1} : \mathrm{Im}(i) \subset Y \to X$ be the inverse of the linear isometric embedding $i : X \hookrightarrow Y$. Notice that, as the restriction map $g \in \mathrm{Im}(\Theta) \mapsto i^{-1} \circ g \circ i \in \mathrm{Iso_L}(X)$ is continuous, the map Θ is automatically a homeomorphic embedding.

Proof of Theorem 1.7.3 and Theorem 1.7.4. Let H be a separable metrizable topological group, and pick a compatible left-invariant metric d on H. If H is locally (OB), we also assume that d is metrically proper, and if H is (OB) generated, we assume that d is maximal.

Consider the Banach space Æ(H, d), i.e., the the Arens-Eells space associated to (H, d) (see [Ro3], Section 3.1, for a precise definition). By Theorem 45 of [Ro3], there exists a continuous homomorphism $\alpha : H \to \mathrm{Aff}(Æ(H, d))$ such that

$$\|\alpha(g)(0) - \alpha(h)(0)\| = d(g, h),$$

for all $h, g \in H$. So, $\alpha : H \to \mathrm{Aff}(Æ(H, d))$ is also a homeomorphic embedding.

By Theorem 3.10 of [Y], there exists a linear g-embedding $i : Æ(H, d) \hookrightarrow \mathbb{G}$. Let $\Theta : \mathrm{Aff}(Æ(H, d)) \to \mathrm{Aff}(\mathbb{G})$ be as in Definition 7.4.3. Define $\varphi : H \to \mathrm{Aff}(\mathbb{G})$ by $\varphi = \Theta \circ \alpha$, so φ is a homorphism and a homeomorphic embedding. Then $g \in H \mapsto \varphi(g)(0) \in \mathbb{G}$ is an isometric embedding of (H,d) into $\mathbb{G}$. Indeed, for all $g,h\in H$, we have

$$\begin{aligned}\|\varphi(g)(0)-\varphi(h)(0)\| &= \|\Theta(\alpha(g))(0)-\Theta(\alpha(h))(0)\|\\ &= \|i(\alpha(g)(0))-i(\alpha(h)(0))\|\\ &= \|\alpha(g)(0)-\alpha(h)(0)\|\\ &= d(g,h).\end{aligned}$$

Therefore, as the map $f\in \mathrm{Aff}(\mathbb{G})\mapsto f(0)\in\mathbb{G}$ is a coarse Lipschitz equivalence (Theorem 1.7.2), the map φ is a coarse Lipschitz embedding of (H,d) into $\mathrm{Aff}(\mathbb{G})$. This completes the proof. $\square$

At last, let us prove Corollary 1.7.5.

Proof of Corollary 1.7.5. By Theorem 1.7.2, $\mathrm{Aff}(\mathbb{G})$ has property (OB). By [Lu] (see the theorem and Remark (ii) in page 633), $(\mathbb{G},\|\cdot\|)$ is almost transitive, i.e., the action $\mathrm{Iso}_{\mathrm{L}}(\mathbb{G})\curvearrowright\mathbb{G}$ induces a dense orbit on the unit sphere of $\mathbb{G}$. Hence, by Proposition 70 and Proposition 79 of [Ro4], the existence of an isometric action $\mathrm{Aff}(\mathbb{G})\curvearrowright M$ with an unbounded orbit gives us that $\mathbb{G}$ maps into M by a coarse solvent map. $\square$

Appendix A

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In this appendix, I include the permissions of the copyright holders of my two published papers which I present in this thesis ([Br2] and [Br3]).

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It is hereby confirmed that the paper "Asymptotic structure and coarse Lipschitz geometry of Banach spaces", published in Studia Mathematica 237 (2017), 71-97, can be used as part of Mr. Bruno de Mendonca Braga's thesis.

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