Completeness of finite-rank differential varieties

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WILLIAM D. SIMMONS
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THESIS

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Defense Committee:

David Marker, Chair and Advisor
John Baldwin
Isaac Goldbring
Jan Verschelde
Carol Wood, Wesleyan University

To my family,

for all their love and support.

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SUMMARY

Algebra, geometry, and mathematical logic are major areas of mathematics that interact with one another in fascinating ways. This thesis looks at a particular question, which we call the δ-completeness problem, that lies at the crossroads of differential algebra, classical algebraic geometry, and model theory. This combination produces significant challenges as well as the potential for novel solutions.

The fundamental theorem of elimination theory states that projective varieties over an algebraically closed field K are *complete*: If V is such a variety and W is an arbitrary variety over K, then the second projection map $\pi_2 : V \times W \to W$ takes Zariski-closed sets to Zariski-closed sets. This property is tightly linked to projectiveness, as shown by the example of the affine hyperbola $xy - 1 = 0$. The images of the projections to either axis lack 0 but contain every other point of the affine line. We must "close up" the variety with a point at infinity to ensure a closed projection.

The geometric picture is clear and standard algebraic tools readily prove the theorem. However, this is not the case with the same question in differential algebraic geometry.

A *differential ring* is a commutative ring R with 1 and a finite set of maps Δ such that for each $\delta \in \Delta$ (*derivations* on R) and $x, y \in R$, $\delta(x+y) = \delta(x) + \delta(y)$ and $\delta(xy) = \delta(x)y + x\delta(y)$. We consider differential fields of characteristic zero with a single derivation δ. Over such a field, a *differential polynomial equation* defines a closed set in the Kolchin topology, the differential

analogue of the Zariski topology. For instance, one of our major examples will be the affine δ-variety defined by $xx'' = x'$, along with its δ-projective closure.

Differential algebraic geometry is much helped by a good replacement for algebraically closed fields, and we find one in the differentially closed fields of characteristic zero. These objects, first studied by model theorists, are the models of a first-order theory DCF_0. Differentially closed fields possess a Nullstellensatz and have many other useful properties; they are a natural choice for studying completeness of δ-varieties.

So our question, first looked at in essentially this form by E.R. Kolchin (15) and later by W.Y. Pong (35), is this: If V is a projective δ-variety over a differentially closed field K of characteristic zero and W is an δ-variety over K, then does the second projection map $\pi_2 : V \times W \to W$ take Kolchin-closed sets to Kolchin-closed sets?

The answer is "not necessarily", but Pong showed one reason for the counterexamples: δ-completeness requires a variety to have *finite rank* (in one of several equivalent senses). In this paper we take the basic model-theoretic and algebraic setup used by Pong and further develop it into several strategies for attacking the δ-completeness problem. Our work includes new examples of complete δ-varieties over DCF_0. Importantly, we also give the first example of an incomplete finite-rank projective δ-variety; hence we now know that this class is not empty.

Our methods are as follows:

1. Modify Pong's "valuative criterion" for δ-completeness and produce alternative versions in terms of the Kolchin closure of the image of the projection as well as differential

elimination ideals. We use these results to give multiple explicit elimination algorithms proving completeness of several new varieties.

2. Reduce from the differential setting to the algebraic by showing how the modified valuative criteria transfer the problem to a sequence of complex algebraic varieties. This enables one to use tools from analysis or standard (non-differential) commutative algebra on the δ-completeness problem.

3. More speculatively, we isolate two conjectural properties (interesting in their own right as questions of algebraic geometry) of these complex varieties, both depending on the notion of generically perturbing the coefficients of the associated systems of equations. We explain how asymptotic properties of the complex varieties might imply δ-completeness of the original variety, given the above conjectures.

The paper concludes with an evaluation of these methods and their prospects for classifying complete δ-varieties. We also discuss applications of δ-completeness. The appendices contain a new case of the differential algebraic phenomenon of non-trivial projective δ-varieties contained entirely in a single affine chart, as well as algorithmic elimination proofs of δ-completeness.

While the story of δ-completeness remains incomplete itself, we make a substantial theoretical and computational contribution to a surprisingly rich problem.

CHAPTER 1

NECESSARY CONCEPTS FROM MODEL THEORY

1.1 Basic notions

We assume that the reader is familiar with first-order formulas, structures, theories, and models. For all the model theory used in this paper, (22) is an excellent reference.

The basic tool we need is the compactness theorem:

Theorem 1.1.1. *Let S be a set of first-order formulas in the language $\mathscr{L}$. Then S has a model if and only if every finite subset of S has a model.*

Our work revolves around various systems (sometimes infinite) of polynomial equations so compactness is always in the background assuring us that if there are any inconsistencies, we can find them by examining a finite collection of equations.

Definable sets in an $\mathscr{L}$-structure $\mathcal{M}$ are the "solution sets" making a first-order formula φ with free variables true in $\mathcal{M}$. The most important example for us of a definable set is that of the image of a projection map. For instance, let $\mathscr{L}$ be the language of rings having signature $(+, -, \cdot, 0, 1)$, let $\mathcal{M}$ be the real numbers $\mathbb{R}$, and let $\varphi(y)$ be the formula $\exists x(x \cdot y = 1)$. Then $\varphi(y)$ defines the set $\{y \in \mathbb{R} \mid y \neq 0\}$. The asymptote at $y = 0$ forming a "hole" in the y-axis is the key feature we address in later chapters.

We are principally interested in the definable sets in models of two particular theories: the theory of algebraically closed fields, ACF_p (where p is either 0 or prime), and the theory of differentially closed fields of characteristic zero, DCF_0. In the case of algebraically closed fields, for our purposes we sometimes write ACF and do not specify the characteristic. Regardless, we focus on a particular model of ACF_0, the complex numbers $\mathbb{C}$. The theory DCF_0 has no such natural representative, but we fix an arbitrary base model $\mathcal{F}$ to work with.

Definable sets find generalization in the notion of a *type*.

Definition 1.1.1. *Let $\mathcal{M}$ be an $\mathscr{L}$-structure and A a subset of $\mathcal{M}$. Let p be a set of $\mathscr{L}(A)$-formulas (i.e., $\mathscr{L}$-formulas with parameters, new constant symbols representing the elements of A) each having at most the free variables $x_1, \ldots, x_n$. Then we say that p is a* type *(over A) of arity n if there exist $\overline{b} = b_1, \ldots, b_n$ in some elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that every formula in p is true of $\overline{b}$ in $\mathcal{N}$ (written $\mathcal{N} \models \varphi(\overline{b})$ for all $\varphi \in p$; recall that an elementary extension is a structure containing a smaller one such that precisely the same formulas with parameters from the smaller structure are true in both). The tuple $\overline{b}$ is a* realization *of p.*

We will have occasion to use types of infinite arity; infinitely many free variables appear as one ranges over all formulas of the type (but only finitely many in each formula).

"Rich" models realizing many types over subsets of their universes are said to be *saturated*. More precisely,

Definition 1.1.2. *Let κ be an infinite cardinal and $\mathcal{M}$ an $\mathscr{L}$-structure. Then $\mathcal{M}$ is κ-*saturated *if $\mathcal{M}$ realizes every n-type over any subset A of $\mathcal{M}$ having cardinality strictly less than κ. A structure $\mathcal{M}$ is* saturated *if it is $|\mathcal{M}|$-saturated, where $|\mathcal{M}|$ is the cardinality of $\mathcal{M}$.*

Compactness implies that as long as the infinite arity is less than κ, then infinite types over small subsets (also cardinality less than κ) are realized in κ-saturated structures (33). The critical example of saturation for us is the fact that in the language of rings, uncountable algebraically closed fields (in particular, $\mathbb{C}$) are saturated. (The reason is that such fields possess uncountably many independent transcendentals over any countable subset. See (22).)

A large, sufficiently saturated (sufficient for whatever proof one has in mind) model of a theory is often referred to as a *monster model* of the theory. One may safely assume that all objects under consideration are contained in a monster model (see, e.g., (42) for set-theoretic details).

We consider two final properties of theories that undergird much of this thesis.

Definition 1.1.3. *An $\mathscr{L}$-theory T is* model complete *if every model $\mathcal{N}$ of T containing a submodel $\mathcal{M}$ of T is an elementary extension of $\mathcal{M}$ (i.e., $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$).*

Model completeness allows one to go "up and down" between models of particular theories. For instance, one statement of Hilbert's Nullstellensatz for algebraically closed fields says that if a system of polynomial equations with parameters from one model of ACF has a solution in a larger model, then the system is solvable in the original (23).

A strictly stronger property is that of *quantifier elimination.*

Definition 1.1.4. *An $\mathscr{L}$-theory T* has quantifier elimination *if every $\mathscr{L}$-formula $\varphi(\overline{x})$ is equivalent modulo T to a formula (with free variables among $\overline{x}$) with no quantifiers. (Having $\varphi(\overline{x})$ and $\psi(\overline{x})$ equivalent modulo T means that for every model $\mathcal{M}$ of T, $\mathcal{M} \models \varphi(\overline{x}) \leftrightarrow \psi(\overline{x})$.)*

Both ACF_p (any characteristic) and DCF_0 have quantifier elimination (23).

L. van den Dries noted a model-theoretic characterization of a special kind of quantifier elimination (43). A *positive formula* is one containing no negation symbols. For example, finite systems of polynomial equations may be represented this way. In general it does not make sense to talk of a theory T having "positive quantifier elimination", but we may ask whether a given formula is equivalent to a positive quantifier-free formula modulo T. Before stating van den Dries' criterion (which we henceforth call the *positive quantifier elimination test*), we recall the definition of a homomorphism of structures.

Definition 1.1.5. *A homomorphism h of $\mathscr{L}$-structures $\mathcal{M}, \mathcal{N}$ is a map from $\mathcal{M}$ to $\mathcal{N}$ that preserves the constant, function, and relation symbols of $\mathscr{L}$. That is,*

1. $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$ *for constant symbols c of $\mathscr{L}$ (where $c^{\mathcal{M}}, c^{\mathcal{N}}$ are the elements of $\mathcal{M}, \mathcal{N}$ assigned to c),*
2. $h(f^{\mathcal{M}}(\overline{a})) = f^{\mathcal{N}}(h(\overline{a}))$ *for function symbols $f \in \mathscr{L}$ and elements $\overline{a}$ of $\mathcal{M}$,*
3. $R^{\mathcal{M}}(\overline{a})$ *implies* $R^{\mathcal{N}}(h(\overline{a}))$ *for relation symbols $R \in \mathscr{L}$ and elements $\overline{a}$ of $\mathcal{M}$.*

If h has an inverse that is also a homomorphism, then h is called an isomorphism.

Induction on complexity of formulas proves that isomorphisms preserve truth of all quantifier-free formulas, while homomorphisms preserve truth of positive quantifier-free formulas. The positive quantifier elimination test essentially says that the converse is true.

Theorem 1.1.2. *(van den Dries) Let T be an $\mathscr{L}$-theory and $\varphi(\bar{x})$ an $\mathscr{L}$-formula. Then φ is equivalent modulo T to a positive quantifier-free formula if and only if for all $\mathcal{M}, \mathcal{N} \models T$,* *substructures* $A \subseteq \mathcal{M}$*,* $\bar{a} \in A$*, and* $\mathscr{L}$*-homomorphisms* $f : A \to \mathcal{N}$ *we have* $\mathcal{M} \models \varphi(\bar{a}) \implies \mathcal{N} \models \varphi(f(\bar{a}))$*.*

(The proof is a standard-but-clever argument involving compactness and proof by contradiction; one shows that the collection of positive quantifier-free formulas implying φ must contain a disjunction of such formulas that is implied by φ.)

Van den Dries as well as some later authors have used this observation to study the positive quantifier-free formulas in various algebraic settings. This thesis takes as its departure point the work of W.Y. Pong in (35), who applied it to DCF_0.

1.2 The main example: DCF_0

The language is that of differential rings with a single derivation δ: $(+, -, \cdot, \delta, 0, 1)$; δ is a unary function symbol. Sums of products of powers of variables and their derivatives, along with coefficient parameters, are *differential polynomials* or δ-polynomials; equations involving differential polynomials are the basic positive quantifier-free formulas. The theory of differential fields of characteristic zero, DF_0, is axiomatized by the usual first-order axioms for fields in addition to an axiom asserting the Leibniz product rule: $\forall x, y(\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y))$. A. Robinson drew on algorithmic work of A. Seidenberg to show that DF_0 extends to a special model-complete theory of differential fields, that of *differentially closed fields of characteristic zero* or DCF_0 (39). (There are many model-complete theories of differential fields; see (13). However, only DCF_0 is a *model companion* of DF_0.)

Later, L. Blum simplified the description of DCF_0 with a powerful first-order axiom schema (2). Before stating it, we recall that the *order* of a δ-polynomial is the highest derivative present.

Non-zero elements of the field over which we form a δ-polynomial have order 0 by convention, and 0 has undefined order.

Blum's axioms for DCF_0 are:

1. The axioms of DF_0

2. Let $f(x)$, $g(x)$ be non-zero δ-polynomials in one variable (allowing parameters as coefficients). If f has strictly higher order than g, then there exists a such that $f(a) = 0 \wedge g(a) \neq 0$.

The following are the major properties of DCF_0 that we use (see (23) for proofs of most of the facts in this section):

1. DCF_0 has quantifier elimination (and hence is model complete). Moreover, the elimination can be done with an explicit algorithm. This shows that DCF_0 is a *decidable theory*. See (8) for a specification of the algorithm; the authors of that paper have implemented quantifier elimination for several theories, including DCF_0, in the logic package REDLOG of the computer algebra system REDUCE. (The current author found REDLOG useful for getting intuition and understanding specific examples.)

2. Models of DCF_0 are algebraically closed fields (in the second group of axioms, let f be a non-zero, non-differential polynomial and g a non-zero field element).

3. Definable sets in models of DCF_0 have well-defined ordinal dimensions provided by several different *ranks* (Morley, Lascar or U-rank, etc.). The existence of these ranks may be

traced to the fact that DCF_0 is ω-*stable*, meaning that a countable set of parameters only gives rise to countably many distinct types. In turn, ω-stability is a consequence of quantifier elimination.

4. Any model of DF_0 is contained in a model of DCF_0; the proof is essentially the same as the corresponding construction of algebraically closed fields from smaller ones.

5. As in the case of ACF, model completeness is equivalent to a Nullstellensatz. Both weak and strong versions carry over with essentially no change; we give the statement in the next chapter.

6. DCF_0 is in many respects a good differential analogue of ACF, but DCF_0 is not *strongly minimal*. This means that it is not the case that every definable set (in one variable, but allowing parameters) in a model of DCF_0 is either finite or cofinite. (ACF is strongly minimal because it has quantifier elimination and non-differential polynomials in one variable have only finitely many solutions.) For instance, consider the δ-polynomial equation $x' = 0$. The solutions comprise the *constant field*, which we denote by $\mathcal{C}$. It is easy to check using Blum's axioms that $\mathcal{C}$ is an algebraically closed field (and hence infinite) such that infinitely many elements of the larger differentially closed field do not belong to $\mathcal{C}$.

The aforementioned ranks come into play in the δ-completeness problem because dimension/rank provides a dividing line between those objects that potentially have the δ-completeness property and those that do not. This dividing line occurs at the first infinite ordinal ω, so we do not enter into details; each rank we consider on models of DCF_0 is finite when the others are.

Outside of model theory, there are also multiple notions of dimension in differential algebra; see (21) and (18). The particular utility of U-rank is that it satisfies relations known as the *Lascar inequalities* that Pong used to study the δ-completeness problem for DCF_0 (35).

Prior to moving on, we note a couple of items that play no further role. First, there are analogous theories $DCF_{0,m}$ and DCF_p for differential fields of characteristic zero with m commuting derivations and differential fields of prime characteristic p, respectively (24),(44). In most respects, whether model theoretic, algebraic, or combinatorial, these theories are more complicated than DCF_0. Secondly, there is a geometric axiomatization of DCF_0 in terms of certain sections of bundles on algebraic varieties; this is due to Pierce and Pillay (31).

CHAPTER 2

NECESSARY CONCEPTS FROM ALGEBRAIC GEOMETRY AND DIFFERENTIAL ALGEBRA

As indicated in the summary, one of our aims is to replace δ-varieties over a differentially closed field with algebraic varieties over $\mathbb{C}$ for the purpose of understanding δ-completeness. In this chapter we review the main definitions and facts we need about these objects.

First, though, we recall a standard concept from commutative algebra that plays an important role in the next chapter.

A *valuation ring* R is an integral domain such that for every element x of the field of fractions $K = \mathrm{Frac}(R)$, either x or $\frac{1}{x}$ belongs to R. Equivalent formulations are (see (1)):

1. R is a maximal element of the set of local (i.e., having only one maximal ideal $\mathfrak{m}$) subrings of a given field K under the *domination partial order*: $(R_1, \mathfrak{m}_{R_1}) \geq (R_2, \mathfrak{m}_{R_2})$ if and only if $R_1 \supseteq R_2$ and $\mathfrak{m}_1 \cap R_2 = \mathfrak{m}_{R_2}$. K is the fraction field of R.

2. For a field K, there is an ordered abelian group Γ (the *value group* of K) and a surjective map $v : K\setminus\{0\} \to \Gamma$ (a *valuation*) such that for all $x, y \in K\setminus\{0\}$, both $v(xy) = v(x)+v(y)$ and $v(x+y) \geq \min\{v(x), v(y)\}$; R is the subring of K whose non-zero elements have non-negative *value* in Γ; K is the fraction field of R.

The crucial property of the second characterization is that it gives us a way of obtaining elements of R from elements of K simply by dividing by an element of K of equal or lesser value.

2.1 Affine and projective varieties over $\mathbb{C}$

We assume the reader is familiar with the notions of the Zariski topology and algebraic varieties in affine space $\mathbb{A}^n(K)$ and projective space $\mathbb{P}^n(K)$ over a field K. By *variety* we mean a closed, but not necessarily irreducible, set in the Zariski topology. A *constructible set* is a finite Boolean combination of Zariski-open and closed sets. We use frequently and without mention the fact that polynomial maps between varieties (i.e., morphisms) are continuous with respect to the Zariski topology.

We generally use the concrete description of a product variety: a closed subset of $V \times W$ is defined by systems of polynomial equations drawing on two sets of variables, one corresponding to each factor of the product. In the event that a given factor is projective, the defining polynomials must be homogeneous with respect to the factor's variables (41).

The Euclidean topology on $\mathbb{C}^n$ refines the Zariski topology; i.e., every Zariski-closed set is also closed in the Euclidean topology. If $X \subseteq \mathbb{C}^n$ is a set, we denote the closures in the Zariski and Euclidean topologies by $\overline{X}^{euc}$ and $\overline{X}^{zar}$, respectively. Two important facts about the connection between the Zariski and Euclidean topologies are:

Theorem 2.1.1. *Let $X \subseteq \mathbb{C}^n$ be a constructible set. Then $\overline{X}^{zar} = \overline{X}^{euc}$.*

Theorem 2.1.2. *Let $Z \subsetneq W$ where W is either an irreducible affine variety or irreducible projective variety over $\mathbb{C}$ and Z is a proper Zariski-closed subset of W. Then for every point $z \in Z$ there is a sequence of points in $W \setminus Z$ converging (in the Euclidean sense) to z.*

(See Theorem 2.33 of (28) and Proposition 7, p.502, of (7).)

The dimension of an algebraic variety is a central notion. There are many equivalent ways of defining dimension for irreducible affine varieties V defined over an algebraically closed field K:

1. Krull dimension given by counting lengths of chains of prime ideals in the coordinate ring $K[V] = K[x_1, \ldots, x_n]/\mathbf{I}(V)$, where $\mathbf{I}(V)$ is the set of all polynomials that vanish at each point of V
2. transcendence degree over K of the function field $K(V) = \text{Frac}(K[V])$
3. vector space dimension of the tangent space at points of V
4. the degree of the Hilbert polynomial, which encodes information about how many polynomials up to a given degree do not belong to $I(V)$
5. Morley rank of V viewed as a definable set.

(See (7) for a concrete treatment of most of these; see (22) for the model-theoretic perspective.) The complex affine varieties we deal with later are generally defined by highly-underdetermined systems of equations and consequently have large dimension.

2.2 Differential polynomial rings and ideals

J.F. Ritt, followed by his student E.R. Kolchin, established the foundations of differential algebra around the middle of the twentieth century (37),(16),(17). Their results have much in common with standard commutative algebra, but possess additional features due to one or more abstract operators satisfying the Leibniz identity for the derivative of a product.

Recall from the summary the definition of a differential ring: a commutative ring R with 1 and a finite set of maps Δ such that for each $\delta \in \Delta$ (*derivations* on R) and $x, y \in R$, $\delta(x+y) = \delta(x)+\delta(y)$ and $\delta(xy) = \delta(x)y+x\delta(y)$. Some immediate examples are commutative rings with 1 and δ the zero map (elements whose image under each derivation is 0 are called *constants*; they form an algebraically closed field in a model of DCF_0) and the ring of holomorphic functions with δ the usual complex derivative (if we extend to the meromorphic functions, then we obtain a differential field) (23).

Differential indeterminates are adjoined to a differential ring to form a *differential polynomial ring.* For instance, in the case that Δ has only one derivation δ (the *ordinary* case; we often write $'$ instead of δ) and there is a single differential indeterminate, the differential polynomial ring $R\{x\}$ is given by $R[x, x_1, x_2, \ldots]$ where δ on R extends via $x_1 = \delta(x), x_2 = \delta(x_1)$, and so on. (Note that this is compatible with our use of the term "differential polynomial" in the previous chapter.) If f is a given δ-polynomial in $R\{x\}$, we write f' for the derivative polynomial of f (the product rule, etc., extend to the whole polynomial ring). The notation $\overline{x}^{(k)}$ indicates the k-th derivatives $(x_1^{(k)}, \ldots, x_n^{(k)})$ of indeterminates $\overline{x} = (x_1, \ldots, x_n)$. In this paper, a monomial (whether in an algebraic or differential polynomial ring) is the product of powers of some finite subset of the indeterminates; by convention we do not include the scalar coefficient from the field. (Aptly, this notion is sometimes called a *power product*; see, e.g., (25). If we wish to indicate the monomial with its scalar coefficient, we will usually call the whole expression a *term*.) We work extensively with projections and thus have blocks of variables corresponding to different factors of a product variety, so we give a name to different parts of a monomial. Given a monomial t with variables from sets of indeterminates $\overline{x}$ and $\overline{y}$, we say that the product of variables from $\overline{x}$ is a $\overline{x}$*-monomial*, and likewise for $\overline{y}$. The $\overline{y}$-monomial is the $\overline{y}$*-coefficient* of the $\overline{x}$-monomial.

Example 2.2.1. Let $\alpha x' x^2 y^3 + \beta$ be a δ-polynomial with coefficients α, β from a differential field. The product $x' x^2$ in the first term is the $\overline{x}$-coefficient of y^3, and y^3 is the $\overline{y}$-coefficient of $x' x^2$. We refer the second summand as a *constant term* of the polynomial, not to be confused with elements of the constant field whose derivatives are 0.

One rates the complexity of differential polynomials according to orderings on differential monomials. We are concerned principally with the case of differential polynomials in one variable over a differential field of characteristic zero, so we restrict our attention to this simplified situation.

The ordering we assume in this paper is lexicographic in the sense that if t_1 and t_2 are δ-monomials in x, then $t_1 > t_2$ if the highest derivative appearing in t_1 is greater than the highest one appearing in t_2. If the order is the same, $t_1 > t_2$ if the degree of the maximal derivative $x^{(k)}$ is higher in t_1. Ties thereafter are decided in the same way using the variables $x^{(k-1)}, \ldots,$ down to x. Following (23) we say that t_2 is *simpler* than t_1 if $t_2 < t_1$ in this ordering. Note that $<$ is a well-ordering, so we can use it for induction on complexity of δ-monomials. If f is a δ-polynomial having highest derivative $x^{(k)}$ in degree d, we call the maximal monomial l_f appearing in f the *leading monomial* (or leading term). The $(x^{(k-1)}, \ldots, x)$-coefficient, along with the scalar coefficient, of $(x^{(k)})^d$ is the *initial* i_f of f. The initial of f' is the *separant* s_f of f. Finally, the polynomial $f - l_f$ is the *tail* t_f of f. Observe that i_f, s_f, and t_f are all simpler (have simpler leading monomials; strictly speaking, we cannot order the polynomials because of of the presence of coefficients) than f itself.

Example 2.2.2. Consider the δ-polynomial $f = 3x(x')^2 + x' - x$. The derivative f' is $6xx'x'' + 3(x')^3 + x'' - x' = (6xx' + 1)x'' + 3(x')^3 - x'$, so the initial i_f is $3x$, the separant s_f is $6xx' + 1$, and the tail t_f is $x' - x$. Note that the *leading variable* of highest derivative in $f', f'', \ldots,$ always has degree 1.

A *differential ideal* of a differential polynomial ring $R\{\overline{x}\}$ in several variables is an ideal (in the usual sense) of $R\{\overline{x}\}$ that is closed under application of the members of Δ. Given F a subset of $R\{\overline{x}\}$, standard notation writes (F) for the non-differential ideal (i.e., linear combinations over $R\{\overline{x}\}$ of elements of F) generated by F, $[F]$ for the differential ideal generated by F (linear combinations over $R\{\overline{x}\}$ of derivatives of all orders of elements of F), and $\{F\}$ for the *radical differential ideal* generated by F (the radical, in the usual sense, of $[F]$; it turns out to be a differential ideal (23)). For $K \models DCF_0$ the strong version of the differential Nullstellensatz states that given a differential ideal $I \subseteq K\{\overline{x}\}$, the radical differential ideal $\{I\}$ consists of all differential polynomials vanishing on the common zero locus $\mathbf{V}(I)$ of I. Like in the non-differential case, membership in a differential polynomial ideal via a specific linear combination of monomials with only the scalar coefficients undetermined is first-order definable. We later use this fact in a transfer argument to pull the δ-completeness problem down from the realm of differential fields and differential polynomials to non-differential polynomials over $\mathbb{C}$.

Example 2.2.3. The following sentence in the language of δ-rings expresses one way that the differential ideal $[x' - x^2, yx - 1]$ could have a δ-polynomial lacking x and its derivatives but having a non-zero constant term:

$$\exists \overline{c} \left(1 + c_1 y' = c_2 y(yx' + y'x) + c_3 y^2(x' - x^2) + (c_4 xy + c_5 y' + c_6)(yx - 1)\right)$$

The sentence is true; $c_1 = c_2 = 1$ and $c_3 = c_4 = c_5 = c_6 = -1$ work.

Later we consider order-bounded subsets of differential ideals. Let $I \subseteq K\{\overline{x}\}$, where K is a differential field, be a differential ideal. In this paper we call the collection of all differential polynomials in I having order less than or equal to k the *k-th order truncated differential ideal of I*, denoted $I^{(k)}$. The set $I^{(k)}$ is not an ideal in $K\{\overline{x}\}$, but it is an ideal in the non-differential polynomial ring over K having an algebraic indeterminate for every distinct variable (including derivatives) appearing in $I^{(k)}$.

The presence of derivations lends significant complexity to differential algebra. Problems arise in positive characteristic; for instance, the derivative of a^n can be zero even if a itself is not a constant.

Another fundamental difficulty is that differential polynomial rings have infinitely many algebraically independent indeterminates (e.g., $x, x', x'', \ldots$). This interferes with Noetherianity and the corresponding inductive arguments. Even when there are theoretical finiteness results (like the Ritt-Raudenbush basis theorem, which implies that *radical differential ideals* in a differential polynomial ring over R satisfy the ascending chain condition if R contains $\mathbb{Q}$ and the radical differential ideals of R satisfy the condition; see (23), (16)), our understanding is often frustrated by the relentless expression swell produced by repeated differentiation. Hence such basic problems as algorithmic decidability of membership in finitely-generated differential polynomial ideals (solved for non-differential polynomial rings by algorithms that generate Gröbner bases) remain open (11). We should mention, though, that researchers have conceived and implemented procedures such as the Rosenfeld-Gröbner algorithm that do allow for some practical calculations in differential polynomial rings (4).

2.3 Affine and projective δ-varieties; differential elimination ideals

As in the algebraic case, it is very enlightening to look at the geometric objects associated with differential polynomial rings. Let K be a differential field. Then *differential varieties* in affine and projective space over K are defined as zero loci of finite systems of Δ-polynomial equations, albeit with the complication of Δ-homogeneity in the projective case.

A polynomial in $K\{x_0, \ldots, x_{n-1}\}$ can be Δ-homogenized by replacing each x_i with x_i/x_n, applying the quotient rule to compute any derivatives that appear, and clearing denominators. For instance, $x_0' \in K\{x_0\}$ becomes $x_1x_0' - x_0x_1'$. This polynomial has a well-defined zero locus in $\mathbb{P}^1$, being independent of the choice of homogeneous coordinates for the points.

Let $f \in K\{x_0, \ldots, x_{n-1}\}$ be a δ-polynomial and $g \in K\{x_0, \ldots, x_{n-1}, x_n\}$ a δ-homogeneous δ-polynomial. In this paper we write $f^{\delta h}$ to denote the δ-homogenization of f and f^h to denote its homogenization as a non-differential polynomial (i.e., treating all variables, including derivatives of variables, as independent algebraic indeterminates; there is no application of the quotient rule). We write $g_{\delta h}, g_h$ to denote the dehomogenization of g achieved by setting $x_n = 1$ (any derivatives of x_n that appear in g vanish in $g_{\delta h}$ since $\delta(1) = 0$).

As in the algebraic case, δ-homogenization and δ-dehomogenization are essentially inverse operations; $(f^{\delta h})_{\delta h} = f$ and $g = x_n^d (g_{\delta h})^{\delta h}$ for some non-negative integer d (15).

The *projective closure* $\overline{V} \subseteq \mathbb{P}^n(K)$ of an affine δ-variety $V \subseteq \mathbb{A}^n(K)$ is the intersection of all projective δ-varieties in $\mathbb{P}^n(K)$ that contain V. As in the algebraic case, the δ-homogenizations of a set of defining equations for V might define a projective δ-variety strictly larger than $\overline{V}$.

One must be careful in making conclusions about the projective closure of given δ-variety. The author is aware of at least two anomalous situations. The first, which he encountered in (35), is that δ-homogenizing a δ-polynomial need not add points at infinity; i.e., an affine δ-variety may be projectively closed. For instance, the δ-homogenization of $f = x_0' - x_0^2$ is $f^{\delta h} = x_1 x_0' - x_0 x_1' - x_0^2$. The point at infinity $(1 : 0)$ does not cause $f^{\delta h}$ to vanish and thus does not lie on $\overline{\mathbf{V}(f = 0)}$ (which we often write as $\overline{f = 0}$).

The second arose in the course of studying the δ-completeness problem. We found that the δ-homogenization of $x'' - x^2$, which *does* contain the point at infinity in $\mathbb{P}^1$, does not define the projective closure; $x'' - x^2 = 0$ is already a projective δ-variety. See Appendix A for the calculation of the projective closure of $x'' - x^2$.

In both affine and projective space, varieties comprise the closed sets of a differential analogue of the Zariski topology, often referred to as the *Kolchin topology.* We denote the Kolchin closure of a set X by $\overline{X}^{kolch}$.

Our final topic is that of *differential elimination ideals.* The definitions and proofs are essentially the same as for non-differential polynomial rings (see Theorem 3, p.193, and Proposition 5, p.397, of (7)).

Theorem 2.3.1. *1. Let $K\{\overline{x},\overline{y}\} = K\{x_1,\ldots,x_n,y_1,\ldots,y_m\}$ be a differential polynomial ring over a differential field K. If I is a differential ideal of $K\{\overline{x},\overline{y}\}$, we define the differential elimination ideal $I_{\overline{x}}$ eliminating $\overline{x}$ to be the ideal of $K\{\overline{y}\}$ given by $I \cap K\{\overline{y}\}$. If $K \models DCF_0$, then $\mathbf{V}(I_{\overline{x}}) = \overline{\pi_2(\mathbf{V}(I))}^{kolch}$.*

2. Let $K\{\overline{x},\overline{y}\} = K\{x_0,\ldots,x_n,y_1,\ldots,y_m\}$ be a differential polynomial ring over a differential field K. If I is a differential ideal of $K\{\overline{x},\overline{y}\}$ generated by δ-polynomials δ-homogeneous with respect to $x_0,\ldots,x_n$, we define the projective differential elimination ideal of I eliminating $\overline{x}$ to be $\hat{I}_{\overline{x}} = \{g \in K\{\overline{y}\} \mid$ for all $0 \leq i \leq n$ there is $e_i \geq 0$ such that $x_i^{e_i} g \in I\}$. If $K \models DCF_0$, then $\mathbf{V}(\hat{I}_{\overline{x}}) = \overline{\pi_2(\mathbf{V}(I))}^{kolch}$.

We occasionally distinguish elimination ideals from projective elimination ideals by referring to the former as "affine elimination ideals".

By analogy with classical algebraic geometry, we would like to use the algebra of differential polynomial rings to understand affine and projective differential varieties. However, the greater complexity of these rings in comparison with their non-differential counterparts gives differential algebraic geometry a unique flavor. To date the theory is at a more rudimentary stage and tends to be less intuitively geometric. (For a survey, see (6).) On the positive side, this state of affairs yields opportunities for some non-traditional tools to play relatively more prominent roles; these tools take center stage in the remaining chapters.

CHAPTER 3

δ-COMPLETENESS

3.1 δ-completeness: definition and basic properties

The following property is the focus of this thesis.

Definition 3.1.1. *Let V be a projective δ-variety defined over a differential field $\mathcal{F}$. We say V is* complete *if for every projective δ-variety W over $\mathcal{F}$ and Kolchin-closed subset Z of $V \times W$, the projection $\pi_2 : V \times W \to W$ maps Z to a Kolchin-closed set.*

As stated in the preceding chapter, we always assume that $\mathcal{F} \models DCF_0$ and that $V \subseteq \mathbb{P}^n(\mathcal{U})$, where $\mathcal{U}$ is a monster model of DCF_0 containing $\mathcal{F}$.

The "fundamental theorem of elimination theory" states that projective algebraic varieties over arbitrary algebraically closed fields are complete with respect to projection of Zariski-closed subsets of products with other projective varieties. (See (41),(7) for two well-written arguments.) We will look at the proof of this theorem later in this chapter and explain why it fails to generalize to projective δ-varieties.

To distinguish the notions in the algebraic and differential cases we sometimes refer to δ-completeness and algebraic completeness, though we usually rely on context and simply write "completeness".

The motivation for the definition of completeness is to isolate a condition that mimics the functionality of compactness in analysis (29). The usual definition in terms of finite open subcovers applies too widely; every δ-variety over a differentially closed field is compact (or quasi-compact, as the notion is often called in algebraic geometry). The definition of completeness uses another characterization of compactness in general topological spaces (for products of algebraic and differential varieties we use stronger topologies than the product topology, so not everything is complete) (30).

Closely related is the concept of *properness.* In algebraic geometry it has an algebraic meaning that generalizes completeness (29). In general, a continuous function f from topological space X to topological space Y is *proper at a point* $p \in Y$ if $f^{-1}(C)$ is compact whenever C is a compact set containing p (see, e.g., (14)). In a later chapter we reduce the problem of identifying complete δ-varieties to discerning properness of projections from $\mathbb{C}^n \times \mathbb{C}^m$ to $\mathbb{C}^m$ at the origin in $\mathbb{C}^m$.

We now enumerate some of basic properties of δ-completeness. Most of them are the same as for algebraic completeness, but there is one notable exception. (For references not given below, see (35).)

1. δ-completeness is a local property, meaning that there is no loss of generality in assuming that the second factor W in the definition is affine space $\mathbb{A}^m(\mathcal{F})$. This follows from the general topological fact that if finitely many open sets U_i cover a space, then a subset X is closed if and only if each $X \cap U_i$ is closed in U_i with respect to the subspace topology (41).

2. One could extend the definition of δ-completeness to more general subsets of $\mathbb{P}^n$ to admit the possibility of complete, non-closed sets (e.g., non-projective quasiprojective δ-

varieties), but this is unnecessary; any complete subset of projective space must be defined by δ-homogeneous differential polynomials. (The proof is exactly the same as in the algebraic case; the homogeneous polynomial equations $\{X_iT_j = X_jT_i\}_{i\neq j}$ define the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$, so if we restrict the $X_0, \ldots, X_n$ to values of a non-closed subset of $\mathbb{P}^n$, then the image of the second projection is the same non-closed set.)

3. Kolchin-closed subsets of δ-complete varieties are δ-complete.
4. A finite union of δ-complete varieties is δ-complete. This is because for any function the image of a union is the union of the images, and finite unions of closed sets are closed.
5. Products of δ-complete varieties are δ-complete.
6. If $\varphi : V \to W$ is a morphism of δ-varieties and V is δ-complete, then $\varphi(V)$ is δ-complete.
7. However, morphisms of irreducible δ-complete varieties into affine space need not be constant. The basic reason is that models of DCF_0, unlike ACF, have infinite proper closed sets (whence the absence of strong minimality). For instance, we saw earlier that there are infinite projectively-closed δ-varieties that lie entirely in a single affine chart.

3.2 Difficulties in generalizing the fundamental theorem of elimination theory

In spite of the similarities we have seen thus far, all is not well in generalizing the theory of algebraic completeness to δ-completeness. In addition to the anomaly about non-constant morphisms, one must soon contend with the following example of Kolchin (15),(35):

Example 3.2.1. Define a Kolchin-closed subset V of $\mathbb{A}^1 \times \mathbb{A}^1$ with two δ-polynomial equations

$$y(x')^2 + x^4 - 1 = 0, 2yx'' + y'x' + 4x^3 = 0,$$

where x corresponds to the first factor and y to the second. Note that the point at infinity $(1 : 0)$ does not satisfy the δ-homogenizations with respect to x of these equations, so V is actually closed in $\mathbb{P}^1 \times \mathbb{A}^1$. The second equation is the derivative of the first, followed by division by x'. It follows, using also the fact that V is defined over a differentially closed field, that the image of the second projection consists of all points in $\mathbb{A}^1$ except 0. By Kolchin irreducibility of $\mathbb{A}^1$, the image is not closed.

This means that $\mathbb{P}^1$ is not δ-complete! However, the very nature of the example suggests a workaround. The two equations together imply that $x' \neq 0$, so the second equation is close to a consequence of the first. So we morally have one δ-polynomial equation in two variables; the system is underdetermined and the second factor takes advantage to make the system solvable for all but one value of x.

Suppose that instead we had an additional δ-polynomial equation "independent" from the original system; for instance, we could have a relation purely in terms of x and its derivatives. As an example, suppose we also had $x''' = (x')^2$. Then $y = 1$ no longer lies in the image of the second projection. For if so, then the second equation implies that $2x'' = -4x^3$, whence $x''' = -6x^2x'$. Combined with $x''' = (x')^2$ and $x' \neq 0$ we get $x' = -6x^2$. We conclude that $37x^4 = 1$, which is impossible because then either x or x' equals 0. So introducing a constraint on x averts the conclusion that the projective closure of $x''' = (x')^2$ is not δ-complete. (This doesn't prove δ-completeness, either, but shows how to short-circuit the bad example.)

The preceding example shows that the standard proof of the fundamental theorem of elimination theory cannot work for projective δ-varieties. We make a few observations about what goes wrong; we use a paraphrased version of the proof of the fundamental theorem from (41).

Consider a projective algebraic variety $V \subseteq \mathbb{P}^n(K)$, where $K \models ACF$. Let $Z \subseteq V \times \mathbb{A}^m$ be defined by a set $\{f_i(x_0, \ldots, x_n, y_1, \ldots, y_m)\}_{1 \leq i \leq r}$ of polynomials, including those defining V, that are homogeneous with respect to $\overline{x}$. We must prove that the image $\pi_2(Z) \subseteq \mathbb{A}^m$ of the second projection is Zariski closed. So far the setup is completely analogous to the differential case.

Consider the variables $\overline{y}$ of the second factor to be parameters of a family of systems of equations (we use $\overline{a}$ to represent some specific value of $\overline{y}$). We claim that $\overline{y}$ being in $\pi_2(Z)$ is a closed condition (i.e., defined by polynomial equations in $\overline{y}$ with coefficients from K). To show this, we show that $\mathbb{A}^m \setminus \pi_2(Z)$ is an open set.

One version of the projective Nullstellensatz states that a collection of homogeneous polynomials in $\overline{x}$ has no solution in $\mathbb{P}^n(K)$ if and only for some positive integer e the polynomials generate the ideal $(x_0^e, \ldots, x_n^e)$ (this is called an *irrelevant ideal*). Suppose $\overline{a} \in \mathbb{A}^m \setminus \pi_2(Z)$. Then it must be that for some e large enough, every $\overline{x}$-monomial t of degree e is generated by a linear combination of $\sum_{i=1}^{r} g_i(\overline{x}) f_i(\overline{x}, \overline{a})$ of polynomials in $\overline{x}$. We may assume that each monomial of g_i has total degree $e-($ homogeneous degree of $f_i)$; all other terms do not interact with the monomials of degree e and are unnecessary to generate t. Multiplying out the polynomials g_i, f_i and grouping the monomials, we have a finite linear system of equations (i.e., the solutions are the field coefficients necessary to generate t from the monomials available in the linear combination).

It is here that the process breaks down for δ-varieties. In general, derivatives of the f_i would be needed to generate t, but δ-homogeneity is poorly behaved; a derivative of a homogeneous δ-polynomial is usually not δ-homogeneous. Moreover, there are arbitrarily many variables $x, x', \ldots,$ that could appear in the constituents of the linear combination. It is not at all clear how to bound the complexity of the linear combination generating t from the $f_i, f_i' \ldots,$ so we apparently do not have the finite-dimensional vector space structure that lies at the heart of the proof of the fundamental theorem. This failure points in the same direction as Kolchin's example showing that $\mathbb{P}^1$ is not δ-complete: without sufficiently many relations on the derivatives, one cannot hope to achieve completeness. This intuition might lead one to look at projective δ-varieties whose defining equations give "non-trivial information" about each variable. More precisely, finite rank (in any of the various model-theoretic or differential algebraic senses mentioned previously) should be a necessary property of complete δ-varieties. The projective line $\mathbb{P}^1$ has infinite rank because it contains δ-varieties of arbitrarily high finite rank: $x' = 0 \subseteq x'' = 0 \subseteq \cdots \subseteq x^{(n)} = 0 \subseteq \ldots.$

Pong proved just that in (35), a paper that will guide our discussion for the remainder of this chapter.

Theorem 3.2.1. *(Pong) Every δ-complete set has finite U-rank.*

The proof uses the Lascar inequalities to show that any set of infinite U-rank can be mapped by a composition of projections into $\mathbb{P}^1$ and still retain infinite rank. Thus the image must be all of $\mathbb{P}^1$, which is not δ-complete. Projections are morphisms, so if the original set had been complete, then the image would have been as well.

Similarly, Pong showed that any δ-complete projective δ-variety maps via a definable δ-isomorphism into $\mathbb{A}^1$ (not surjectively, though, lest the rank be infinite). The same thing holds for any finite-rank δ-variety, affine or projective. However, unless a given variety is known beforehand to be complete, we cannot say *a priori* that the image is closed in $\mathbb{A}^1$.

For the rest of this paper we follow Pong's general conventions in (35) when working with δ-varieties:

1. All δ-varieties considered are defined by δ-polynomials with coefficients from a fixed arbitrary field $\mathcal{F} \models DCF_0$. In particular, completeness means completeness with respect to varieties defined over $\mathcal{F}$. This does not limit the generality of our results because $\mathcal{F}$ is arbitrary.

2. The first-order language $\mathscr{L}$ we use is the language of differential rings, augmented with constant symbols for each element of $\mathcal{F}$. Our basic theory is $Th_{\mathscr{L}}(\mathcal{F})$; i.e., the elementary diagram of $\mathcal{F}$. This ensures that all models of our theory contain contain an isomorphic copy of $\mathcal{F}$.

3. Though we make no specific use of this hypothesis, we may assume that $\mathcal{F}$ is contained in a large saturated model $\mathcal{U}$ of DCF_0. Unless stated otherwise, $\mathbb{P}^n$ and $\mathbb{A}^m$ mean $\mathbb{P}^n(\mathcal{U})$ and $\mathbb{A}^m(\mathcal{F})$. We consider all the $\mathcal{U}$-points satisfying the defining δ-polynomial equations of a δ-variety V to belong to V.

We now describe a proof of the fundamental theorem that points in the direction we take later in this paper.

Recall van den Dries' positive quantifier elimination test from the first chapter. In his note (43), van den Dries showed that the criterion gives quick proofs of algebraic completeness as well as completeness of projective varieties over real closed fields. In another use, Prestel employed the technique to prove completeness of projective varieties over algebraically closed and real closed valued fields (36).

For a projective algebraic variety $V \subseteq \mathbb{P}^n$ the second projection of a subvariety of $Z \subseteq V \times \mathbb{A}^m$ is definable by an existential formula $\varphi_Z(\overline{y})$ (with free variables for the coordinates of points in $\mathbb{A}^m$) in the language of rings. Hence if all such formulas are equivalent to positive quantifier-free formulas (i.e., define affine algebraic varieties in $\mathbb{A}^m$), then V is complete.

The hypotheses of the positive quantifier elimination criterion give us a subring $A \subseteq K$, a field homomorphism $f : A \to L$, and elements $\overline{a} \in A^m$ such that $K, L \models ACF$ and $K \models \varphi_Z(\overline{a})$. We must determine if $L \models \varphi_Z(f(\overline{a}))$. Van den Dries uses a standard "place extension theorem" that implies that we may assume A to be a valuation ring. (Given a subring A of a field K and a homomorphism $f : K \to L$, Zorn's lemma yields a maximal extension of f having domain a subring $\hat{A}$ of K and codomain some field. Then after several algebraic lemmas one concludes that $\hat{A}$ is a valuation ring. See (1).)

This is the key step, because by homogeneity (with respect to the variables $\overline{x}$ of the first factor $\mathbb{P}^n$) of the polynomials defining Z, we may divide the homogeneous coordinates of any non-trivial solution by any non-zero value and obtain new homogeneous coordinates. $K \models \varphi(\overline{a})$ means that there exist $\overline{x} \in K^n$, not all 0, such that $(\overline{x}, \overline{a}) \in Z$. We do not know, however, whether these values of $\overline{x}$ belong to A; they very well might not. However, since A is a valuation ring, we may divide by the element of minimal value. This gives us one coordinate, 1, that is definitely not in the kernel of f. The remaining coordinates have non-negative value; i.e., they belong to the valuation ring A. So an algebraic result gives us a representative of the point $(\overline{x}, \overline{a}) \in V$ whose coordinates all lie in A. This allows f to map all the parameters to elements of L; the coordinate 1 goes to 1, so points of projective space go to points of projective space as needed for preservation of φ. Homomorphisms preserve positive quantifier-free formulas, which is what $\varphi(\overline{a})$ is once we get rid of the quantifier and assign the values of the divided coordinates to $\overline{x}$. The outcome is that $L \models \varphi(\overline{a})$.

Pong followed this same pattern in (35) for the case of DCF_0. The necessary algebraic results are contained in work of P. Blum and S. Morrison, among others (3),(26),(27), and they *almost* allow for the same conclusion. The one difference lies in the nature of the maximal subring produced by the Zorn's lemma argument.

Following Pong's terminology and notation we make an important definition:

Definition 3.2.1. *Let R be a δ-subring of a δ-field K.Let $f : R \to L$ be a δ-ring homomorphism into a δ-field L. We say R is a maximal δ-ring (with respect to K, or simply K-maximal) if f does not properly extend to a δ-ring homomorphism with domain a δ-subring of K and codomain a δ-field. In this paper, K and L are always taken to be models of $Th_{\mathscr{L}}(\mathcal{F})$ contained in $\mathcal{U}$.*

The crucial fact (3) is that a K-maximal δ-ring R is a local differential ring with unique maximal δ-ideal $\mathfrak{m}$ (i.e., $\mathfrak{m}$ is the unique maximal ideal of R, and $\mathfrak{m}$ is also closed under the derivation); for conciseness we may write $(R, \mathfrak{m}), (R, f, \mathfrak{m})$, or even $(R, K, f, \mathfrak{m})$ to denote all the data.

Three additional results due to Morrison are critical to our work on δ-completeness; they are Proposition 4 and its corollary in (26) and Corollary 3.2 in (27).

Theorem 3.2.2. *(Morrison) Let R be a maximal differential ring of a differential field K, and let x be an element of K. Suppose that $\frac{1}{x}$ is not in R and that there is a linear relation*

$$x^{(k)} = \sum_{i=0}^{k-1} a_i x^{(i)} + r,$$

where r and a_i $(0 \leq i \leq k-1)$ are elements of R. Then $x \in R$.

Corollary 3.2.1. *(Morrison) Let R be a maximal differential ring of a differential field K, and x a non-zero element of K. If for some positive integer k either $x^{(k)}$ or $x^{(k)}/x$ is in R, then either x or $\frac{1}{x}$ is in R.*

Theorem 3.2.3. *(Morrison) If $\phi_0 : R \to \Omega$ is a differential specialization and if x is integral over R, then ϕ_0 extends differentially to x. Thus a maximal differential ring of a differential field L is integrally closed in L.*

(A differential specialization is simply a non-zero Δ-homomorphism from a Δ-ring containing $\mathbb{Q}$ into a Δ-field.)

In spite of these nice properties, being a K-maximal δ-ring is not as strong as being a valuation ring. (Indeed, if K-maximal δ-rings were valuation rings in general, then van den Dries' algebraic completeness proof would go through without change and $\mathbb{P}^1$, for instance, would be δ-complete.) In general the most important property of a K-maximal δ-ring R is that if $x \in K \setminus R$, then $1 \in \mathfrak{m}\{x\}$, where $\mathfrak{m}\{x\}$ is the differential subring of K generated by $\mathfrak{m}$ and x. The next chapter is largely devoted to exploiting this fact.

Lacking a valuation, Pong did the next best thing and formulated a criterion for δ-completeness in terms of the missing property: being able to assume that parameters in K actually lie in R so that van den Dries' argument goes through.

Theorem 3.2.4. *(Pong) Let $V \subseteq \mathbb{A}^n$ be an affine δ-variety defined over $\mathcal{F}$. Then the following are equivalent:*

1. *V is δ-complete.*
2. *For any K-maximal δ-ring R, we have $V(K) = V(R)$ (i.e., if a tuple $\overline{a} \in K^n$ has all its coordinates in K and $\overline{a} \in V$, then every coordinate of $\overline{a}$ belongs to R).*

Note that the hypothesis of finite rank does not appear, so the algebraic property of the K-points of V descending to R guarantees finite rank; we find this quite interesting.

Pong used this affine "valuative criterion" to produce new examples of δ-complete varieties. In particular, he proved completeness of the projective closure of any δ-variety in $\mathbb{A}^1$ defined by $x' = P(x)$, where P is a non-differential polynomial in the single variable x. (To our knowledge, the only positively-identified complete δ-variety prior to Pong's work was $\mathbb{P}^n(\mathcal{C})$, the projective closure of the constant field; this insight was due to Kolchin (15).) We discuss and significantly extend Pong's method in the next chapter.

Before leaving this chapter behind, though, we must report a significant recent development. We would be pleased to announce that all finite-rank projective δ-varieties over a model of DCF_0 are δ-complete, or even that there is a possibility that such is the case. Indeed, we suspected this for a long time. Unfortunately, we have found the following counterexample:

Proposition 3.2.1. *$\overline{x'' = x^3}$ is not δ-complete.*

Proof. Let $\mathcal{F} \models DCF_0$. Consider the subset W of $\mathbb{A}^1(\mathcal{F}) \times \mathbb{A}^1(\mathcal{F})$ defined by $x'' = x^3$ and $2yx^4 - 4y(x')^2 = 1$. We claim that $\pi_2(W) = \{y \mid y' = 0 \text{ and } y \neq 0\}$. That set is not δ-closed because $y' = 0$ is an irreducible δ-variety containing the point 0. (Note that y' is irreducible as a polynomial. By Corollary 1.7 (p.44) of (23), it follows that the differential ideal $I(y') = \{g \in \mathcal{F}\{x\} \mid s_{y'}^k g \in [y'] \text{ for some } k\}$ is prime. The separant $s_{y'}$ of y' is 1, so $I(y') = [y']$ is prime and $\mathbf{V}([y']) = \mathbf{V}(y' = 0)$ is an irreducible δ-variety.) This suffices to prove incompleteness of $\overline{x'' = x^3}$ because $x'' = x^3$ is already projective (the point at infinity does not lie on the δ-homogenization of $x'' = x^3$).

For the first containment, suppose $y \in \pi_2(W)$. Let $(x, y) \in W$, and differentiate the equation $2yx^4 - 4y(x')^2 = 1$. Substituting x^3 for x'', we find $2y'x^4 - 4y'(x')^2 + 8yx^3x' - 8yx^3x' = 2y'x^4 - 4y'(x')^2 = y'(2x^4 - 4(x')^2) = 0$. Multiplying both sides by y and applying the relation $2yx^4 - 4y(x')^2 = 1$ gives $y' = 0$; clearly $y \neq 0$.

For the other containment let y be a non-zero constant. We need to find x such that $x'' = x^3$ and $2x^4 - 4(x')^2 = \frac{1}{y}$. Blum's axioms for DCF_0 imply that there exists x such that

$2x^4 - 4(x')^2 = \frac{1}{y}$ and $2x^4 \neq \frac{1}{y}$. Therefore $x' \neq 0$. Differentiation produces $8x^3x' - 8x'x'' = 0$. Because $x' \neq 0$, we may divide and obtain $x'' = x^3$. Hence $y \in \pi_2(W)$. □

Notice that this easily generalizes to all equations of the form $x'' = x^n$ for $n \geq 2$. Moreover, note the similarity with Kolchin's example proving incompleteness of $\mathbb{P}^1$. We see that not only is having too few relations deleterious to completeness, but having too many of the wrong kind is also an obstacle. For now we focus on developing more tools for verifying completeness; we discuss the implications of this bad example later.

CHAPTER 4

MODIFIED VALUATIVE CRITERIA AND REDUCTION OF THE δ-COMPLETENESS PROBLEM TO COMPLEX VARIETIES

We have seen how Pong used his differential valuative criterion to demonstrate δ-completeness of δ-varieties in $\mathbb{P}^1$ whose restrictions to $\mathbb{A}^1$ are defined by equations of the form $x' = P(x)$ (where P is a non-differential polynomial). Can this technique be extended to prove completeness results for larger classes of δ-varieties? The answer is yes, but it is desirable to modify the valuative criterion before undertaking the project.

There are at least a couple of reasons for enhancing the valuative criterion. First of all, as stated the criterion is highly non-constructive. A maximal δ-ring in an arbitrary model of DCF_0 is an ethereal object presenting few tools for its manipulation. The author's experience in imitating Pong's proof indicates that it is only necessary to treat the elements of $\mathfrak{m}$ as differential indeterminates. The calculations boil down to formal symbolic elimination arguments. This suggests that it would be more appropriate to have a "syntactic" version of the valuative criterion that deals with polynomials instead of maximal δ-rings. Second, the valuative criterion only applies to δ-varieties in $\mathbb{A}^n$. Of course, finite-rank varieties miss generic hyperplanes (35), so we may assume that up to projective equivalence a given finite-rank projective δ-variety is contained in a single standard affine chart. But what do such varieties look like? Even in $\mathbb{A}^1$ the example of $x'' = x^2$ (see Appendix A) shows that it can be non-obvious whether a specific affine δ-variety is projective. We nonetheless record an observation that does shed some light:

Proposition 4.0.2. *Let $V \subseteq \mathbb{A}^1(K)$ be a projectively-closed δ-subvariety in the affine line over a differential field K of characteristic 0. If V has infinitely many points, then there is a projectively-closed δ-variety $W \subseteq \mathbb{A}^1(K)$ that contains V and is defined by a single δ-polynomial $f(x)$ such that f has a monomial of the form x^n for some $n \geq 2$.*

Proof. Since V is projectively closed, there exist δ-homogeneous polynomials $f_1(x_0, x_1), \ldots,$ $f_r(x_0, x_1)$ defining V. As their common zero locus lies in a standard affine chart, we may assume that for some i, f_i (call it f from now on) misses the point at infinity $(1 : 0)$. Hence upon setting $x_1 = 1$ the dehomogenized $f_{\delta h}$ defines a projectively-closed affine variety W in $\mathbb{A}^1$. (We refer to x_0 as x when discussing W.) The point at infinity does not cause f to vanish, so x_1 does not divide f. By the near-inverse relationship between homogenization and dehomogenization we then see that $(f_{\delta h})^{\delta h} = f$. Since V has infinitely many points, a derivative of x_0 or x_1 must appear in f. In fact a derivative of x_0 must be present; the δ-homogenization of a non-differential polynomial is still non-differential, so $f_{\delta h}$ must contain a derivative of x lest $(f_{\delta h})^{\delta h} = f$ define a finite set. Rehomogenizing $f_{\delta h}$ requires clearing denominators with at least a power of two in x_1 because of the derivative of x that appears. Finally, if $f_{\delta h}$ contained no monomial x^n for $n \geq 2$, then every monomial of $(f_{\delta h})^{\delta h} = f$ would be divisible by x_1 or one of its derivatives. That would force f to vanish at $(1 : 0)$, contrary to hypothesis. $\square$

This result explains how $x' = 0$ can fail to be projective while $x'' = x^2$ is, even though δ-homogenization adds the point at infinity to both. It is only a necessary condition, however, and the author is not certain that a δ-polynomial $f(x)$ having a monomial x^n for $n \geq 2$ forces the affine variety $\mathbf{V}(f)$ to be projective.

Even if we knew that, there would still be advantages to having a valuative criterion that applies directly to projective δ-varieties. For instance, later in this chapter we show that to understand complete δ-varieties, it suffices to understand δ-varieties V in $\mathbb{P}^1$ whose restrictions to a standard affine chart are defined by polynomials of the form $Q(x, x', \ldots, x^{(n-1)})x^{(n)} = P(x, x', \ldots, x^{(n-1)})$. Such V are generally not affine, so by focusing only on the affine projectively-closed varieties we would be forced to consider a wider assortment of δ-polynomials, complicating our analysis.

4.1 Modified valuative criteria

We begin our development of valuative criteria for projective δ-varieties with an important definition.

Definition 4.1.1. *Let $p = (p_0 : p_1 : \cdots : p_n) \in \mathbb{P}^n$ and let S be a δ-ring. With only slight abuse of terminology, we say p is in S (denoted $p \in S$) if for some $0 \leq i \leq n$ we have $p_i \neq 0$ and $\frac{p_j}{p_i} \in S$ for all $0 \leq j \leq n$. If V is a differential subvariety of $\mathbb{P}^n$ such that $p \in V$ and $p \in S$, we write $p \in V(S)$.*

Notice that p being in S or not is independent of the choice of homogeneous coordinates for p, so this definition is reasonable. Also note that it is not sufficient for p to have a representative such that every coordinate lies in S; such a representative must be the result of dividing every coordinate of some other representative by one of the latter representative's coordinates. In particular, $p \in S$ implies that p has a representative such that at least one coordinate is 1 and every other coordinate also belongs to S. Requiring some coordinate to be a unit of S ensures that not every coordinate maps to 0 under a ring homomorphism on S; we will need this feature in the upcoming proof.

The following is a modification of Pong's affine valuative criterion for differential completeness. It gives us a version of the criterion that works for differential subvarieties of $\mathbb{P}^n$ (instead of just affine varieties). The proof strategy is the same as Pong's, but some care is required to make the argument work in projective space.

Theorem 4.1.1. *Let V be a differential subvariety of $\mathbb{P}^n$. Then V is δ-complete if and only if for every K-maximal δ-ring $(R, f, \mathfrak{m})$ and point $p \in V(K)$ we have $p \in V(R)$.*

Proof. Forward direction: We show the contrapositive. Suppose V has a point p such that $p \in K$ but $p \notin R$. We show that V is not δ-complete by finding a projective δ-subvariety of V that is not complete. (This suffices because completeness is closed under containment.)

By assumption there is a point $p = (p_0 : \cdots : p_n) \in V(K)$ such that $p \notin R$; our notion of $p \in K$ is independent of representative, so we may assume each coordinate of p belongs to K. Let $I \subseteq \{0, \ldots, n\}$ denote the set of indices of non-zero coordinates of p. The fact that $p \notin R$ implies for each $i \in I$ there is an index j_i such that $\frac{p_{j_i}}{p_i} \notin R$. (Necessarily $j_i \neq i$ because $\frac{p_i}{p_i} = 1 \in R$.) It follows from the properties of maximal δ-rings that $1 \in \mathfrak{m}\{\frac{p_{j_i}}{p_i}\}$, so there exist elements $m_{i_k} \in \mathfrak{m}$ satisfying an equation $\sum_k m_{i_k} t_{i_k} \left(\frac{p_{j_i}}{p_i}\right) = 1$, where the t_{i_k} are δ-monomials in $\frac{p_{j_i}}{p_i}$.

Let $P_i(\bar{x}, \bar{y})$ be the δ-polynomial resulting from clearing denominators in $\sum_k y_{i_k} t_{i_k} \left(\frac{x_{j_i}}{x_i}\right) - 1$, where $\bar{x}, \bar{y}$ are differential indeterminates. Note that P_i is δ-homogeneous in $\bar{x}$ and so defines a δ-subvariety of a product variety where the $\bar{x}$ are projective coordinates and the $\bar{y}$ are affine coordinates. Also observe that every $\bar{x}$-monomial in P_i has a coefficient from $\bar{y}$ with the exception of one: the monomial obtained by multiplying -1 by a positive power of x_i when clearing denominators.

By design, the following formula $\varphi(\bar{y})$ is true in K when y_{i_k} is interpreted as m_{i_k}:

$$\exists \bar{x}\,(\bar{x} \in V \wedge (\wedge_{i \notin I} x_i = 0) \wedge (\vee_{i \in I} x_i \neq 0) \wedge (\wedge_{i \in I} P_i(\bar{x}, \bar{y}) = 0))\,.$$

(That is, p witnesses the truth of $\varphi(\overline{m})$.) The formula $\varphi(\bar{y})$ defines a projection whose failure to be closed would imply that the projective δ-variety $V \cap \wedge_{i \notin I}(x_i = 0)$ is not complete. By van den Dries' positive quantifier-elimination test, we simply need to verify that $\varphi(f(\overline{m}))$ is not true in the codomain of f.

Because $\mathfrak{m} = \ker(f)$, for every $i \in I$ all monomials of $P_i(\bar{x}, f(\overline{m}))$ vanish except for $-x_i^{r_i}$ for some positive integer r_i. Then $\varphi(f(\bar{m}))$ asserts that

$$\exists \bar{x}\,(\bar{x} \in V \wedge (\wedge_{i \notin I} x_i = 0) \wedge (\vee_{i \in I} x_i \neq 0) \wedge (\wedge_{i \in I}(-x_i^{r_i} = 0)))\,,$$

which is contradictory (one subformula requires x_i to be non-zero for some $i \in I$, but another forces all such to be zero). This proves that V is not complete.

Reverse direction: Let a collection $\{P_j(\bar{x}, \bar{y})\}$ of δ-polynomials define an arbitrary subvariety of $V \times \mathbb{A}^m$, where $\bar{x}, \bar{y}$ are respectively projective and affine coordinates. To prove V is δ-complete we must show that the following $\varphi(\bar{y})$ is equivalent to a positive quantifier-free formula:

$$\exists \bar{x}\,(\bar{x} \in V \wedge (\vee_{0\le i\le n}(x_i \neq 0)) \wedge (\wedge_j P_j(\bar{x},\bar{y}) = 0))\,.$$

It is enough to prove that if $(R, f, \mathfrak{m})$ is a K-maximal differential ring such that $f : R \to L$ and $K \models \varphi(\bar{a})$ for elements $\bar{a}$ of R, then $L \models \varphi(f(\bar{a}))$.

Let $\psi(\bar{x},\bar{y})$ be the subformula of φ such that $\varphi(\bar{y}) = \exists\bar{x}\psi(\bar{x},\bar{y})$. Since $K \models \varphi(\bar{a})$, there is $p = (p_0 : \cdots : p_n) \in K$ such that $K \models \psi(p_0,\ldots,p_n,\bar{a})$. (What $K \models \varphi(\bar{a})$ actually tells us is that each $p_i \in K$ and some $p_i \neq 0$, but K is a field so we may divide by any non-zero coordinate and still have every coordinate in K; thus the definition of $p \in K$ is satisfied.) We have $p \in V(K)$, so it follows by hypothesis that $p \in V(R)$. Dividing by coordinate p_i for some $0 \le i \le n$ ensures that every resulting coordinate belongs to R; importantly, the resulting i-th coordinate is 1.

Hence $\psi(\frac{p_0}{p_i},\ldots,1,\ldots,\frac{p_n}{p_i},\bar{a})$ has parameters from R and is true in K. The map f is a homomorphism with domain R, homomorphisms preserve satisfaction of positive quantifier-free formulas, and the only non-positive subformula of $\psi(\bar{x},\bar{y})$ is the disjunction $\vee_{0\le i\le n}(x_i \neq 0)$, so we conclude $L \models \psi(f(\frac{p_0}{p_i}),\ldots,f(1) = 1$ (i.e., a non-zero value) $,\ldots,f(\frac{p_n}{p_i}),f(\bar{a}))$. This implies the desired $L \models \varphi(f(\bar{a}))$, finishing the proof.

□

(For future reference, we will call this result the *projective valuative criterion.*)

The following result makes use of the projective valuative criterion and even mimics the proof of the criterion, but it eliminates explicit reference to maximal δ-rings. That is an important step toward making the valuative criterion more concrete.

Theorem 4.1.2. *Let V be a differential subvariety of $\mathbb{P}^n$; call the projective coordinates $x_0, \dots, x_n$. Denote by I a non-empty subset of $\{0, \dots, n\}$, and for each $i \in I$ choose some $j_i \neq i$ between 0 and n. Let W be a subvariety of $V \times \mathbb{A}^m$ consisting of the zero loci of δ-polynomials of the form x_j for all j such that $0 \leq j \leq n$, $j \notin I$, and of the form P_i for all $i \in I$, where P_i is the δ-homogeneous polynomial resulting from clearing denominators in $\sum_k y_{i_k} t_{i_k} \left(\frac{x_{j_i}}{x_i}\right) - 1$ (the t_{i_k} are δ-monomials in $\frac{x_{j_i}}{x_i}$ and the $\bar{y}$ are affine coordinates corresponding to $\mathbb{A}^m$). Then V is δ-complete if and only if for every such W the Kolchin closure $(\overline{\pi_2(W)})^{kolch}$ of the image of the projection of W into $\mathbb{A}^m$ does not contain the point $\bar{0}$.*

Proof. If V is δ-complete, then for W as described we have $\pi_2(W) = \overline{\pi_2(W)}^{kolch}$. The point $\bar{0}$ does not belong to $\pi_2(W)$ because if all the $\bar{y}$ were 0, then by the form of the P_i (every term except a lone positive power of x_i has a coefficient from $\bar{y}$), x_i would be 0 for every $i \in I$; we already know that each $x_j = 0$ for $j \notin I$. Thus $\bar{x}$ would not be a point of projective space, and we conclude that $\bar{0}$ does not belong to $\overline{\pi_2(W)}^{kolch}$.

Conversely, assume the second condition. Suppose toward contradiction that V is not δ-complete. The projective valuative criterion implies that there exists a K-maximal δ-ring $(R, f, \mathfrak{m})$ and point $p = (p_0 : \cdots : p_n) \in V(K) \setminus V(R)$. As in the proof of the projective valuative criterion itself, there are elements $\overline{m}$ in $\mathfrak{m}$ such that for each non-zero coordinate p_i

and corresponding coordinate p_{j_i}, $j_i \neq i$, we have $\sum_k m_{i_k} t_{i_k}\left(\frac{p_{j_i}}{p_i}\right) - 1 = 0$ for some δ-monomials t_{i_k} in $\frac{p_{j_i}}{p_i}$.

Inserting differential indeterminates $\bar{x}$ and $\bar{y}$ in place of the coordinates of p and $\overline{m}$, respectively, and adding to the collection the monomials x_j for each $p_j = 0$, we obtain a set of δ-homogeneous polynomials having the form described in the hypotheses. Let W be the subvariety of $V \times \mathbb{A}^m$ defined by these polynomials.

Consider the image set $\pi_2(W)$. By hypothesis, its Kolchin closure $\overline{\pi_2(W)}^{kolch}$ does not contain the point $\bar{0}$. Hence the ideal $\mathbf{I}(\overline{\pi_2(W)}^{kolch})$ of all polynomials (in $\mathcal{F}\{y_1, \ldots, y_m\}$) that vanish at every point of $\overline{\pi_2(W)}^{kolch}$ contains a polynomial f with non-zero constant term $a \in \mathcal{F}$. Since $\overline{m}$ belongs to $\pi_2(W)$, we conclude $f(\overline{m}) = 0$. Since f has coefficients in $\mathcal{F}$ and every term except a has a variable for which we substitute an m_{i_k} (or some derivative thereof), it follows from $f(\overline{m}) = 0$ that $0 \in a + \mathfrak{m}$; i.e., $a \in \mathfrak{m}$. That is impossible because $\mathfrak{m}$, being maximal, is a proper ideal but a is a unit of R.

$\square$

Recall from the second chapter our discussion of projective differential elimination ideals. We refer to the following as the *syntactic version* of the projective valuative criterion because it characterizes completeness by the form of δ-polynomials in the projective differential elimination ideal.

Theorem 4.1.3. *Let V and W be as in the preceding theorem, and let I be the differential ideal in $\mathcal{F}\{x_0, \ldots, x_n, y_1, \ldots, y_m\}$ generated by the defining polynomials of both V and W. Then V is*

δ-complete if and only if for all such W the projective differential elimination ideal $\hat{I}_{\overline{x}}$ contains a δ-polynomial in $\bar{y}$ having non-zero constant term.

Proof. If V is δ-complete, the geometric version of the projective valuative criterion asserts that $\bar{0} \notin \overline{\pi_2(W)}^{kolch}$. Because $\mathbf{V}(\hat{I}_{\overline{x}}) = \overline{\pi_2(W)}^{kolch}$, $\hat{I}_{\overline{x}}$ must contain a polynomial with non-zero constant term. Conversely, if V is not complete, then $\bar{0} \in \overline{\pi_2(W)}^{kolch}$ and so no member of $\hat{I}_{\overline{x}}$ can have a non-zero constant term. □

We now have three different perspectives on δ-completeness: the original projective valuative criterion with its use of maximal δ-rings, the geometric version focusing on the Kolchin closure of the image under projection of subvarieties of a special form, and the syntactic version concerning the δ-polynomials that show up in elimination ideals. The latter two results give more concrete ways of showing that a specific δ-variety is complete, but they are still difficult to work with given the well-known complexity of differential polynomial rings.

We describe a way of getting around some of that complexity by replacing finite-rank δ-varieties over a model of DCF_0 with algebraic varieties defined over $\mathbb{C}$. Before doing that, though, we consider what the modified valuative criteria tell us in the special case of $\mathbb{P}^1$.

4.2 The modified valuative criteria in $\mathbb{P}^1$

Restricting to the projective line has multiple advantages. Obviously, the fewer variables we have to keep track of, the more pleasant our analysis is. It also becomes trivial to identify finite-rank δ-varieties (all proper closed subsets of $\mathbb{P}^1$ are finite rank because non-zero δ-polynomials have finite order, which bounds the other usual ranks in the one-variable case; see Lemma 5.8, p.67, of (23)). Moreover, we get helpful simplification in the valuative criterion. In $\mathbb{P}^1$, if U is a standard affine chart and $V_a = V \cap U$, we only need project an *affine* subvariety of the product $V_a \times \mathbb{A}^m$ and thus are able to use differential elimination ideals instead of having to invoke *projective* differential elimination ideals. The reason is that if $p = (x_0 : x_1) \in V(K) \setminus V(R)$ for $V \subseteq \mathbb{P}^1$, we may write p as $(x : 1)$ where $x \neq 0$ and both $x, \frac{1}{x}$ belong to $K \setminus R$. The product subvariety $\left(\sum_k y_{0_k} t_{0_k} \left(\frac{x_1}{x_0}\right) - 1\right)^{\delta h} \cap \left(\sum_k y_{1_k} t_{1_k} \left(\frac{x_0}{x_1}\right) - 1\right)^{\delta h}$ misses $(1 : 0)$ (and $(0 : 1)$, for that matter), so we may dehomogenize and view it as an affine product subvariety $\left(\sum_k y_{0_k} \left(\frac{1}{x}\right) - 1\right) \cap \left(\sum_k y_{1_k} (x) - 1\right)$ of $V_a \times \mathbb{A}^m$. (Basically we are taking advantage of the fact that the gap between projective and affine is narrow because each affine chart nearly fills up the space.) Here are the three versions of the projective valuative criterion restated for δ-subvarieties of $\mathbb{P}^1$:

Theorem 4.2.1. *Let V be a differential subvariety of $\mathbb{P}^1$. Then V is δ-complete if and only if for every K-maximal δ-ring $(R, f, \mathfrak{m})$ and point $p = (x : 1)$ or $(1 : x) \in V(K)$ we have either $x \in R$ or $\frac{1}{x} \in R$.*

Theorem 4.2.2. *Let V be a differential subvariety of $\mathbb{P}^1$. Let x be the affine coordinate corresponding to a standard affine chart U and denote by V_a the affine restriction $V \cap U$. Let W be the affine subvariety of $V_a \times \mathbb{A}^m$ defined by (in addition to the defining polynomials of V_a) two δ-polynomials of the form $\sum_k y_k s_k(x) - 1$ and $\sum_l z_l t_l \left(\frac{1}{x}\right) - 1$ (the s_k and t_l are δ-monomials in x and $\frac{1}{x}$, respectively, and the $\bar{y}, \bar{z}$ are affine coordinates partitioning the coordinates of $\mathbb{A}^m$). Then V is δ-complete if and only if for every such W the Kolchin closure $\overline{\pi_2(W)}^{kolch}$ of the image of the projection of W into $\mathbb{A}^m$ does not contain the point $\bar{0}$.*

Theorem 4.2.3. *Let V and W be as in the preceding theorem, and let I be the differential ideal in $\mathcal{F}\{x, \bar{y}, \bar{z}\}$ generated by the defining polynomials of both V_a and W. Then V is δ-complete if and only if for all such W the (affine) differential elimination ideal I_x contains a δ-polynomial in $\bar{y}, \bar{z}$ having non-zero constant term.*

Proof. Using the observations preceding these statements, check that the hypotheses of the projective valuative criteria for $\mathbb{P}^n$ are satisfied. $\square$

Note that the first defining equation of W implies that $x \neq 0$, so it does not matter whether we actually clear the denominator in the second or just treat $\frac{1}{x}$ as a variable satisfying the relation $x \cdot \left(\frac{1}{x}\right) = 1$. In other words, we work in $\mathbb{A}^1$ with affine coordinate x, but we freely divide our δ-polynomials by x when desired. For convenience, we use the following *ad hoc* terminology:

Definition 4.2.1. *Polynomials of the form $\sum_k y_k s_k(x) - 1$ and $\sum_l z_l t_l \left(\frac{1}{x}\right) - 1$ are called* 1-witnesses in x *and* 1-witnesses in $\frac{1}{x}$*, respectively.*

Example 4.2.1.

$$x' = 0 \wedge yx - 1 = 0 \wedge z\left(\frac{1}{x}\right) - 1 = 0$$

We saw earlier that the projective closure $\overline{x' = 0}$ is δ-complete. So the syntactic version of the valuative criterion for $\mathbb{P}^1$ assures us that the differential elimination ideal obtained by eliminating x and its derivatives from $[x', yx - 1, z\left(\frac{1}{x}\right) - 1]$ contains a δ-polynomial in y, z with a non-zero constant term. That is easy to verify in this simple case: $(yx - 1)(z\left(\frac{1}{x}\right) - 1) = yz - yx - z\left(\frac{1}{x}\right) + 1 = yz - 1 - 1 + 1 = yz - 1$. In a more involved example we would need to differentiate the generators $x', yx - 1, z\left(\frac{1}{x}\right) - 1$ and use them as well to find a δ-polynomial in y, z having non-zero constant term.

Next we use our modified valuative criteria to do more: prove completeness of a class of examples not covered by Pong's or Kolchin's results.

Theorem 4.2.4. *The projective closure in $\mathbb{P}^1$ of a δ-variety defined by a linear ordinary differential equation in one variable is complete.*

Proof. We use the first version of the projective valuative criterion for $\mathbb{P}^1$ to prove that $\overline{x^{(k)} = \sum_{i=0}^{k-1} a_i x^{(i)} + b}$, for $a_i, b \in \mathcal{F}$, is complete. We must show that for any $x \in K \models DCF_0$ satisfying the equation and R a K-maximal δ-ring, either x or $\frac{1}{x}$ belongs to R. But this is exactly what Proposition 4 of (26) says. $\square$

At this point the natural question is to what extent the δ-completeness problem in $\mathbb{P}^1$ represents the problem in all $\mathbb{P}^n$. In other words, if we could classify/identify exactly which δ-subvarieties of $\mathbb{P}^1$ are complete, could we necessarily do the same for $\mathbb{P}^n$? (For instance, we might ask whether an effective decision procedure for δ-completeness in $\mathbb{P}^1$ extends to $\mathbb{P}^n$.) The author has an proof showing that there are no incomplete finite-rank δ-varieties in $\mathbb{P}^n$ if and only if there are no such varieties in $\mathbb{P}^1$. Of course, the proof was written before the discovery of the incomplete example $x'' = x^3$. That example makes the aforementioned equivalence (designed to prove completeness of all finite-rank projective δ-varieties in $\mathbb{P}^n$ in the event that all finite-rank projective δ-varieties in $\mathbb{P}^1$ are complete) true but moot.

Given the more complicated situation of some finite-rank projective δ-varieties being complete and others incomplete, it is less clear how much information about higher dimensions we get from the case of the projective line. We make two observations.

First, to understand the δ-complete subvarieties of $\mathbb{P}^n$ it suffices to understand the δ-complete subvarieties of $\mathbb{A}^2$. As discussed earlier, Pong showed that finite-rank δ-varieties in $\mathbb{P}^n$ embed in $\mathbb{A}^1$ via a series of projections. However, the image is not necessarily closed (though it *is* a δ-constructible set) in $\mathbb{A}^1$ if the variety is not δ-complete. As the example in Appendix A shows, Kolchin closedness of a δ-constructible set is not always obvious. Hence a decision procedure for completeness of δ-varieties in $\mathbb{A}^1$ need not automatically result in a decision procedure for δ-varieties in $\mathbb{P}^n$. What we can do is isomorphically map the image of the projection to a closed set of $\mathbb{A}^2$. Suppose the isomorphic image of a finite-rank δ-variety $V \subseteq \mathbb{P}^n$ in $\mathbb{A}^1$ is defined by $f_1(x) = \cdots = f_r(x) = 0 \wedge g(x) \neq 0$ for δ-polynomials $f_1, \ldots, f_r, g$ in x. This possibly-non-closed set of $\mathbb{A}^1$ is δ-isomorphic to the closed set $\widetilde{V}$ of $\mathbb{A}^2$ defined by $f_1(x) = \cdots = f_r(x) = 0 \wedge yg(x) - 1 = 0$, where y is a new differential indeterminate. Since V and $\widetilde{V}$ are isomorphic as δ-varieties, one is δ-complete if and only if the other is.

The second observation depends on the following fact (see, e.g., the lemma on p.181 of (38)):

Proposition 4.2.1. *Let K be an algebraically closed field. Let $\Sigma_n : \mathbb{A}^1(K) \times \cdots \times \mathbb{A}^1(K) \to \mathbb{A}^n(K)$ be the morphism taking $\bar{x} = (x_1, x_2, \ldots, x_n)$ to $(\sigma_1(\bar{x}) = x_1 + x_2 + \cdots + x_n, \ldots, \sigma_n(\bar{x}) = x_1 x_2 \cdots x_n)$, where σ_i is the i-th elementary symmetric function. Then Σ_n extends to a surjective, finite-to-one morphism from $\mathbb{P}^1(K) \times \cdots \times \mathbb{P}^1(K)$ to $\mathbb{P}^n(K)$.*

Proof. Let $y_1, y_2, \ldots, y_n$ be the second homogeneous coordinates of each factor of $\mathbb{P}^1$. The homogenized extension map is $\Sigma_n^h : ((x_1 : y_1), \ldots, (x_n : y_n)) \mapsto (x_1 y_2 \cdots y_n + y_1 x_2 \cdots y_n + \cdots + y_1 y_2 \cdots x_n : \cdots : x_1 x_2 \cdots x_n : y_1 y_2 \cdots y_n)$. An inductive argument shows that it suffices to prove surjectivity and finite-to-oneness for the original affine map Σ_n. (For example, if $y_1 = 0$ and all other $y_i \neq 0$, we may assume $x_1 = y_2 = \cdots = y_n = 1$. Substituting into Σ_n^h we get $(1 : x_2 + x_3 + \cdots + x_n : \cdots : x_2 x_3 \cdots x_n : 0)$; i.e., the coordinates not on the ends constitute the image of Σ_{n-1}, so we may use induction.)

Surjectivity of the affine map is described in the reference. The main idea is that solutions of the system of equations $x_1 + \cdots + x_n = a_1, \ldots, x_1 \cdots x_n = a_n$ (where a_i is an arbitrary element of K) are precisely the additive inverses of the roots of the polynomial $x^n + a_1 x^{n-1} + \cdots + a_n = (x + x_1) \cdots (x + x_n)$. Since K is algebraically closed, there are such roots; this shows surjectivity. Finite-to-oneness now follows from the observation that there are only finitely many permutations of the roots, so only finitely many preimages in $\mathbb{A}^1 \times \cdots \times \mathbb{A}^1 \to \mathbb{A}^n$ of $(a_1, \ldots, a_n) \in \mathbb{A}^n$. $\square$

The point is that if we can identify the complete δ-varieties contained in $\mathbb{P}^1$, then we immediately identify many complete δ-varieties in $\mathbb{P}^n$. Specifically, all closed subsets of $\Sigma_n^h(V_1 \times \cdots \times V_n)$ are δ-complete if the V_i are δ-complete subsets of $\mathbb{P}^1$. While there may be complete varieties in $\mathbb{P}^n$ not contained in such an image, at least Σ_n^h is surjective. Thus we potentially capture more complete varieties than we would with the Segre embedding (see (41)), which does not even map $\mathbb{P}^1(\mathcal{C}) \times \mathbb{P}^1(\mathcal{C})$ onto $\mathbb{P}^3(\mathcal{C})$ (note, for instance, that $(1:1:1:0) \in \mathbb{P}^3(\mathcal{C})$ is not in the image); in contrast $\Sigma_n^h(\mathbb{P}^1(\mathcal{C}) \times \mathbb{P}^1(\mathcal{C})) = \mathbb{P}^2(\mathcal{C})$.

These remarks explain our present focus on the one-variable case. As we will see, this is still challenging. To get more control over the situation, we would like to restrict further the δ-polynomials we must consider. We concentrate on minimal incomplete examples of a certain form.

Proposition 4.2.2. *Let $V \subseteq \mathbb{P}^1$ be a finite-rank, incomplete δ-variety. Then for some $n > 0$ there is an incomplete δ-variety (contained in $\mathbb{P}^1$ and containing V) whose restriction to a standard affine chart is defined by a single n-th order equation $Q(x, \dots, x^{(n-1)})x^{(n)} = P(x, \dots, x^{(n-1)})$, where Q and P are δ-polynomials having at most order $n-1$ and possessing no common factors.*

Proof. V is defined by some δ-homogeneous equations $\wedge_i f_i = 0$. Because completeness is closed under containment, we may assume the projective δ-variety defined by some single f_i is not complete. Dehomogenize to get $f(x, \dots, x^{(r)}) = (f_i)_{\delta h}$ for some order r.

Again by completeness' closure under containment, the projective δ-variety defined by $(f')^{\delta h}$ is not complete. The derivative f' has the form $s_f x^{(r+1)} + t$, and the separant s_f and tail t have order at most r. Let Q be s_f, P be $-t$, and n be $r+1$.

This completes the proof if Q and P have no common factor. Otherwise, suppose their greatest common divisor is a δ-polynomial $g(x, \dots, x^{(n-1)})$. Then $Q(x, x', \dots, x^{(n-1)})x^{(n)} = P(x, x', \dots, x^{(n-1)})$ if and only if either $\frac{Q}{g}x^{(n)} = \frac{P}{g}$ or $g(x, \dots, x^{(n-1)}) = 0$. It follows that either $\overline{\frac{Q}{g}x^{(n)} = \frac{P}{g}}$ is not complete or $\overline{g}$ is not. The polynomials $\frac{Q}{g}$ and $\frac{P}{g}$ have no common factor, so we are done if $\overline{\frac{Q}{g}x^{(n)} = \frac{P}{g}}$ is incomplete. If $\overline{g}$ is not complete, repeat the entire argument (note that the leading monomial of s_g is strictly simpler than that of Q). By well-ordering of the δ-monomials, this process terminates with an incomplete variety of the desired form. $\square$

We could now launch into a study of specific examples of the form $Q(x,\ldots,x^{(k-1)})x^{(k)} = P(x,\ldots,x^{(k-1)})$ using the various valuative criteria directly. With sufficient patience and ingenuity one can find elimination algorithms for particular δ-polynomials. This is essentially (though they emphasized the algebraic setting of maximal δ-rings more than the formal elimination aspect) what Pong did for the case $x' = P(x)$ and what Morrison did in her result on linear differential equations.

The basic idea is this: Suppose we suspect that the projective closure $\overline{V}$ of a finite-rank δ-variety $V \subseteq \mathbb{A}^1$ defined by $f(x) = 0$ is complete. We argue by contradiction. If $\overline{V}$ is not complete, then by the valuative criterion for $\mathbb{P}^1$ there exists $(x:1) \in \overline{V}(K) \setminus \overline{V}(R)$ for some K-maximal δ-ring R with unique maximal δ-ideal $\mathfrak{m}$. As discussed above, it follows that there are polynomials of the form $P_1(x) = \sum_k y_k s_k(x) - 1$ and $P_2(\frac{1}{x}) = \sum_l z_l t_l\left(\frac{1}{x}\right) - 1$ such that the coefficients $\overline{y}, \overline{z} \in \mathfrak{m}$. The goal is to work in the δ-ideal generated by f, P_1, P_2 (but never dividing by the $\overline{y}, \overline{z}$ so we don't leave R) in order to eliminate x and its derivatives while obtaining an equation of the form (elements of $\mathfrak{m}$) $- 1 = 0$. If we can do this then we have shown that $\mathfrak{m}$ contains a unit, contradicting its status as a maximal ideal. Hence if a successful "1-preserving elimination algorithm" exists for each possible pair of equations $P_1(x) = 1, P_2(\frac{1}{x}) = 1$, then we know no counterexample to completeness of $\overline{V}$ can exist (i.e., $\overline{V}$ is complete).

The task of tracking orders and degrees of an exploding number of δ-monomials soon becomes difficult. Indeed, much of the rest of this paper is dedicated to searching for an alternative to the approach described in the preceding paragraph. However, the painstaking construction of elimination algorithms bears fruit both for positive and negative results. See Appendix B for new examples of complete δ-varieties obtained by these (detailed) arguments. On the negative side, the example from the last chapter proving $x'' = x^3$ to be incomplete arose precisely from analyzing how a certain algorithm could fail.

Regardless, each elimination seems to require case-specific details, so it is difficult to see how one could obtain general results about the class of complete δ-varieties. We need a widely-applicable method for verifying the valuative criterion.

We opt to translate our problem out of the realm of differential polynomial rings and into that of algebraic varieties over $\mathbb{C}$. Classical and computational algebraic geometry have a wealth of tools for studying such objects, and the Euclidean topology allows us to use metric arguments. The remainder of this chapter explains how the transfer works.

4.3 Reduction to complex algebraic varieties

Lemma 4.3.1. *Let $\bar{a} = \{a_i\}_{i \in \mathbb{N}}$ be a sequence of parameters from $\mathcal{F} \models DCF_0$. Then there exists a sequence of complex numbers $\bar{b} = \{b_i\}_{i \in \mathbb{N}}$ such that $\bar{b}$ has the same type (infinite, without parameters, and restricted to the language of rings) over $\mathbb{Q}$ as $\bar{a}$.*

Proof. Let $p(\bar{x})$ be the type in the language of rings of the sequence $\bar{a}$ in $\mathcal{F}$. The differentially closed field $\mathcal{F}$ is *a fortiori* an algebraically closed field of characteristic 0, so $ACF_0 \cup p(\bar{x})$ is consistent. As $\mathbb{C}$ is an uncountable saturated model of ACF_0, there are elements $\bar{b} = \{b_i\}_{i \in \mathbb{N}}$ in $\mathbb{C}$ that realize p. $\square$

We use this observation as follows. Suppose $f \in \mathcal{F}\{\bar{x}\}$ has coefficients $a_1, \ldots, a_r$ from $\mathcal{F}$. The set of first-order formulas (in the language of rings, with no derivation) that are satisfied in $\mathcal{F}$ by $a_1, \ldots, a_r, a_1', \ldots, a_r', a_1'', \ldots$ form a type in countably many variables over the countable set $\mathbb{Q}$. By the lemma, there are elements $b_{1,0}, \ldots, b_{r,0}, b_{1,1}, \ldots, b_{r,1}, b_{1,2} \ldots$ of $\mathbb{C}$ that realize the type; the b_{ij} from $\mathbb{C}$ mirror perfectly all the algebraic relations (over $\mathbb{Q}$) between $a_1, \ldots, a_r$ and their subsequent derivatives.

Definition 4.3.1. *A non-differential polynomial obtained from a δ-polynomial $f \in \mathcal{F}\{\bar{x}\}$ by replacing the $\mathcal{F}$-coefficients with complex numbers having the same type over $\mathbb{Q}$ in the language of rings is a* replacement *of f, and we denote it by $\tilde{f}$.*

(We do not care which particular replacement we use; what matters is that the type of the choice of coefficients is the same.) Along with changing the coefficients, we reinterpret the variables as ranging over $\mathbb{C}$, not $\mathcal{F}$, as illustrated by the following example.

Example 4.3.1. Consider the system

$$y_2(ax^2) + y_1 x - 1 = 0, y_2(a'x^2 + 2a^2x^3) + y_1(ax^2) + y_2'(ax^2) + y_1'x = 0,$$

where $a \in \mathcal{F}$ satisfies $a'' + a = 0$ but no lower-order equations over $\mathbb{Q}$. (To prove completeness of the affine projective δ-variety $x' = ax^2$ using the valuative criterion one must rule out this system, among others, as a potential counterexample.) By hypothesis on a, the set $\{a, a'\}$ is algebraically independent over $\mathbb{Q}$, so any pair of algebraically independent complex numbers b_0, b_1 may replace a and a', respectively. However, we must replace a'' with $-b_0$ to preserve the relation satisfied by a'' and a, a''' must be $-b_1$ because $a'' + a = 0$ implies $a''' + a' = 0$, and so on.

We now view the replacements

$$y_2(b_0x^2) + y_1x - 1 = 0, y_2(b_1x^2 + 2b_0^2x^3) + y_1(b_0x^2) + y_2'(b_0x^2) + y_1'x = 0$$

as defining an *algebraic* variety in $\mathbb{C}^5$, where the (algebraic, despite their names) variables are x, y_2, y_1, y_2', y_1'. If we need to consider further derivatives of the polynomials in the system, we add variables $(y_2'', y_1'', \dots)$ and substitute parameters $(b_2 = -b_0, b_3 = -b_1, \dots)$ to produce an algebraic variety in higher dimensional affine space.

The only thing lost in replacing δ-polynomials over $\mathcal{F}$ with non-differential polynomials over $\mathbb{C}$ is the heritage of being linked by the derivation δ. But this is unimportant; to use the valuative criterion we only need to preserve existence/non-existence of polynomials of the right form (non-zero constant term), and the type does this (see the upcoming proposition). What we gain is the ability to apply analytic arguments to study the question of what kinds of polynomials can be in the elimination ideal.

Let $I \subseteq \mathcal{F}\{\bar{x}, \bar{y}\}$ be the differential ideal generated by δ-polynomials $f_1, \dots, f_r$. For natural number k let $\widetilde{I}^{(s_k)} \subseteq \mathbb{C}[\bar{x}, \bar{y}, \dots, \bar{x}^{(s_k)}, \bar{y}^{(s_k)}]$ be the non-differential ideal generated by replacement polynomials $\tilde{f}_1, \dots, \tilde{f}_r, \dots, \widetilde{f_1}^{(k)}, \dots, \widetilde{f_r}^{(k)}$, where s_k is the highest-order variable appearing in the replacement polynomials. (For typographical reasons we write $\widetilde{f_i}^{(j)}$ for the replacement polynomial of the δ-polynomial $f_i^{(j)}$; it does *not* mean a derivative of the replacement polynomial. The same applies to the notation $\widetilde{I}^{(s_k)}$; it does not denote the s_k-th order truncation differential ideal corresponding to some undefined object $\widetilde{I}$.) We think of $\widetilde{I}^{(k)}$ as being a "replacement ideal" over $\mathbb{C}$ for the s_k-th order truncation differential ideal $I^{(s_k)}$, except that we only require the generators $\widetilde{f_i}^{(j)}$ to actually be replacement polynomials for elements of I.

Proposition 4.3.1. *Let $I \subseteq \mathcal{F}\{\bar{x}, \bar{y}\}$ be the differential ideal generated by δ-polynomials $f_1(\bar{x}, \bar{y}), \ldots, f_r(\bar{x}, \bar{y})$. The differential elimination ideal $I_{\bar{x}} \subseteq \mathcal{F}\{\bar{y}\}$ contains a δ-polynomial in $\bar{y}$ having non-zero constant term if and only if there exist natural numbers k, s_k such that for $\widetilde{I}^{(s_k)}$ as described above, the non-differential elimination ideal $\widetilde{I}^{(s_k)}_{\bar{x},\ldots,\bar{x}^{(s_k)}} \subseteq \mathbb{C}[\bar{y}, \ldots, \bar{y}^{(s_k)}]$) contains a polynomial having non-zero constant term.*

Proof. We give the forward direction; the reverse is completely analogous. Let $f \in I_{\bar{x}}$ have non-zero constant term. For some natural numbers k, s_k there are $g_{i,j} \in \mathcal{F}\{\bar{x}, \bar{y}\}$ such that

$$f = \sum_{j=0}^{k} \sum_{i=1}^{r} g_{i,j} f_i^{(j)}$$

and every polynomial appearing in the linear combination has order less than or equal to s_k. Both existence of $g_{i,j}$ of the given form and the fact that the result has non-zero constant term are expressible by a sentence in the first-order language of rings using as parameters only the coefficients of $f_i^{(j)}$. (As always in transferring from $\mathcal{F}$ to $\mathbb{C}$, we treat derivatives as separate algebraic coefficients or indeterminates.) Since those coefficients have complex number substitutes with the same type over $\mathbb{Q}$, the sentence remains true in $\mathbb{C}$ using the replacements

$\widetilde{f}_i^{(j)}$ instead of $f_i^{(j)}$. The polynomial now asserted to exist by the modified sentence still has non-zero constant term and variables only in $\bar{y}, \ldots, \bar{y}^{(s_k)}$, as desired. $\square$

Now we can pivot back to the geometric setting, but this time over $\mathbb{C}$. We call this final version the *transferred valuative criterion.*

Theorem 4.3.1. *Let V be a differential subvariety of $\mathbb{P}^1(\mathcal{F})$ for $\mathcal{F} \models DCF_0$ and let x be the affine coordinate corresponding to a standard affine chart U. Denote by V_a the affine restriction $V \cap U$ and label the defining δ-polynomials of V_a as $f_1(x), \ldots, f_r(x)$. Let W be an affine δ-subvariety of $V_a \times \mathbb{A}^m(\mathcal{F})$ defined by, along with the defining polynomials of V_a, two δ-polynomials of the form $g_0 = \sum_i y_i t_{0,i}(x) - 1$ and $g_1 = \sum_j z_j t_{1,j}\left(\frac{1}{x}\right) - 1$ (the $t_{0,i}$ and $t_{1,j}$ are δ-monomials in x and $\frac{1}{x}$, respectively, and the $\bar{y}, \bar{z}$ are affine coordinates partitioning the coordinates of $\mathbb{A}^m(\mathcal{F})$). Let $k \in \mathbb{N}$ and let s_k be the highest order of any variable in $f_1, \ldots, f_r, \ldots, f_1^{(k)}, \ldots, f_r^{(k)}, g_0, g_1 \ldots, g_0^{(k)}, g_1^{(k)}$. Denote by $\widetilde{V}_a^{(s_k)}$ the complex algebraic variety defined by the replacement polynomials $\widetilde{f}_1, \ldots, \widetilde{f}_r, \ldots, \widetilde{f}_1^{(k)}, \ldots, \widetilde{f}_r^{(k)}$ in $\mathbb{C}[x, \ldots, x^{(s_k)}]$. Finally, $\widetilde{W}^{(s_k)}$ is the complex algebraic subvariety of $\widetilde{V}_a^{(s_k)} \times \mathbb{C}^{m(k+1)}$ defined by the replacement polynomials $\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_0^{(k)}, \widetilde{g}_1^{(k)}$ (along with the defining polynomials of $\widetilde{V}_a^{(s_k)}$) in $\mathbb{C}[x, \ldots, x^{(s_k)}, \bar{y}, \bar{z}, \ldots, \bar{y}^{(k)}, \bar{z}^{(k)}]$. Then V is δ-complete if and only if for every such W there exist k, s_k such that the Euclidean closure $\overline{\pi_2(\widetilde{W}^{(s_k)})}^{euc}$ of the image of the projection of $\widetilde{W}^{(s_k)}$ into $\mathbb{C}^{m(k+1)}$ does not contain the point $\bar{0}$.*

Proof. First characterize δ-completeness with the syntactic valuative criterion for $\mathbb{P}^1$ and then apply the transfer principle given by the preceding proposition to translate δ-completeness into a statement about non-differential polynomials with non-constant terms in the non-differential elimination ideal $\widetilde{I}^{(s_k)}_{x,\ldots,x^{(s_k)}}$, where I is the δ-ideal generated by $f_1,\ldots,f_r,g_0,g_1$. As non-differential elimination ideals define the Zariski closure of the image of a projection, we conclude $\bar{0} \notin \overline{\pi_2(\widetilde{W}^{(s_k)})}^{zar}$ if and only if $\widetilde{I}^{(s_k)}_{x,\ldots,x^{(s_k)}}$ contains an element with a non-zero constant term. In $\mathbb{C}^n$ the Euclidean closure of a constructible set is the same as the Zariski closure, so $\overline{\pi_2(\widetilde{W}^{(s_k)})}^{zar} = \overline{\pi_2(\widetilde{W}^{(s_k)})}^{euc}$.

□

In applying the transferred valuative criterion to particular varieties, we generally use the relations given by the defining δ-polynomials of V_a to write the equations for W. In particular, the δ-monomials $t_{0,i}, t_{1,j}$ in g_0, g_1 may become polynomials (or even fractions; in doing so, we must take any possible vanishing of denominators into account) after the substitution. Fractions arise, for instance, in the case $x^{(n)} = P(x,\ldots,x^{(n-1)})/Q(x,\ldots,x^{(n-1)})$.

Example 4.3.2. Let $V \subseteq \mathbb{P}^1$ be the projective closure of the affine δ-variety defined by $x' = x$. Observe that after substitution δ-monomials in x have the form x^r for $r > 0$ and that δ-monomials in $\frac{1}{x}$ are only those of the form $\frac{1}{x^s}$ for $s > 0$ (e.g., $\left(\frac{1}{x}\right)' = -\frac{x'}{x^2} = -\frac{1}{x}$). The transferred valuative criterion concerns projections of systems of the form:

$$g_0 = \sum_{i=1}^{r} y_i x^i = 1$$
$$g_1 = \sum_{j=1}^{s} z_j \frac{1}{x^j} = 1.$$

(We have ignored the integer coefficients of the monomials.) To prove completeness of V, we must show that for some positive integer k the image of the projection of the variety defined by replacement polynomials $\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_0^{(k)}, \widetilde{g}_1^{(k)}$ does not contain the origin $\bar{0}$ in the Euclidean closure.

For concreteness, consider the δ-variety $W \subseteq \mathbb{A}^1(\mathcal{F}) \times \mathbb{A}^4(\mathcal{F})$:

$$g_0 = y_2 x^2 + y_1 x = 1$$
$$g_1 = z_2 \frac{1}{x^3} + z_1 \frac{1}{x} = 1.$$

If we allow ourselves one derivative of each of g_0, g_1, then we have the complex algebraic variety $\widetilde{W}^{(1)}$:

$$\widetilde{g}_0 = y_2 x^2 + y_1 x = 1$$
$$\widetilde{g}_0' = 2y_2 x^2 + y_1 x + y_2' x^2 + y_1' x = 0$$
$$\widetilde{g}_1 = z_2 \frac{1}{x^3} + z_1 \frac{1}{x} = 1$$
$$\widetilde{g}_1' = -3z_2 \frac{1}{x^3} - z_1 \frac{1}{x} + z_2' \frac{1}{x^3} + z_1' \frac{1}{x} = 0$$

projecting from $\mathbb{C}^1 \times \mathbb{C}^8$ into $\mathbb{C}^8$. It suffices to show that if every y_2, y_2', y_1, y_1', z_2, z_2', z_1, z_1' (henceforth written simply as $\bar{y}, \bar{z}$) has sufficiently small Euclidean norm as a complex number, then there is no complex number x such that the equations of $\widetilde{W}^{(1)}$ are all satisfied.

That is easy in this case, and we do not even need to use $\widetilde{g}_0'$ or $\widetilde{g}_1'$; simply note that if all the $\bar{y}, \bar{z}$ are very small compared to 1, then $\widetilde{g}_0$ and $\widetilde{g}_1$ respectively force both x and $\frac{1}{x}$ to be much larger than 1. This contradiction applies to any possible equations for W, so without making explicit elimination arguments we have re-proven the fact that $\overline{x' = x}$ is δ-complete.

It should be noted that every version of the valuative criterion we have given in this chapter has an analogue for affine δ-varieties. The statements and proofs are virtually identical, though they do not require a 1-witness in $\frac{1}{x}$. In particular, the affine transferred valuative criterion states that $V \subseteq \mathbb{A}^1$ is δ-complete if and only if for every 1-witness (in x) and its derivatives up to order k, the origin does not belong to the Euclidean closure of the image of the projection of the complex variety defined by the corresponding replacement polynomials.

The next chapter explores a proposed strategy for applying the transferred valuative criterion to a broader spectrum of δ-varieties than was practical with earlier versions.

CHAPTER 5

AN ASYMPTOTIC APPROACH

The transferred valuative criterion states that completeness of a δ-variety $V \subseteq \mathbb{P}^1$ is determined by the behavior of certain *algebraic* varieties defined over $\mathbb{C}$. In particular, we differentiate the defining polynomials of $V \cap U$ (where U is a standard affine chart) as well as arbitrary 1-witnesses in x and $\frac{1}{x}$ an arbitrary finite number of times and treat the result as a complex affine variety. Now we must determine whether or not the variety has an "asymptote" at the origin. More formally, we need to ascertain if the projection is proper at the origin; if it is for all 1-witnesses and their derivatives up to arbitrary order, then V is δ-complete. Otherwise V is not complete.

We may use any means necessary to decide properness, and every tool of complex algebraic geometry, analysis, etc., is fair game. (See (14) and(40) for recent work on the problem.) In spite of such tools we have been unable to locate any general result that applies to the special scenario of the transferred valuative criterion. For this reason the author has relied on extensive computational experimentation in order to gain insight. The outcome is a heuristic approach that appears to be valid for many finite-rank δ-varieties in $\mathbb{P}^1$. Given the bad example of the incomplete $x'' = x^3$, the tactic cannot prove completeness in all cases; we discuss the situation later in this chapter. We will delineate what is conjectural and what we actually know as we outline the procedure. The goal is to explain *why* completeness occurs when it does and provide suggestions for future work on the problem.

To make the exposition easier to follow, we interweave two running examples. One of them is the δ-complete (due to Pong) variety $x' = x^2$, which is projectively closed though affine. The second is $xx'' = x'$, which was not covered by the earlier work of Pong or Kolchin but for which Appendix B gives an explicit elimination algorithm proving δ-completeness of the projective closure.

5.1 The asymptotic strategy

Our proposed strategy employs proof by contradiction. Given a δ-variety in $\mathbb{A}^1$, we suppose its projective closure in $\mathbb{P}^1$ is not complete. By the transferred valuative criterion, there are 1-witnesses in x and $\frac{1}{x}$ defining a complex affine variety in $\mathbb{C}^n \times \mathbb{C}^m$ with an "asymptote" at the origin of $\mathbb{C}^m$. We analyze the terms of the 1-witnesses and attempt to show that in reality the system cannot have this feature.

To avoid any confusion, we record what we mean by an "asymptote" of a complex algebraic variety.

Definition 5.1.1. *Let $V \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a complex algebraic variety. We say V has an* asymptote *at a point $\bar{y}_0 \in \mathbb{C}^m$ if*

1. *$\bar{y}_0$ does not belong to the image of the projection π_2 of V into $\mathbb{C}^m$ and*
2. *for every $\epsilon > 0$ there exists $\bar{y} \in \mathbb{C}^m$ such that $|\bar{y} - \bar{y}_0| < \epsilon$ (in the usual Euclidean metric on $\mathbb{C}^m$) and $\bar{y} \in \pi_2(V)$.*

The strategy of contradicting existence of an asymptote can only work if the variety actually is complete. If we fail to obtain the contradiction, then we are not certain of the variety's status.

We fix some notation for the systems of equations given by the transferred valuative criterion. Because we are not now concerned with the original system or the process of transfer from $\mathcal{F}$ to $\mathbb{C}$, we omit some symbols and use different names for the constituent parts of the equations.

We mean by a *1-witness system in* x (of order k) the complex algebraic variety defined by the equations we called $\tilde{g}_0, \ldots, \tilde{g}_0^{(k)}$ in the statement of the transferred valuative criterion and write

$$\sum_{i=1}^{r} y_i f_i(x) = 1$$
$$\sum_{j=0}^{1} \sum_{i=1}^{r} \binom{1}{j} y_i^{(j)} f_i^{(1-j)}(x) = 0$$
$$\vdots$$
$$\sum_{j=0}^{k} \sum_{i=1}^{r} \binom{k}{j} y_i^{(j)} f_i^{(k-j)}(x) = 0.$$

The f_i are replacement monomials (i.e, we now view $x, x', \ldots,$ as algebraic indeterminates), and each row after the first is simply the derivative of the preceding one (the binomial coefficients come from the generalized Leibniz rule for higher-order derivatives of products). Strictly speaking, we must also include the replacement polynomials representing the equations of the original δ-variety V. In practice, though, we usually make the substitutions automatically and so do not include the equations of V in the system. For conciseness, we often give the label $V_{x,\overline{y},k}$ to a k-th order 1-witness system in x. To emphasize the δ-polynomial f defining the restriction of the original δ-variety to a standard affine chart we write $V_{x,\overline{y},k}^{f}$.

Our notation is exactly analogous for a *1-witness system in* $\frac{1}{x}$ (of order k), the complex algebraic variety previously defined by $\tilde{g}_1, \ldots, \tilde{g}_1^{(k)}$:

$$\sum_{i=1}^{s} z_i g_i \left(\frac{1}{x}\right) = 1$$
$$\sum_{j=0}^{1} \sum_{i=1}^{s} \binom{1}{j} z_i^{(j)} g_i^{(1-j)} \left(\frac{1}{x}\right) = 0$$
$$\vdots$$
$$\sum_{j=0}^{k} \sum_{i=1}^{s} \binom{k}{j} z_i^{(j)} g_i^{(k-j)} \left(\frac{1}{x}\right) = 0.$$

We frequently call this variety $V_{1/x,\overline{z},k}$ (or $V^f_{1/x,\overline{z},k}$). If the systems in x and $\frac{1}{x}$ have different orders, it does no harm to include additional derivatives, so we may assume they have the same number of equations. Furthermore, the two systems are linked; i.e., the value of $\frac{1}{x}$ is the reciprocal of the value of x. The intersection $V_{x,\overline{y},k} \cap V_{1/x,\overline{z},k}$ (which we call simply a 1-witness system and denote by V_k or V_k^f) defines the variety we earlier called $\widetilde{W}^{(s_k)}$. We have written the systems separately for convenience in discussing their respective behavior.

For future reference, we also give a name to the union of all equations in V_k^f for all $k \in \mathbb{N}$: V_∞^f. We call this set of equations and/or their zero locus (which is not an algebraic variety because there are infinitely many distinct variables; label them $\overline{x}_\infty, \overline{y}_\infty$) the *limit* of the 1-witness systems V_k^f.

Since the variables $\overline{y}, \overline{z}$ all have degree 1 and are not multiplied by each other in a 1-witness system, we may represent the systems by matrices with polynomial entries.

Example 5.1.1. $x' = x^2$: As seen earlier, this affine δ-variety is also δ-projective. Hence to prove completeness it suffices for every 1-witness in x to generate for some k a 1-witness system V_k^f such that the projection of V_k^f is proper at the origin.

We present a sample system that we refer to as $V_{x,\overline{y},3}^{x'=x^2}$:

$$y_2x^2 + y_1x = 1$$

$$2y_2x^3 + y_1x^2 + y_2'x^2 + y_1'x = 0$$

$$6y_2x^4 + 2y_1x^3 + 4y_2'x^3 + 2y_1'x^2 + y_2''x^2 + y_1''x = 0.$$

Writing just the coefficients of the $\overline{y}$ (and the output column with 1 in the first entry and zeros elsewhere) we have a 1-witness matrix $M_{x,\overline{y},3}^{x'=x^2}$:

$$\begin{bmatrix} x^2 & x & 0 & 0 & 0 & 0 & 1 \\ 2\,x^3 & x^2 & x^2 & x & 0 & 0 & 0 \\ 6\,x^4 & 2\,x^3 & 4\,x^3 & 2\,x^2 & x^2 & x & 0 \end{bmatrix}$$

To facilitate our future discussion, we introduce terminology for several components of 1-witness systems and their associated matrices. We speak of *blocks* of monomials to identify their location in a 1-witness matrix. In the first row, we refer to the r entries ($r = 2$ in the matrix above) from the 1-witness as comprising the *initial block* of the system and the first (or leading) block of the row. In later rows, the first r entries form the first block, the next r entries form the second block, and so on. In addition to the $\overline{x}$-monomials, we may speak of their $\overline{y}$-coefficients as being $\overline{y}$-coefficients of the first block (of a given row), the second block, etc. (In the case of a 1-witness system in $\frac{1}{x}$, we refer to $\overline{z}$ coefficients.) So y_2'' and y_1'' are the $\overline{y}$-coefficients of the third block of the third row. When it is clear from context, we also say that the polynomial $y_2''x^2 + y_1''x$ is the third block of the third row of the system.

Consider two $\overline{x}$-monomials that appear in the same column. We say that the lower monomial, including its $\overline{y}$-coefficient y, is a *y-descendant* of the original monomial (along with the coefficient y). For instance, the $(3,3)$-entry of the above system, $4y_2'x^3$, is a y_2'-descendant of the $(2,3)$-entry $y_2'x^2$.

Example 5.1.2. $xx'' = x'$:

The Poizat example is apparently not projective in $\mathbb{A}^1$, so we use two 1-witness systems.

Here is a system $V_{x,\overline{y},3}^{xx''=x'}$ in x:

$$\begin{aligned}
y\frac{x'}{x} &= 1\\
y\left(\frac{x'}{x^2} - \frac{(x')^2}{x^2}\right) + y'\frac{x'}{x} &= 0\\
y\left(\frac{-4(x')^2}{x^3} + \frac{x'}{x^3} + \frac{2(x')^3}{x^3}\right) + y'\left(\frac{2x'}{x^2} - \frac{2(x')^2}{x^2}\right) + y''\frac{x'}{x} &= 0.
\end{aligned}$$

The associated matrix is $M_{x,\overline{y},3}^{xx''=x'}$:

$$\begin{bmatrix} \frac{x'}{x} & 0 & 0 & 1 \\ \frac{x'}{x^2} - \frac{(x')^2}{x^2} & \frac{x'}{x} & 0 & 0 \\ -4\,\frac{(x')^2}{x^3} + \frac{x'}{x^3} + 2\,\frac{(x')^3}{x^3} & 2\,\frac{x'}{x^2} - 2\,\frac{(x')^2}{x^2} & \frac{x'}{x} & 0 \end{bmatrix}$$

A system in $\frac{1}{x}$ is $V^{xx''=x'}_{1/x,\bar{z},3}$:

$$\begin{aligned} z\frac{1}{x^2} &= 1 \\ z\left(\frac{-2x'}{x^3}\right) + z'\left(\frac{1}{x^2}\right) &= 0 \\ z\left(\frac{6(x')^2}{x^4} - \frac{2x'}{x^4}\right) - z'\left(\frac{4x'}{x^3}\right) + z''\frac{1}{x^2} &= 0. \end{aligned}$$

The 1-witness matrix $M^{xx''=x'}_{1/x,\bar{z},3}$ is:

$$\begin{bmatrix} x^{-2} & 0 & 0 & 1 \\ -2\,\frac{x'}{x^3} & x^{-2} & 0 & 0 \\ 6\,\frac{(x')^2}{x^4} - 2\,\frac{x'}{x^4} & -4\,\frac{x'}{x^3} & x^{-2} & 0 \end{bmatrix}$$

Note that the initial term $\frac{x'}{x}$ in the first system is a δ-polynomial in x modulo the substitution $x'' = \frac{x'}{x}$.

Whether or not they have an asymptote, 1-witness systems possess several nice properties. The defining system of equations has a triangular form, with each subsequent equation adding variables not present in the preceding ones. We also have the following observation:

Proposition 5.1.1. *Let $V_k^f \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a 1-witness system such that f has the form $Q(x, x', \dots, x^{(n-1)})x^{(n)} = P(x, x', \dots, x^{(n-1)})$ and $V_{x,\overline{y},k}^f, V_{1/x,\overline{z},k}^f$ are as explained previously. Let $(\overline{x}, \overline{y})$ be a point of V_k^f such that $Q(\overline{x}) \neq 0$ and $x \neq 0$. Then V_k^f is non-singular at $(\overline{x}, \overline{y})$ and the dimension of V_k^f at $(\overline{x}, \overline{y})$ is (total number of variables)$-$ (total number of equations) $=$ $(k+1)(r+s-2)+n$, where r and s denote the number of variables $\overline{y}$ and $\overline{z}$ in the 1-witnesses in x and $\frac{1}{x}$. In particular, V_k^f has positive dimension.*

Proof. Note that there are $2(k+1)$ equations in $(k+1)(r+s)+n$ unknowns. By Theorem 1.16 of (28), it suffices to show that the Jacobian matrix of V_k^f has full rank (i.e., $2(k+1)$). The first partial derivative with respect to $y_i^{(l)}$ of the l-th equation of $V_{x,\overline{y},k}^f$ is simply f_i. Clear denominators and make the substitution $x^{(n)} = \frac{P(x,\dots,x^{(n-1)})}{Q(x,\dots,x^{(n-1)})}$. Since $Q(\overline{x}) \neq 0$, $f_i(\overline{x}) \neq 0$ for some $1 \leq i \leq r$. (Here is where we use the fact that we have a 1-witness.) Moreover, by the triangular form of our system the entries above the l-th row in the column of partial derivatives with respect to $y_i^{(l)}$ are 0. The argument is the same for the (separate) columns corresponding to partial derivatives with respect to $z_i^{(l)}$ ($x \neq 0$, along with $Q \neq 0$, ensures that some $g_i(\overline{x}) \neq 0$). Hence the matrix contains $2(k+1)$ linearly independent columns and has full rank. $\square$

How could we go about proving that the systems in the examples do not have asymptotes? Of course, we could always clear denominators and compute the elimination ideal in $\overline{y}, \overline{z}$. But such a direct approach does not scale well as we increase the size of the 1-witness systems. What we need is a method that explains *why* all 1-witnesses fail for a given finite-rank δ-variety. Our proposal is to consider what happens to solutions of the 1-witnesses for very small $\bar{y}, \bar{z}$.

Suppose we have a 1-witness system with an asymptote at the origin. This assumption gives us an infinite collection of points on the variety:

Definition 5.1.2. *Let $V \subseteq \mathbb{C}^n \times \mathbb{C}^m$ be a complex algebraic variety having an asymptote at $\bar{0} \in \mathbb{C}^m$. For each $i \in \mathbb{N}$ let σ_i be a pair $(\overline{x}_i \in \mathbb{C}^n, \overline{y}_i \in \mathbb{C}^m)$ such that $(\overline{x}, \overline{y}) \in V$ and $\overline{y}_i \to \overline{0}$ as $i \to \infty$. We call the sequence $\sigma = \{\sigma_i\}_{i \in \mathbb{N}}$ an* asymptotic solution sequence *(or just solution sequence) of V for $\overline{y}$ approaching $\overline{0}$.*

Using this terminology, the transferred valuative criterion states that a finite-rank projective δ-variety in $\mathbb{P}^1$ is complete if and only if for some k the induced complex 1-witness system V_k does not have an asymptotic solution sequence.

Note that every infinite subsequence of a solution sequence is also a solution sequence for V. Because of this, in the course of analyzing a system we may restrict our attention to subsequences with some desirable property; a contradiction using only a subsequence is still a contradiction and thus proves completeness. We explain this further following our discussion of asymptotic notation.

One desirable property of subsequences merits special mention here. It explains why we have felt free to divide by Q, x, etc., in earlier examples of 1-witness systems.

Proposition 5.1.2. *Let f have the form $Q(x, x', \ldots, x^{(n-1)})x^{(n)} = P((x, x', \ldots, x^{(n-1)})$. If the 1-witness system $V_k^f \subseteq \mathbb{C}^n \times \mathbb{C}^m$ has an asymptote for all $k \in \mathbb{N}$, then either*

1. *For all k the 1-witness system V_k^f has an asymptotic solution sequence $\sigma(k)$ consisting of points $\sigma_{ik} = (\overline{x}_{ik}, \overline{y}_{ik})$ such that $Q(\overline{x}_{ik}) \neq 0$ and $(x^{(j)})_{ik} \neq 0$ for all $i \in \mathbb{N}$ and $0 \leq j \leq n-1$ or*

2. *$\overline{Q}$ is not δ-complete.*

Proof. Suppose the first option does not hold; we show that then $\overline{Q}$ is not δ-complete. By definition the equations of $V_{k_1}^f$ are a subset of the equations of $V_{k_2}^f$ for $k_2 > k_1$, so a solution sequence for $V_{k_2}^f$ is also a solution sequence for $V_{k_1}^f$. It follows from the negation of the first option and the fact that each V_k^f has an asymptote that for all k there are $\overline{y}$ arbitrarily close to $\overline{0}$ such that for some $\overline{x}$, $(\overline{x}, \overline{y})$ satisfies the equations of V_k^f but either $Q(\overline{x}) = 0$ or $x^{(j)} = 0$ for some $0 \leq j \leq n-1$. By the pigeonhole principle, one of these happens for arbitrarily large, and thus all, k. So either the 1-witness system V_k^Q (which has the equations of V_k^f as well as $Q = 0, \ldots, Q^{(k)} = 0$) or $V_k^{x^{(j)}}$ for some $0 \leq j \leq n-1$ has an asymptotic solution sequence for all k. However, $\overline{x^{(j)}}$ is complete by 4.2.4, so it must be that V_k^Q has asymptotic solution sequences for all k and is incomplete.

□

As we saw in Proposition 4.2.2, if $\overline{Q}$ is not δ-complete, then we may differentiate Q (switching back to the differential field setting temporarily) and obtain an incomplete variety with defining equation of the standard form $\widetilde{Q}(x, \ldots, x^{(n-1)})x^{(n)} = \widetilde{P}(x, \ldots, x^{(n-1)})$ but with $\widetilde{Q}$ having simpler leading term than Q. Since we are primarily interested in minimal examples of incompleteness, if we are studying the equation $Q(x, x', \ldots, x^{(n-1)})x^{(n)} = P(x, x', \ldots, x^{(n-1)})$ we are justified in vetting only asymptotic solution sequences whose $\overline{x}$ values do not cause

Q or any $x^{(j)}$ to vanish. Obtaining a contradiction just for such sequences will not prove completeness of $Q(x, x', \dots, x^{(n-1)})x^{(n)} = P(x, x', \dots, x^{(n-1)})$, then, but it assures us that if $Q(x, x', \dots, x^{(n-1)})x^{(n)} = P(x, x', \dots, x^{(n-1)})$ is not complete, then it is because there is a simpler incomplete example at hand. Ultimately, the hope (not always possible, due to examples like $x'' = x^3$) is to start from the bottom and inductively rule out incompleteness. The proposition roughly means that if we know everything below a given level of complexity and having a certain form is complete, then we don't have to worry about vanishing separants of that form or zero values of $x^{(j)}$ in doing calculations at the current level. In other words, we may divide by Q and $x, \dots, x^{(n-1)}$ in trying to understand asymptotes of a 1-witness system V_k^f. The rest of this section gives an informal sketch of an asymptotic strategy that we develop further in the next. We want to study the relative magnitudes of terms in the equations of a 1-witness system V_k^f. In order to do this, we introduce several notions modeled on standard asymptotic notation but modified for our purposes.

Definition 5.1.3. *Let $\sigma = \overline{x}_1, \overline{x}_2, \dots,$ be a sequence of tuples (not necessarily an asymptotic solution sequence) in $\mathbb{C}^n$ and let p, q be functions from $\mathbb{C}^n$ to $\mathbb{C}$. (For us, these are usually represented by polynomials in $\mathbb{C}[x_1, \dots, x_n]$.)*

1. *We write $p \in o_\sigma(q)$ to mean that for all $\epsilon > 0$ there exists i_0 such that if $i \geq i_0$, then $|p(\overline{x}_i)| \leq \epsilon |q(\overline{x}_i)|$. (Informally, p is asymptotically smaller than q as p, q are evaluated at later and later members of σ.)*

2. *We write* $q \in \Omega_\sigma(p)$ *(or, equivalently,* $p \in O_\sigma(q)$*; note the switch) to mean that there exists* $\epsilon > 0$ *and* i_0 *such that if* $i \geq i_0$*, then* $|q(\overline{x}_i)| \geq \epsilon|p(\overline{x}_i)|$*. (Informally,* q *asymptotically at least "keeps up" with* p*. We also say that* q *asymptotically has at least order* p*.)*

3. *If for* $r : \mathbb{C}^n \to \mathbb{C}$ *we also have both* $p \in o_\sigma(q), p \in o_\sigma(r)$*, we write* $p \in o_\sigma(q, r)$ *(and similarly for* Ω_σ*).*

4. *We write* $q \in \Omega_{\sigma\exists}(p)$ *to mean that there exists* $\epsilon > 0$ *and an infinite subsequence* $\tilde{\sigma}$ *of* σ *such that if* $\overline{x}_i$ *in* σ *also belongs to* $\tilde{\sigma}$*, then* $|q(\overline{x}_i)| \geq \epsilon|p(\overline{x}_i)|$*.*

5. *Lastly, we write* $q \geq_\sigma p$ *to mean that there exists* i_0 *such that if* $i \geq i_0$*, then* $|q(\overline{x}_i)| \geq |p(\overline{x}_i)|$*.*

(See, e.g., (12) for basic asymptotic properties, which we use freely.)

Note that the negation of $q \in \Omega_{\sigma\exists}(p)$ is $q \in o_\sigma(p)$. In using $\Omega_{\sigma\exists}$ we must be careful to check that the various subsequences asserted to exist harmonize with each other. For instance, if we know that $q \in \Omega_{\sigma\exists}(p)$ and $r \in \Omega_{\sigma\exists}(p)$ and we wish to say something about the behavior of both q and r, we need to restrict to a common subsequence on which both asymptotic claims apply.

This is particularly important in the situation of assumed δ-incompleteness, where we have an infinite list of 1-witness systems V_k^f each supposed to have an asymptotic solution sequence σ_k. This hypothesis does not guarantee that the sequence σ_{k_1} for a large index $k_1 > k_0$ has the same values as σ_{k_0} of the variables appearing in $V_{k_0}^f$ (though the equations of the system are all still satisfied, of course).

For this reason we typically invoke the following restrictions on asymptotic solution sequences σ_k:

Proposition 5.1.3. *Let V_∞^f be a limit of 1-witness systems V_k^f each having an asymptotic solution sequence σ_k. Denote by $\sum_{i=1}^r y_i f_i(x) = 1$ the 1-witness in x and $\sum_{i=1}^s z_i g_i\left(\frac{1}{x}\right) = 1$ the 1-witness in $\frac{1}{x}$. Choose any finite set of monomials $T = \{t_1, \ldots, t_N\}$ in the variables of V_∞^f such that T includes all the monomials of the initial blocks of both 1-witnesses. Then there exist asymptotic solution sequences $\{\tilde{\sigma}_k\}_{k\in\mathbb{N}}$ such that*

1. *There is a permutation p of $\{1, \ldots, N\}$ such that $t_{p(1)} \geq_{\tilde{\sigma}_k} t_{p(2)} \geq_{\tilde{\sigma}_k} \cdots \geq_{\tilde{\sigma}_k} t_{p(N)}$ for all k.*

2. *There is a monomial $y_{i_0} f_{i_0}$ in the initial block of the 1-witness in x such that for all k, $y_{i_0} f_{i_0} \geq_{\sigma_k} y_i f_i$ for all $1 \leq i \leq r$. Likewise, there is a monomial $z_{i_1} g_{i_1}$ in the initial block of the 1-witness in $\frac{1}{x}$ such that for all k, $z_{i_1} g_{i_1} \geq_{\sigma_k} z_i g_i$ for all $1 \leq i \leq s$. Moreover, $|y_{i_0} f_{i_0((\overline{x},\overline{y})_\alpha)}| \geq \frac{1}{r}$ and $|z_{i_1} g_{i_1}((\overline{x},\overline{y})_\alpha)| \geq \frac{1}{s}$ for all members $(\overline{x},\overline{y})_\alpha$ of all sequences $\tilde{\sigma}_k$.*

Proof. There are only finitely many ordered sequences of the members of T. For each k and each element of σ_k, some particular order holds. The pigeonhole principle implies that there is a permutation p of $\{1, \ldots, N\}$ such that for infinitely many $k_0, k_1, \ldots$ and infinitely many elements $(\overline{x},\overline{y})_\alpha$ of each σ_{k_l}, $|t_{p(1)}((\overline{x},\overline{y})_\alpha)| \geq |t_{p(2)}((\overline{x},\overline{y})_\alpha)| \geq \cdots \geq |t_{p(N)}((\overline{x},\overline{y})_\alpha)|$. For $k = k_l$, let $\tilde{\sigma}_k$ be an infinite subsequence of σ_k on which the chosen order holds. (This is even stronger than $\geq_{\tilde{\sigma}_k}$.) For values of $k \neq k_l$ for any l, let $\tilde{\sigma}_k = \tilde{\sigma}_{k_l}$, where l is minimal such that $k_l > k$.

The chosen order fixes an ordering of the monomials from the initial block; the maximal one from the 1-witness in x is $y_{i_0} f_{i_0}$ and likewise for the 1-witness in $\frac{1}{x}$. By definition of a 1-witness and the triangle inequality, for each element of a solution sequence one of the summands must have magnitude at least equal to 1 divided by the number of summands. If any summand does, the maximal one does. $\square$

We call the maximal terms $y_{i_0} f_{i_0}$ and $z_{i_0} g_{i_0}$ *leading monomials* of the initial blocks of their respective 1-witnesses. By the proposition, we make the convention that for any V_∞^f we consider, there are chosen leading monomials as well as a fixed ordering on the variables $\overline{x}$. (We do not know beforehand what those choices are, so our arguments necessarily involve cases.)

Example 5.1.3. $x' = x^2$:

Recall the 1-witness matrix $M_{x,\overline{y},3}^{x'=x^2}$

$$\begin{bmatrix} x^2 & x & 0 & 0 & 0 & 0 & 1 \\ 2\,x^3 & x^2 & x^2 & x & 0 & 0 & 0 \\ 6\,x^4 & 2\,x^3 & 4\,x^3 & 2\,x^2 & x^2 & x & 0 \end{bmatrix}$$

corresponding to the system of equations $V_{x,\overline{y},3}^{x'=x^2}$

$$y_2x^2 + y_1x = 1$$

$$2y_2x^3 + y_1x^2 + y_2'x^2 + y_1'x = 0$$

$$6y_2x^4 + 2y_1x^3 + 4y_2'x^3 + 2y_1'x^2 + y_2''x^2 + y_1''x = 0.$$

The exponent pattern is clear. Suppose this 1-witness system had an asymptote witnessed by solution sequence σ. By definition of an asymptotic solution sequence, $\overline{y}_i \to \overline{0}$ as $i \to \infty$. The first equation then shows that $|x| \to \infty$ as $i \to \infty$. (For conciseness we may write $x \to \infty$ when the context is evident.) As a result, we might be tempted to say that $y_2x^2 \in \Omega_\sigma(y_1x)$. It certainly is true that because $x \to \infty$, then $x^2 \in \Omega_\sigma(x)$; even more, it is true that $x \in o_\sigma(x^2)$. However, the asymptotic relation of y_2x^2 and y_1x depends also on the values of $(y_2)_i$ and $(y_1)_i$ as $i \to \infty$. If the trend in σ is for y_2 to be increasingly small relative to y_1 it is possible, for instance, that $y_2x^2 \in o_\sigma(y_1x)$.

The key is that the equations put constraints on the asymptotic behavior of the variables. The relation provided by $x' = x^2$ ensures that we get repetition in the equations. As an example, suppose that $y_2x^2 \in o_\sigma(y_1x)$. This implies $y_1x \in \Omega_\sigma(1)$. (In fact, eventually $y_1x_1 \to 1$ because y_2x^2 becomes too small to contribute much in comparison, but the 1-witness must still be satisfied. For the sake of the example we refrain from using the stronger relation.) Then because $y_2x^3 \in o_\sigma(y_1x^2)$ and $y_1x \in \Omega_\sigma(1)$ the second equation tells us that either $y_2'x^2 + y_1'x \in \Omega_\sigma(x)$; i.e., $y_2'x + y_1' \in \Omega_\sigma(1)$. We conclude that either $y_2' \in \Omega_{\sigma\exists}(\frac{1}{x})$ or $y_1' \in \Omega_{\sigma\exists}(1)$. The latter would contradict the claim that σ is an asymptotic solution sequence. That is because $\overline{y} \to \overline{0}$, but $y_1' \in \Omega_{\sigma\exists}(1)$ means that there is a positive magnitude below which $|y_1'|$ never falls for some infinite subsequence of σ. So we know that $y_2' \in \Omega_{\sigma\exists}(\frac{1}{x})$. At this point we restrict to the subsequence on which y_2' is at least as large asymptotically as $\frac{1}{x}$.

This illustrates our statement about the equations imposing constraints on the variables. We started with a hypothesis on the relative growth of the monomials in the first equation and ended up with a bound on how fast y_2' can shrink in terms of x.

Continuing, it appears (we discuss potential obstacles in the next section; for now we focus on communicating the general idea) that the magnitude of the initial segment $6y_2x^4+2y_1x^3+4y_2'x^3$ should be at least of order x^2. But the remainder of the third equation only contains monomials whose degree in x is less than or equal to 2. So at least one of y_1', y_2'', y_1'' must asymptotically have order at least 1. This contradicts the assumption that $V_{x,\overline{y},3}^{x'=x^2}$ has an asymptote.

The preceding example suggests a general way in which 1-witness systems fail to have asymptotes: continued substitution using the defining equation of the original finite-rank δ-variety leads to growth in the total degree of successive equations. Potentially this causes "upward pressure" on the variables $\overline{y}^{(j)}$; the monomials in which they appear have relatively lower degree in $\overline{x}$, so the only way the equations can be satisfied is if for larger j the $\overline{y}^{(j)}$ have asymptotically greater magnitude. Eventually this magnitude is of order 1, and we contradict existence of an asymptote.

The author's computational studies of specific δ-polynomials indicate that this phenomenon is the chief driver of completeness in finite-rank projective δ-varieties. We now investigate what is required to make the idea a rigorous tool for studying completeness.

5.2 Conjectural ingredients for verifying the asymptotic strategy

We begin this section on conjectures with a proof.

Proposition 5.2.1. *Let V_k^f be a 1-witness system and for $k > 0$ let h_k be the equation $\sum_{j=0}^k \sum_{i=1}^r \binom{k}{j} y_i^{(j)} f_i^{(k-j)}(x) = 0$ from $V_{x,\overline{y},k}^f$ (or the equation $\sum_{j=0}^k \sum_{i=1}^s \binom{k}{j} z_i^{(j)} g_i^{(k-j)}(\frac{1}{x}) = 0$ from $V_{1/x,\overline{z},k}^f$). Denote the partial sum of h_k from $j = 0$ to some $l < k$ by $h_{k,1}$ and the remaining sum from $l + 1$ to k by $h_{k,2}$. Suppose that V_k^f has an asymptotic solution sequence σ and that $h_{k,1}$ is eventually non-zero (i.e. $h_{k,1}$ is non-zero when evaluated at any member of σ having sufficiently large index). Then $h_{k,1} \notin \Omega_{\sigma\exists}(\{f_i^{(k-j)}\}_{max})$ as i ranges from 1 to r and j from $l+1$ to k (or $h_{k,1} \notin \Omega_{\sigma\exists}(\{g_i^{(k-j)}\}_{max})$ if h_k belongs to $V_{1/x,\overline{z},k}^f$).*

Proof. Nothing about the argument changes if h_k belongs to $V_{1/x,\overline{z},k}^f$, so we only write one case. Suppose toward contradiction that $h_{k,1} \in \Omega_{\sigma\exists}(\{f_i^{(k-j)}\}_{\max})$. (This means that $h_{k,1} \in \Omega_{\sigma\exists}(f)$, where f is the function whose value at any given member of σ is the maximal value of all the $f_i^{(k-j)}$ evaluated at that member.) Restrict to a subsequence witnessing this relation; for convenience we redefine σ to be said subsequence. Note that for any $1 \leq i \leq r$ and $l+1 \leq j \leq k$ we have $h_{k,1} \in \Omega_\sigma(f_i^{(k-j)})$.

Because $h_{k,1} + h_{k,2} = h_k = 0$ for all members of σ, we claim that for some i and some $j \geq l + 1$ we must have $y_i^{(j)} f_i^{(k-j)} \in \Omega_{\sigma\exists}(h_{k,1})$. This is because $h_{k,2}$ only has n summands $y_i^{(j)} f_i^{(k-j)}$ for some natural number n, so for each $(\overline{x}, \overline{y})_\alpha \in \sigma$, one of those summands must have magnitude at least $\left|\left(\frac{1}{n}\right) h_{k,1}((\overline{x},\overline{y})_\alpha)\right|$. By the pigeonhole principle, this happens infinitely often for a single summand. Pick out the corresponding subsequence and call it $\tilde{\sigma}$; using $\frac{1}{n}$ as ϵ we have met the condition for $y_i^{(j)} f_i^{(k-j)} \in \Omega_{\tilde{\sigma}}(h_{k,1})$. Restrict to $\tilde{\sigma}$, relabel it as σ, and proceed.

The combined relations $y_i^{(j)} f_i^{(k-j)} \in \Omega_\sigma(h_{k,1})$ and $h_{k,1} \in \Omega_\sigma(f_i^{(k-j)})$ imply that for some $\tilde{\epsilon} > 0$ we have $|y_i^{(j)} f_i^{(k-j)}| \geq \tilde{\epsilon}|f_i^{(k-j)}|$ for all members of σ beyond some point. Observe that $h_{k,1}$ being eventually non-zero forces $f_i^{(k-j)}$ to be eventually non-zero. Dividing we find that asymptotically $|y_i^{(j)}| \geq \tilde{\epsilon}$, contradicting the fact that σ is an asymptotic solution sequence as $\overline{y} \to \overline{0}$. □

The preceding result, which we call the *asymptotic test for δ-completeness*, simply formalizes the tactic from Example 5.1.3. That is, to prove completeness by the asymptotic test and transferred valuative criterion it suffices to show that for some k and partition $h_{k,1}, h_{k,2}$ of h_k we actually have $h_{k,1} \in \Omega_{\sigma\exists}(\{f_i^{(k-j)}\}_{\max})$ (respectively $h_{k,1} \in \Omega_{\sigma\exists}(\{g_i^{(k-j)}\}_{\max})$) as i ranges from 1 to r and j from $l+1$ to k. This gives us a goal to work towards, but there are several potential obstructions.

Example 5.2.1. $x' = x^2$:

Consider the 1-witness matrix $M^{x'=x^2}_{x,\overline{y},4}$:

$$\begin{bmatrix} x^2 & x & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2\,x^3 & x^2 & x^2 & x & 0 & 0 & 0 & 0 & 0 \\ 6\,x^4 & 2\,x^3 & 4\,x^3 & 2\,x^2 & x^2 & x & 0 & 0 & 0 \\ 24\,x^5 & 6\,x^4 & 18\,x^4 & 6\,x^3 & 6\,x^3 & 3\,x^2 & x^2 & x & 0 \end{bmatrix}$$

Gaussian elimination produces the following row-reduced matrix:

$$\begin{bmatrix} x^2 & x & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -x^2 & x^2 & x & 0 & 0 & 0 & 0 & -2\,x \\ 0 & 0 & 0 & -2\,x^2 & x^2 & x & 0 & 0 & 2\,x^2 \\ 0 & 0 & 0 & 0 & 0 & -3\,x^2 & x^2 & x & 0 \end{bmatrix}$$

In particular, note the zeros on the diagonal in the third and fourth rows. However, the most substantial issue is the 0 in the last entry of the fourth row. What has happened is that various determinants (e.g., that of the 3×3 submatrix in the upper left corner of $M^{x'=x^2}_{x,\overline{y},4}$) are identically zero. By Cramer's rule, such cancellations determine the form of the reduced matrix.

The consequence is that the asymptotic argument we made in 5.1.3 is potentially "cut off" in the fourth row. Since the coefficients of y_2, y_1, y_2', y_1', and y_2'' are all 0, the constraints imposed by previous rows on the those variables do not apply. Hence it may very well be that y_1'', y_2''', y_1''' are all asymptotically very small without contradiction from the fourth row.

In spite of this, the asymptotic approach still works in this case. The growth in the degree of x was sufficient to place $2x^2$ in the last entry of the third row, and no coefficient of $\overline{y}$ has degree greater than 2 in this row. Hence the system is unsolvable unless either y_1', y_2'', or $y_1'' \in \Omega_{\sigma\exists}(1)$. This shows that this 1-witness system cannot have an asymptote.

How do we know that such fortunate things will happen? It seems difficult to anticipate the outcome for all 1-witness systems and finite-rank δ-varieties. Probably the greatest challenge in understanding complete δ-varieties is the danger of unforeseen relations among terms of a 1-witness system. This can happen because either

1. there is too much repetition (as in the example above for the complete δ-variety $x' = x^2$) or
2. differentiation and substitution immediately cause cancellation modulo the δ-ideal generated by the defining equations of the variety (exemplified by the incomplete δ-variety $x'' = x^3$).

As for the first problem, intuitively such "accidents" ought to be special; slight changes to the system should avert them. (After all, the vanishing of a determinant is a closed condition.) This leads to the idea of replacing the coefficients of the 1-witness system with values very nearby that form an algebraically independent set over $\mathbb{Q}$. We refer to this change as *generically perturbing* the coefficients of the system; we explain this further in a moment.

We admit that we currently do not know how to resolve the second problem. It seems qualitatively different to perturb the coefficients of a non-zero polynomial (say, $x^2 - y$) than those of the zero polynomial (e.g., $x^2 - x^2$; the behavior of $1.027x^2 - .994y$ is more similar to that of $x^2 - y$ than the behavior of $1.027x^2 - .994x^2 = .033x^2$ is to that of 0). As indicated earlier, membership in differential polynomial ideals is usually difficult to ascertain, so it may not be evident for a given variety whether outcome-altering cancellations can arise. Our response is to defer the issue to future research and continue discussing the asymptotic approach with the understanding that it only applies to varieties lacking "bad cancellations", with the exact meaning of that yet to be determined. Barring such a criterion, the asymptotic approach can only indicate why a given variety "should" be complete in the absence of special relations.

Example 5.2.2. $x' = x^2$:

Using Maple's (pseudo-)random number generator `rand` we obtain a numerically perturbed version of $M^{x'=x^2}_{x,\overline{y},4}$:

$$\begin{bmatrix} 0.998\,x^2 & 1.002\,x & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1.995\,x^3 & 0.997\,x^2 & 1.009\,x^2 & 0.995\,x & 0 & 0 & 0 & 0 & 0 \\ 6.041\,x^4 & 2.004\,x^3 & 4.020\,x^3 & 2.003\,x^2 & 1.009\,x^2 & 1.006\,x & 0 & 0 & 0 \\ 23.826\,x^5 & 5.963\,x^4 & 17.883\,x^4 & 6.006\,x^3 & 5.954\,x^3 & 3.000\,x^2 & 0.990\,x^2 & 0.996\,x & 0 \end{bmatrix}$$

This time, Gaussian elimination produces the following:

$$\begin{bmatrix} 0.998\,x^2 & 1.002\,x & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1.005\,x^2 & 1.009\,x^2 & 0.995\,x & 0 & 0 & 0 & 0 & -1.998\,x \\ 0 & 0 & -0.057\,x^3 & -2.015\,x^2 & 1.009\,x^2 & 1.006\,x & 0 & 0 & 2.021\,x^2 \\ 0 & 0 & 0 & -6.619\,x^3 & 3.378\,x^3 & 0.433\,x^2 & 0.990\,x^2 & 0.996\,x & 6.673\,x^3 \end{bmatrix}$$

Even though we haven't achieved true algebraic independence, the small change was enough. No determinants vanished, and we are able to read off easily from the fourth row that asymptotically at least one of $y_1', y_2'', y_1'', y_2''', y_1'''$ belongs to $\Omega_{\sigma\exists}(1)$. It is true that there was a "near miss" in the $(3,3)$ entry; -0.057 is small relative to the other coefficients in the matrix. However, asymptotically $-0.057x^3$ is just as large as x^3, so non-zero constants do not affect the analysis.

Algebraically independent coefficients by definition avoid cancellations, so any generically perturbed row-reduced 1-witness matrix will have upper triangular form in the left-most square submatrix. Therefore the central matter is the following question:

Question 5.2.1. *Let V_k^f be a 1-witness system having an asymptote. Is it always possible to replace the coefficients of the monomials in the defining equations of V_k^f with a collection of coefficients algebraically independent over $\mathbb{Q}$ such that the resulting system $\widetilde{V}_k^f$ still has an asymptote?*

If so, then non-existence of an asymptote for the perturbed variety implies that the original variety likewise could not have had an asymptote. We emphasize again that relations modulo the equations of the δ-variety are very important and that perturbing the coefficients of a polynomial whose value is identically zero may lead to changes in asymptotic behavior. The issue here is whether or not asymptotes are preserved when there are no such "bad cancellations".

It is helpful to think of the situation projectively to see more clearly what is at stake. Let $V_k^f \subseteq \mathbb{C}^n \times \mathbb{C}^m$ have an asymptote witnessed by solution sequence $\sigma = \{(\overline{x}, \overline{y})_i\}_{i \in \mathbb{N}}$. If we homogenize the defining equations of V_k^f with respect to $x_1, \ldots, x_n$ (but not $\overline{y}$) we obtain a projective variety $(V_k^f)^h \subseteq \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^m$. It is well known, however, that $(V_k^f)^h$ is not necessarily the projective closure of V_k^f; $(V_k^f)^h$ could have additional components contained in the hyperplane at infinity $x_{n+1} = 0$. In any case, if we stay in the standard affine chart U where $x_{n+1} \neq 0$, the solution sequence σ gives a sequence of points $((x_1 : \cdots : x_n : 1), \overline{y})_1, \ldots,$ in $(V_k^f)^h \cap U$ such that the magnitude of $\overline{y}_i$ goes to 0 as $i \to \infty$.

Taking advantage of homogeneity with respect to $\overline{x}$, we may represent the sequence as $((\frac{x_1}{x_{\max}} : \cdots : \frac{x_n}{x_{\max}} : \frac{1}{x_{\max}}), \overline{y})_1, \ldots,$ where for each $i \in \mathbb{N}$ the coordinate $x_{\max}$ is the coordinate of $(x_1 : \cdots : x_n : x_{n+1})_i$ having largest magnitude. The coordinates given by $\overline{y}_i$ individually converge to 0 and the normalized coordinates $((\frac{x_1}{x_{\max}} : \cdots : \frac{x_n}{x_{\max}} : \frac{1}{x_{\max}}), \overline{y})_1, \ldots,$ all have magnitude bounded above by 1, so by compactness of the unit box in $\mathbb{C}^{n+1} \times \mathbb{C}^m$ there is a subsequence that converges to a point in $\mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^m$. (At least one of the normalized values $\frac{x_1}{x_{\max}}, \ldots, \frac{x_{n+1}}{x_{\max}}$ is equal to 1, so the sequence of tuples $(\frac{x_1}{x_{\max}}, \ldots, \frac{x_{n+1}}{x_{\max}})_1, \ldots,$ in $\mathbb{C}^{n+1}$ converges to a tuple whose entries are not all 0, giving the homogeneous coordinates of a point in $\mathbb{P}^n(\mathbb{C})$.)

Consequently an asymptote in the affine chart U picks out a limit point in the projective closure of $V_k^f \subseteq (V_k^f)^h$. We may then interpret the question as asking whether there are sufficiently small generic perturbations of the coefficients of V_k^f that only slightly change the projective closure of V_k^f. For if the projective closure (not just the homogenization) of the perturbed $\widetilde{V}_k^f$ contains a point p having $\overline{y} = \overline{0}$, then Theorem 2.1.2 implies that p is a limit of points on $\widetilde{V}_k^f$ in U. In other words, p being a limit point witnesses that $\widetilde{V}_k^f$ has an asymptote. The author has thus far been unable to resolve the question. However, it is easy to see that one can at least generically perturb a 1-witness in x and retain an asymptote.

Proposition 5.2.2. *Generically perturbing the coefficients of a 1-witness $\sum_{i=1}^r y_i f_i(x) - 1$ preserves asymptotes.*

Proof. Suppose the variety defined by $\sum_{i=1}^r y_i f_i(x) - 1$ has an asymptote as $\overline{y} \to \overline{0}$. If all the monomials in $\overline{x}$ are the same (i.e., only the $\overline{y}$ variables differ from term to term), then pick the same small value for all $\overline{y}$ and rely on algebraic independence of the perturbed coefficients to avoid cancellation. Otherwise, specializing the variables $\overline{y}$ with the value $\overline{y}_i$ given by an asymptotic solution sequence for the original unperturbed 1-witness yields a non-constant polynomial in the variables $\overline{x}$. Since $\mathbb{C}$ is algebraically closed, we can obviously perturb the coefficients and retain a non-constant (and hence solvable) polynomial equation. Solutions $\overline{\widetilde{x}}_i$ pair with the original $\overline{y}_i$ to form an asymptotic solution sequence for the perturbed 1-witness. $\square$

The next potential obstruction to realizing the hypotheses of the asymptotic test is related to the first, and we conjecture (again barring "bad cancellations") that the workaround is the same: generic perturbation of coefficients. The problem is that even if cancellation (of either kind discussed above) does not occur, it is not obvious that the asymptotic growth rate of a polynomial is as large as one would expect from looking at the individual monomials.

Example 5.2.3. $xx'' = x'$:

Consider the 1-witness matrix $M^{xx''=x'}_{x,\overline{y},3}$ (we ignore the 1-witness system in $\frac{1}{x}$ for the moment):

$$\begin{bmatrix} \frac{x'}{x} & 0 & 0 & 1 \\ \frac{x'}{x^2} - \frac{(x')^2}{x^2} & \frac{x'}{x} & 0 & 0 \\ -4\,\frac{(x')^2}{x^3} + \frac{x'}{x^3} + 2\,\frac{(x')^3}{x^3} & 2\,\frac{x'}{x^2} - 2\,\frac{(x')^2}{x^2} & \frac{x'}{x} & 0 \end{bmatrix}$$

Suppose the system has an asymptotic solution sequence σ such that $\frac{x'}{x} \geq_\sigma \frac{1}{x}$. In light of earlier considerations, we assume that we may replace the coefficients of the monomials in the matrix with values that are algebraically independent. Rewrite the $(2,1)$-entry as $\alpha_1 \frac{x'}{x^2} + \alpha_2 \frac{(x')^2}{x^2}$ for some complex α_1, α_2 close to 1 and -1, respectively, but generic.

However, independence of α_1 and α_2 does not guarantee that $\alpha_1 \frac{x'}{x^2} + \alpha_2 \frac{(x')^2}{x^2} \in \Omega_{\sigma\exists}(\{\frac{x'}{x^2}, \frac{(x')^2}{x^2}\}_{\max} = \frac{(x')^2}{x^2})$. Factoring, we see that the expression is $\left(\frac{x'}{x^2}\right)(\alpha_1 + \alpha_2 x')$. If, for instance, $\alpha_2 x'$ asymptotically approaches $-\alpha_1$, then *a priori* the overall expression may be small even if separately $\frac{x'}{x^2}$ and $\frac{(x')^2}{x^2}$ are not.

Since by the first row we have $y\frac{x'}{x} = 1$, $y \cdot (2,1)$ is equal to $\frac{\alpha_1}{x} + \alpha_2\frac{x'}{x}$. Suppose that $\frac{\alpha_1}{x} + \alpha_2\frac{x'}{x} \notin \Omega_{\sigma\exists}(\frac{x'}{x})$; i.e., $\frac{\alpha_1}{x} + \alpha_2\frac{x'}{x} \in o_\sigma(\frac{x'}{x})$. Hence there is a function, which we do not name but simply denote as $o_\sigma(\frac{x'}{x})$, such that $\frac{1}{x} = \frac{-\alpha_2 x'}{\alpha_1 x} + o_\sigma(\frac{x'}{x})$. (The α_i are fixed complex numbers that do not change as we proceed along the solution sequence; hence we may divide by them without changing the asymptotic size of expressions like $o_\sigma(\frac{x'}{x})$.)

Now consider the entry $(3,1)$ written with generic coefficients as $\alpha_3\frac{(x')^2}{x^3} + \alpha_4\frac{x'}{x^3} + \alpha_5\frac{(x')^3}{x^3}$. Applying the relation $y\frac{x'}{x} = 1$, we have $y \cdot (3,1) = \alpha_3\frac{x'}{x^2} + \alpha_4\frac{1}{x^2} + \alpha_5\frac{(x')^2}{x^2}$. Combining this with the equation $\frac{1}{x} = \frac{-\alpha_2 x'}{\alpha_1 x} + o_\sigma(\frac{x'}{x})$ gives us the following calculation:

$$\begin{aligned}
\alpha_3\frac{x'}{x^2} + \alpha_4\frac{1}{x^2} + \alpha_5\frac{(x')^2}{x^2} &= \\
\frac{\alpha_4}{\alpha_1 x}\left(\frac{\alpha_3\alpha_1}{\alpha_4}\frac{x'}{x} + \frac{\alpha_1}{x}\right) + \alpha_5\frac{(x')^2}{x^2} &= \\
\frac{\alpha_4}{\alpha_1 x}\left(\frac{\alpha_3\alpha_1}{\alpha_4}\frac{x'}{x} - \frac{\alpha_2 x'}{x} + o_\sigma\left(\frac{x'}{x}\right)\right) + \alpha_5\frac{(x')^2}{x^2} &= \\
\frac{\alpha_4}{\alpha_1 x}\left(\left(\frac{\alpha_3\alpha_1 - \alpha_2\alpha_4}{\alpha_4}\right)\frac{x'}{x} + o_\sigma\left(\frac{x'}{x}\right)\right) + \alpha_5\frac{(x')^2}{x^2} &= \\
\left(\frac{-\alpha_2 x'}{\alpha_1 x} + o_\sigma(\frac{x'}{x})\right)\left(\frac{\alpha_4}{\alpha_1}\right)\left(\left(\frac{\alpha_3\alpha_1}{\alpha_4} - \alpha_2\right)\frac{x'}{x} + o_\sigma\left(\frac{x'}{x}\right)\right) + \alpha_5\frac{(x')^2}{x^2} &= \\
\alpha\frac{(x')^2}{x^2} + o_\sigma\left(\frac{(x')^2}{x^2}\right)&,
\end{aligned}$$

where α is a rational function in the values $\alpha_1, \ldots, \alpha_5$. Genericity ensures that α is non-zero, so we may choose $0 < \epsilon < |\alpha|$. By definition o_σ allows us to find an index i_0 such that if $i \geq i_0$, then the magnitude of $o_\sigma(\frac{(x')^2}{x^2})$ is less than or equal to $\epsilon|\frac{(x')^2}{x^2}|$. We see that $\alpha_3\frac{(x')}{x^2} + \alpha_4\frac{1}{x^2} + \alpha_5\frac{(x')^2}{x^2} = \alpha\frac{(x')^2}{x^2} + o_\sigma(\frac{(x')^2}{x^2}) \in \Omega_{\sigma\exists}(\frac{(x')^2}{x^2})$ (in $\Omega_\sigma(\frac{(x')^2}{x^2})$, even).

As a result, the "smallness relation" imposed by $\frac{\alpha_1}{x} + \alpha_2 \frac{x'}{x} \in o_\sigma(\frac{x'}{x})$ making $y \cdot (2,1)$ asymptotically smaller than its constituent monomials is overcome by genericity in $y \cdot (3,1)$, which *is* asymptotically as large as its monomials. Thus the asymptotic argument does not stall but proceeds with new relations imposed on the rest of the third row (because now we know that the remaining terms must cancel values of the order of $\frac{(x')^2}{x^2}$).

We have used nothing very particular about $xx'' = x'$; we have only performed elimination on a restricted subset of the polynomials in a 1-witness system and invoked genericity to show that smallness relations on some of the polynomials cannot force others with algebraically independent coefficients to be asymptotically small as well. This leads to the following question:

Question 5.2.2. *Let $\widetilde{V}^f_\infty$ be a generically perturbed limit (i.e., the infinite set of all coefficients in all equations is algebraically independent over $\mathbb{Q}$) of 1-witness systems V^f_k having asymptotic solution sequences σ_k. Let $H = \{h_1, h_2, \dots\}$ be a collection of infinitely many subpolynomials from the equations of $\widetilde{V}^f_\infty$ such that only finitely many distinct variables appear in H. Is it true that for some $k_0 \in \mathbb{N}$, $k \geq k_0$ implies that $h_k \in \Omega_{\sigma_k \exists}(\{\text{terms of } h_k\}_{max})$?*

(In the question, by "subpolynomials" we simply mean that each equation of $\widetilde{V}^f_\infty$ is given by some polynomial and h_i is the sum of a subset (eventually proper, since we only have finitely many distinct variables) of the monomials of that polynomial.)

The question asks (roughly) whether there is a kind of asymptotic Hilbert's basis theorem for generically perturbed polynomial systems. Our intuition is that smallness relations are analogous to generators of ideals in polynomial rings, and that one cannot have infinitely many such that are independent.

A plausible hypothesis should at least be true in simple cases. Here is one:

Proposition 5.2.3. *Let $t_1, \ldots, t_m$ be monomials in variables $x_1, \ldots, x_n$. By σ we denote a sequence of tuples $\{\overline{x}_l\}_{l \in \mathbb{N}}$ (not necessarily an asymptotic solution sequence) in $\mathbb{C}^n$. Let $\{\alpha_{i,j}\}_{1 \le i,j \le m}$ be a set of complex numbers algebraically independent over $\mathbb{Q}$. Then for some $1 \le i \le m$ we have $\alpha_{i,1}t_1 + \cdots + \alpha_{i,m}t_m \in \Omega_{\sigma\exists}(\{t_1, \ldots, t_m\}_{max})$.*

Proof. As explained earlier, we may find a subsequence of σ for which a single t_i is always maximal; hence we assume that σ has this property to begin with. Without loss of generality the maximal monomial is t_1. If t_1 is eventually 0 (i.e., for all sufficiently large l, $t_1(\overline{x}_l) = 0$), then by maximality every t_i is eventually 0 and the claim is trivially true.

Otherwise, suppose that t_1 is not eventually 0. We assume toward contradiction that for no $1 \le i \le m$ do we have $\alpha_{i,1}t_1 + \cdots + \alpha_{i,m}t_m \in \Omega_{\sigma\exists}(t_1)$. Then the following holds:

$$\alpha_{1,1}t_1 + \cdots + \alpha_{1,m}t_m \in o_\sigma(t_1)$$

$$\vdots$$

$$\alpha_{m,1}t_1 + \cdots + \alpha_{m,m}t_m \in o_\sigma(t_1).$$

But the $\alpha_{i,j}$ are the coefficients of a non-singular (by genericity) $m \times m$ matrix. Inverting, we obtain $t_i \in o_\sigma(t_1)$ for each i (because the $\alpha_{i,j}$ are constant and any $\mathbb{C}$-linear combination of polynomials in $o_\sigma(t_1)$ remains in $o_\sigma(t_1)$). Since t_1 is not eventually 0, $t_1 \notin o_\sigma(t_1)$; this contradicts the previous sentence.

□

Independent asymptotic monomial growth (our phrase for the conclusion of the question), in conjunction with confirmation that a sufficiently small generic perturbation preserves asymptotes, would significantly assist in verifying the asymptotic test for δ-completeness. It does so by placing the burden on the combinatorics of *monomials* under repeated differentiation and substitution. We conclude the section with examples of the third stage of the asymptotic approach: using the two conjectural ingredients to contradict the asymptotic test for specific δ-varieties.

Example 5.2.4. $x' = x^2$:

We argue asymptotically, assuming generic preservation of asymptotes as well as independent asymptotic monomial growth, that $x' = x^2 = \overline{x' = x^2}$ is δ-complete. (That is, we assume that our systems $V^{x'=x^2}_{x,\overline{y},k}$ have asymptotes, that the set of constant coefficients of monomials is algebraically independent over $\mathbb{Q}$, and that for large enough k there are subsequences of σ_k for which large enough sets of subpolynomials in finitely many variables are eventually as large as their leading terms.) Any 1-witness has the form $y_m x^m + \cdots + y_1 x = 1$. Let $y_j x^j$ be the fixed leading term of the 1-witness. Consider the $(m+1)$-st row of the 1-witness matrix $M^{x'=x^2}_{x,\overline{y},m}$. The coefficients of $y_m, \ldots, y_j, \ldots, y_1$ in this row are (ignoring constant factors) $x^{2m}, \ldots, x^{j+m}, \ldots x^{m+1}$, respectively.

Since $y_j x^j \in \Omega_{\sigma_k}(1)$ for all k, we have $y_j x^{j+m} \in \Omega_{\sigma_k}(x^m)$ for all k. In other words, the single monomial $y_j x^{j+m}$ is asymptotically as large as any coefficient of $y_i^{(l)}$ for $l \geq m$ in the $(m+1)$-st row. Now, we restrict our attention for a moment to the first m^2 columns (i.e., the coefficients of $y_m, \ldots, y_1, \ldots, y_m^{(m-1)}, \ldots, y_1^{(m-1)}$). By the assumption of independent asymptotic monomial growth, for some natural number N the $(m+1+N)$-th row has the property that the sum (again ignoring constant factors)

$$y_m x^{2m+N} + \cdots + y_j x^{j+m+N} + \cdots + y_1 x^{m+1+N} \;\; + \cdots + y_m^{(m-1)} x^{m+1+N} + \cdots + y_1^{(m-1)} x^{2+N}$$

belongs to $\Omega_{\sigma_{m+N}\exists}(x^{m+N})$. Beyond the first m^2 columns, however, the largest degree of x in a coefficient of $y_m^{(m)}, \ldots, y_m^{(m+N)}, \ldots, y_1^{(m+N)}$ is $m+N$. Since $y_j x^{m+N}$ is eventually non-zero (always, in fact), we have a contradiction to the asymptotic test. Hence the 1-witness $y_m x^m + \cdots + y_1 x = 1$, which was arbitrary, cannot give rise to systems with asymptotes. By the transferred valuative criterion, $x' = x^2$ is δ-complete.

What we have just presented is not a proof of completeness of $x' = x^2$ (which in any case was shown by Pong) because of the two conjectural ingredients (even discounting the background issue of "bad cancellations" discussed earlier). However, it illustrates the basic mechanism by which δ-completeness of a finite-rank projective δ-variety $\overline{f = 0} \subseteq \mathbb{P}^1$ seems to arise.

In particular, repeated differentiation and substitution lead to *monomials* in the leading block that are asymptotically larger than the *coefficients* of the $\overline{y}$ in the initial block of the 1-witness. (One could colloquially say that this allows the monomials of the leading block to "get large.") Then further differentiation and substitution preserve the size advantage of the monomials of the leading block over the coefficients of subsequent blocks ("stay large"). Lastly, independent asymptotic monomial growth eventually assures that the leading block is asymptotically as large as its monomials. At this point we violate the asymptotic test's necessary condition for an asymptote to exist, so the posited 1-witness system cannot provide a counterexample to completeness.

Example 5.2.5. $xx'' = x'$:

This example is more difficult than the previous one for two main reasons. First, we must now deal with two variables, x and x'. Second, the affine δ-variety $xx'' = x'$ is not apparently not projective, so we require the 1-witness system in $\frac{1}{x}$ as well as the one in x.

The presence of multiple variables $\overline{x}$ complicates the classification of 1-witnesses; we explain how to do it in this case. First consider an arbitrary monomial $x^a(x')^b$ such that a, b are integers. Differentiating and applying the substitution $x'' = \frac{x'}{x}$, we obtain $ax^{a-1}(x')^{b+1} + bx^{a-1}(x')^b$. Notice that up to a constant multiple, the first summand is the result of multiplying $x^a(x')^b$ by $\frac{x'}{x}$. The second summand results from multiplying $x^a(x')^b$ by $\frac{1}{x}$. Since we work asymptotically and with generic constant coefficients, it is convenient to focus only on the degrees of variables and consider differentiation followed by substitution as a rewrite process in which we transform one monomial into several others via multiplication by various factors. It is important, though, to note that we can only multiply by a given factor when there is a non-zero power of the corresponding variable in the original monomial. For example, differentiation and substitution for $x^{-2}x'$ allows us to use both $\frac{x'}{x}$ and $\frac{1}{x}$, so we count on having both $x^{-3}(x')^2$ and $x^{-3}x'$ in the derivative after applying the relation $x'' = \frac{x'}{x}$. However, x^3 only yields x^2x' because x' does not appear in the original monomial.

We need to understand which monomials can appear in the 1-witnesses in x and $\frac{1}{x}$. A monomial in x and its derivatives must be (after substitution) a polynomial in $x, x', \frac{1}{x}$, and $\frac{x'}{x}$. However, we can only apply the multiplication factor $\frac{1}{x}$ if there is already an x' present. Thus we characterize the monomials that can appear in a 1-witness in x (call them *admissible monomials*) as $x^a(x')^b$ where b is always non-negative and if a is negative, then $b > 0$.

The admissible monomials of a 1-witness in $\frac{1}{x}$ are more restricted because x and x' cannot appear unaccompanied. A monomial in $\frac{1}{x}$ and its derivatives has, after substitution, the form $x^a(x')^b$ where a is always a negative integer, b is non-negative, and $|a| > b$.

Since there are more restrictions on the admissible monomials of the $\frac{1}{x}$ system, it is a good idea to start searching for contradictions there. However, it turns out that we need the x-system as well to cover all cases.

There are two scenarios for an arbitrary 1-witness in $\frac{1}{x}$: Either the leading monomial contains x' or it does not.

1. (*leading monomial of* $\sum_{i=1}^{s} z_i g_i\left(\frac{1}{x}\right) = 1$ *contains* x') Denote the leading monomial by $z\frac{(x')^b}{x^a}$, with $a > b > 0$ (for convenience we have placed x in the denominator and so the exponent a of $\frac{1}{x}$ is now positive). We follow the "get large, stay large" strategy from the preceding example. Let $\tilde{z}\frac{(x')^c}{x^d}$ be an arbitrary monomial from the 1-witness in $\frac{1}{x}$. We show that sufficiently many derivatives of $z\frac{(x')^b}{x^a}$ will produce among the z-descendant monomials one that is asymptotically as large as $\frac{(x')^c}{x^d}$. (This is not immediate, even though $z\frac{(x')^b}{x^a}$ is the leading monomial of the initial block, because we are comparing to $\frac{(x')^c}{x^d}$, not $\tilde{z}\frac{(x')^c}{x^d}$.)

Because $b > 0$, each derivation of $\frac{(x')^b}{x^a}$ can apply both $\frac{x'}{x}$ and $\frac{1}{x}$. Choosing $\frac{x'}{x}$ c times and $\frac{1}{x}$ $d-c$ times we obtain a monomial $z\frac{(x')^b}{x^a}\left(\frac{x'^c}{x^d}\right) \in \Omega_{\sigma_k}(\frac{(x')^c}{x^d})$, as desired. (A descendant monomial of $z\frac{(x')^b}{x^a}$ has "gotten large" relative to $\frac{(x')^c}{x^d}$.)

By definition of an asymptotic solution sequence, $z \in o_{\sigma_k}(1)$ for all k. Since $z\frac{(x')^b}{x^a} \in \Omega_{\sigma_k}(1)$ for all k, it follows that $1 \in o_{\sigma_k}(\frac{(x')^b}{x^a})$ and thus either $\frac{x'}{x} \geq_{\sigma_k} 1$ or $\frac{1}{x} \geq_{\sigma_k} 1$. Note that with each succeeding row any monomial can change at most by a multiplicative factor of $\frac{x'}{x}$ or $\frac{1}{x}$; moreover, every descendant of $\frac{(x')^b}{x^a}$ has x' and thus can apply both factors when transitioning to the next row. This, along with $\frac{x'}{x} \geq_{\sigma_k} 1$ or $\frac{1}{x} \geq_{\sigma_k} 1$, implies that on each row, among the z-descendants of $z\frac{(x')^b}{x^a}\left(\frac{x'^c}{x^d}\right)$ is one that is $\Omega_{\sigma_k}(\{$descendants of $\frac{x'^c}{x^d}$ in the $(d+1)$-st and
later blocks$\}_{\max})$. Thus for arbitrarily large k, the descendants of $z\frac{x'^b}{x^a}$ are "staying large" relative to the descendants of $\frac{(x')^c}{x^d}$ in the later blocks.

The two preceding paragraphs show that for some k_0, after k_0 derivatives we have a monomial (a z-descendant of $z\frac{(x')^b}{x^a}$ or perhaps something larger) that is larger than any $\overline{x}$-coefficient of the variables $\overline{z}$ in the original 1-witness. At this point we consider the columns containing all non-zero entries of the first k_0+1 rows except the final block (whose entries are the $\overline{x}$-coefficients of $\overline{z}$ in the 1-witness) of the k_0+1-st row. The size advantage of the descendants of $z\frac{(x')^b}{x^a}$ in the first block is preserved for $k \geq k_0$, so it still holds after we apply the assumption of independent asymptotic monomial growth. Therefore we have a natural number k_1, a subsequence $\tilde{\sigma}_{k_1}$ of σ_{k_1}, and a partition $h_{k_1,1}, h_{k_1,2}$ of h_{k_1} such that $h_{k_1,1} \in \Omega_{\tilde{\sigma}_{k_1}}(t)$ for every $\overline{x}$-monomial t from $h_{k_1,2}$. This contradicts the asymptotic test for completeness.

2. (*leading monomial of* $\sum_{i=1}^{s} z_i g_i \left(\frac{1}{x}\right) = 1$ *does not contain* x') Denote by $z\frac{1}{x^a}$ $(a > 0)$ the leading monomial of the initial block. This implies $1 \in o_{\sigma_k}\left(\frac{1}{x}\right)$. Consider another term $\tilde{z}\frac{(x')^c}{x^d}$ of the 1-witness. If $c > 0$, then the argument is virtually the same as before (here we are forced to use $\frac{x'}{x}$ at least once since $\frac{1}{x^a}$ lacks x'; there is no problem because $c > 0$). A significant difference arises if $c = 0$. At issue is the fact that differentiating $\frac{1}{x^a}$ the first time introduces an x', and the multiplication factors $\frac{x'}{x}$ and $\frac{1}{x}$ provide no mechanism for removing x'. Hence the z-descendants of $z\frac{1}{x^a}$) only have monomials containing x'. If x' shrinks at too fast a rate, the descendants of $z\frac{1}{x^a}$ might be asymptotically smaller than the $\overline{x}$-monomials of the initial block. In other words, "getting large" might be impossible.

   There are two cases:

   (a) It is possible that for some positive integer n the following holds: for infinitely many k (refer to them as $\tilde{k}$) we have $\frac{x'}{x^n} \in \Omega_{\sigma_{\tilde{k}}\exists}(1)$. Restrict to subsequences $\tilde{\sigma}_{\tilde{k}}$ such that $\frac{x'}{x^n}$ belongs to $\Omega_{\tilde{\sigma}_{\tilde{k}}}(1)$. Then $z\frac{1}{x^a}\left(\frac{x'}{x^n}\right) \in \Omega_{\tilde{\sigma}_{\tilde{k}}}(1)$ for all $\tilde{k}$. Using the same tactics as usual, we find a z-descendant $z\frac{x'}{x^N}$ of $z\frac{1}{x^a}$ such that for some $\tilde{k}_0$ we have $z\frac{x'}{x^N} \in \Omega_{\tilde{\sigma}_{\tilde{k}_0}}(\frac{1}{x^d})$. Thus z-descendants of $z\frac{1}{x^a}$ "get large" and the argument proceeds as before.

   (b) If this does not hold, then for all n there is k_n such that $k \geq k_n$ implies $\frac{x'}{x^n} \in o_{\sigma_k}(1)$. Since $1 \in o_{\sigma_k}\left(\frac{1}{x}\right)$, it follows that $\frac{(x')^l}{x^m} \in o_{\sigma_k}(1)$ for all $l > 0$ and $m \leq n$ (including negative powers of m; i.e., placing x in the numerator). Choose n large enough to exceed the power of any $\frac{1}{x}$ in the 1-witness of the x-system. Since pure powers of $\frac{1}{x}$ are not admissible in the x-system, every monomial of the 1-witness in x is $o_{\sigma_{k_n}}(1)$, which is impossible. This contradiction completes the argument.

CHAPTER 6

PROGNOSIS FOR THE δ-COMPLETENESS PROBLEM AND APPLICATIONS

We begin this concluding chapter with a list of the known δ-complete δ-varieties in $\mathbb{P}^1(\mathcal{F})$, where $\mathcal{F} \models DCF_0$:

1. General classes:

   - All projective closures of varieties defined by linear differential polynomials in one variable.
   - The projective closure of $x^{(n)} = P(x^{(n-1)})$ (P a non-differential polynomial in $x^{(n-1)}$).
   - The projective closure of $P(x^{(n)}) = 0$ for any polynomial $P(x^{(n)}) \in \mathcal{F}[x^{(n)}]$ (e.g., $(x^{(n)})^2 - 3x^{(n)} + 2 = 0$).

2. First-order:

   - The projective closure of $Q(x)x' = P(x)$ (Q and P non-differential polynomials in x).
   - The projective closure of $(x')^n = x + \alpha$ (α is an arbitrary element of $\mathcal{F}$).

3. Second-order:

   - The projective closure of $xx'' = x'$.

All of these are either new or are strict generalizations of earlier examples from the literature. Consult Theorem 4.2.4 and Appendix B for proofs of these results, which all depend on the projective valuative criterion for δ-varieties and consist of explicit elimination algorithms augmented by algebraic properties of maximal δ-rings.

This list illustrates that many finite-rank projective δ-varieties are δ-complete. However, the existence of counterexamples such as $x'' = x^3$ shows that classifying the complete δ-varieties is not likely to be easy. Explicit elimination algorithms like those in Appendix B remain the gold standard for determining δ-completeness, but so far they must be custom built; this makes it difficult to tell from a variety's equation whether or not an elimination will go through. On the other hand, the known counterexamples are quite special. The reason that elimination fails in the case of $x'' = x^3$ for a 1-witness $1 = yx^4 + z(x')^3$ is that x^4 and $(x')^2$ induce a cycle; differentiation and substitution to eliminate one regenerates the other, so induction stalls at $(x')^2$ if x^4 is also present. Another way of expressing this coupling is to note that the derivative of $2x^4 - 4(x')^2$ belongs to the δ-ideal $[x'' - x^3]$, though $2x^4 - 4(x')^2$ itself does not. Further investigation is needed to understand the prevalence of this phenomenon as well as the ultimate potential of elimination algorithms for deciding completeness. In the previous chapter we sketched an asymptotic strategy stemming from a valuative criterion for differential varieties over DCF_0 transferred to algebraic varieties over $\mathbb{C}$. As noted, it focuses on individual monomials with generic coefficients and hence can be undermined by the aforementioned relations. However, even if we can predict when those problems arise, substantial work remains on all three components of the approach.

Regarding the preservation of asymptotes under sufficiently small generic perturbations, a central difficulty not mentioned earlier is the singular nature of asymptotes. It does not suffice for a perturbation of the coefficients to keep the system solvable in a neighborhood of a given point; we need to know that the change works arbitrarily close to the asymptote.

To date, the author has expended less effort on the second conjectural ingredient, independent asymptotic monomial growth, so he has less intuition regarding its difficulty or ultimate correctness.

Lastly, even given the above hypotheses it remains to be seen whether the asymptotic approach admits general arguments that dispatch large numbers of cases at once. This likely depends on theorems giving reductions in the complexity of equations and 1-witnesses that must be analyzed in order to draw conclusions about a class of δ-varieties. More work also needs to be done on understanding admissible monomials and what exactly are the necessary features for making arguments like those in the $x' = x^2$ and $xx'' = x'$ examples succeed.

We now step back from the asymptotic strategy and consider a number of more general issues pertaining to the δ-completeness problem. Having mentioned the importance of complexity-controlling reductions to future progress, we should cite J. Freitag's argument proving that it suffices to consider second factors that have finite rank (10). That is, if V is a δ-variety, then V is not δ-complete if and only if there is a finite-rank δ-variety W and δ-closed subset Z of $V \times W$ whose image in W under projection is not δ-closed. We do not know if this is useful from the perspective of the asymptotic approach, as that method apparently does not use relations on the second-factor variables $\overline{y}$ and $\overline{z}$ other than those given by the 1-witness system.

The completeness question for partial differential fields is even more complicated than the single-derivation case considered in this thesis. Freitag (9) has translated Pong's paper (35) to the setting of $DCF_{0,m}$ and found that most results transfer over with little change. However, in the partial case the arguments only show that Δ-complete Δ-varieties must have U-rank strictly less than ω^m. The examples in (9) of Δ-complete varieties (e.g., the several-derivation analogues of the projective closures of the field of constants and $x' = P(x)$, where P is a non-differential polynomial) all have finite U-rank, so Freitag poses the question of whether there are any complete examples in the gap from ω to ω^m.

We expect that our methods likewise require little adaptation to work in the partial case, though we do not foresee much progress there barring at least as much progress on the question in DCF_0.

We should also mention a category that has a particularly close relationship to finite-rank δ-varieties, namely that of *algebraic D-varieties*. Algebraic D-varieties are algebraic varieties possessing a regular section of the projection map from the prolongation (the tangent bundle, but "twisted" by the derivation); Kowalski and Pillay (19) studied quantifier-elimination for D-varieties, and Pillay made an initial examination of complete D-varieties (32), even using model-theoretic language to prove a weak form of completeness for modular algebraic D-groups.

Lastly, one might wonder about the general applicability of the differential completeness notion. As mentioned in the third chapter, δ-completeness does not enjoy all of the properties of algebraic completeness. One example is the use of completeness to prove commutativity of abelian varieties as algebraic groups (41). The argument relies on the fact that that irreducible projective algebraic varieties only have constant morphisms into affine space; the existence of affine projectively-closed δ-varieties shows this is false in the differential setting. Nonetheless, there are mathematically interesting results having δ-completeness as a central ingredient. For instance, Freitag has shown the following (10):

Theorem 6.0.1. *(Freitag) Let $V \subseteq \mathbb{P}^n(\mathcal{U})$ be a Δ-complete Δ-variety defined over a differential field $k \subseteq \mathcal{U}$, where $\mathcal{U}$ is a monster model of $DCF_{0,m}$. Let*

$$ld(V) := \{\overline{x} \in \mathbb{P}^n(\mathcal{U}) \,|\, \overline{x} \textit{ is linearly dependent over } V\}.$$

Then $ld(V)$ is an irreducible projective Δ-variety defined over k.

(Linear dependence of $\overline{x} \in \mathbb{P}^n$ over V means that there is a point $\overline{v} \in V$ such that $\sum_{i=0}^{n} v_i x_i = 0$.) This extends a theorem of Kolchin's from (15). Under certain circumstances the Δ-variety of linearly dependent points over V is defined by Δ-polynomials that generalize the usual Wronskian determinant (10).

We conclude by mentioning a natural object from differential algebraic geometry whose completeness status is unknown but would be valuable to determine. *Manin kernels* are Kolchin closures of the torsion points of abelian varieties defined over differential fields (alternatively, they are the kernels of particular differential-algebraic homomorphisms from abelian varieties to linear algebraic groups) (20). Manin kernels are finite-rank δ-varieties that are also differential-algebraic groups; they appear in the study of the Mordell-Lang conjecture from diophantine geometry (including Hrushovski's model-theoretic proof) (5). In his thesis Pong found explicit

δ-polynomial definitions for the Manin kernels of a family of elliptic curves (34), so one could examine them for δ-completeness using the methods of the current paper.

APPENDICES

Appendix A

APPENDIX A: THE AFFINE δ-VARIETY $X'' = X^2$ IS PROJECTIVELY CLOSED

Here we show that the projective closure of $x'' = x^2$ in $\mathbb{P}^1$ is simply the set $V = \{(x:y) \in \mathbb{P}^1 \mid (x:y) \in (x'' - x^2)^{\delta h}$ and $y \neq 0\}$; that is, the affine δ-variety $x'' = x^2$ contained in the first chart is already projectively closed. This is counterintuitive because the point at infinity $(1:0)$ satisfies the δ-homogeneous equation $(x'' - x^2)^{\delta h} = 0$, where $(x'' - x^2)^{\delta h}$ is

$$y^2 x'' - 2yy'x' - yy''x + 2(y')^2 x - yx^2.$$

It is easy to show (e.g., Lemma 1, p.49, (41)) that V is closed in $\mathbb{P}^1$ if and only if its affine restrictions are closed in the subspace Kolchin topology on both affine charts. This is obviously the case for the first chart; set $y = 1$, and we get the original equation $x'' = x^2$. It is not obvious for the second chart because we have insisted on excluding the point $(1:0)$. Indeed, letting $x = 1$ in the equation $y^2 x'' - 2yy'x' - yy''x + 2(y')^2 x - yx^2 = 0$ we obtain $-yy'' + 2(y')^2 - y = 0$; it is not clear that we may remove the solution $y = 0$ and still have a Kolchin-closed affine set. Nonetheless, we will find δ-polynomials whose common zero locus is precisely the set $-yy'' + 2(y')^2 - y = 0 \wedge y \neq 0$.

Consider the δ-elimination ideal I_t of the differential ideal $I = [-yy'' + 2(y')^2 - y, yt - 1]$. I_t defines the Kolchin closure of $-yy'' + 2(y')^2 - y = 0 \wedge y \neq 0$, but we will see that the closure adds no new points. We use Maple's `EliminationIdeal` command to find generators of I_t having at most order 3. For the purpose of giving input to Maple, we use the variables y, y_1, y_2, y_3, and t. Consider the (non-differential) polynomial ideal `I1`:

$$(-yy_2 + 2(y_1)^2 - y, 3y_1y_2 - yy_3 - y_1, yt - 1).$$

The elimination ideal we obtain from EliminationIdeal(I1, {`y_3,y_2,y_1,y`}) is

$$(-3y_1y_2 + yy_3 + y_1, y_2y - 2y_1^2 + y, 3y_2^2 - 1 - 2y_3y_1 + 2y_2).$$

The first generator is the derivative of the second and so is redundant as a differential polynomial. The third has a non-zero constant term -1, so $y = 0$ is not in the Kolchin closure of the image of the projection. Since the polynomial $-y''y + 2(y')^2 - y$ belongs to the δ-elimination ideal, we conclude that the image equals the closure. Thus the affine restriction $-yy'' + 2(y')^2 - y = 0 \wedge y \neq 0$ is equivalent to the closed condition $-y''y + 2(y')^2 - y = 0 \wedge 3(y'')^2 - 1 - 2y'''y' + 2y'' = 0$.

If desired, one may confirm the calculation by hand. Let $p_1 = 3(y'')^2 - 1 - 2y'''y' + 2y''$, $p_2 = -y''y + 2(y')^2 - y$, and $p_3 = 3y'y'' - yy''' - y'$. Observe that $yp_1 = (-3y'' + 1)p_2 + 2y'p_3$. This proves that the zero locus of the proposed elimination ideal indeed contains the set $-yy'' + 2(y')^2 - y = 0 \wedge y \neq 0$; the reverse containment follows from the term -1 in p_1.

Appendix B

APPENDIX B: NEW EXAMPLES OF COMPLETE δ-VARIETIES

Here we prove δ-completeness for several finite-rank projective δ-varieties not covered by the work of Kolchin or Pong. In particular, we consider the projective closures of $xx'' = x'$, $(x')^n = x + \alpha$, $x^{(n)} = P(x^{(n-1)})$, $P(x^n) = 0$, and $Q(x)x' = P(x)$.

The reader may wonder why the following exposition uses the original projective valuative criterion rather than the apparently cleaner syntactic version. The reason is that the algebraic results around maximal δ-rings significantly reduce the amount of actual elimination that one must perform. We mainly use two facts due to Morrison (cited in Chapter 3): 1. If $x^{(k)} \in R$ (where $x \neq 0$, $k > 0$, and R is a K-maximal δ-ring), then either $x \in R$ or $\frac{1}{x} \in R$. 2. K-maximal δ-rings are integrally closed in K.

To begin, we give a concrete illustration of a "1-preserving δ-operation", that is, a sequence of differential and algebraic transformations that take an equation of the form $1 = f_1$ and produce an equation (implied by the first) of the form $1 = f_2$ for some δ-polynomials f_1, f_2.

Example B.0.6. Let $(R, \mathfrak{m})$ be a maximal δ-ring. Consider the equation $1 = c_1 x + c_2 x'$ with $c_i \in \mathfrak{m}$. We will replace x'' with x^3 in subsequent derivatives (we would make this choice to study the δ-variety $x'' = x^3$). The derivative of $c_2 x'$ modulo this substitution is $c_2' x' + c_2 x^3$. Pick $c_2 x'$ as a target for elimination using $c_2 x^3$. Differentiating and substituting, we obtain $0 = c_1' x + (c_1 + c_2')x' + c_2 x^3$. Multiplying, we get $0 = (-x'/x^3)(c_1' x + (c_1 + c_2')x' + c_2 x^3) =$

$-c_1'x'/x^2 - (c_1 + c_2')(x')^2/x^3 - c_2x'$. Add to the first equation to obtain $1 = c_1x - c_1'x'/x^2 - (c_1 + c_2')(x')^2/x^3$, from which the targeted term has been eliminated while preserving 1.

Next we describe a 1-preserving operation that does not involve differentiation but is important for simplifying the results of those that do. We state it more generally for rings (commutative, with 1).

Proposition B.0.4. *Let A be a ring, x a unit in A, and B a subring (not necessarily containing 1) of A. If $1 \in B[x]$ and $1 \in B[1/x]$, then $1 \in B$.*

Proof. Use induction. Suppose $1 = \sum_{i=0}^{m} b_i x^i$ in $B[x]$ and $1 = \sum_{j=0}^{n} c_j (1/x)^j$ in $B[1/x]$. We may assume $m \geq n$ (by symmetry, since $1/x$ is a unit in A if and only if x is).

1. Subtract c_0 from both sides of the second equation to get $1 - c_0 = \sum_{j=1}^{n} c_j (1/x)^j$. Then multiply both sides by x^n to obtain $(1 - c_0)x^n = \sum_{j=1}^{n} c_j x^{n-j}$.

2. Multiply the first equation by $1 - c_0$ and split up the right-hand side based on whether the exponent is less than n: $1 - c_0 = (1 - c_0)(\sum_{i=n}^{m} b_i x^i) + (1 - c_0)(\sum_{i=0}^{n-1} b_i x^i)$.

3. By step 1, we may replace $(1 - c_0)x^n$ (which divides each term of the first sum in step 2 after distributing $1 - c_0$) with a polynomial in $B[x]$ of degree strictly less than n.

4. Add c_0 to both sides of the resulting expression. Note that the polynomial is still in $B[x]$ because $(1 - c_0)b_i \in B$. The new expression witnesses $1 \in B[x]$ with a polynomial of degree less than m. By induction we can continue reducing the degrees of x and $1/x$ until both are eliminated, leaving $1 \in B$.

Appendix B (Continued)

□

Definition B.0.1. *We call the procedure used to eliminate x and $1/x$ in the preceding proposition the* reciprocal elimination algorithm.

Next we prove a useful variant of the reciprocal elimination algorithm that we use to prove that $\overline{xx'' = x'}$ is δ-complete.

Proposition B.0.5. *Let A be a ring, x, y units in A, and B a subring (not necessarily containing 1) of A. Suppose 1 satisfies the following equations in which the polynomials on the right have coefficients from B:*

1. $1 = \sum b_{vw} y^v / x^w + \sum b_z / x^z$ *where $0 < v < w$ and z is non-negative.*
2. $1 = \sum c_{rs} x^r / y^s + \sum c_q / x^q$ *where $r < s$, $s > 0$ and q is non-negative.*

Then $1 \in B[\frac{1}{x}]$.

Proof. We refer to expressions having the form of the first equation as *type I equations*; monomials satisfying the corresponding conditions are *type I terms.* The label *type II* analogously applies to the second equation and its terms. We are only concerned with preservation of these conditions, so we do not need to track each coefficient. In keeping with this, we use a placeholder symbol c for the coefficients; this does not mean that they are identical, but simply that some coefficient from B is present. Similarly, once we observe that a condition is preserved, we may reuse notation for terms of the same form. For example, rather than expanding $(1 + \sum c/x^q)\,(1 + \sum c/x^q)$, we simply write $1 + \sum c/x^q$ again to represent the product, which is a sum of 1 and terms having coefficients from B and a non-negative power of x in the denominator.

Let u be the maximal exponent of y (in the numerator) in the type I equation, and t the maximal exponent of y (in the denominator) in the type II equation. If there is no y in an equation, then respectively u or t is 0. There are two cases:

1. $u \leq t$: Subtract the y-free terms in the type I equation from both sides and divide by y^u to obtain $(1 + \sum c/x^z)/y^u = \sum cy^{v-u}/x^w = \sum c/x^w y^{u-v}$. Multiply the type II equation by $(1 + \sum c/x^z)$. Note that multiplying a type II term by $1 + \sum c/x^z$ preserves the type II requirements on the exponents of x and y. Hence we may reuse the notation for the terms not divided by $1/y^u$. Substituting for $(1 + \sum c/x^z)/y^u$ we obtain

$$\begin{aligned} 1 &= \sum c/x^q + \sum_{0<s<u} cx^r/y^s + \left(\sum_{u \leq s} cx^r/y^{s-u}\right)\left(\sum c/x^w y^{u-v}\right) \\ &= \sum c/x^q + \sum_{0<s<u} cx^r/y^s + \sum_{u \leq s} \sum cx^{r-w}/y^{s-v} \end{aligned}$$

   It remains to verify the requirements on the exponents of terms for which $u \leq s$. For those, $v < w$ and $r < s$, so $r - w < s - v$. If $s = v$, the term becomes y-free with x in the denominator. Otherwise, because $0 < v$ the exponent of y decreases but remains positive. Therefore we have a new type II equation but with a smaller maximal exponent of y (the exponents of y in $\sum_{0<s<u} cx^r/y^s$ are less than u and hence than t). Thus after finitely many iterations there will no longer be an s such that $u \leq s$ and we are guaranteed to move to the second case.

2. $t < u$: Subtract the y-free terms in the type II equation from both sides and multiply by y^t to obtain $(1 + \sum c/x^q)y^t = \sum cx^r y^{t-s}$. Multiply the type I equation by $(1 + \sum c/x^q)$. As before, this multiplication does not disturb the required balance of exponents and so we only alter the notation for terms divisible by y^t. We find

$$\begin{aligned} 1 &= \sum c/x^z + \sum_{0<v<t} cy^v/x^w + \left(\sum_{t \le v} cy^{v-t}/x^w\right)\left(\sum cx^r y^{t-s}\right) \\ &= \sum c/x^z + \sum_{0<v<t} cy^v/x^w + \sum_{t \le v} \sum cy^{v-s}/x^{w-r} \end{aligned}$$

For the terms such that $t \le v$, the exponent of y is still non-negative because $s \le t \le v$. Again, $v < w$ and $r < s$, so $v - s < w - r$. Since $s > 0$, we have a new type I equation with a lower maximal exponent of y (the exponents of y in $\sum_{0<v<t} y^v/x^w$ are less than t and hence than u). Consequently, after finitely many iterations, there will no longer be a v such that $t < v$ and we return to the first case.

At the end of each step, we still have equations of type I and II. Because either t or u decreases with each iteration of the algorithm, either u or t eventually becomes 0. (The types do not allow u or t to be negative.) If u or t is 0, then in the remaining terms x only appears in the denominator. This proves $1 \in B[1/x]$.

□

We proceed to the proofs of δ-completeness.

Proposition B.0.6. $\overline{xx'' = x'}$ *is δ-complete.*

Appendix B (Continued)

Proof. We use the projective valuative criterion for $\mathbb{P}^1$. Let $(R, \mathfrak{m})$ be a K-maximal δ-ring and let $p = (x : 1) \in V(K)$, where $V = \overline{xx'' = x'}$. We must show that either $x \in R$ or $\frac{1}{x} \in R$. This is immediate if $x = 0$; otherwise, if $x' = 0$ then Morrison's result implies that either $x \in R$ or $\frac{1}{x} \in R$. So suppose that x, x' are non-zero and also that neither x nor $\frac{1}{x}$ belongs to R. As a result, we know that both $1 \in \mathfrak{m}\{x\}$ and $1 \in \mathfrak{m}\{1/x\}$.

We focus on $1 \in \mathfrak{m}\{1/x\}$. Because x and x' are non-zero, we may divide by both of them, as well as substitute $\frac{x'}{x}$ for x''. Differentiating a monomial $(x')^j/x^i$ such that $i > j \geq 0$ results in a sum of monomials such that each is a multiple of $(x')^j/x^i$ times either x'/x (when differentiating $1/x$) or $1/x$ (when differentiating x' and applying the substitution). This preserves the excess of x in the denominator. As $\mathfrak{m}$ is a local ring, we may always subtract terms that are elements of $\mathfrak{m}$ from 1 and divide (we continue to do so without comment for any free elements of $\mathfrak{m}$ that appear during the upcoming elimination). Hence we may assume that there are coefficients $c_{ij} \in \mathfrak{m}$ and pairs of exponents $i > j \geq 0$ such that $1 = \sum c_{ij}(x')^j/x^i$. Note that this satisfies the definition of a type I equation (with x' playing the role of y) in the preceding proposition. Our strategy is to show that modulo differentiation and substitution of $\frac{x'}{x}$ for x'', this equation implies $1 \in \mathfrak{m}[\frac{1}{x}]$ (and hence x is integral over R). That will provide a contradiction because we know from Morrison that R is integrally closed, so $x \in R$.

We eliminate x' in favor of $\frac{1}{x'}$ and show that the resulting relation $1 \in \mathfrak{m}[x, \frac{1}{x}, \frac{1}{x'}]$ is witnessed by a type II equation. Then the reciprocal elimination variant applied to the original type I equation and the resulting type II equation will imply that $1 \in \mathfrak{m}[\frac{1}{x}]$, as desired.

Appendix B (Continued)

Differentiate $1 = \sum c_{ij}(x')^j/x^i$ and invoke the substitution to get

$$0 = \sum c'_{ij}(x')^j/x^i + \sum(-i)c_{ij}(x')^{j+1}/x^{i+1} + \sum(j)c_{ij}(x')^j/x^{i+1}.$$

We target the term $c_{IJ}(x')^J/x^I$ of the original equation that is maximal with respect to the monomial ordering $lex(\deg(x'), \deg(x))$, where the degrees of x', x are taken with respect to x' in the numerator and x in the denominator. That is, first eliminate the term with the highest degree I of x in the denominator out of those terms having the highest overall degree $J > 0$ of x' in the numerator. (If x' does not appear, $1 \in \mathfrak{m}[\frac{1}{x}]$ already.). Do this by adding the following equation to the original:

$$\begin{aligned} 0 &= \left(\frac{x}{Ix'}\right)\left(\sum c'_{ij}(x')^j/x^i + \sum(-i)c_{ij}(x')^{j+1}/x^{i+1} + \sum(j)c_{ij}(x')^j/x^{i+1}\right) \\ &= \sum \frac{c'_{ij}}{I}(x')^{j-1}/x^{i-1} + \sum \frac{-i}{I}c_{ij}(x')^j/x^i + \sum \frac{j}{I}c_{ij}(x')^{j-1}/x^i. \end{aligned}$$

We carefully examine the consequences of this elimination step.

- Claim: 1 is preserved. Confirmation: We added 0 to the original equation, and 1 does not appear among the added terms.
- Claim: $c_{IJ}(x')^J/x^I$ was eliminated. Confirmation: Since $I > J > 0$, dividing by I was valid and the summands included the canceling term $-c_{IJ}(x')^J/x^I$.
- Claim: Every remaining term is strictly less than $c_{IJ}(x')^J/x^I$ with respect to $lex(\deg(x'), \deg(x))$. Confirmation: The remaining terms from the original equation were already less than

$c_{IJ}(x')^J/x^I$ in this ordering, and they didn't change. The new terms that were added all saw the exponent of x' decrease, except for $\frac{-i}{I}c_{ij}(x')^j/x^i$. But these stayed the same as before, and so for $i \neq I$ are still less than $c_{IJ}(x')^J/x^I$. In particular, among the new terms there is only one copy of the monomial $(x')^J/x^I$, so after cancellation of that term only lesser ones remain. Note that relations leading to cancellations among the other terms cause no problem; they can only hasten the elimination of non-type-II terms.

- Claim: All exponents of x have stayed the same except those in the terms $\frac{c'_{ij}}{I}(x')^{j-1}/x^{i-1}$. Confirmation: This is immediate from the expression. The important thing to note is that the exponent of x' in the numerator has decreased by the same amount. Since $0 \leq j < i$, repetition of this operation will place x' in the denominator before it places x in the numerator. In other words, the descendants of $c_{ij}(x')^j/x^i$ that have x in the numerator must have a larger power of x' in the denominator; i.e., they will be type II terms.

Repeat until all terms with x' in the numerator are eliminated. This process terminates because we started with a finite $J > 0$, we only add finitely many terms at each stage (and they are lesser in the ordering than the eliminated term), and for each monomial $(x')^j/x^i$ with $0 < j < i$, there are only finitely many monomials $(x')^j/x^k$ such that $k < i$. The terms that are left are all type II terms; there are none with x' in the numerator, those with x in the numerator were already discussed, and terms of the form $c/(x^i(x')^j)$ for i, j non-negative and $c \in \mathfrak{m}$ are automatically type II. Since 1 is preserved at each stage, we have a type II equation. This completes the proof.

$\square$

Appendix B (Continued)

Proposition B.0.7. $\overline{(x')^n = x + \alpha}$ *is* δ*-complete for all natural numbers* n *and* $\alpha \in \mathcal{F} \models DCF_0$.

Proof. We use the projective valuative criterion as in the previous proof. If $n = 0$ or 1, then $\overline{(x')^n = x + \alpha}$ is linear and we have already verified δ-completeness. As before, if $x' = 0$, then $x \in R$ or $\frac{1}{x} \in R$ and we are done. So suppose $n \geq 2$, $x' \neq 0$ and that the variety is not δ-complete so that we have $1 \in \mathfrak{m}\{x\}$ and $1 \in \mathfrak{m}\{1/x\}$.

Differentiating, we get $n(x')^{n-1}x'' = x' + \alpha'$. Divide to obtain $x'' = 1/(n(x')^{n-2}) + \alpha'/(n(x')^{n-1})$. We may also divide the original equation to see that $x' = x/(x')^{n-1} + \alpha/(x')^{n-1})$. It follows that $\mathfrak{m}\{x\} \subseteq \mathfrak{m}[x, \frac{1}{x'}]$, so for some elements $c_{ij} \in \mathfrak{m}$ there is an equation of the form $1 = \sum c_{ij}x^i/(x')^j$ such that both $i, j \geq 0$. We use the same conventions as before of dropping the subscripts, reusing notation when the form is the same, and dividing by $1 - c$ for any free $c \in \mathfrak{m}$ that arises in the elimination.

The derivative is

$$
\begin{aligned}
0 = &\sum c'(x)^i/(x')^j + \sum (i)cx^{i-1}/(x')^{j-1} + \sum((-j)cx^i/(x')^{j+1})(1/(n(x')^{n-2}) \\
&+ \alpha'/(n(x')^{n-1}))
\end{aligned}
$$

$$
\begin{aligned}
= &\sum c'(x)^i/(x')^j + \sum (i)cx^{i-1}/(x')^{j-1} + \sum(-j)cx^i/(n(x')^{j+n-1}) \\
&+ \sum(-j)c\alpha' x^i/(n(x')^{j+n})).
\end{aligned}
$$

Again we order lexicographically, but we consider the degrees of x' in the denominator and x in the numerator (whereas for $xx'' = x'$ we considered the exponent of x' to be positive if

Appendix B (Continued)

there was a positive integer power of x' in the numerator). This is because we want to eliminate $\frac{1}{x'}$ and obtain $1 \in \mathfrak{m}[x, x']$. Once we have done that, we may use reciprocal elimination with the equations witnessing $1 \in \mathfrak{m}[x, \frac{1}{x'}]$ and $1 \in \mathfrak{m}[x, x']$ to conclude $1 \in \mathfrak{m}[x]$. That will imply that $\frac{1}{x}$ is integral over R, and hence $\frac{1}{x} \in R$; by this contradiction to hypothesis we will conclude that $\overline{(x')^n = x + \alpha}$ is δ-complete.

So let J (greater than 0, lest we already be done) be the maximal degree of x' in any denominator, and let I be the largest power of x in a numerator for those terms featuring $1/(x')^J$. There are two possibilities: either $\alpha' = 0$ or $\alpha' \neq 0$. We give the details for the case $\alpha' \neq 0$; the other is virtually identical (see below), except that instead of multiplying by $n(x')^n/(J\alpha')$ before adding to the original, we would multiply by $-n(x')^{n-1}/J$.

$$\begin{aligned} 0 = \left(\frac{n(x')^n}{J\alpha'}\right) \Big(& \sum c' x^i/(x')^j + \sum (i) c x^{i-1}/(x')^{j-1} + \\ & + \sum (-j) c x^i/(n(x')^{j+n-1}) + \sum (-j) c \alpha' x^i/(n(x')^{j+n}) \Big) \end{aligned}$$

$$\begin{aligned} &= \sum n c' x^i / J\alpha'(x')^{j-n} + \sum nic x^{i-1}/J\alpha'(x')^{j-n-1} + \\ &+ \sum -jcx^i/(J\alpha'(x')^{j-1}) + \sum -jcx^i/(J(x')^j). \end{aligned}$$

- Claim: 1 is preserved. Confirmation: Same as before.
- Claim: $c_{IJ}x^I/(x')^J$ was eliminated. Confirmation: Same as before, because $J > 0$.
- Claim: Every remaining term is strictly less than $c_{IJ}x^I/(x')^J$ with respect to $lex(\deg(x'), \deg(x))$. Confirmation: Same as before, because $n > 0$ (if $\alpha' = 0$, then we need the fact that $n > 1$).

- Claim: For every term, x appears in the numerator (if at all). Confirmation: The exponent of x only changes in $nicx^{i-1}/J\alpha'(x')^{j-n-1}$. If $i = 0$, the term is 0, so we only get a decrease if there is a positive power of x in the numerator to begin with. Even then, the decrease is only by one, so the new exponent is again 0 or positive.

  By induction, we may repeat this procedure finitely many times and obtain an equation witnessing $1 \in \mathfrak{m}[x, x']$. As explained, this completes the proof.

□

Proposition B.0.8. $\overline{x^{(n)} = P(x^{(n-1)})}$ *is δ-complete for all $n > 0$ and polynomials $P \in \mathcal{F}[x^{(n-1)}]$, where $\mathcal{F} \models DCF_0$.*

Proof. Having given two similar proofs, we abbreviate here and only cite the distinct features of this case. Write $P(x^{(n-1)})$ as $\sum_{j=0}^{J} b_j (x^{(n-1)})^j$ where $b_j \in \mathcal{F}$ (and $b_J \neq 0$). It suffices to consider the non-linear case $J \geq 2$.

At this point there is a small wrinkle. To control the complexity of the elimination, we do not want to deal with all the variables $x, x', \ldots, x^{(n-1)}$. Since $x^{(n-1)} \in R$ implies $x \in R$ or $\frac{1}{x} \in R$ for any K-maximal δ-ring $(R, \mathfrak{m})$, we may assume that $x^{(n-1)} \notin R$, so that $1 \in \mathfrak{m}\{x^{(n-1)}\}$. Observe that $\mathfrak{m}\{x^{(n-1)}\} \subseteq \mathfrak{m}[x^{(n-1)}]$, so for some $c_i \in \mathfrak{m}$ we have

$$1 = \sum_{i=1}^{I} c_i (x^{(n-1)})^i.$$

We show that repeated differentiation and substitution produce an equation witnessing $1 \in \mathfrak{m}[\frac{1}{x^{(n-1)}}]$. Then reciprocal elimination will imply that $1 \in \mathfrak{m}$, which is absurd.

Appendix B (Continued)

The derivative is

$$0 = \sum_{i=1}^{I} c_i'(x^{(n-1)})^i + \sum_{i=1}^{I} \left(ic_i(x^{(n-1)})^{i-1} \left(\sum_{j=0}^{J} b_j(x^{(n-1)})^j \right) \right).$$

Multiply by $-1/(Ib_J(x^{(n-1)})^{J-1})$ to obtain

$$0 = \sum_{i=1}^{I} \frac{-c_i'}{Ib_J}(x^{(n-1)})^{i-J+1} + \sum_{i=1}^{I} \left(\frac{-ic_i}{Ib_J}(x^{(n-1)})^{i-J} \left(\sum_{j=0}^{J} b_j(x^{(n-1)})^j \right) \right).$$

This expression has maximal degree I in $x^{(n-1)}$ (in the numerator), and there is only one term with that degree (because $J \geq 2$). Adding to the original, we cancel the leading term $c_I(x^{(n-1)})^I$ and leave terms in $x^{(n-1)}$ with lesser exponents (including possibly pushing $x^{(n-1)}$ into the denominator). After finitely many steps, we eliminate $x^{(n-1)}$ from all numerators and produce an equation witnessing $1 \in \mathfrak{m}[\frac{1}{x^{(n-1)}}]$. This completes the proof.

□

We take a brief break from involved eliminations and make the following easy observation.

Proposition B.0.9. *Let $P(x^{(n)}) \in \mathcal{F}[x^{(n)}]$ be a non-differential polynomial in $x^{(n)}$, where n is a natural number and $\mathcal{F} \models DCF_0$. Then $\overline{P(x^{(n)}) = 0}$ is δ-complete.*

Proof. As long as P is not simply an element of $\mathcal{F}$ (making the claim trivially true), there are only finitely many $\alpha \in \mathcal{F}$ that satisfy the equation $P(\alpha) = 0$. But the projective closure of $x^{(n)} = \alpha$ is δ-complete for each. The entire variety is comprised of the union of the $\overline{x^{(n)} = \alpha}$ and is therefore δ-complete. □

Appendix B (Continued)

Our final result generalizes Pong's first-order examples.

Proposition B.0.10. *Let $P(x), Q(x) \in \mathcal{F}[x]$ for $\mathcal{F} \models DCF_0$. Then $\overline{Q(x)x' = P(x)}$ is δ-complete.*

Proof. We continue to use the conventions established earlier in this appendix. Because $\mathcal{F}$ is algebraically closed, we may factor Q as $\Pi_{i=1}^{m}(x+\alpha_i)^{l_i}$ for distinct $\alpha_i \in \mathcal{F}$ and positive integers l_i. We may also assume that P and Q have no common factors (as polynomials, with x viewed as an indeterminate). If they have $x+\alpha$ in common and $x = -\alpha$ (with x now viewed as a field element, not as an indeterminate), then $x \in R$ automatically; otherwise, simply divide out the factor $x+\alpha$. Likewise we may assume that x does not cause Q to vanish.

Suppose that $\overline{Q(x)x' = P(x)}$ is not δ-complete and so $1 \in \mathfrak{m}\{x\}$ and $1 \in \mathfrak{m}\{1/x\}$. We use the first relation. Note that by repeated differentiation and substitution with $x' = P/Q$, we may write $(P/Q)^{(k)}$ in the form (polynomial in $\mathcal{F}[x]$)$/Q^l$ for some positive integer l. The powers $(x+\alpha_i)^{l_i}, (x+\alpha_i)^{l_j}$ are relatively prime for $i \neq j$, and $x+\alpha_i$ is linear, so by taking a partial fraction decomposition and simplifying we have an equation of the form

$$1 = \sum cx^r + \sum_{i=1}^{m} \sum c/(x+\alpha_i)^{s_i}$$

in $\mathfrak{m}[x, 1/(x+\alpha_1), \ldots, 1/(x+\alpha_m)]$ where $r, s_i > 0$. (Simplification here refers to reducing the numerator and denominator as much as possible, e.g.: $(x+2)/(x+\alpha)^2 = (x+\alpha-\alpha+2)/(x+\alpha)^2 = 1/(x+\alpha) + (2-\alpha)/(x+\alpha)^2$.)

Appendix B (Continued)

We explain our strategy before differentiation unleashes a barrage of notation that might obscure the aim. We wish to convert the original equation into one that witnesses $1 \in \mathfrak{m}[x]$; then $\frac{1}{x}$ is integral over R and we contradict the assumption that $\frac{1}{x} \notin R$. To do this, we must remove the $x + \alpha_i$ from all denominators. Proceed sequentially, starting with the powers $(x + \alpha_1)^{s_i}$. Let $S_1 > 0$ be the largest power of $x + \alpha_1$ in any denominator.

The derivative is

$$\begin{aligned} 0 = &\sum c' x^r + \sum_{i=1}^{m} \sum c'/(x+\alpha_i)^{s_i} + \sum rcx^{r-1}(P/Q) \\ &+ \sum_{i=1}^{m} \sum \left((-s_i)c/(x+\alpha_i)^{s_i+1}\right)\left((P/Q) + \alpha_i'\right). \end{aligned}$$

Since $x + \alpha_1$ is not a factor of P, we may divide and write $P = (x + \alpha_1)\widetilde{P} + \beta_1$ for some $\widetilde{P} \in \mathcal{F}[x]$ and $\beta_1 \in \mathcal{F} \setminus \{0\}$. This gives us

$$\begin{aligned} 0 = &\sum c' x^r + \sum_{i=1}^{m} \sum c'/(x+\alpha_i)^{s_i} + \sum rcx^{r-1}(P/Q) \\ &+ \sum_{i=1}^{m} \sum \left((-s_i)c/(x+\alpha_i)^{s_i+1}\right)\left((((x+\alpha_1)\widetilde{P} + \beta_1)/Q) + \alpha_i'\right) \end{aligned}$$

$$\begin{aligned} = &\sum c' x^r + \sum_{i=1}^{m} \sum c'/(x+\alpha_i)^{s_i} + \sum rcx^{r-1}(P/Q) \\ &+ \sum_{i=1}^{m} \sum \Big((-s_i)(x+\alpha_1)\widetilde{P}c/(Q(x+\alpha_i)^{s_i+1}) + \\ &+ (-s_i)\beta_1 c/(Q(x+\alpha_i)^{s_i+1}) + (-s_i)\alpha_i' c/((x+\alpha_i)^{s_i+1})\Big). \end{aligned}$$

Appendix B (Continued)

Multiply both sides of this equation by $(x+\alpha_1)Q/(S_1\beta_1)$ to obtain

$$= \sum (x+\alpha_1)Qc'x^r/(S_1\beta_1) + \sum_{i=1}^{m}\sum (x+\alpha_1)Qc'/(S_1\beta_1(x+\alpha_i)^{s_i})$$
$$+ \sum (x+\alpha_1)Prcx^{r-1}/(S_1\beta_1) + \sum_{i=1}^{m}\sum \Big((-s_i)(x+\alpha_1)^2\widetilde{P}c/(S_1\beta_1(x+\alpha_i)^{s_i+1})$$
$$+ (-s_i)(x+\alpha_1)c/(S_1(x+\alpha_i)^{s_i+1}) + (-s_i)(x+\alpha_1)Q\alpha_i'c/(S_1\beta_1(x+\alpha_i)^{s_i+1})\Big)$$

Add this to the original equation and evaluate the results.

- Claim: 1 is preserved. Confirmation: Same as before.
- Claim: $c/(x+\alpha_1)^{S_1}$ was eliminated. Confirmation: The canceling term is $(-s_i)(x+\alpha_1)c/(S_1(x+\alpha_i)^{s_i+1})$ when $i=1$ and $s_i=S_1$. Everything is well defined because $\beta_1 \neq 0$ and $S_1 > 0$.
- Claim: In every remaining term either $x+\alpha_1$ does not appear in the denominator, or the exponent of $x+\alpha_1$ in the denominator is strictly between 0 and S_1. Confirmation: By maximality of S_1, only one copy of the canceling term $-c/(x+\alpha_1)^{S_1}$ appears among the terms $(-s_i)(x+\alpha_1)c/(S_1(x+\alpha_i)^{s_i+1})$; the rest have strictly fewer than S_1 factors of $x+\alpha_1$ in the denominator. Among the other terms, either there are no $x+\alpha_i$ in the denominator, or at least two powers of $x+\alpha_1$ have been placed in the numerator (explicitly in the case of the terms $(-s_i)(x+\alpha_1)^2\widetilde{P}c/(S_1\beta_1(x+\alpha_i)^{s_i+1})$ and implicitly in the cases of $(x+\alpha_1)Qc'/(S_1\beta_1(x+\alpha_i)^{s_i})$ and $(-s_i)(x+\alpha_1)Q\alpha_i'c/(S_1\beta_1(x+\alpha_i)^{s_i+1})$ because Q is divisible by at least one power of $x+\alpha_1$). This guarantees that these other

Appendix B (Continued)

terms either had no $x + \alpha_1$ in the denominator to begin with or have seen its exponent reduced.

Simplifying as before, we may put the resulting equation in the same form $1 = \sum cx^r + \sum_{i=1}^{m} \sum c/(x + \alpha_i)^{s_i}$ as the original, but such that the maximal exponent of $x + \alpha_1$ in any denominator is strictly less than S_1.

Repeat until $x + \alpha_1$ is eliminated from all denominators; then do the same thing for $x + \alpha_2, \ldots, x + \alpha_m$. After finitely many steps this algorithm produces an equation witnessing $1 \in \mathfrak{m}[x]$, as needed.

$\square$

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VITA

William D. Simmons

Curriculum Vitae

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Address: Dept. of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Science and Engineering Offices 322 (M/C 249)
851 S. Morgan Street, Chicago, IL 60607-7045
Phone: 1-(312)-996-1804
Email: simmons@math.uic.edu

EDUCATION

Ph.D. University of Illinois at Chicago, Mathematics, August 2013
Advisor: David Marker
Thesis: *Completeness of finite-rank differential varieties*

M.S. Brigham Young University, Mathematics, August 2007
Master's Project: *Super models: Several concepts and applications of model theory*

B.S. Brigham Young University, Mathematics, August 2004

RESEARCH INTERESTS

Model theory (particularly model theory of fields), differential algebra, definability and effectiveness in algebra, mathematical logic

TALKS AND PRESENTATIONS

ASL Model Theory Special Session (invited)	University of Waterloo	May 8-11, 2013
Graduate Student Colloquium	University of Illinois at Chicago	February 25, 2013
Joint Math. Meetings of AMS/MAA (contributed)	San Diego, CA	January 9, 2013
Mathematics Colloquium	Brigham Young University	July 10, 2012
Model Theory Seminar	McMaster University	March 27, 2012
Kolchin Seminar in Differential Algebra	Graduate Center, CUNY	March 9, 2012
Logic Seminar	University of Illinois at Chicago	February 28, 2012
Logic Seminar	University of Illinois at Chicago	May 4, 2010
Louise Hay Logic Seminar	University of Illinois at Chicago	April 22, 2010
Joint student logic seminar	University of Chicago	April 24, 2008
Graduate student logic seminar	University of Illinois at Chicago	September 5, 2007
Spring Research Conference	Brigham Young University	March 17, 2007

PUBLICATIONS

1. W. Simmons, *Completeness of finite-rank differential varieties*, 2013, in preparation.

SPECIAL MEETINGS AND WORKSHOPS ATTENDED

Model Theory of Fields	June 19-25, 2010	AMS Mathematics Research Community
Model Theory	August 9-14 2009	ESF Mathematics Conference
Winter School in o-minimal Geometry	January 12-16, 2009	Fields Institute
Proof Theory Workshop	June 6-17, 2005	University of Notre Dame

TEACHING

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago:

Visiting Lecturer/Instructor	Fall 2012, Spring 2013	Calculus I (Math 180)
Teaching Assistant	Spring 2010, Fall 2011	Calculus for Business (Math 165)
Instructor	Fall 2010	Calculus III (Math 210)
Teaching Assistant	Fall 2009	Intermediate Algebra (Math 090)
Teaching Assistant	Spring 2009	Finite Mathematics for Business (Math 160)
Teaching Assistant	Fall 2008	Introduction to Differential Equations (Math 220)
Teaching Assistant	Spring 2008	Calculus II (Math 181)
Teaching Assistant	Fall 2007	Precalculus Mathematics (Math 121)

Department of Mathematics, Brigham Young University:

Instructor	Summer and Fall 2006	Quantitative Reasoning (Math 102)
Teaching Assistant	Spring 2006	Number Theory (Math 387)
Instructor	Fall 2005, Winter 2006	Calculus I (Math 112)
Instructor	Summer 2005	Trigonometry (Math 111)
Teaching Assistant	Fall 2004, Winter 2005	Introduction to Calculus (Math 119)

OTHER EXPERIENCE

Research Assistant (mathematics)	Spring 2011, Spring 2012	University of Illinois at Chicago
Completed Teaching Mathematics course	Fall 2007	University of Illinois at Chicago

DISTINCTIONS AND SERVICE

Honorable Mention for performance as a teaching assistant, Spring 2009 (the Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, awards this designation to approximately 10% of teaching assistants each semester)

Selected by peers to serve in 2006 on the Graduate Advisory Committee of the Department of Mathematics, Brigham Young University

SKILLS

Technical software:	Maple, LaTeX
Languages:	Spanish (written and spoken; fluent)

MEMBERSHIPS

American Mathematical Society
Association for Symbolic Logic
Mathematical Association of America