Hyperbolic 3-manifolds with k-free fundamental group

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Chicago, 2011

Chicago, Illinois

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For my mother,

Margeurite Louise Guzman.

Thank you for loving me.

ACKNOWLEDGMENTS

I would like to thank my advisor, Peter B. Shalen, for his amazing perseverance with me beyond what could have ever been expected; Assistant Director of Graduate Studies, Kari Dueball, for our many talks; and my committee, Richard D. Canary, Marc Culler, Benson Farb, and Brooke Shipley, for their continued aegis. Especially noteworthy is Marc Culler for bringing relevant work to my attention.

I am thankful for my mother, whose dedication and love cultivated the person I needed to be to make this possible.

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SUMMARY

The results of Marc Culler and Peter Shalen for $2,3$ or 4-free hyperbolic 3-manifolds are contingent on properties specific to and special about rank two subgroups of a free group. Here we determine what construction and algebraic information is required in order to make a geometric statement about M a closed, orientable hyperbolic manifold with k-free fundamental group for any value of k greater than four. Main results are both to show what the formulation of the general statement should be, for which Culler and Shalen's result is a special case, and that it is true modulo a group-theoretic conjecture. A major result is in the $k = 5$ case of the geometric statement. Specifically, I show that the required group-theoretic conjecture is in fact true in this case, and so the proposed geometric statement when M is 5-free is indeed a theorem. One can then use the existence of a point and knowledge about $\pi_1(M, P)$ resulting from this theorem to attempt to improve the known lower bound on the volume of M, which is currently 3.44 (6, Theorem 1.5).

CHAPTER 1

INTRODUCTION

The goal of this paper is to explore how the geometry of a closed, orientable hyperbolic 3-manifold and its topological properties, especially its fundamental group, interact to provide new information about the manifold.

A hyperbolic n-manifold is a complete metric space that is locally isometric to the classical non-euclidean space $\mathbb{H}^n$ in which the sum of the angles of a triangle is less than π, or, equivalently, a complete Riemannian manifold of constant sectional curvature -1. Furthermore, one can express a hyperbolic n-manifold as the quotient of hyperbolic n-space modulo a discrete torsion-free group Γ of orientation-preserving isometries, in turn Γ is isomorphic to $\pi_1(M)$; it is this vantage point that I take in this paper.

Definition 1.0.1. A group Γ is said to be k-*free*, where k is a given positive integer, if every finitely generated subgroup of Γ of rank less than or equal to k is free. (Recall that the rank of a finitely generated group G is the minimal cardinality of a generating set for G.)

A recurring theme here is the interplay between classical topological properties of a hyperbolic 3-manifold and its geometric invariants, such as volume. The property of having k-free fundamental group bridges these ideas via the $\log(2k-1)$-Theorem (2, Main Theorem) which uses geometric data about the manifold in regards to displacements of points under elements of $\pi_1(M)$ in $\mathbb{H}^3$ and forms the basis for the ideas of Chapter 2. One connection with topology is given by the first homology groups of M with coefficients in Z_p see (2). Also, by a theorem of Jaco and Shalen (7), the fundamental group of a hyperbolic 3-manifold M is 2-free. If $\pi_1(M)$ is not 2-free, then M has a finite cover, $\widetilde{M}$, with the rank of $\pi_1(\widetilde{M})$ equal to 2.

Note that a closed hyperbolic manifold with k-free fundamental group, for $k \geq 2$, is in particular $k-1, k-2, k-3, \ldots, 2$- free. So, in particular, the results of Culler and Shalen in (6) in conjunction with those here show that for a closed, orientable, hyperbolic 3-manifold M with 5-free fundamental group, we have vol $M \geq 3.44$. A long range goal of the present work is to improve this bound with the added topological and geometric information that is gotten by virtue of the 5-free assumption.

To contextualize some previous results on the topic, we require a few definitions. We say that a point P of a hyperbolic 3-manifold M is λ-*thin*, where λ is a given positive number, if there exists a homotopically non-trivial loop of length less than or equal to λ with basepoint P. A point which is not λ-thin is said to be λ-*thick*. If $P \in M$ is λ-thick, it is equivalently the center of a hyperbolic ball in M of radius $\lambda/2$. Further, a point $P \in M$ is λ-*doubly thin* if there are two non-commuting loops that represent elements of $\pi_1(M, P)$ of length less than λ. If P is not λ-doubly thin, we say that it is λ-*semithick*.

It has been shown that if $\pi_1(M)$ is 2-free, M contains a point whose injectivity radius is at least $(\log 3)/2$, and by (1, Corollary 9.3), if $\pi_1(M)$ is 3-free, M contains a point of injectivity radius $(\log 5)/2$. That is to say if M is closed, orientable, and $\pi_1(M)$ is 3-free, M contains a $\log 5$-thick point. By (6, Theorem 1.4) with the additional hypothesis that M has 4-free fundamental group, they show the existence of a $\log 7$-semithick point of M.

It appears then that a naive guess might be to say that if M is a closed, orientable hyperbolic 3-manifold with k-free fundamental group, there exists a point of M that is $\log(2k-1)$ "partly"-thick, and some of the exercise is in determining what"partly" should be in this new context. However, while all of the arguments showing this in the 2, 3, and 4-free cases involve the $\log(2k-1)$ Theorem (6, Main Theorem), they use different subtle topological arguments involving coverings of $\mathbb{H}^3$ by tubes. In the 4-free case, while the proof begins with the same viewpoint as in (1, Corollary 9.3), the topological and geometric arguments following from the assumption that M contains *no* $\log 7$-semithick point to derive a contradiction, are much deeper. I began with the viewpoint of (6) and as I tried to determine what topological arguments and group-theoretic statements were needed to show, if possible, that a closed, orientable hyperbolic 3-manifold M with 5-free fundamental group contains a $\log 9$-"thickish" point, I also attempted to see what assumptions were required in general, to show how if $\pi_1(M)$ were to be k-free, under what conditions one might expect M to contain a $\log(2k-1)$-point of a yet-to-be-quantified thickness. But, because many of Culler and Shalen's results depended on Kent's result (8) that a rank two subgroup of a free group has rank two join, and more specifically that a rank two subgroup cannot be generated from a cyclic group, these results do not extend. For example, a rank three subgroup *can* of course be contained in a two *or* three generator group, and so it was clear that additional group-theoretic assumptions might be required.

I now proceed to state some requisite preliminaries followed by my main result, whose proof is contained in Chapter 5.

Remark 1.0.2. Given M a closed, orientable hyperbolic 3-manifold, we may write M as the quotient $\mathbb{H}^3/\Gamma$, where Γ is a discrete group of orientation-preserving isometries of $\mathbb{H}^3$ that is torsion-free. Every isometry of Γ is loxodromic since M is closed (and so cannot be parabolic or elliptic). Every non-trivial element γ of Γ is contained in a unique maximal cyclic subgroup $C(\gamma)$ of Γ which is the centralizer of γ in Γ, which means that non-trivial elements of distinct maximal cyclic subgroups do not commute.

Definition 1.0.3. Supposing $M = \mathbb{H}^3/\Gamma$ is given as above, let $C(\Gamma)$ be the set of maximal cyclic subgroups of Γ. After fixing a positive real number λ, let $C_\lambda(\Gamma)$ denote the set of maximal cyclic subgroups $C = C(\gamma)$ of Γ having at least one (loxodromic) generator of C with *translation length* less than λ (see Definition 2.0.13). For each maximal cyclic subgroup $C \in C_\lambda(\Gamma)$, we consider the hyperbolic cylinder $Z_\lambda(C)$ of points in $\mathbb{H}^3$ that are displaced by a distance less than λ by some non-trivial element of C.

Definition 1.0.4. Given a point $P \in \mathbb{H}^3$, let $C_P(\lambda)$ denote the set of all C in $C_\lambda(\Gamma)$ for which P is an element of $Z_\lambda(C)$. We then associate to each point P in $\mathbb{H}^3$ a group, $G_P(\lambda)$, which is defined by $G_P(\lambda) = \langle C : C \in C_P(\lambda)\rangle$. If $C_P(\lambda) = \emptyset$, then set $G_P(\lambda) = \langle 1\rangle$, and define rank $G_P(\lambda) = 0$. Also, if the value of λ is understood to be fixed, we may refer to $G_P(\lambda)$ simply as G_P.

Definition 1.0.5. Suppose H is a subgroup of a group G. Then we define the *minimum enveloping rank of* H, or r_H to be the smallest rank among the ranks of groups for which H is a subgroup, if such a number exists. If H is not contained in a finitely generated subgroup of G, then we define r_H to be ∞. More formally, when H is contained in a finitely gerenated subgroup of G, we may define r_H as the smallest positive integer among the set $\{\text{rank } K : H \leq K \leq G\}$.

1.0.6. Note that if H is non-trivial and non-cyclic, $r_H \geq 2$. Furthermore, if h denotes the rank of H, since H is in particular a subgroup of itself, by definition we have $r_H \leq h$.

Definition 1.0.7. Suppose $M = \mathbb{H}^3/\Gamma$ is a closed, orientable, hyperbolic 3-manifold ($\Gamma \leq \text{Isom}_+(\mathbb{H}^3)$ is discrete and purely loxodromic). Given a number $\lambda > 0$, we define the number $r_M(\lambda) \in \mathbb{N} \cup \{0\}$ to be the infinum of the set $\{r_{G_P(\lambda)} : P \in \mathbb{H}^3\}$. If the value of λ is understood to be fixed, we may refer to $r_M(\lambda)$ simply as r_M.

A main goal was the formulation of the following geometric statement:

Conjecture 1.0.8. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k-free with $k \geq 5$. Then when $\lambda = \log(2k-1)$, we have $r_M \leq k-3$.*

1.0.9. Simply stated, this says that if M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k-free with $k \geq 5$, then when $\lambda = \log(2k-1)$, there exists a point P in M such that the class of all homotopically non-trivial loops of $\pi_1(M, P)$ of length less than λ is contained in a subgroup of Γ of rank $\leq k-3$.

In retrospect, Culler and Shalen's work established the truth of Conjecture 1.0.8 for values of k equal to $2, 3$ and 4, but it was not at all obvious what the generalization should be. Notice that the manifold M does not contain a λ-thick point (i.e. a point where the injectivity radius is at least $\lambda/2$) if and only if the family of cylinders $\mathcal{Z} = Z_\lambda(C)$ form an open cover of $\mathbb{H}^3$. A locally finite family $\mathcal{Z}$ of cylinders has a natural association to the set of maximal cyclic subgroups of Γ, and it is how this family of cylinders covers $\mathbb{H}^3$ that is of particular importance, and we encode this information in the nerve. New challenges and many refinements, for instance determining "connectedness" arguments for certain skeleta of the nerve in order to show homotopy-equivalence to $\mathbb{H}^3$ (and therefore contractibility), were involved in extending the 4-free arguments to the k-free arguments, and are detailed in Chapter 3. To prove Conjecture 1.0.8, I have shown that we require a group-theoretic conjecture that gives a bound on the rank of the join of two rank-m subgroups of a free group for $m = k - 2$:

Conjecture 1.0.10. *Given two rank m subgroups of a free group whose intersection has rank greater than or equal to m, their join must have rank less than or equal to m ($m \geq 2$).*

This statement is the subject of Chapter 4 and was motivated by combining known results in the area as proved by Kent (8), Louder, and McReynolds (9). As noted previously, Culler and Shalen used Kent's result that if two rank-2 subgroups of a free group have rank-2 intersection then they have a rank-2 join (8), but there were many details required to extend it. For $m = 3$, following the suggestion of Marc Culler and using an argument in Kent's paper (8), I recently was able to show the group-theoretic Conjecture 1.0.10 is in fact valid for $m = 3$ which establishes Conjecture 1.0.8 for the value of $k = 5$; this is the topic of Chapter 6. Therefore, the statement of Conjecture 1.0.8 when $k = 5$ gives the following Theorem:

Theorem 1.0.11. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k-free with $k = 5$. Then when $\lambda = \log 9$, we have $r_M \leq 2$.*

Furthermore, the work in Chapter 5 establishes that Conjecture 1.0.8 is true modulo Conjecture 1.0.10 and we have the following theorem:

Theorem 1.0.12. *Conjecture 1.0.10 with $m = k - 2$ implies Conjecture 1.0.8.*

The proof of Theorem 1.0.12 is contained in Chapter 5. In the proof we consider the action of Γ on the sets of components of two disjoint subsets X_i, X_j of a simplicial complex K, and using (6, Lemma 5.12) and (6, Lemma 5.13), we show by way of contradiction that $\Gamma \leq \mathrm{Isom}_+\mathbb{H}^3$ admits a simplicial action without inversions on a tree $T = \mathcal{G}(X_i, X_j)$ with the property that the stabilizer in $\pi_1(M)$ of every vertex of T is a locally free subgroup of $\pi_1(M)$.

As corollaries, I state some geometric properties for particular values of r_M in Chapter 5. As a special case, the Theorem of Culler and Shalen that if M is a closed, orientable hyperbolic 3-manifold with 4-free fundamental group then M must contain a $\log 7$-semithick point (6, Theorem 1.4), is precisely the $k = 4$ case of Conjecture 1.0.8; hence the conclusion can be reinterpreted from the present notation as saying $r_M(\log 7)$ is less than or equal to one, which by definition implies the existence of a point P of M such that the rank of the group $G_p(\log 7)$ is less than or equal to one. One can then give lower bounds for the nearby volume, i.e. the volume of the $\log(7)/2$ neighborhood of P and for the distant volume, i.e. the volume of the complement of this neighborhood. Culler and Shalen applied their result to studying the relationship between homology and volume by estimating the nearby volume. My result shows that in the 5-free case, $r_M \leq 2$ and so M contains a point P for which $G_p(\log 9)$ has minimum enveloping rank less than or equal to two. I am now investigating the problem of estimating the nearby and global volume of the manifold M in this situation.

CHAPTER 2

PRELIMINARIES AND DEFINITIONS

Definitions 2.0.13. Suppose we are given a positive real number $\lambda > 0$ and that the subgroup $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^3)$ is discrete and cocompact. For $\gamma \in \Gamma$ we define the hyperbolic cylinder $Z_\lambda(\gamma)$ to be the set of points $P \in \mathbb{H}^3$ such that $d(P, \gamma \cdot P) < \lambda$. Recall that since γ is loxodromic, there is a γ-invariant line, $A(\gamma) \subset \mathbb{H}^3$, called the *axis* of γ, such that γ acts on the points of $A(\gamma)$ as a translation by a distance $l > 0$, called the *translation length* of γ. For any point $P \in \mathbb{H}^3$, we have $d(P, \gamma \cdot P) \geq l$ with equality only when $P \in A(\gamma)$. Then as long as $l < \lambda$, the cylinder $Z_\lambda(\gamma)$ is non-empty (the radius of this cylinder is computed by a simple application of the hyperbolic law of cosines and is a monotonically increasing function for λ in the interval (l, ∞); see, for example, (5) for further details).

Given a cyclic subgroup C of Γ, we define the cylinder $Z_\lambda(C) = \bigcup_{1 \neq \gamma \in C} Z_\lambda(\gamma)$. Recall the definitions of $C(\Gamma)$ and $C_\lambda(\Gamma)$ of 1.0.3. Then for each maximal cyclic subgroup $C \in C_\lambda(\Gamma)$, there is a non-trivial element $\gamma \in C$ such that $Z_\lambda(C) = Z_\lambda(\gamma)$. Notice that if $C \in C(\Gamma) - C_\lambda(\Gamma)$, we have $Z_\lambda(C) = \emptyset$.

Note that the family of cylinders $(Z_\lambda(\gamma))_{1 \neq \gamma \in \Gamma}$ is *locally finite* as Γ is discrete; i.e. for every point P in $\mathbb{H}^3$, there is a neighborhood of P which has non-empty intersection with only finitely many of the subsets $Z_\lambda(\gamma)$. Further, because the family $(Z_\lambda(\gamma))_{1 \neq \gamma \in \Gamma}$ is locally finite, so then is the family $(Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$.

The following Lemma is an application of the $\log(2k-1)$ Theorem (2, Main Theorem).

Lemma 2.0.14. *Suppose $\Gamma \le Isom_+(\mathbb{H}^3)$ is discrete, loxodromic, k-free ($k \ge 2$) and torsion-free. If there exists a point $P \in Z_{\log(2k-1)}(C_1) \cap \cdots \cap Z_{\log(2k-1)}(C_n)$, then the rank of $\langle C_1, \ldots, C_n \rangle$ is $\le k-1$.*

Proof. (by induction on n)

Base case: If $n = 1$, then $P \in Z_{\log(2k-1)}(C)$. Because rank $C = 1$ and $k \ge 2$, rank $C \le k-1$ is satisfied.

Induction assumption: If $n = q$ then $X_q = \langle C_1, \ldots, C_q \rangle$, and so we assume that rank $X_q \le k-1$.

Induction step: Notice that $X_{q+1} = \langle X_q, C_{q+1} \rangle = \langle C_1, \ldots, C_q, C_{q+1} \rangle$. We must show that rank $X_{q+1} \le k-1$. To simplify notation, let $r =$ rank X_q. First, consider when rank $\langle X_q, C_{q+1} \rangle = r$. Since $r \le k-1$ by our induction assumption, we are done.

Next, consider the case when rank $\langle X_q, C_{q+1} \rangle >$ rank $X_q = r$.

Remark 2.0.15. Now $X_q \le \Gamma$ which is k-free, rank $X_q < k$, $C_{q+1} = \langle t \rangle$ is cyclic, and rank $(X_q \vee C_{q+1}) >$ rank $X_q = r$, so $(X_q \vee C_{q+1})$ is the free product of X_q and C_{q+1} by (6, Lemma 4.3).

By the remark and our induction assumption, rank $\langle X_q, C_{q+1} \rangle = r + 1 \le (k-1) + 1 = k$. Therefore rank $\langle X_q, C_{q+1} \rangle \le k$, leaving two subcases to consider. First, if $r < k-1$, then rank $\langle X_q, C_{q+1} \rangle < k$ and we are done.

In the second subcase, suppose $r = k-1$. The remark then gives that rank $\langle X_q, C_{q+1}\rangle = r+1 = k$; we proceed to prove that rank $\langle X_q, C_{q+1}\rangle \leq k-1$ by way of contradiction.

Since $n = q+1$, by hypothesis $P \in Z_{\log(2k-1)}(C_1) \cap \cdots \cap Z_{\log(2k-1)}(C_{q+1})$. Choose a generator γ_i for each C_i, where $1 \leq i \leq q+1$. For each i, there exists a number $m_i \in \mathbb{N}$ with $d(P, \gamma_i^{m_i} \cdot P) < log(2k-1)$ by definition of the cylinders; denote this property (*).

Now the rank of $\langle \gamma_1, \ldots, \gamma_{q+1}\rangle$ is k, and so this group is free (being a subgroup of Γ) which is k-free. In particular, $\{\gamma_1, \ldots, \gamma_{q+1}\}$ is a generating set of a free group of rank k, and so it must contain a subset S of k independent elements whose span has rank k. So let $S = \{\gamma_{i_1}, \ldots, \gamma_{i_k}\} \subseteq \{\gamma_1, \ldots, \gamma_{q+1}\}$ be as described. Furthermore, the set $S' = \{\gamma_{i_1}^{m_{i_1}}, \ldots, \gamma_{i_k}^{m_{i_k}}\}$ is also a set of k independent elements whose span has rank k. Then as $S' \subseteq \mathrm{Isom}_+(\mathbb{H}^3)$ is a set of k freely-generating (loxodromic) generators with rank $\langle S'\rangle = k$, the $\log(2k-1)$ Theorem of (2) applies here to give that $\max_{1\leq j\leq k} d(P, \gamma_{i_j}^{m_{i_j}} \cdot P) \geq \log(2k-1)$, thereby contradicting property (*) above. Therefore, rank $\langle X_q, C_{q+1}\rangle \leq k-1$ as required, and in particular is equal to $k-1$ in this subcase. □

Now recall from the Introduction the Definitions 1.0.4 and 1.0.7 of $G_P(\lambda)$ and $r_M(\lambda)$, respectively.

Corollary 2.0.16. *Given* $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^3)$ *is discrete, purely loxodromic, and k-free with $k \geq 2$, then for any point $P \in \mathbb{H}^3$, we have* $\mathrm{r}k\ G_P(\log(2k-1)) \leq k-1$.

Proof. This result is a direct consequence of Lemma 2.0.14 along with the preceding definitions.

□

Given $M = \mathbb{H}^3/\Gamma$ a closed, orientable, hyperbolic 3-manifold, we now make a few observations regarding the number r_M:

2.0.17. Given a point P in $\mathbb{H}^3$, it follows from the definitions that $r_M \leq r_{G_P(\lambda)} \leq \text{rank } G_P(\lambda)$.

2.0.18. When $\lambda = \log(2k-1)$, as a direct consequence of Corollary 2.0.16 and (2.0.17), we have $r_M \leq k-1$.

When $r_M \geq 1$, we claim:

2.0.19. $\mathbb{H}^3 = \bigcup_{C_1,\ldots,C_{r_M} \in \mathcal{C}_\lambda(\Gamma)} Z_\lambda(C_1) \cap \cdots \cap Z_\lambda(C_{r_M})$.

Proof. Suppose P is a point of $\mathbb{H}^3$. As (2.0.17) says that rank $G_P \geq r_M$, there exist maximal cyclic subgroups $C_1^P, \ldots, C_{r_M}^P$ of Γ such that $\langle C_1^P, \ldots, C_{r_M}^P \rangle \leq G_P$ with $P \in Z_\lambda(C_1^P) \cap \cdots \cap Z_\lambda(C_{r_M}^P)$ (keeping in mind that P may be in additional cylinders). The statement follows. $\square$

CHAPTER 3

CONTRACTIBILITY ARGUMENTS AND Γ-LABELED COMPLEXES

Definition 3.0.20. An indexed covering $\mathcal{U} = (U_i)_{i \in I}$ of a topological space by non-empty open sets defines an abstract simplicial complex, *the nerve of* $\mathcal{U}$, whose vertices are in bijective correspondence with the elements of the index set I and whose simplices $\{v_{i_0}, \ldots, v_{i_n}\}$ correspond to the non-empty intersections $U_{i_0} \cap \cdots \cap U_{i_n}$ of sets of $\mathcal{U}$. In particular, if $\mathcal{Z}(\lambda) = (Z_\lambda(C_i))_{i \in I, C_i \in C_\lambda(\Gamma)}$ is a cover of $\mathbb{H}^3$ by cylinders, then the family $\mathcal{Z}(\lambda)$ defines a Γ-*labeled complex*, namely a pair $(K, (C_v)_v)$ where K is the nerve of $\mathcal{Z}(\lambda)$ and the maximal cyclic subgroup C_v corresponds to the element $Z_\lambda(C_v)$ of $\mathcal{Z}(\lambda)$ as indexed by the vertex v of K. For purposes of notation, we may refer to this vertex v by v_C.

Definition 3.0.21. Given a group Γ and $(K, (C_v)_v)$ a Γ-labeled complex, we say the labeling defines a *labeling-compatible* Γ-action on $(K, (C_v)_v)$ if for every vertex v of K, the action defined by $C_{\gamma \cdot v} = \gamma C_v \gamma^{-1}$ is simplicial.

Remark 3.0.22. Note that if $\Gamma \leq \mathrm{Isom}_+\mathbb{H}^3$ is discrete and torsion-free, if the family $\mathcal{Z}(\lambda) = (Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$ covers $\mathbb{H}^3$, and if K is the nerve of $\mathcal{Z}(\lambda)$, then the Γ-labeled complex $(K, (C_v)_v)$ admits a labeling-compatible Γ-action. Let $V = \{v_0, \ldots, v_n\}$ be the set of vertices of an n-simplex of K; by definition $\cap_{0 \leq i \leq n} Z_\lambda(C_{v_i}) \neq \emptyset$. Given $1 \neq \gamma \in \Gamma$ and $v_i \in V$, let $w_i = \gamma \cdot v_i$. First, note that w_i is well-defined as a vertex of K, as w_i corresponds to the maximal cyclic subgroup C_{w_i} of Γ as given by $C_{w_i} = \gamma \cdot C_{v_i}$, and so we have $w_i \in V$. Furthermore notice

$C_{\gamma^{-1}\cdot w_i} = \gamma^{-1}C_{w_i}\gamma = \gamma^{-1}\gamma C_{v_i}\gamma^{-1}\gamma = C_{v_i}$. Now we only need to show that the set $W = \{w_0, \ldots, w_n\}$ of vertices of K is in fact the vertex set of an n-simplex of K; this is equivalent to showing that $\cap_{0\leq i\leq n} Z_\lambda(C_{w_i})$ is non-empty. Observe that $\cap_{0\leq i\leq n} Z_\lambda(C_{w_i}) = \cap_{0\leq i\leq n} Z_\lambda(C_{\gamma\cdot v_i}) = \cap_{0\leq i\leq n} Z_\lambda(\gamma C_{v_i}\gamma^{-1}) = \cap_{0\leq i\leq n} Z_\lambda(C_{v_i}) \neq \emptyset$.

Definition 3.0.23. Given a Γ-labeled complex $(K, (C_v)_v)$ and σ an open simplex in K, define the subgroup $\Theta(\sigma)$ of Γ to be the group $\langle C_v : v \in \sigma\rangle$.

3.0.24. Suppose K is given to be the nerve of a family $\mathcal{Z}(\lambda) = (Z_\lambda(C_i))_{i\in I, C_i\in C_\lambda(\Gamma)}$ which is a cover of $\mathbb{H}^3$ by cylinders. If there exists a point $P \in \mathbb{H}^3$ in the intersection $Z_\lambda(C_0)\cap\cdots\cap Z_\lambda(C_n)$, it follows that $\{v_{C_0}, \ldots, v_{C_n}\}$ is an n-simplex σ of K, and by the Definitions 1.0.4 and 3.0.23, we have $\theta(\sigma) \leq G_P(\lambda)$.

Definitions 3.0.25. Suppose $(K, (C_v)_v)$ is a Γ-labeled complex. Given an open simplex σ in K, the *minimum enveloping rank of* σ will denote the minimum enveloping rank of the associated subgroup $\Theta(\sigma)$ in Γ. Notice that if $\tau \in K$ is a face of $\sigma \in K$, then we have $r_{\theta(\tau)} \leq r_{\theta(\sigma)}$; i.e. the minimum enveloping rank of a face of σ is less than or equal to that of σ. We may then define a subcomplex $K_{(n)}$ of K to be the subcomplex that consists of the non-trivial open simplices σ for which $r_{\theta(\sigma)} \leq n$.

Lemma 3.0.26. *Let $M = \mathbb{H}^3/\Gamma$. Suppose $\mathcal{Z}(\lambda) = (Z_\lambda(C_i))_{i\in I, C_i\in C_\lambda(\Gamma)}$ is a cover of $\mathbb{H}^3$ by cylinders and that $r_M \geq k-2$.*

Let $|K|$ denote the geometric realization of the nerve of $\mathcal{Z}(\lambda)$. Then $|K| - |K_{(k-3)}|$ is homotopy-equivalent to $\mathbb{H}^3$ and therefore contractible.

Proof. The family $(Z_\lambda(C_i))_{i\in I, C_i\in C_\lambda(\Gamma)}$ covers $\mathbb{H}^3$ and has the property that every finite intersection of (open) cylinders is contractible, as any such intersection is either empty or convex. Thus Borsuk's Nerve Theorem (3) applies, and we have $|K|$ is homotopy-equivalent to $\mathbb{H}^3$. It is only left to show that $|K| - |K_{(k-3)}|$ is homotopy-equivalent to $|K|$. Suppose σ is a non-trivial open simplex of $|K_{(k-3)}|$, which by definition is to say that the minimum enveloping rank of $\theta(\sigma)$ is $\leq k-3$. Let $v^\sigma_{i_0}, ..., v^\sigma_{i_l}$ be the vertices of σ, and set $I_\sigma = \{i \in I : v_i \in \sigma\}$.

Let U_i for $i \in I$ denote the cylinder $Z_\lambda(C_i)$ associated with the vertex v_i as defined by the nerve of the cover $\mathcal{Z}(\lambda)$. In particular U_{i_m} will denote the cylinder $Z_\lambda(C_{v_{i_m}})$ associated with the vertex $v^\sigma_{i_m}$ of K for $0 \leq m \leq l$. Define the intersection $\mathcal{U}_\sigma$ to then be $U_{i_0} \cap \cdots \cap U_{i_l}$. Let $J_\sigma = \{j \in I - I_\sigma : U_j \cap \mathcal{U}_\sigma \neq \emptyset\}$. Define the set $V_{j,\sigma} = \{U_j \cap \mathcal{U}_\sigma : j \in J_\sigma\}$ and the family $\mathcal{V}_\sigma = (V_{j,\sigma})_{j\in J_\sigma}$.

We proceed to show that:

3.0.26.1. *$\mathcal{V}_\sigma$ is a cover for $\mathcal{U}_\sigma$.*

Proof. Suppose on the contrary that $\mathcal{V}_\sigma$ is in fact *not* a cover for $\mathcal{U}_\sigma$. Then there exists a point P of $\mathcal{U}_\sigma$ such that $P \notin U_i$ for any $i \in I - I_\sigma$. In particular, $G_P(\lambda) \leq \theta(\sigma)$. However by 3.0.24 we also have $\theta(\sigma) \leq G_P(\lambda)$, and so $\theta(\sigma) = G_P(\lambda)$. Then because $r_{\theta(\sigma)} \leq k-3$, we have $r_{G_P(\lambda)} \leq k-3$. But, the minimum enveloping rank of $G_P(\lambda)$ is $\geq k-2$ as $r_M \geq k-2$ by hypothesis, providing a contradiction. Therefore, $\mathcal{V}_\sigma$ covers $\mathcal{U}_\sigma$ as claimed. □

So $\mathcal{V}_\sigma$ is in fact a cover of $\mathcal{U}_\sigma$ which inherits the subspace topology, and so it follows from the definitions that the nerve of $\mathcal{V}_\sigma$ is simplicially isomorphic to the link of σ in K. Note that two different indices in J_σ may define the same set in $\mathcal{V}_\sigma$ but they will define different sets in $\mathcal{Z}(\lambda)$; this is why it is essential to define the nerve of $\mathcal{V}_\sigma$ using J_σ: so that the map from the vertex set of the nerve of $\mathcal{V}_\sigma$ to the vertex set of the link of σ in K is not only simplicial but bijective; that the inverse of this map is simplicial is straightforward. To see this, suppose v_j is a vertex in the nerve of $\mathcal{V}_\sigma$, then by definition $U_j \cap \mathcal{U}_\sigma \neq \emptyset$, i.e. $(U_{i_0} \cap \cdots \cap U_{i_l}) \cap U_j$ is non-empty. In particular, $U_{i_0} \cap U_j$, $U_{i_1} \cap U_j, \ldots, U_{i_l} \cap U_j$ are all non-empty, so that $\{v_{i_0}, v_j\}, \{v_{i_1}, v_j\}, \ldots, \{v_{i_l}, v_j\}$ are all edges of K (v_j is distinct from the vertices of σ), and v_j is in the link of σ in K. The reverse inclusion is similar.

Applying Borsuk's Nerve Theorem to $\mathcal{V}_\sigma$ in place of $\mathcal{Z}$, we see the underlying space of the nerve of $\mathcal{V}_\sigma$ is homotopy-equivalent to $\mathcal{U}_\sigma$. Since $\mathcal{U}_\sigma$ is a finite, non-empty intersection of convex open sets, it is contractible. We conclude that the link in K of every simplex of minimum enveloping rank m with $0 \leq m \leq k-3$ is contractible and non-empty.

We now show that the inclusion $|K| - |K_{(k-3)}| \to |K|$ is a homotopy equivalence.

By local finiteness of the cover $\mathcal{Z}$ from which its nerve $|K|$ is defined, we may index the vertices of $|K_{(k-3)}|$, and therefore we may index the simplices of $|K_{(k-3)}|$ and partially order them in the following way: if σ_i, σ_j are such that σ_i is a proper face of σ_j, then $j < i$.

Define $F_n = \sigma_1 \cup \cdots \cup \sigma_n$. We may regard $|K| - |K_{(k-3)}|$ as the topological direct limit of the subspaces $|K| - |K_{(k-3)}| \cup F_n$. Thus it suffices to show that the inclusion $|K| - |K_{(k-3)}| \cup F_n \to (|K| - |K_{(k-3)}|) \cup F_{n+1}$ is a homotopy equivalence, where $F_{n+1} = F_n \cup \{\sigma_{n+1}\}$.

Let S denote the open star of σ_{n+1} in K. By how we've listed the simplices in $|K_{(k-3)}|$, we have $S \subset K_{F_{n+1}}$. Then $K_{F_{n+1}} - S \cong K_{F_n}$ since $K_{F_{n+1}} - S$ is a deformation retract of K_{F_n}. And

$K_{F_{n+1}} - S \cong K_{F_{n+1}}$ because the link of σ_{n+1} in K is contractible by our work above. Hence the inclusion $K_{F_n} \to K_{F_{n+1}}$ is a homotopy equivalence as required. $\square$

CHAPTER 4

GROUP-THEORETIC PRELIMINARIES

We will say that W is a *saturated* subset of the geometric realization $|K|$ of a simplicial complex K, if W (endowed with the subspace topology) is a union of open simplices of $|K|$ (endowed with the weak topology).

Given a Γ-labeled complex $(K, (C_v)_v)$ and saturated subset $W \subseteq |K|$, we define the subgroup $\Theta(W)$ of Γ to be the group $\langle C_v : v \in \sigma, \sigma \subset W \rangle$.

We now restate Conjecture 1.0.10 from the Introduction which is necessary to prove Proposition 4.0.29, which is an essential ingredient in the proof of 1.0.8. Let $H \vee K = \langle H, K \rangle$.

Conjecture 4.0.27. *Suppose H, K are two rank h subgroups of a free group with $h \geq 3$. If the rank of $H \cap K$ is greater than or equal to h, then the rank of $H \vee K$ must be less than or equal to h.*

Definition 4.0.28. We say a group Γ has *local rank* $\leq k$ where k is a positive integer, if every finitely generated subgroup of Γ is contained in a subgroup of Γ which has rank less than or equal to k. The local rank of Γ is the smallest k with this property. If there does not exist such a k then we define the local rank of Γ to be ∞. Note that if Γ is finitely generated, its local rank is simply its rank.

Proposition 4.0.29. *Assume Conjecture 4.0.27. Let $k, r \in \mathbb{Z}^+$ with $k > r \geq 3$ and $k \geq 5$. Suppose Γ is a k-free group, $(K, (C_v)_v)$ a Γ-labeled complex, and W a saturated, connected subset of $|K|$ such that* rank $\Theta(\sigma) = r$ *for all $\sigma \subset W$. Assume additionally that either*

(i) there exists a positive integer n such that for all open simplices σ in W, the dimension of σ is n or $n-1$, or

(ii) $r = k-2$ and $\sigma \in |K^{(k-1)}| - |K_{(k-3)}|$ for all $\sigma \in W$.

Then the local rank of $\Theta(W)$ is at most r.

Proof. By definition, we are required to show that every finitely generated subgroup of $\Theta(W)$ is contained in a finitely generated subgroup of $\Theta(W)$ which has rank less than or equal to r. So suppose that $E \leq \Theta(W)$ is a finitely generated subgroup of $\Theta(W)$. Then $E \leq \Theta(V_0)$ for some saturated subset V_0 of W that contains finitely many open simplices. Because W is connected and V_0 contains only finitely many open simplices, there is a smallest connected subset V of W that is a union of finitely many open simplices such that $V_0 \subseteq V$; clearly $E \leq \Theta(V)$ and V is finitely generated. We will show by induction on the number of simplices in V that $\Theta(V)$ has rank at most r.

Proceeding as in (6, Proposition 4.4), by connectedness we may list the (finitely many) open simplices of V in the following way: $\sigma_0, \ldots, \sigma_m$, ($m \geq 0$ since V is non-empty) where for any i with $0 \leq i \leq m$, there is an index l with $0 \leq l < i$ such that σ_l is a proper face of σ_i or σ_i is a proper face of σ_l. Define the saturated subset $V_i = \sigma_0 \cup \cdots \cup \sigma_i$ for $0 \leq i \leq m$; by induction on i, we will show rank $\Theta(V_i) \leq r$. The base case is straightforward as $\Theta(V_0) = \Theta(\sigma_0)$ and

σ_0 is an open simplex of W, and so has rank r by hypothesis. For the induction step assume rank $\Theta(V_{i-1}) = r$; we want to show that rank $\Theta(V_i) = r$. By how we have arranged the list of simplices in V, there is an index l with $0 \leq l < i$ such that σ_l is a proper face of σ_i or σ_i is a proper face of σ_l.

Case (i): First consider the case when σ_i is a proper face of σ_l. Then $\Theta(V_i) = \Theta(V_{i-1})$ as $\sigma_i < \sigma_l \in V_{i-1}$. By our induction assumption, rank $\Theta(V_{i-1}) \leq r$, and so rank $\Theta(V_i) \leq r$ as required.

Case (ii): Next, consider the case when σ_l is a proper face of σ_i. Let $P = \Theta(V_{i-1})$, $Q = \Theta(\sigma_i)$ and $R = \Theta(\sigma_l)$. Then rank $P \leq k-2$ by the induction hypothesis and rank $Q =$ rank $R = k-2$ by assumption. We want to show $\Theta(V_i) = P \vee Q$ has rank less than or equal to r.

Subcase (i): Assume first that property (i) holds. Then since σ_l is a proper face of σ_i, we must have $dim\sigma_i = n$ and $dim\sigma_l = dim\sigma_i - 1 = n-1$. Let v denote the vertex of σ_i such that $span\{\sigma_l, v\} = \sigma_i$ and let $C = C_v$. Then $Q = R \vee C$, and $P \vee C = P \vee Q$. So we proceed to show that rank$(P \vee C) \leq r$.

By way of contradiction, assume rank$(P \vee C) > r$. Then since C is infinite cyclic, $P \vee C$ has rank at most rank $P + 1 = r + 1$ and so $P \vee C$ has rank exactly $r + 1$. As Γ is k-free and $r < k$ (and hence $r + 1 \leq k$), it follows that $P \vee C$ is free as a subgroup of Γ *and* in particular is the free product of the subgroups P and C ((6, Lemma 4.3)). But, since $R \leq P$, in particular $Q = R \vee C$ is the free product of R and C, and so has rank equal to rank $R + 1 = r + 1$, which is a contradiction as the rank of Q is exactly r. We conclude that $P \vee C$ has rank $\leq r$ as required for this subcase.

Subcase (ii): Next we assume property (ii). Then $r = \text{rank } Q = \text{rank } R = k-2$ (rank $P \leq k-2$ by induction assumption). As $r = \text{r}k\ Q = \text{rank } R = k-2$, both $\mathrm{d}im\sigma_l$ and $\mathrm{d}im\sigma_i$ are at least $k-3$. Also $\sigma_l, \sigma_i \in K^{(k-1)}$, so both $\mathrm{d}im\sigma_l$ and $\mathrm{d}im\sigma_i$ are $\leq k-1$. Finally, since our Case (ii)-assumption is that σ_l is a proper face of σ_i, possible pairs $(\mathrm{d}im\sigma_l, \mathrm{d}im\sigma_i)$ are $(k-3, k-2)$, $(k-2, k-1)$, and $(k-3, k-1)$. Let $C \leq \Gamma$ denote the subgroup of Q such that $Q = R \vee C$; then $P \vee Q = P \vee C$ since $R \leq P$.

4.0.30. First, we look at the rank of P. A priori we know that rank $P \geq 2$ (i.e. P cannot be cyclic and is non-trivial) since P contains the rank-$(k-2)$ subgroup R.

In particular, as $R = \Theta(\sigma_l)$ is a subgroup of P, and as σ_l is an element of $K^{(k-1)} - K_{(k-3)}$, we know that the minimum enveloping rank of R is strictly greater than $k-3$. Along with our induction assumption that rank $P \leq k-2$, we conclude the rank of P is exactly $k-2$. (Note that for this reason in the case when $k = 4$, it is enough only to say in (ii) that $r = 2$, since what is required for the rest of the argument is that P have rank exactly $2 = k-2$ in this case, an immediate consequence of P containing the rank 2 subgroup R. Specifically, in the $k = 4$ case, we see that a group containing a rank two subgroup certainly cannot have rank one; whereas in cases for $k \geq 5$, one observes that a group that contains a rank three (or more) subgroup *can* have rank two or more, and so that $r_{\theta(\sigma)} \geq k-2$ is required in the statement of (ii)). Next observe that we must have rank $C = 1$ or 2 as demonstrated by the possible pairs $(\mathrm{d}im\sigma_l, \mathrm{d}im\sigma_i)$ above. All together, this gives that rank$(P \vee C) \leq k$ and so $P \vee C$ is free as a subgroup of Γ.

4.0.31. Next, notice that because $Q \leq P \vee Q$ and Q has minimum enveloping rank $\geq k-2$, $P \vee Q$ cannot have rank less than $k-2$. Along with the bound rank$(P \vee C) \leq k$ of (4.0.30), we conclude there are only three possibilities for the rank of the group $P \vee Q (= P \vee C)$: these are $k, k-1, and\ k-2$.

4.0.32. As we have $R \leq P$, $R \leq Q$, and $R \leq P \cap Q$, then for the same reason as outlined in (4.0.30) with $P \cap Q$ taking the place of P, we conclude rank$(P \cap Q) > k-3$. Therefore, we may apply Conjecture (4.0.27) which gives that rank$(P \vee Q) \leq k-2$, and so must be equal to $k-2$ by (4.0.31), completing this final Subcase and proving the Proposition.

□

As a sidenote, after performing some computations with known bounds on the ranks of intersections and joins of free groups, we have the following refinement of Conjecture 4.0.27:

Conjecture 4.0.33. *Suppose P, Q are two rank $(k-2)$ subgroups of a free group with $k \geq 5$. If $k-2 \leq \text{rank } (P \cap Q) \leq (k-4)(k-3)+1$, then $\text{rank } (P \vee Q)$ is $\leq k-2$.*

For reference, we will show how to compute the positive integer $c = (k-4)(k-3)+1$ of Conjecture 4.0.33.

Computing $c = (k-4)(k-3)+1$:

To provide an outline for what follows, a priori we assume the lower bound $l = k-2$ on the rank of $P \cap Q$ in Conjecture 4.0.27; we then use Burns' inequality (4) to give an upper bound u on the rank of $P \cap Q$; and finally we consider for which subintervals of $[l, u]$ we can deduce that rank $(P \vee Q)$ is $\leq k-2$ by using the strong form of Burns' inequality (8, Theorem 2), which relates the rank of the intersection to the rank of the join.

4.0.34. Burns' inequality (4) (see also (10)) gives rank $(P \cap Q) - 1 \leq 2(\text{rank } P - 1)(\text{rank } Q - 1) - \min(\text{rank } P - 1, \text{rank } Q - 1)$, so that rank $(P \cap Q) \leq 2(k-3)^2 - (k-3) + 1 = 2k^2 - 13k + 22$. As the hypotheisis of Conjecture 4.0.27 is that rank $(P \cap Q) \geq k-2$, we have that rank $(P \cap Q)$ can take values between $k-2$ and $2(k-3)^2 - (k-3) + 1$.

Using the strong form of Burns' inequality proved in (8, Theorem 2), since P, Q are subgroups of a free group with the rank of P less than or equal to the rank of Q (here they are both equal to $k-2$), then the following inequality holds:

4.0.35. rank $(P \cap Q) - 1 \leq 2(\text{rank } P - 1)(\text{rank } Q - 1) - (\text{rank } P - 1)(\text{rank } (P \vee Q) - 1)$. This gives an upper bound on the rank of $P \vee Q$: rank $(P \vee Q) \leq 2(k-3) + 1 - \frac{\text{rank } (P \cap Q) - 1}{k-3} = 2k - 5 - \frac{\text{rank } (P \cap Q) - 1}{k-3}$.

Let $y = \lceil \frac{\text{rank } (P \cap Q) - 1}{k-3} \rceil$ (because rank $(P \vee Q)$ must be an integer). In light of the previous inequality and the possible rank-values of $P \vee Q$ given in (4.0.31) of the last Proposition for which the Conjecture will be applied, it is helpful to know for what values of y we have $2k-5-y$ equal to k, $k-1$, and $k-2$ (for if rank $(P \vee Q) \leq k-2$ we are done); these are the cases when y is equal to $k-5$, $k-4$, and $k-3$ respectively.

First, if $y = k-5$, the definition of y implies that rank $(P \cap Q)$ must be an integer in the half-open interval $((k-6)(k-3)+1, (k-5)(k-3)+1]$. Setting $a = (k-6)(k-3)+1$ and $b = (k-5)(k-3)+1$, then as $a <$ rank $(P \cap Q) \leq b$, we have by (4.0.35) that rank $(P \vee Q) \leq k$.

Next, if $y = k-4$, then $(k-5)(k-3)+1 < \text{rank } (P \cap Q) \leq (k-4)(k-3)+1$. Letting $c = (k-4)(k-3)+1$, (4.0.35) shows that when $b < \text{rank } (P \cap Q) \leq c$, we have $\text{rank } (P \vee Q) \leq k-1$.

Finally, when $y = k-3$, then $(k-4)(k-3)+1 < \text{rank } (P \cap Q) \leq (k-3)^2+1$. Let $d = (k-3)^2+1$. Now (4.0.35) gives that in particular that if $c < \text{rank } (P \cap Q) \leq d$, then $\text{rank } (P \vee Q) \leq k-2$.

Therefore in the case when $y = k-3$, the conclusion of Conjecture 4.0.27 holds, and so we only need appeal to the conclusion of Conjecture 4.0.27 in the last Proposition to assert that $\text{rank } (P \vee Q) \leq k-2$ in the cases when $y = k-5$ and $y = k-4$; this is to say that we apply Conjecture 4.0.27 to prove Proposition 4.0.29 only when $\text{rank } (P \cap Q)$ is an integer $\in (a,b] \cup (b,c] = (a,c] = (\ (k-6)(k-3)+1, (k-4)(k-3)+1\ ]$.

Moreover, if $\text{rank } (P \cap Q) > c$ then (4.0.35) shows that $\text{rank } (P \vee Q) \leq k-2$, and so we have the following result:

4.0.35.1. *If P, Q are two rank $(k-2)$ subgroups of a free group with $k \geq 5$ and if the rank of $P \cap Q$ is greater than $(k-4)(k-3)+1$, then the rank of the join of P and Q is less than or equal to $k-2$.*

CHAPTER 5

THEOREM AND GENERAL BOUND ON R_M

We now restate formally and prove the implicative statement of 1.0.12 given in the Introduction. For the proof we require a few basic definitions about graphs.

Definitions 5.0.36. We say that $\mathcal{G}$ is a *graph* if $\mathcal{G}$ is at most a one-dimensional simplicial complex (and so $\mathcal{G}$ has no loops or multiple edges). A *tree* T is a connected graph with no cycles; i.e. T is a graph which is simply connected. Further, if X_i and X_j are disjoint, saturated subsets of a simplicial complex $|K|$, we will make use of the concept of an abstract bipartite graph $\mathcal{G} = \mathcal{G}(X_i, X_j)$ constructed in the following way. Let $\mathcal{W}_i$, $\mathcal{W}_j$ be the sets of connected components of X_i and X_j respectively. Then the vertices of $\mathcal{G}$ are the elements of $\mathcal{W}_i \cup \mathcal{W}_j$, and a pair $\{v_{W_i}, v_{W_j}\}$ is an edge if there exist simplices $\sigma \in W_i$ and $\tau \in W_j$ for which $\sigma \leq \tau$ or $\tau \leq \sigma$. Finally, we say that the simplicial action of a group Γ on a graph $\mathcal{G}$ is without inversions if for every $\gamma \in \Gamma$ that stabilizes an edge $e = \{v_1, v_2\} \in \mathcal{G}$, we have $\gamma \cdot v_1 = v_1$ and $\gamma \cdot v_2 = v_2$.

The following two Lemmas taken directly from (6) will provide the contradiction necessary to prove Theorem 1.0.12:

Lemma 5.0.37. *Suppose that K is a simplicial complex and that X_i and X_j are saturated subsets of $|K|$. Then $|\mathcal{G}(X_i, X_j)|$ is a homotopy-retract of the saturated subset $X_i \cup X_j$ of $|K|$.*

Proof. This is (6, Lemma 5.12). □

Lemma 5.0.38. *Let M be a closed, orientable, aspherical 3-manifold. Then $\pi_1(M)$ does not admit a simplicial action without inversions on a tree T with the property that the stabilizer in $\pi_1(M)$ of every vertex of T is a locally free subgroup of $\pi_1(M)$.*

Proof. This is (6, Lemma 5.13) □

Finally, we will appeal to the property stated in the next Remark in the proof of Theorem 5.0.40.

Remark 5.0.39. Suppose a group Γ admits a labeling-compatible action on a Γ-labeled complex $(K, (C_v))_v$, as is defined in 3.0.21. If W is a saturated subset of $|K|$ and γ is any element of Γ, it follows that $\Theta(\gamma \cdot W) = \gamma\Theta(W)\gamma^{-1}$. (Since by the definitions, $\Theta(\gamma \cdot W) = \langle C_v : v \in \gamma \cdot W\rangle = \langle C_{\gamma \cdot v} : v \in W\rangle = \langle \gamma C_v \gamma^{-1} : v \in W\rangle = \gamma\langle C_v : v \in W\rangle\gamma^{-1} = \gamma\Theta(W)\gamma^{-1}$). So if an element γ of Γ is invariant on W, then it is in the normalizer of $\Theta(W)$. More generally, the stabilizer in Γ of W is a subgroup of the normalizer of $\Theta(W)$.

The following Theorem is stated as Theorem 1.0.12 in the Introduction.

Theorem 5.0.40. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k-free with $k \geq 5$. Then if one assumes the Conjecture of 1.0.10 with $m = k - 2$, setting $\lambda = \log(2k-1)$ we have $r_M \leq k - 3$.*

Proof. We have $M = \mathbb{H}^3/\Gamma$, where $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^3)$ is discrete, compact, and torsion-free.

We will assume that $r_M \geq k-2$ and proceed by way of contradiction. Equivalently, suppose that for all points P in $\mathbb{H}^3$, the minimum enveloping rank of G_P is $\geq k-2$. Then in particular, $\mathbb{H}^3 = \bigcup_{C_1^P,\ldots,C_{k-2}^P \in C_{\log(2k-1)}(\Gamma), P\in\mathbb{H}^3} Z_{\log(2k-1)}(C_1^P) \cap \cdots \cap Z_{\log(2k-1)}(C_{k-2}^P)$ as described in (2.0.19). Without loss of generality we write

$\mathbb{H}^3 = \bigcup_{C_1,\ldots,C_{k-2} \in C_{\log(2k-1)}(\Gamma)} Z_{\log(2k-1)}(C_1) \cap \cdots \cap Z_{\log(2k-1)}(C_{k-2})$, and define the family

$\mathcal{Z} = (Z_{\log(2k-1)}(C_i))_{C_i \in C_{\log(2k-1)}(\Gamma), 1\le i\le k-2}$.

We have that $\mathcal{Z}$ is an open cover of $\mathbb{H}^3$ which satisfies the hypothesis of Lemma 3.0.26. Then if K denotes the nerve of $\mathcal{Z}$, the result gives that $|K| - |K_{(k-3)}| \cong \mathbb{H}^3$. Since the inclusion $|K^{(n)}| - |K_{(k-3)}| \to |K| - |K_{(k-3)}|$ induces isomorphisms on π_0 and π_1 for $n \ge k-1$ (see (6, Lemma 5.6)), it follows that $|K^{(k-1)}| - |K_{(k-3)}|$ is connected and simply connected.

Let σ be an open simplex in $|K^{(k-1)}| - |K_{(k-3)}|$. Applying Lemma (2.0.14) with $n = \mathrm{d}im(\sigma) + 1$ (i.e. n is the number of vertices of σ and therefore the number of associated maximal cyclic subgroups of Γ whose associated cylinders have nonempty intersection, as is determined by the nerve), we have that the rank of $\Theta(\sigma)$ is less than or equal to $k-1$. Now since σ is in $|K^{(k-1)}| - |K_{(k-3)}|$, by definition the minimum enveloping rank of $\Theta(\sigma)$ is at least $k-2$. In particular, the rank of $\Theta(\sigma)$ is at least $k-2$ by (1.0.6).

5.0.41. All together, we conclude that for any open simplex σ in $|K^{(k-1)}| - |K_{(k-3)}|$, the rank of $\Theta(\sigma)$ is $k-2$ or $k-1$. So, we may write $|K^{(k-1)}| - |K_{(k-3)}|$ as a disjoint union of the saturated subsets X_{k-2} and X_{k-1}, where X_i for $i = k-2, k-1$ is the union of all open simplices σ of $K^{(k-1)}$ for which $\Theta(\sigma)$ has rank i.

We claim the following:

5.0.42. For $i \in \{k-2, k-1\}$ and for any component W of X_i, the local rank of $\Theta(W)$ is at most i.

Proof. First we consider the case when $i = k-2$. Then W is a component of X_{k-2}, and for any open simplex σ of X_{k-2}, rank $\Theta(\sigma)$ is exactly $k-2$. Taking $r = k-2$ in Proposition (4.0.29), specifically in item (ii), we have that the local rank of $\Theta(W)$ is at most $k-2$. The proof in the case of (ii) shows that the local rank of $\Theta(W)$ is *exactly* $k-2$.

Suppose next that $i = k-1$. Then W is a component of X_{k-1}, and so for each open simplex σ of X_{k-1}, we have rank $\Theta(\sigma)$ is exactly $k-1$. If d denotes the dimension of σ, then the subgroup $\Theta(\sigma)$ is generated by $d+1$ cyclic groups which are elements of $C_{\log(2k-1)}(\Gamma)$. Hence rank $\Theta(\sigma) \leq d+1$ and in particular $d \geq$ rank $\Theta(\sigma) - 1$. As rank $\Theta(\sigma) = k-1$, we have $d \geq k-2$. But because σ is a simplex contained in $K^{(k-1)}$, d is less than or equal to $k-1$, and so we must have $d = k-2$ or $k-1$. Letting $r = k-1$ and $n = k-1$ in item (i) of Proposition (4.0.29), we satisfy the hypotheses and the conclusion gives that $\Theta(W)$ has local rank at most $r = k-1$ as desired. $\square$

Next, we claim:

5.0.43. The local rank of $\Theta(W)$ is exactly $k-2$ or $k-1$.

Proof. Let l_W be the *local* rank of $\Theta(W)$. Our previous claim shows that $l_W \leq k-1$. If in fact $l_W \leq k-3$, then by definition any finitely generated subgroup of $\Theta(W)$ is contained in a finitely generated subgroup of rank less than or equal to $k-3$. As $\Theta(\sigma) \leq \Theta(W)$, this says that $\Theta(\sigma)$ is contained in a subgroup of rank less than or equal to $k-3$ and so the minimum enveloping rank of $\Theta(\sigma)$ would be $\leq k-3$ in this situation. However, given an open simplex σ in W, in particular σ is a simplex of $|K^{(k-1)}| - |K_{(k-3)}|$ and so $\Theta(\sigma)$ has minimum enveloping rank $\geq k-2$, providing a contradiction. Therefore, l_W is $k-2$ or $k-1$. □

5.0.44. (The analogue of (6, Claim 5.13.2)) If W is a component of X_{k-2} or X_{k-1}, the normalizer of $\Theta(W)$ in Γ has local rank at most $k-1$.

Proof. As a subgroup of Γ, the normalizer of $\Theta(W)$ is k-free. Clearly $\Theta(W)$ is a normal subgroup of its normalizer, and since by the result of (5.0.43) we have $l_W = k-2$ or $k-1$ which are strictly less than k, it follows by (6, Proposition 4.5) that the normalizer of $\Theta(W)$ has local rank at most l_W. □

Set $T = \mathcal{G}(X_{k-2}, X_{k-1})$ (see Definitions 5.0.36). By Lemma 5.0.37, T is a homotopy-retract of $X_{k-2}\dot{\cup}X_{k-1}$, which is equal to $|K^{(k-1)}| - |K_{(k-3)}|$ by (5.0.41). Since $|K^{(k-1)}| - |K_{(k-3)}|$ is connected and simply connected, T is a tree.

By Definition 3.0.21 of the Γ-labeling compatible action of Γ on K, we see that for any $\gamma \in \Gamma$ and σ in $K^{(k-1)}$, $\Theta(\sigma)$ and $\Theta(\gamma \cdot \sigma)$ are conjugates in Γ (see Remark 5.0.39), and so have equal rank. Consequently, X_{k-2} and X_{k-1} are invariant under the action of Γ. Note that if w is a vertex of T, the stabilizer Γ_w of w in Γ is really the stabilizer of the associated component W in X_{k-2} or X_{k-1}, and so by Remark 5.0.39, $\Gamma_w \leq$ normalizer $\Theta(W)$.

5.0.45. By our work above in (5.0.44), the local rank of normalizer $\Theta(W)$ is at most $k-1$, and given that it contains Γ_w as a subgroup, Γ_w must also have local rank at most $k-1$, and, in particular, is locally free being a subgroup of Γ which is k-free.

Therefore we have constructed an induced action by Γ on the tree T *without* inversions. Since the stabilizer of any vertex of T is locally free as a subgroup of Γ by (5.0.45), our construction admits a contradiction to Lemma 5.0.38. $\square$

The following Propositions and Definitions will be used to explain the geometry of the cases when $r_M(\lambda) = 0$ and 1, and in particular will be used when $\pi_1(M)$ is 5-free and $\lambda = \log 9$ in Corollary 6.0.56.

Proposition 5.0.46. *Suppose $\lambda > 0$ and $M = \mathbb{H}^3/\Gamma$ is a closed, orientable hyperbolic 3-manifold with Γ discrete and purely loxodromic. If $r_M = 0$, then M contains an embedded ball of radius $\lambda/2$.*

Proof. As $r_M = 0$, rank $G_P \geq 0$ for all $P \in \mathbb{H}^3$, and in particular, the choice of r_M means there is a point $P_0 \in \mathbb{H}^3$ with rank $G_{P_0} = 0$. Then $P_0 \notin Z_\lambda(C)$ for any $C \in C_\lambda(\Gamma)$, and so $d(P_0, \gamma \cdot P_0) \geq \lambda$ for all $\gamma \in \Gamma - \{1\}$, and more generally, $\mathbb{H}^3 \neq \bigcup_{C \in C_\lambda(\Gamma)} Z_\lambda(C)$. If $B_{P_0}(\lambda/2)$ denotes the hyperbolic open ball of radius $\lambda/2$ with center P_0, in particular this says that the injectivity radius of $B_{P_0}(\lambda/2)$ in M is $\lambda/2$; namely $B_{P_0}(\lambda) \cap \gamma \cdot B_{P_0}(\lambda) = \emptyset$. To see this, consider a point P' in $B_{P_0}(\lambda)$. If in fact it was true that $\gamma(P')$ is also in $B_{P_0}(\lambda)$, it would then follow that $d(P_0, \gamma \cdot P_0) \leq d(P_0, \gamma \cdot P') + d(\gamma \cdot P', \gamma \cdot P_0) < \lambda/2 + \lambda/2 = \lambda$, giving a contradiction. Therefore if $q : \mathbb{H}^3 \to M$ is the projection map, $q|B : B \to M$ is injective and the conclusion follows. $\square$

Definitions 5.0.47. Let $\mathfrak{X}_M$ be the set of points P in M such that if l_P denotes the length of the shortest, homotopically non-trivial loop based at P, then there is a maximal cyclic subgroup

D_P of $\pi(M, P)$ such that for every homotopically non-trivial loop c based at P of length l_P, we have $[c] \in D_P$. Note that the loop c of length l_P may represent a proper power of a generator of D_P. Let $\mathfrak{s}_M(P)$ be the smallest length of any loop c based at P such that $[c] \notin D_P$.

Proposition 5.0.48. *Suppose $\lambda > 0$ and $M = \mathbb{H}^3/\Gamma$ is a closed, orientable hyperbolic 3-manifold with Γ discrete and torsion-free. If $r_M = 1$, then there exists a point $P^* \in \mathbb{H}^3$ with $P* \in \mathfrak{X}_M$ and $\mathfrak{s}_m(P*) = \lambda$.*

Proof. As $r_M = 1$, the definition of r_M gives that rk $G_P \geq 1$ for all $P \in \mathbb{H}^3$ (more generally that $\mathbb{H}^3 = \bigcup_{C \in C_\lambda(\gamma)} Z_\lambda(C))$ *and* that there is a point $P_0 \in \mathbb{H}^3$ with rk $G_{P_0} = 1$. Hence $P_0 \in Z_\lambda(C_0)$ for some $C_0 \in C_\lambda(\Gamma)$ and $P_0 \notin Z_\lambda(C)$ for any other $C \in C_\lambda(\Gamma) - C_0$, namely $G_{P_0} = \langle C_0 \rangle$. Set $Z_0 = Z_\lambda(C_0)$ and $Y = \bigcup_{C \in C_\lambda(\Gamma) - C_0} Z_\lambda(C)$. Then $\mathbb{H}^3 = Y \cup Z_0$. Since $\mathbb{H}^3$ is connected, $(Z_\lambda(C))_{C \in C_\lambda(\Gamma)}$ is an open cover, and Γ is discrete, we must have the intersection $Y \cap Z_0$ is nonempty and open. Notice $P_0 \notin Y$ means $Z_0 \not\subseteq Y$. As Z_0 is connected, we conclude that the frontier of the set $Y \cap Z_0$ relative to Z_0 is nonempty; let F denote this set. Let us choose a point P^* in F. In particular, this says that (i) $P^* \in Z_0$ *and* (ii) P^* is in the frontier of Y (relative to $\mathbb{H}^3$)). (In concluding (ii), recall that the collection of cylinders in Y comprises a locally finite collection because Γ is *discrete*, and so P^*, a limit point of Y, does not belong to this open collection). If γ_0 is a generator for C_0, (i) implies that $d(P^*, \gamma_0^m \cdot P^*) < \lambda$ for some integer $m \geq 1$. By (ii), we know that $d(P^*, \gamma_1 \cdot P^*) = \lambda$ for some $\gamma_1 \in \Gamma - \gamma_0$ *and* that $d(P^*, \gamma \cdot P^*) \geq \lambda$ for all $\gamma \in \Gamma - C_0$. Using the base point $P^* \in \mathbb{H}^3$ to identify $\pi(M, q(P^*))$ with Γ, we have that γ_0^m is represented by a loop of length less than λ based at $q(P^*)$, and any other homotopically non-trivial loop of length less than λ based at $q(P^*)$ is identified with an element of C_0. Therefore, we have shown the existence of a point $P^* \in \mathfrak{X}_M$ for which the smallest length of any loop represented by $[c]$ in M based at P^* with the property that $[c]$ is not in D_{P^*}, is exactly λ. $\square$

CHAPTER 6

MATRICES AND THEOREM FOR THE CASE K=5

We will now restate some of Kent's constuction and results regarding joins and intersections of free groups; in particular, we incorporate the background (6.0.49 and 6.0.50) as discussed in (8) which is needed to apply (8, Lemma 7) and (8, Lemma 8) to prove Proposition 6.0.52 that follows. Subsequently, Conjecture 1.0.10 for rank-3 subgroups H and K is affirmed in Corollary 6.0.53.

6.0.49. Given a free group F free on the set $\{a, b\}$, we associate with F the wedge W of two circles based at the wedge point, and we orient the (two) edges of W. Then for any subgroup H of F there is a unique choice of basepoint $*$ in the covering space $\widetilde{W_H}$ such that $\pi_1(\widetilde{W_H}, *)$ is exactly H. Then Γ_H will denote the smallest subgraph containing $*$ of $\widetilde{W_H}$ that carries H. By this construction, Γ_H inherits a natural oriented labeling, i.e. each edge is labeled with one of $\{a, b\}$ and initial and terminal vertices (not necessarily distinct) are determined by the orientation. Hence $\mathrm{rank}\pi_1(\Gamma_H) = \mathrm{rank}\ H$. Also by this construction, any vertex of Γ_H is at most 4-valent. Define a 3- or more valent vertex of Γ_H to be a *branch vertex.* We will from here on assume that all graphs in our duscussion are normalized so that all branch vertices are 3-valent (see the beginning of (8, Section 3) for this explanation).

6.0.50. If Γ is a graph, let $b(\Gamma)$ denote the number of branch vertices in Γ. Note that if Γ is 3-regular, i.e. all branch vertices are 3-valent, then we have $\overline{\chi}(\Gamma) = \mathrm{rank}(\pi_1(\Gamma)) - 1 = b(\Gamma)/2$.

By 6.0.49 this says that if H, K are subgroups of F, then $\operatorname{rank}(\pi_1(\Gamma_{H\vee K}))-1 = \operatorname{rank}(H\vee K)-1$. If $V(\Gamma_H)$ and $V(\Gamma_K)$ denote the vertex sets of Γ_H and Γ_K respectively, then we can define the graph $\mathcal{G}_{H\cap K}$ whose vertex set is the product $V(\Gamma_H)\times V(\Gamma_K)$ and for which $\{(a,b),(c,d)\}$ is an edge if there are edges $e_1 = \{a,c\}$ in Γ_H and $e_2 = \{b,d\}$ in Γ_K for which e_1 and e_2 have the same label, and e_1 is oriented from a to c and e_2 is oriented from b to d. The graph $\mathcal{G}_{H\cap K}$ is the pullback of the maps $\Gamma_H \to W$ and $\Gamma_K \to W$ in the category of oriented graphs, and $\Gamma_{H\cap K}$ is a subgraph of $\mathcal{G}_{H\cap K}$ that carries the fundamental group. We then have the projections $\Pi_H : \mathcal{G}_{H\cap K} \to \Gamma_H$ and $\Pi_K : \mathcal{G}_{H\cap K} \to \Gamma_K$. Let the graph $\mathcal{T}$ denote the topological pushout of the maps $\Gamma_{H\cap K} \to \Gamma_H$ and $\Gamma_{H\cap K} \to \Gamma_K$ in the category of not properly labeled oriented graphs. Hence the graph $\mathcal{T}$ is defined as the quotient of the disjoint union $\Gamma_H \cup \Gamma_K$ modulo $\sim_{\mathcal{R}}$ where $x \in H$ is equivalent to $y \in K$ if $x \in \Pi_H(\Pi_K^{-1}(y))$ or $y \in \Pi_K(\Pi_H^{-1}(x))\Pi_H$. Now since the map $\mathcal{T} \to \Gamma_{H\vee K}$ factors into a series of folds (which is surjective at the level of π_1), it follows that $\chi(\mathcal{T}) \le \chi(\Gamma_{H\vee K})$. Equivalently $\overline{\chi}(\Gamma_{H\vee K}) \le \overline{\chi}(\mathcal{T})$.

6.0.51. As outlined in Kent's Section 3.2 (8), we consider the $(2h-2)\times(2k-2)$ matrix, $M = (f(x_i, y_j))$, where $f : X \times Y \to \{0,1\}$ is the function defined on the sets $X = \{x_1, \ldots, x_{2h-2}\}$ and $Y = \{y_1, \ldots, y_{2k-2}\}$ of branch vertices of Γ_H and Γ_K, respectively, by letting $f(x_i, y_j) = 1$ if (x_i, y_j) is a branch vertex of $\Gamma_{H\cap K}$ and 0 otherwise. Then the number of ones in M is equal to the number of valence-3 vertices in $\Gamma_{H\cap K}$; i.e. $b(\Gamma_{H\cap K}) = \Sigma_{i,j} f(x_i, y_j)$. From 6.0.50, we have $\overline{\chi}(\Gamma_{H\cap K}) = b(\Gamma_{H\cap K})/2$. By (8, Lemma 8), after permuting rows and columns of M, we may write M in the block form: $(M_1, \ldots, M_l, O_{(p\times q)})$ where $O_{(p\times q)}$ is the $p\times q$ zero matrix, every row and every column of each of the M_i has a 1. Here, every block M_i of M represents a unique equivalence class of $\sim_{\mathcal{R}}$ with representatives in Γ_H and Γ_K; all-zero rows represent the $\leq p$ equivalence class(es) of Γ_H which do not have representatives in Γ_K; and all-zero column(s) represent the $\leq q$ equivalence class(es) of Γ_K which do not have representatives in Γ_H.

Proposition 6.0.52. $\mathrm{rank}(H \vee K) \leq 1 + \frac{1}{2}(l + p + q)$.

Proof. Note that $2h - 2 \geq l + p$ and $2k - 2 \geq l + q$ as $2h - 2$ is $\#\{\text{rows of } M\}$ and $2k - 2$ is $\#\{\text{columns of } M\}$ where M is the block matrix of (6.0.51), and so l is bounded above by the positive integer $\min((2h-2) - p, (2k-2) - q)$. We have $\mathrm{rank}(H \cup K) - 1 \leq \overline{\chi}(\mathcal{T}) \leq \frac{1}{2}(l+p+q)$ by combining 6.0.50 and 6.0.49 along with (8, Lemma 7) for the first inequality and (8, Lemma 8) for the second. In particular, $\mathrm{rank}(H \vee K) \leq 1 + \frac{1}{2}(l + p + q)$ as required. $\square$

Corollary 6.0.53. *Suppose* $h = k = 3$ *and* $\mathrm{rank}(H \cap K) \geq 3$. *Then* $\mathrm{rank}(H \vee K) \leq 3$.

Proof. As $h = k = 3$, we consider the 4×4 block matrix M of 6.0.51 where each row and each column of each of the M_i has a 1. So the number of ones, which is the number of valence-3 vertices in $\Gamma_{H \cap K}$, is ≥ 4.

6.0.54. Note that l is bounded above by $min(4 - p, 4 - q)$ by the proof of Proposition 6.0.52, and so in particular, $l + p \leq 4$ and $l + q \leq 4$. Thus we may rewrite Proposition 6.0.52 to read $\mathrm{rank}(H \vee K) \leq 1 + min(2 + \frac{p}{2}, 2 + \frac{q}{2})$. Using this formula, we see that $\mathrm{rank}(H \vee K) \leq 3$ unless p and q are ≥ 2, and so we need only consider the following cases:

Case $p = 4$ *or* $q = 4$*:* This case is an impossibility, as this would imply $M = O_{4\times 4}$, and hence the number of branch vertices of $\Gamma_{H \cap K}$ is zero, a contradiction, and so we must have $p, q \leq 3$.

Case $p = 3$ or $q = 3$: Suppose first that $p = 3$. Then (6.0.54) says that $l \leq 1$, implying that $l = 1$ and the top row of M has 4 ones, and so $q = 0$. In this case, the inequality of (6.0.54) gives $\mathrm{rank}(H \vee K) \leq 1 + \mathrm{m}in(2 + \frac{3}{2}, 2)$, and so $\mathrm{rank}(H \vee K) \leq 3$. When $q = 3$ the argument is symmetric, and so the conclusion is satisfied.

Case $p = 2$ or $q = 2$: By symmetry assume $p = 2$. This gives $l \leq 2$ by the bound on l of (6.0.54). Now if $l \leq 1$, then q must be equal to 3 to satisfy the requirement on the number of ones in M (i.e. the valence-3 vertices in $\Gamma_{H\cap K}$), which is the previous case. Next, if $l = 2$, then as $l \leq \mathrm{m}in(4-p, 4-q)$, we have $q \leq 2$. First, if $p = q = 2$, then the requirement that the number of ones in M is ≥ 4 fails as the values of p, q, and l would force M to have the form $(M_1, M_2, O_{2\times 2})$ where $M_1 = M_2 = (1)$, and so M would only contain two ones. Next, if $q = 1$, then again we apply (6.0.54) to give $\mathrm{rank}(H \vee K) \leq 1 + \mathrm{m}in(2 + \frac{2}{2}, 2 + \frac{1}{2}) = 1 + \mathrm{m}in(3, 2.5) = 3.5$. Of course, this says that $\mathrm{rank}(H \vee K) \leq 3$ as ranks must be integral and the conclusion is established. □

We now restate Theorem 1.0.11 from the Introduction:

Theorem 6.0.55. *Suppose M is a closed, orientable, hyperbolic 3-manifold such that $\pi_1(M)$ is k-free with $k = 5$. Then when $\lambda = \log 9$, we have $r_M \leq 2$.*

Proof. This is a direct result of Corollary 6.0.53 along with Theorem 5.0.40. □

For the final Corollary recall Definitions 5.0.47.

Corollary 6.0.56. *Suppose $M = \mathbb{H}^3/\Gamma$ is a closed, orientable hyperbolic 3-manifold with $\Gamma \leq \mathrm{Isom}_+(\mathbb{H}^3)$ discrete, purely loxodromic and 5-free. Then when $\lambda = \log 9$, one of the following three alternatives holds:*

(i) M *contains an embedded ball of radius* $\log(9/2)$*,*

(ii) There exists a point $P^* \in \mathbb{H}^3$ *with* $P^* \in \mathfrak{X}_M$ *such that* $\mathfrak{s}_m(P^*)$*, is equal to* $\log 9$*, or*

(iii) $\mathbb{H}^3 = \bigcup_{C_1, C_2 \in \mathcal{C}_{\log 9}(\Gamma)} Z_{\log 9}(C_1) \cap Z_{\log 9}(C_2)$ *(i.e.* $\text{rank } G_{\widetilde{P}} \geq 2$ *for all* $\widetilde{P} \in \mathbb{H}^3$*),* and *there exists a point* $\widetilde{P} \in \mathbb{H}^3$ *such that* $\text{rk } G_{\widetilde{P}} = 2$*.*

Let $q : \mathbb{H}^3 \to \mathbb{H}^3/\Gamma$ *be the projection map. As* $M = \mathbb{H}^3/\Gamma$*, we have* $\Gamma \cong \pi_1(M)$*. We may then equivalently restate (iii) to say there exists a point* $P = q(\widetilde{P})$ *in* M *such that the class of all homotopically non-trivial loops of* $\pi_1(M, P)$ *of length* $\leq \log 9$ *is contained in a rank-*2 *subgroup of* Γ*.*

Proof. The result of Theorem 6.0.55 is that $r_M \leq 2$; so the only possible values for r_M are 0, 1 and 2.

Case (i) follows when $r_M = 0$ and is the result of Proposition 5.0.46, and Case (ii) follows when $r_M = 1$ and is the result of Proposition 5.0.48. Case (iii) occurs when $r_M = 2$ and is merely restating that definition. $\square$

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