Topics in Knowledge Representation: Belief Revision and Conditional Knowledge Bases

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Chapter 2 contains published work. Chapter 3 contains work submitted for publication. Chapter 4 contains work to be submitted for publication in the future. I am the primary author of the research in all three chapters. My thesis advisor, Professor György Turán contributed to the research in all three chapters, and Professor Jan Verschelde contributed to the research in Chapter 4.

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SUMMARY

Results in two topics within knowledge representation and reasoning are presented. The first, belief revision, concentrates on incorporation of new knowledge into previous knowledge. The objective of research presented herein is to provide a formal logical framework to evaluate whether a finite set of logical statements, postulates, can be used to characterize a class of belief revision operators. A connection between characterizability by postulates and definability in a fragment of second order logic is established. This connection allows tools from finite model theory to be employed to identify classes of belief revision operators which are not characterizable. This framework is developed in Chapter 2 and extended to the case of Horn belief revision in Chapter 3. In addition, examples of characterizable and non-characterizable classes of belief revision operators are given to demonstrate the application of this framework.

The second topic is a method for inference on conditional knowledge bases proposed by Kern-Isberner et al. A set of conditional statements with associated probabilities form a conditional knowledge base. However, conditional knowledge bases may not contain enough information to fully determine a probability distribution. Therefore, the maximum entropy principle is applied, allowing probabilities of statements outside the knowledge to be inferred. The technique explored in Chapter 4 represents knowledge bases as polynomials and uses methods from computational algebra for inference on new conditional statements. Several examples and experiments were conducted to determine the viability of using this method for inference with arbitrary conditional knowledge bases. Additionally, explanations regarding the algebraic geometry of these examples are given.

CHAPTER 1

INTRODUCTION

1.1 Knowledge Representation and Reasoning

Acquiring, conceptualizing, evaluating, and modifying knowledge are fundamental parts of the human experience. For this reason, the subfield of knowledge representation and reasoning is also an essential part of artificial intelligence research. As a consequence of the integral part knowledge plays in our everyday lives, the challenge of translating these processes to computational models and procedures can be easily underestimated. Delving into modeling knowledge and reasoning systems reveals a wealth of sophisticated compelling applications and active multidisciplinary research areas.

Due to the ubiquitous nature of knowledge and its use, a variety of fields have begun developing applications using knowledge representation. The need to organize, manipulate, and draw conclusions from information plays a vital role in many of these applications. Increases of computational power and availability of data have been accompanied by the drive to implement the theoretical models of knowledge representation. However, theoretical and applied models may differ greatly in their focus and methodology. For instance, a theoretical approach may employ techniques which are not computationally efficient, rendering it impractical. Conversely, an application may have an efficient implementation which does not adhere to widely accepted theoretical principles, causing behavior which may be unexpected or undesirable. Therefore, frameworks bridging between the theoretical and the practical serve an important function in the field's growth.

Another consequence of knowledge representation being applicable in many fields is that unified research from a range of disciplines is required. Application areas include inferring information from text or video, automating computerized decision making, merging conflicting information from multiple sources, and evaluating context of natural language input. For such applications, knowledge representation must integrate techniques developed by engineers, computer scientists, mathematicians, cognitive scientists, philosophers, and others to design pertinent solutions. These solutions readily apply techniques from mathematics, statistics, logic, and computer science with the goal of devising models for representing knowledge and algorithms for reasoning on that knowledge. Moreover, one may anticipate frameworks connecting theoretical and practical aspects of knowledge representation to also be multidisciplinary in nature.

1.2 Background

Knowledge representation can be roughly divided into two major approaches – logic and probability. A broad survey of knowledge representation may give the impression these are incompatible approaches. Each approach contributes by addressing pressing issues which exist in knowledge representation. Despite the weaknesses of each approach, compelling reasons remain to integrate research from these two approaches.

Logic has an established role in representation and reasoning in mathematics. A natural next step would be to extend formal logic to a system which is capable of handling human knowledge. Even though logic has been an indispensable tool in knowledge representation for representing knowledge and changes in that knowledge, several of the shortcomings of traditional logic are defining features of human reasoning. For example, in mathematical logic, the addition of new information always results in the expansion of knowledge. In particular, the addition of contradictory information results in the ability to conclude anything. Human reasoning does not seem to conform to this property of classical logic. As a result, nonmonotonic logic was developed to describe the situation in which obtaining new information does not necessarily cause expansion of previously known information.

Furthermore, classical logic does not offer an adequate solution for the representation of uncertainty. Within many applications, information is known with some level of confidence. Probability is well equipped to resolve representation problems which involve uncertainty. Moreover, probability has proven quite successful in fields such as natural language processing and machine learning. Even though uncertainty remains only one aspect of knowledge, instances occur in which logic representation would be more appropriate.

Merging logic and probability is an active area of research with a myriad of exciting open problems and worthwhile application. As noted recently in [37], incorporation of logical and statistical approaches may be vital for implementations of knowledge representation such as semantic web applications. Therefore, the future may include applications which employ multiple integrated techniques from both approaches to knowledge representation.

1.2.1 Belief Change

Belief change studies the operations of extending, revising, and retracting information from a knowledge base. The first portion of this thesis will focus will on belief revision, the incorporation of new information into currently held beliefs. More formally, given a knowledge base K, a belief revision operator $*$ assigns a formula $K * \varphi$ to every formula φ. Here φ is called the revising formula, and $K * \varphi$ is called the revised knowledge base. If the revising formula is consistent with K, then revision is equivalent to adjoining the knowledge base by the revising formula. In this case, the expansion operator $+$ is written instead of $*$.

While different approaches to belief revision exist, one of the most influential approaches was introduced by Alchourrón, Gärdenfors, and Makinson (AGM). In [3], they propose an axiomatic approach by requiring a rational revision operator to satisfy eight postulates [3].

AGM1 $K * \varphi$ is logically closed.

AGM2 $\varphi \in K * \varphi$.

AGM3 $K * \varphi \subseteq K + \varphi$.

AGM4 If $\neg\varphi \notin K$, then $K * \varphi = K + \varphi$.

AGM5 $K * \varphi$ is inconsistent only if φ is inconsistent.

AGM6 If $\mu = \varphi$, then $K * \mu = K * \varphi$.

AGM7 $K * (\mu \wedge \varphi) \subseteq (K * \mu) + \varphi$.

AGM8 If $\neg\mu \notin K * \varphi$, then $(K * \mu) + \varphi \subseteq K * (\mu \wedge \varphi)$.

An epistemic state is a knowledge base with additional epistemic information used in the belief change operations. Although additional information may be represented in various forms, the current work is primary concerned with preference relations over possible worlds or truth assignments. In general, there may be numerous ways to revise a given knowledge base. The purpose of the preference relation is to contribute information which aids in determining the revised knowledge base. One contribution area is the work by Katsuno and Mendelzon. In [27], they defined a revision operator by minimization on total preorders of truth assignments.

Theorem 1.2.1. *(Katsuno, Mendelzon* [27]*) A belief revision operator satisfies the AGM postulates iff it can be obtained from a faithful total preorder using minimization.*

This characterization is a special case of the system of spheres characterization introduced by Grove in [22]. Kastuno and Mendelzon also proved a result for revision operators generated by minimization on a partial orders.

Theorem 1.2.2. *(Katsuno, Mendelzon* [27]*) A belief revision operator satisfies a modified set of AGM postulates iff it can be obtained from a faithful partial order using minimization.*

In the past several years, a growing body of results on how to adapt the AGM theory to Horn logic has been published [13] [43]. A Horn clause is a disjunction of literals with at most one positive literal. A Horn formula is a conjunction of Horn clauses. Horn formulas provide an expressive, yet tractable knowledge representation framework. In particular, satisfiability and equivalence of Horn formulas are efficiently computable, a desirable quality for applications. Therefore, development of such operators hold promising potential for practical applications. In [11], Delgrande and Peppas offer a representation result for Horn revision operators which consists of infinitely many postulates by appending a scheme called acyclicity to the AGM postulates.

Theorem 1.2.3. *(Delgrande, Peppas* [11]*) A Horn belief revision operator satisfies the Horn AGM postulates and the acyclicity postulate scheme iff it can be obtained from a Horn compliant faithful ranking using minimization.*

1.2.2 Conditional Knowledge Bases

Horn belief revision confronts the challenge of constructing efficiently computable belief revision operators, while conditional knowledge bases attempt to build models addressing other obstacles to practical applications. More precisely, models which represent knowledge with propositional logic lack the ability to express uncertainty, yet uncertainty is prevalent in human reasoning.

As an example, consider creating a model of information acquired regarding a patient's symptoms and medical history. The goal is to infer a diagnosis of the patient. This information may offer neither a complete description of the patient's health nor certainty of the condition indicated by their symptoms. At most one may be able to relate symptoms and medical conditions by statements such as

Given the patient has symptom A, there is a probability, P, that they have condition B.

Given this set of knowledge with associated measures of uncertainty, there should be a method to infer the likelihood of unknown information. In the example, information may exist linking individual symptoms and medical conditions; however, for diagnosis, the probability of a medical condition A_j given all the symptoms $B_1, ..., B_j$ would need to be inferred. From this information, if a probability distribution were assigned such that known probabilities are preserved, then a probability could be inferred for the A_j given $B_1, ..., B_j$. However, there could be many distributions which satisfy this condition. Therefore, there must be additional criteria for selecting a probability distribution. One way to accomplish this would be to attempt to minimize assumptions made when assigning probabilities.

The framework for conditional knowledge bases developed by [28] does exactly this. A knowledge base, KB, over a propositional logic language, $\mathcal{L}$, is of the form

$$KB = \left\{ B_j | A_j \left[\frac{n_j}{m_j} \right] : A_j, B_j \in \mathcal{L}, m_j, n_j \in \mathbb{N},\ 0 \le j \le s \right\}.$$

Each element in the knowledge base is a conditional statement, B_j given A_j, with a probability $\frac{n_j}{m_j}$. A knowledge base may not provide enough information to determine a unique probability for all possible conditional formulas. For this reason, the method of maximum entropy is borrowed from information theory. Maximum entropy minimizes bias, while determining a unique probability distribution. Once this distribution is found, other statements not included in the knowledge base may be inferred.

In [30], techniques from [16] are adapted to this specific problem to produce a system of polynomials representing a knowledge base. The objective of encoding the maximum entropy principle into a polynomial system was to use tools from computational algebraic geometry. Computationally, this representation of the problem allows for use of techniques from computational algebraic geometry. These algorithms are well known and researched as well as widely

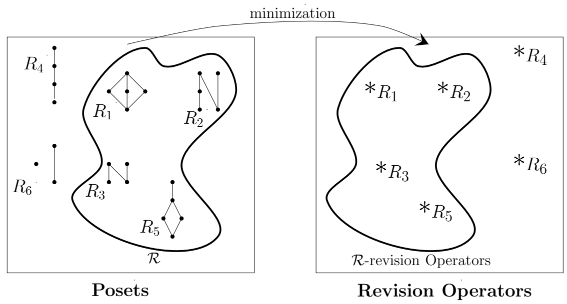

Figure 1: Classes of Posets Generate $\mathcal{R}$-revision Operators

available in common computer algebra systems. Additionally, this approach provides an opportunity to investigate the algebraic structure of conditional knowledge bases.

1.2.3 Overview

Chapter 2 focuses on developing a logical framework for the characterizability of belief revision operators. A fundamental realization of attempts to formalize mathematics was that some logical systems cannot be finitely axiomatized. In other words, there is no finite set of logical statements which fully describe the system. Similarly to these results about formal logic, are there analogous situations in belief revision, when a finite set of logical statements is unable to describe classes of revision operators? Such issues were also studied in modal logic [5] and dynamic epistemic logic [40]. However, much of the belief revision literature has concentrated on providing postulates for various operators, without consideration of negative results.

In order to study this problem, consider classes of belief revision operators, where a class can be defined by a class of posets. This class, then, defines the class of revision operators generated by minimization from posets in the class. The situation is pictured in Figure 1. Given a class of posets over the possible worlds with known properties, the question is what properties of the revision operators are entailed by this information? For example, total orders mean that any two models can be compared, while bounded height means that the length of any chain of comparable models is bounded. Depending on the type of property describing the class of posets, this observation may be used to give a postulate characterization of the corresponding class of revision operators.

The first step in building a framework for this study, one needs a precise definition of a postulate. This issue was not considered so far in the belief revision literature, with the exception of the work of Schlechta [38] and Ben-Naim [4] to be discussed later on. As noted in the survey [19]

> "[T]heories of belief change developed in the AGM tradition are not logics in a strict sense, but rather informal axiomatic theories of belief change. Instead of characterizing the models of belief and belief change in a formalized object language, the AGM approach uses a natural language (ordinary mathematical English) to characterize the mathematical structures under study."

After formalizing the notion of a postulate, a relationship is to be found between the postulate definability classes of revision operators, and the definability classes of posets. The main observation in establishing such a connection is that postulates are implicitly universally quantified over revising formulas, and for posets generating the revision operators this corresponds to universal quantification over subsets of the universe, i.e., to universal monadic second-order quantification.

The formal definitions for postulates are given in Definition 2.5.1. These basic postulates can then be used to formalize the idea of characterizability of revision operators by Theorem 2.7.3. The relationship between postulate characterizability of classes of revision operators and definability of the corresponding classes of posets reduces the problem of proving non-characterizability by postulates to proving undefinability of the poset class. The situation is complicated by the fact that belief revision models are finite structures. Most of the tools from mathematical logic to prove non-axiomatizability are not applicable for finite models. For this reason, techniques are used from finite model theory, a subfield which studies finite logical structures.

The main technique to prove undefinability in finite model theory is model-theoretic games. A version of such games is defined, such that a winning strategy corresponds to characterizability by basic postulates. The game given in Definition 2.8.4 is played on two structures, one in the class $\mathcal{R}$ and the other outside $\mathcal{R}$. The goal for Duplicator/Spoiler, the two players, is to maintain/avoid a partial isomorphism generated by the game moves on the element and subset relations of the underlying structures. The minimal subsets generated by a revision operator using minimization are encoded in a structure called a min-variant given in Definition 2.8.3. This modification reduces the game to a first-order logic game on the min-variant structures.

Typical applications of such games in finite model theory are to determine the monadic second-order (MSO) definability of graph properties [2, 33]. For instance, this method can prove disconnected graphs are not universal monadic second-order definable ($\forall MSO$). In this work, the implicit quantification of a postulate over all formulas mentioned above gives a translation into a fragment, $\forall MSO_{\min}$, of universal monadic-second order logic with a special predicate for minimal sets. Therefore, the characterizability of a class of operators corresponds to $\forall MSO_{\min}$ - definability by Theorem 2.8.5. Using this game, one can prove, for example, that revision operators over disconnected – as well as connected – posets are not $\forall MSO_{\min}$ -definable. Note that this is in contrast to the fact that the class of connected posets is universal monadic second-order definable. As a consequence, in addition to creating a new application of finite model theory, a new fragment of ∀MSO has been obtained, which is shown to be incomparable to first-order logic as well. Our results on characterizability are summarized in Figure 7.

In Chapter 3, the characterizability framework is adapted to apply to Horn belief revision. The material in this chapter is submitted for publication.

In order to adapt the general framework, several modifications must be made to accommodate postulates composed of Horn formulas. It has to be taken into consideration that revision formulas are now restricted to be Horn, and that Horn logic is closed under conjunction, but it is not closed under negation. In the corresponding version of universal monadic second-order logic, universal monadic second-order quantifiers are replaced by universal monadic second-order quantifiers which range over closed sets of truth assignments as opposed to arbitrary sets as in Definition 3.5.2.

In Theorem 1.2.3, Delgrande and Peppas [11] characterize Horn revision operators obtained by minimization from Horn compliant total preorders using an infinite postulate scheme. Using the characterizability framework for Horn revision, it is shown that this characterization cannot be replaced by a finite set of postulates.

Theorem 3.8.1. *The class of Horn belief revision operators obtained from Horn compliant faithful rankings using minimization cannot be characterized by a finite set of postulates.*

This result may also be of interest as the existence of interesting, inherently infinite postulate characterizations appear to be a new phenomenon in belief revision.

The final section, Section 3.9, discusses the case of strictly Horn compliant revision operators for which characterizability remains an open problem. A generalization of two tractable classes of Horn belief revision operators introduced by Delgrande and Peppas [11] is given, showing that strictly Horn compliant revision operators come from Horn compliant total preorders where the levels are obtained by unit resolutions and weakenings. It also follows that strictly Horn compliant revision operators are tractable in general.

Chapter 4 deals with conditional knowledge bases. The work described in this chapter is mainly experimental, supplemented with material from algebraic geometry and computational algebra.

The goal is to expand on the work presented in [28] and [30] described above, which describes a maximum entropy-based computational algebra approach to the inference problem in conditional knowledge bases. Computational examples, mathematical explanations, and experimental data are provided. The following is a brief outline of the issues considered.

[30] suggests that a polynomial may be simplified by removal of a specific GCD from the polynomials presented in [29]. Solutions for the original and GCD-simplified versions of the polynomial system are considered with respect to the computational output when attempting to infer an unknown probability. Situations in which unexpected output may occur are identified

and explained. Algebraically, this corresponds to considering systems which may have multiple roots between $(0, 1)$ and algebraic independence of polynomials in the system. Geometrically, these situations occur when multiple solutions have a projection on the final coordinate which lie in $(0, 1)$ and the solution set contains higher dimensional sets, respectively.

Experiments are conducted to study the effects of different knowledge base structures on the aforementioned features and computation time. Knowledge base structures were chosen from properties in [28] along with one which did not conform to a known property. A thousand examples were run with random formulas input into the KB structures. Both the original and GCD-simplified polynomial systems were analyzed for multiple roots, algebraic independence, and computation time. The results demonstrate clearly the dramatic differences that the KB structure can have on output. Moreover, sensitivity of the KB structure which was not generated by a property was reflected in its variance in computation time.

The overall conclusion of this work on conditional knowledge bases is that the computational algebra approach can have difficulties both in terms of providing the correct answer, and in terms of computational complexity, and it is of interest to explore structural properties of the knowledge bases where the approach is efficient.

CHAPTER 2

CHARACTERIZABILITY IN BELIEF REVISION

The material of the following chapter was published as part of the proceedings of the 24th International Joint Conference on Artificial Intelligence [39].

2.1 Introduction

Few non-characterizability results exist within the belief revision literature. The first known negative results are due to Schlechta and Ben-Naim [4, 32, 38]. Impossibility results are also proved by Reis et al.[36]. Having tools to prove non-characterizability could be useful when one tries to understand the properties of a class of revision operators. While the standard presentations of belief change start with a set of postulates and then find a matching class of revision operators, there may be situations when a class of revision operators comes first. This may happen, for example, when one considers a class of revision operators which is a natural variant of previously considered ones. For example, a natural subclass of Horn revision operators is called strictly Horn compliant. Frameworks capable of assisting with assessing characterizability could be useful when applied to such classes. Also, one may introduce a class of efficiently computable revision operators for practical purposes. In these cases one may inquire what are the properties of the given class of revision operators? For instance, does it form a "nice" class, which can be characterized by postulates?

In this chapter, methods are developed to prove non-characterizability results using tools from finite model theory. Postulates are translated into a fragment of universal monadic second-order logic and Ehrenfeucht-Fraïssé games are used to prove undefinability. The goal is to "characterize postulate characterizability." This is achieved for families of partial orders by showing that the answer to the question is positive iff the family is definable by a $\forall MSO_{\min}$ sentence, which is a special kind of monadic second-order sentence. Interestingly, $\forall MSO_{\min}$ -definability turns out to be incomparable to first-order definability. This characterization is used to give several examples of characterizable and non-characterizable classes of revision operators. The negative results are proved using a "forgetful" version of the Ajtai-Fagin variant of Ehrenfeucht-Fraïssé games [2, 33].

2.2 General Concepts

This section introduces concepts which will be used within the two chapters discussing characterizabiity of belief revision. Material specific to each topic will then be explained when relevant.

Let $\mathcal{L}$ be a language of propositional logic over finitely many variables. A knowledge base, K, is a logically closed set of propositional formulas over $\mathcal{L}$. Due to $\mathcal{L}$ being finite, K may be represented with a single formula. If needed, K_n is used to indicate that K is over n variables. Both representations of K will be used in fashion appropriate for the context. Belief revision considers the situation in which new information in the form of a propositional formula, φ, is obtained. A revision operator $*$ maps from $\mathcal{S}_{\mathcal{L}} \times \mathcal{S}_{\mathcal{L}}$ to $\mathcal{S}_{\mathcal{L}}$ with $\mathcal{S}_{\mathcal{L}}$ the set of all formulas of

	total	**reflexive**	**antisymmetric**	**transitive**
Pseudo Order	✓	✓		
Partial Preorder		✓		✓
Partial Order		✓	✓	✓
Total Preorder	✓	✓		✓

Figure 2: Relations and their Properties

$\mathcal{L}$. In other words, $*$ is considered to assign a revised knowledge base $K * \varphi$ to every knowledge base K and every revising formula φ.

Truth assignments (or interpretations, models, possible worlds) are assignments of truth values to the variables. The set $\{0,1\}^n$ of truth assignments over n variables is denoted by T_n. A Boolean function is a function of the form $f : \{0,1\}^n \to \{0,1\}$. The set of truth assignments a for which $f(a) = 1$ is denoted by $|f|$, and the set of truth assignments satisfying a propositional formula φ is denoted by $|\varphi|$. Given a closed set A of truth assignments, $\langle A \rangle$ denotes some propositional formula φ such that $|\varphi| = A$.

A binary relation $\leq$ is total iff for all a, b $a \leq b$ or $b \leq a$. Otherwise the relation is called partial. A relation is reflexive iff for all a, $a \leq a$. The relation $b < a$ is strict iff $b \leq a$ but $a \not\leq b$. A relation is called antisymmetric iff for all a and b, $a < b$ implies that $b \not< a$. Transitivity requires that for all a, b, c if $a \leq b$ and $b \leq c$, then $a \leq c$.

Structures $R = (X, \leq)$, where X is a finite ground set and $\leq$ is a binary relation will be considered in the following chapters. Figure 2 gives a summary of the properties of binary relations of these structures.

In addition, several concepts will be needed to speak about the relationship of elements in the ground set. Elements a and b are *comparable* if $a \leq b$ or $b \leq a$ holds. Otherwise they are *incomparable*. The comparability graph of R is the undirected graph over X such that for any pair of vertices (a, b) is an edge iff $a \leq b$ or $b \leq a$. Elements a and b in a partial preorder are *twins*, denoted by $a \approx b$, if $a \leq b$ and $b \leq a$. An element a is minimal if there is no b such that $b < a$. If $X' \subseteq X$ then a is minimal in X' iff $a \in X'$ and there is no $b \in X'$ such that $b < a$. The set of minimal elements of X' is denoted by $\min_{\leq} X'$, or simply $\min X'$ if $\leq$ is clear from the context. A total preorder determines a partition $(V_1, \ldots, V_m)$ of its elements into levels: V_1 is the set of minimal elements, V_2 is the set of minimal elements in $V \setminus V_1$, etc.

2.3 Further Concepts

Partial preorders will be used to represent preferences over truth assignments, with truth assignments satisfying the knowledge base being most preferred[1]. This assumes that partial preorders considered have a special structure, referred to as regularity.

Definition 2.3.1. *(Regular) A partial preorder is* regular *if*

1. *every minimal element is smaller than any non-minimal element and*
2. *the number of elements is a power of 2.*

The first assumption means that every truth assignment satisfying the knowledge base is preferred to every truth assignment not satisfying it. This is a standard assumption in belief

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[1]Following standard usage, $a \leq b$ is taken to mean that a is preferred to b.

change theory. The second assumption is made because the number of truth assignments is always a power of two, and the elements of the partial preorder are identified with truth assignments. From the point of view of partial preorders these are mild technical assumptions which do not have an essential effect on definability. An example of a non-regular partial preorder is the 4-element poset with $a < b, c < d$ and no other comparability. Condition 1 is satisfied, for example, if there is a unique minimal element.

In the standard definition, the partial preorder is defined over the set of truth assignments. For this discussion it is more convenient to separate the partial preorder and the labeling of its elements by truth assignments. Similar distinctions are made in modal logic as well [5].

Definition 2.3.2. *(Faithful partial preorder) A* faithful partial preorder *for a knowledge base K_n is a pair $F = (R, t)$, where $R = (X, \leq)$ is a regular partial preorder and $t : X \to T_n$ is a bijection between the elements of X and truth assignments, such that $a \in X$ is minimal iff $t(a)$ satisfies K_n.*

2.4 Revision Using Minimization

One of the basic approaches to belief change is to perform minimization using epistemic states represented by faithful partial preorders.

Definition 2.4.1. *(Revision using minimization) The* revision operator $*_F$ for K, determined by a faithful partial preorder F for K, using minimization *is*

$$K *_F \varphi = \langle\big(t(\min_{\leq} t^{-1}(|\varphi|))\big)\rangle.$$

Thus the revised knowledge base is satisfied by the minimal satisfying truth assignments of the revising formula. Faithfulness implies that if the revising formula is consistent with the knowledge base then the revised knowledge base is the conjunction of the knowledge base and the revising formula.

Consider the situation depicted in Figure 3. Let $n = 2$ and the knowledge base be $K = x_1 \wedge x_2$. Then F_1, F_2 and F_3 are faithful partial preorders for K. In the figure, the elements of X have explicit labels for the purpose of illustrating revision my minimization. The double line in F_3 indicates that $t^{-1}(01)$ and $t^{-1}(00)$ are twins. Consider the revising formula $\varphi = \bar{x}_1$. Elements belonging to $t^{-1}(|\varphi|)$ are shown as black dots and the elements of $\min_{\leq} t^{-1}(|\varphi|)$ are in ovals. Then it holds that

$$K *_{F_1} \varphi = \neg x_1 \wedge x_2 \quad \text{and} \quad K *_{F_2} \varphi = K *_{F_3} \varphi = \neg x_1.$$

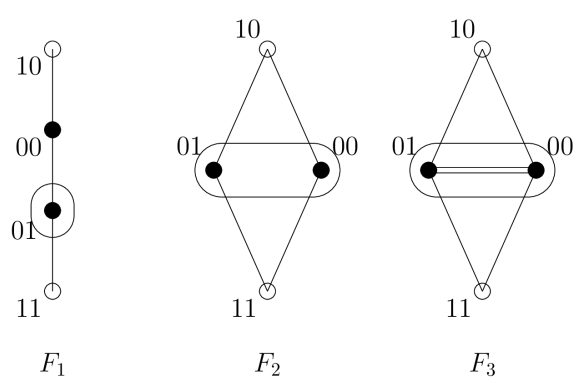

Figure 3: Revision on Posets and Preorders

Note that the regular partial preorders underlying F_1 and F_2 are posets; the first one is total and the second one is not. The revision operator $*_{F_1}$ satisfies the postulate "if K and φ are inconsistent then $|K * \varphi|$ is a singleton". The revision operator $*_{F_2}$ does not satisfy the postulate. Also, note that the regular partial preorders underlying F_2 and F_3 are not isomorphic, but the corresponding revision operators are identical. The difference between revision operators determined by partial preorders and posets, indicated by this example, can be formulated as follows.

Definition 2.4.2. *A revision operator $*$ is* poset-based *if it is of the form $*_F$ for some faithful poset F.*

Lemma 2.4.3. a) *There are non-isomorphic regular partial preorders R_1 and R_2 such that there are faithful partial preorders $F_1 = (R_1, t_1)$ and $F_2 = (R_2, t_2)$ with $*_{F_1} = *_{F_2}$.*

b) *Let K be a knowledge base, and $*$ be a poset-based revision operator. Then there is a unique faithful poset F such that $* = *_F$.*

Proof a) As discussed above, this is shown by the regular partial preorders underlying F_2 and F_3 above.

b) Let $F = (R, t)$, where $R = (X, \leq)$ is a regular poset and $t : X \to T_n$ is a bijection between the elements of X and truth assignments. Then for any two elements $u, v \in X$ it holds that

$$u < v \text{ iff } K_n *_F \langle t(u), t(v)\rangle = \langle t(u)\rangle, \quad \text{and} \quad u \sim v \text{ iff } K_n *_F \langle t(u), t(v)\rangle = \langle t(u), t(v)\rangle.$$

Thus F can be reconstructed from $*$ up to isomorphism of R. $\Box$

2.5 Postulates and Characterizability

Consider the AGM postulate

$$\text{if } K \wedge \varphi \text{ is satisfiable, then } K * \varphi = K \wedge \varphi. \tag{2.1}$$

Here $K, K*\varphi$ and φ can be considered as unary predicates over the set of truth assignments. For example, in Equation 2.1 a unary predicate would be introduced such that $x \in K(x)$ iff $x \in |K|$. Thus the above postulate can be rewritten in such a manner.

$$[\exists x(K(x) \wedge \varphi(x))] \rightarrow [\forall x\big((K*\varphi)(x) \leftrightarrow (K(x) \wedge \varphi(x))\big)]. \tag{2.2}$$

Postulates refer to a fixed knowledge base K, and are implicitly universally quantified over formula symbols such as φ. They express general requirements that are supposed to hold for all revising formulas. The definition for postulate generalizes this example. However, this definition is one possibility to formalize the notion of a postulate. It seems natural and covers most postulates considered for belief revision operators. In order to emphasize that this is just one, though hopefully basic, notion, the definition refers to such postulates as basic.

Definition 2.5.1. *(Basic postulate) A* basic postulate *$\mathcal{P}$ is a first-order sentence with unary predicate symbols $K, \varphi_1, \ldots, \varphi_\ell$ and $K*\mu_1, \ldots, K*\mu_m$, where $\mu_1, \ldots, \mu_m$ are Boolean combinations of $\varphi_1, \ldots, \varphi_\ell$.*

A revision operator satisfies a basic postulate for a knowledge base K if the basic postulate holds for all $\varphi_1, \ldots, \varphi_\ell$, with the variables ranging over the set of sets of truth assignments.

Allowing for Boolean combinations as arguments to the belief revision operator is necessary as many postulates use such constructs. For example, the AGM postulates refer to $K * (\varphi \wedge \psi)$. Definition 2.5.1 covers most postulates in [27] and in Section 7.3 of [23]. Also, note in this context, a finite set of postulates can be represented by one postulate for the same reason that K can be considered a single propositional formula.

This framework was constructed to address the form of a majority of the postulates put forth in the literature. However, there is no inherent "correct" form for a postulate. If one chose to allow quantifier alternation for instance, another framework similar to this could be constructed to accommodate classes of partial preorders defined in that manner.

Characterizability in belief revision is intended to correspond to the formal logic concept of finite axiomizabiity. The definition of characterizability applies to revision operators within the partial preorder minimization framework. Classes of revision operators can be defined by specifying a class of partial preorders.

Definition 2.5.2. *($\mathcal{R}$-revision operator) Let $\mathcal{R}$ be a family of regular partial preorders. Let K be a knowledge base and $*$ be a revision operator for K. Then $*$ is an $\mathcal{R}$-revision operator iff there is a faithful partial preorder $F = (R, t)$ for K with $R \in \mathcal{R}$, representing $*$ using minimization.*

Lemma 2.4.3 shows that there is a bijection between poset-based revision operators and the posets generating them, while this is not always the case for revision operators generated by partial preorders. Therefore, in the rest of this chapter the discussion will be restricted to posets for simplicity.

Definition 2.5.3. *(Characterization, characterizability) Let $\mathcal{R}$ be a family of regular posets. A finite set of basic postulates $\mathcal{P}$* characterizes *$\mathcal{R}$-revision operators if for every knowledge base K and every poset-based revision operator $*$ for K the following holds: $*$ satisfies the basic postulates in $\mathcal{P}$ iff $*$ is an $\mathcal{R}$-revision operator.*

The family of $\mathcal{R}$-revision operators is characterizable *if there is a finite set of basic postulates characterizing $\mathcal{R}$-revision operators.*

2.6 min-formulas and Translation

We define a translation of basic postulates as defined in Definition 2.5.1 into sentences over an extension of the first-order language of posets. The language of posets contains the binary relation symbol $\leq$ and equality. The translated sentences also contain additional unary predicate symbols $A_1, \ldots, A_\ell$. These correspond to propositional formulas $\varphi_1, \ldots, \varphi_\ell$ occurring in the basic postulates. The following is some notation for the translations.

Definition 2.6.1. *(Hat) Given a Boolean combination μ of $\varphi_1, \ldots, \varphi_\ell$, denoted by $\hat{\mu}$ the first-order formula obtained by replacing the φ's with A's.*

For instance, for $\mu(x) = \varphi_1(x)$, one has $\hat{\mu}(x) = A_1(x)$, and for $\mu(x) = \varphi_1(x) \wedge \varphi_2(x)$, one has $\hat{\mu}(x) = A_1(x) \wedge A_2(x)$. In addition, a predicate must be introduced for minimals sets of the $A_1, \ldots, A_\ell$. Given a formula ν over the language $\leq, A_1, \ldots, A_\ell$ with a single free variable x, write $\min^\nu_\leq$ for a formula expressing that x is a minimal element satisfying ν, i.e.,

$$\min^\nu_\leq(x) \;\equiv\; \nu(x) \wedge \forall y(\nu(y) \rightarrow \neg(y < x)). \tag{2.3}$$

When $\leq$ is clear from the context it is omitted as a subscript. Similarly, minimal elements in the poset can be defined by

$$\min(x) \;\equiv\; \forall y(\neg(y < x)).$$

In order to express revision, minimal sets of formulas and their Boolean combinations must be considered. Therefore, special cases when the formula ν is a Boolean combination of the unary predicates $A_1, \ldots, A_\ell$ are needed.

Definition 2.6.2. *(*min*-formula) A* min*-formula over the unary predicate symbols $A_1, \ldots, A_\ell$ is a first-order formula built from the A_is and formulas of the form $\min^\nu_\leq(x)$, where the ν's are arbitrary Boolean combinations of the A_is.*

By combining these definitions, one can define a translation of a postulate to formulas in a restricted version $\forall MSO$. Each step is a straightforward substitution based upon the previous definitions.

Definition 2.6.3. *(Translation) The* translation *$\tau(P)$ of a basic postulate $\mathcal{P}$ is the* min*-sentence obtained from $\mathcal{P}$ by replacing*

1. *every occurrence of $K(x)$ with $\min(x)$,*

2. *every occurrence of* $\varphi_i(x)$ *and* $\mu_i(x)$ *with their "hat" versions and*

3. *every occurrence of* $K * \mu_i$ *with* $\min^{\hat{\mu}_i}(x)$.

Note that Part 2 in the definition is redundant as the definition for φ_i is a special case of the definition for μ_i. The translation of a basic postulate is a first-order sentence over the predicate symbols $\leq, A_1, \ldots, A_\ell$. It is based on the observation that the definition of revision by minimization (Definition 2.4.1) uses the underlying poset in a restricted manner, by simply taking the minimal elements of the revising formula.

Example 2.6.4. *(Translation of basic postulate (Equation 2.2)) Applying Definition 2.6.3 one obtains the* min*-sentence*

$$\left[\exists x(\min(x) \wedge A_1(x))\right] \quad \rightarrow \left[\forall x\big(\min^{A_1}(x) \leftrightarrow \quad (\min(x) \wedge A_1(x))\big)\right]. \tag{2.4}$$

Because τ is a simple syntactic transformation, the correspondence between min-sentences and postulates is a direct consequence of the previous definitions.

Proposition 2.6.5. *The mapping* τ *is a bijection between basic postulates containing revising formulas* $\varphi_1, \ldots, \varphi_\ell$ *and* min*-sentences over unary predicates* $A_1, \ldots, A_\ell$.

In order to interpret min-formulas the following structure is introduced. As an abuse of notation, the same notation for a predicate symbol and its interpretation over a structure will be used assuming that the structure is clear from the context.

Definition 2.6.6. *(ℓ-extension) Let $R = (X, \leq)$ be a partial preorder. An ℓ-*extension *of R is a structure*

$$R' = (X, \leq, A_1, \ldots, A_\ell),$$

where $A_1, \ldots, A_\ell$ are unary relations.

Given $K, \varphi_1, \ldots, \varphi_\ell$ and a faithful partial preorder F for K, the $(\varphi_1, \ldots, \varphi_\ell)$-extension of F is determined in the standard way, by interpreting the unary predicate symbols $A_1, \ldots, A_k$ by $A_i(a) = \varphi_i(t(a))$. Recall that a distinction is made between an element a of the poset, and the truth assignment $t(a)$ assigned to that element. Again, the following proposition is a direct consequence of the definitions.

Proposition 2.6.7. *Let K be a knowledge base, $F = (R, t)$ be a faithful partial preorder for K and let $*_F$ be the revision operator determined by F using minimization. Let $\varphi_1, \ldots, \varphi_\ell$ be propositional formulas and $\mathcal{P}$ be a basic postulate. Then $\mathcal{P}$ is satisfied by $*_F$ for $\varphi_1, \ldots, \varphi_\ell$ iff the $(\varphi_1, \ldots, \varphi_\ell)$-extension of F satisfies $\tau(P)$.*

2.7 $\forall MSO_{\min}$ -definability

In the next two sections the concepts and tools for the logical characterization of postulate characterizability are developed using finite model theory [17, 33]. As propositional logic formulas occurring in the basic postulates are implicitly universally quantified and such formulas are translated into subsets of the partial preorders, it is natural to consider universal monadic second-order logic.

A *universal monadic second-order ($\forall MSO$) sentence* is of the form

$$\Phi = \forall A_1, \ldots, A_\ell \Psi, \tag{2.5}$$

where $A_1, \ldots, A_\ell$ range over unary predicates (or subsets) of the universe, and Ψ is a first-order sentence using the unary predicate symbols $A_1, \ldots, A_\ell$ in addition to the original language (in this case $\leq$ and equality). An existential second-order ($\exists MSO$) sentence is of the form $\Phi = \exists A_1, \ldots, A_\ell \Psi$.

As noted in the previous section, translations of basic postulates, such as (Equation 2.4), have special structure. The translation of a postulate only refers to the underlying order relation $\leq$ in a *restricted manner*, only inside a $\min_{\leq}^{\hat{\mu}}$-formula. This corresponds to using only a fragment of universal monadic second-order logic, defined as follows.

Definition 2.7.1. *($\forall MSO_{\min}$ sentence) A $\forall MSO_{\min}$ sentence is a universal second order sentence where Ψ is a* min*-sentence.*

The definition of $\exists MSO_{\min}$ sentences is analogous.

For example, universally quantifying in (Equation 2.4) over the second-order variable A_1 yields the $\forall MSO_{\min}$ sentence

$$\forall A_1\big(\big[\exists x(\min(x) \wedge A_1(x))\big] \quad \rightarrow [\forall x\big(\min^{A_1}(x) \leftrightarrow \quad (\min(x) \wedge A_1(x))\big)\big]\big).$$

Thus from the formal rewriting of the AGM postulate (Equation 2.1) as (Equation 2.2), one arrives at (Equation 2.6) through the intermediate step (Equation 2.4). Each step is an invertible syntactic transformation.

Definition 2.7.2. *($\forall MSO_{\min}$ -definability) A family $\mathcal{R}$ of regular posets is $\forall MSO_{\min}$ -definable if there is a $\forall MSO_{\min}$ sentence Φ such that for every regular poset R it holds that $R \in \mathcal{R}$ iff R satisfies Φ.*

The translation of a postulate into a $\forall MSO_{\min}$ sentence allows for tools from logic to be applied to postulates. In particular, a primary concern is determining the definability of a class of operators by a set of finite postulates. To this end, the logical characterization of basic postulate characterizability can now be stated. These theorems reduce questions about characterizability to questions about $\forall MSO_{\min}$ -definability. The next section develops tools from finite model theory for proving *undefinability.*

Theorem 2.7.3. *Let $\mathcal{R}$ be a family of regular posets. The family of $\mathcal{R}$-revision operators is characterizable iff the family $\mathcal{R}$ is $\forall MSO_{\min}$ -definable.*

Proof ($\Rightarrow$) Let $\mathcal{R}$ be a family of regular posets such that $\mathcal{R}$-revision operators are characterized by a basic postulate $\mathcal{P}$. The claim is that $\mathcal{R}$ is defined by the $\forall MSO_{\min}$ sentence

$$\Phi = \forall A_1, \ldots A_\ell \; \tau(P).$$

Assume that the regular poset $R = (X, \leq)$ is in $\mathcal{R}$. Let the number of its elements be 2^n. Let $t : X \to \{0,1\}^n$ be an arbitrary bijection between X and the set of truth assignments.

There is a faithful poset $F = (R, t)$ for some knowledge base K_n, and thus the corresponding revision operator $*_F$ is an $\mathcal{R}$-revision operator. Therefore $*_F$ satisfies $\mathcal{P}$. Consider arbitrary unary relations $A_1, \ldots, A_\ell$ over the elements. Applying Proposition 2.6.7 to the propositional formulas $\varphi_1, \ldots, \varphi_\ell$ corresponding to $A_1, \ldots, A_\ell$, it follows that $A_1, \ldots, A_\ell$ and the corresponding min-predicates satisfy $\tau(P)$. Thus R satisfies Φ.

Now assume that the regular poset R is not in $\mathcal{R}$. Again, let $t : X \to \{0,1\}^n$ be an arbitrary bijection between X and the set of truth assignments. Consider the faithful poset $F = (R, t)$ for some knowledge base K_n. This determines a revision operator $*_F$. By Lemma 2.4.3, up to isomorphism, R is the only poset such that some faithful poset determines this $*_F$, thus $*_F$ is not an $\mathcal{R}$-revision operator. Hence $*_F$ does not satisfy $\mathcal{P}$. So there are propositional formulas $\varphi_1, \ldots, \varphi_\ell$ such that the corresponding instance of $\mathcal{P}$ is false. By Proposition 2.6.7 the corresponding unary predicates $A_1, \ldots, A_\ell$ and their min predicates falsify $\tau(P)$. Hence R falsifies Φ.

($\Leftarrow$) Assume that $\mathcal{R}$ is defined by the $\forall MSO_{\min}$ -sentence $\forall A_1, \ldots, A_\ell \Psi$, where Ψ is a min-sentence. By Proposition 2.6.5 there is a basic postulate P such that $\tau(P) = \Psi$. The claim is that $\mathcal{R}$-revision operators are characterized by P. This follows similarly as the other direction.
□

2.8 Games

The q-round first-order Ehrenfeucht - Fraïssé game over two relational structures is played by two players, Spoiler and Duplicator. In each round Spoiler picks one of the structures and an element of that structure. Duplicator responds by picking an element in the other structure.

After q rounds Duplicator wins if the mapping, assigning elements picked in the same round to each other, yields isomorphic substructures. Otherwise Spoiler wins. A basic result is that a class of structures is first-order definable iff there is a q such that if the q-round game is played on two structures, one belonging to the class and the other not, then Spoiler has a winning strategy.

The $\exists MSO_{\min}$ -definability of a class is the same as the $\forall MSO_{\min}$ -definability of its complement. Therefore, in the following definitions, $\exists MSO_{\min}$ is used instead of $\forall MSO_{\min}$ for convenience. Similarly, in Section 2.9 this fact will provide a convenience for proving characterizability results.

The first-order Ehrenfeucht - Fraïssé game has a variant corresponding to $\exists MSO$ definability. Because the objective is to prove undefinability, a modified game, defined in [2], will be used. The form of this game, the Ajtai-Fagin game, is better suited to prove undefinability in this situation.

Definition 2.8.1. *($(\mathcal{R}, \ell, q)$-$\exists MSO$ Ajtai-Fagin game) Given a class $\mathcal{R}$ of structures and parameters ℓ and q, the $(\mathcal{R}, \ell, q)$-$\exists MSO$ Ajtai-Fagin game is played as follows:*

1. *Duplicator picks a structure $R_1 \in \mathcal{R}$,*
2. *Spoiler picks ℓ subsets $A_1, \ldots, A_\ell$ of the universe of R_1,*
3. *Duplicator picks a structure $R_2 \notin \mathcal{R}$ and subsets $B_1, \ldots, B_\ell$ of the universe of R_2,*
4. *Spoiler and Duplicator play a q-round first-order Ehrenfeucht - Fraïssé game on the structures extended with the subsets (i.e., unary relations) selected.*

The connection between ∃-definability and the Ajtai-Fagin game is as follows.

Theorem 2.8.2. [2] *A class $\mathcal{R}$ is $\exists MSO$-definable iff there are ℓ, q such that Spoiler has a winning strategy in the $(\mathcal{R}, \ell, q)$-$\exists MSO$ Ajtai-Fagin game.*

Due to the restriction of ∀MSO to $\forall MSO_{\min}$, the Ajtai-Fagin game must be modified to reflect definability of this specific fragment. In order to accomplish this, the concept of variant is introduced.

Definition 2.8.3. *(ℓ-min-variant) Let $R = (X, \leq)$ be a partial order. Let $A_1, \ldots, A_\ell$ be unary predicate symbols. Denote the $L = 2^{2^\ell}$ logically inequivalent Boolean combinations μ of $A_1, \ldots, A_\ell$ as the unary predicate symbols $M_1, \ldots, M_L$. An ℓ-*min-variant *of R is a structure*

$$R'' = (X, A_1, \ldots, A_\ell, M_1, \ldots, M_L),$$

where $R' = (X, \leq, A_1, \ldots, A_\ell)$ is an ℓ-extension of R, and $M_1, \ldots, M_L$ are the interpretations of the formulas $\min^{\nu}_{\leq}(x)$, for Boolean combinations ν of the A_is.

Note that R'' is a structure with *unary* predicates only, the relation $\leq$ is *not* included, it is "forgotten". This is the "forgetful property" of the game, mentioned in Section 2.1. Even though the order relation $\leq$ is used to interpret the unary predicates $M_1, \ldots, M_L$, this relation is not used in min-formulas. R'' is not an extension of R; therefore, it is referred to as a variant.

Using a variant, a modified version of the Ajtai-Fagin game will be applicable to $\forall MSO_{\min}$ -definability over partial orders. This game is referred to as the Katsuno-Mendelzon (KM) game.

Definition 2.8.4. *(($\mathcal{R}, \ell, q$)-$\exists MSO_{\min}$ game, or Katsuno-Mendelzon game) Given a class $\mathcal{R}$ of posets and parameters ℓ and q, the ($\mathcal{R}, \ell, q$)-$\exists MSO_{\min}$ game is played by Spoiler and Duplicator as follows:*

1. *Duplicator picks a poset $R_1 = (X_1, \leq_1)$ in $\mathcal{R}$,*
2. *Spoiler picks ℓ subsets $A_1, \ldots, A_\ell$ of X_1,*
3. *Duplicator picks a poset $R_2 = (X_2, \leq_2) \notin \mathcal{R}$ and subsets $B_1, \ldots, B_\ell$ of X_2,*
4. *Form the ℓ-min-variant R_1'' of R_1 determined by the ℓ-extension $R_1' = (X_1, \leq_1, A_1, \ldots, A_\ell)$, and the ℓ-min-variant R_2'' of R_2 determined by the ℓ-extension $R_2' = (X_2, \leq_2, B_1, \ldots, B_\ell)$,*
5. *Spoiler and Duplicator play a q-round first-order Ehrenfeucht - Fraïssé game on R_1'' and R_2''.*

Theorem 2.8.5. *A class $\mathcal{R}$ of posets is $\exists MSO_{\min}$ -definable iff there are ℓ and q such that Spoiler wins the ($\mathcal{R}, \ell, q$) - $\exists MSO_{\min}$ game.*

Theorem 2.8.5 follows directly from Theorem 2.8.7. The following concepts are needed for the statement of Theorem 2.8.7. Consider a structure with $\ell + L$ unary predicates. The *type* $t(a)$ of an element a of the structure is the binary vector of length $\ell + L$ describing the behavior of a with respect to the unary predicates. For a number q, the *q-profile* of the structure collects approximate counting information on the number of different types in the form a function

$$\gamma : \{0,1\}^{\ell+L} \to \{0, \ldots, q\},$$

where for every type u it holds that

$$\gamma(u) = \min(|\{a \in X : \gamma(a) = u\}|, q).$$

Thus the q-profile of the structure tells, for every type u, the exact number of elements in the structure having type u if this number is less than q, and it gives the value q if this number is at least q.

For a q-profile γ let

$$\Phi = \bigwedge_{u \in \{0,1\}^{\ell+L}} \Phi_u$$

be a first-order formula describing γ, where for every binary vector u, the formula Φ_u says either that there are exactly i elements with type u for $i < q$, or that there are at least q elements with type u, whichever is the case for $\gamma(u)$. Note that the quantifier rank of Φ is at most q.

The following is a standard fact about the first-order Ehrenfeucht-Fraïssé game for unary structures.

Lemma 2.8.6. *Consider the q-round first-order Ehrenfeucht-Fraissé game played on two structures with unary predicates, such that for every atom it holds that its size in the two structures is either the same, or both are at least q. Then Duplicator wins.*

For a partial order R and numbers ℓ and q, let

$$\Gamma_{\ell,q}(R)$$

be the set of q-profiles of all ℓ-min-variants of R. Thus in this case $L = 2^{2^\ell}$. Let $\Delta_{\ell,q}$ be the set of functions from $\{0,1\}^{\ell+L}$ to $\{0,\ldots,q\}$.

Theorem 2.8.7. *Let $\mathcal{R}$ be a family of partial orders and ℓ, q be numbers. Then the following are equivalent:*

1. *$\mathcal{R}$ is definable by a $\exists MSO_{\min}$-sentence with ℓ second-order quantifiers and quantifier-rank-q first-order part,*
2. *Spoiler has a winning strategy in the $(\mathcal{R}, \ell, q)$-$\exists MSO_{\min}$ game,*
3. *there is a set $\Gamma \subseteq \Delta_{\ell,q}$ such that for every partial order R it holds that*

$$R \in \mathcal{R} \quad \textit{iff} \quad \Gamma_{\ell,q}(R) \cap \Gamma \neq \emptyset.$$

Proof. ($1 \Rightarrow 2$) As usual, Spoiler can use the definition of $\mathcal{R}$ to win, by picking the subsets of R_1 which satisfy the $\exists MSO$-sentence defining $\mathcal{R}$.

($2 \Rightarrow 3$) Given Spoiler's winning strategy, let Γ be the set of all q-types of ℓ-min-variants formed by Spoiler's responses in round 2 to all possible initial choices R_1 of Duplicator.

If $R_1 \in \mathcal{R}$ then by the definition of the game there is an ℓ-min-variant of R_1 with q-type γ in Γ, namely the one formed by Spoiler's response to R as Duplicator's initial move. If $R_1 \in \mathcal{R}$ then $\Gamma_{\ell,q}(R_1) \cap \Gamma \neq \emptyset$ by definition.

Now assume there is a partial order $R_2 \notin \mathcal{R}$ with an ℓ-min-variant having q-type $\gamma \in \Gamma$. As $\gamma \in \Gamma$, there is a partial order $R_1 \in \mathcal{R}$ such that if Duplicator starts with R_1 then Spoiler responds with forming a ℓ-min-variant having q-type γ. But then it follows from Lemma 2.8.6

that Duplicator can pick R_2 and its ℓ-min-variant having q-profile γ in round 3, and win the game.

($3 \Rightarrow 1$) Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$. Then $\exists A_1 \ldots A_\ell(\gamma_1 \vee \ldots \vee \gamma_m)$ defines $\mathcal{R}$.$\square$

For non-characterizability the following corollary will be used. The formulation takes into account that the theorem holds for general posets, but the relevant concept here is regular posets.

Corollary 2.8.8. *Let $\mathcal{R}$ be a class of regular posets. Assume that for every ℓ and q, Duplicator has a winning strategy in the $(\mathcal{R}, \ell, q)$ - $\exists MSO_{\min}$ game such that each of the posets, R_2, are also regular. Then $\mathcal{R}$ is not $\exists MSO_{\min}$ -definable.*

2.9 Classes of Posets

A chain (resp., antichain) is a set of pairwise comparable (resp., incomparable) elements. The *height* (resp., *width*) of a poset is size of a largest chain (resp., antichain). A poset is total (aka linear) if it has width 1. A poset is connected (resp., disconnected) if its comparability graph is connected (resp., disconnected).

For technical reasons explained earlier, only regular posets are considered here. As a result, somewhat modified notions are needed. In particular, the minimal elements of the poset will be disregarded. The reason is that for the current purposes the contribution of the minimal elements is not essential, as they relate to the other elements in a trivial way, but they may interfere with the structure of the remaining elements. For example, minimal elements are always incomparable. Therefore, if minimal elements are included, then a poset with a knowledge base such that $|K|$ consists of more than one element would never be total. For this reason, modified definitions for regular posets are introduced.

The r-height (resp. r-width) of a regular poset is the height (resp., width) of the poset obtained by removing the minimal elements. A regular poset is r-total (resp., r-connected, r-disconnected) if the poset obtained by removing the minimal elements is total (resp., connected, disconnected). Denote by $\mathcal{H}_{\leq k}$ the class of regular posets with r-height at most k. The classes $\mathcal{H}_{\geq k}, \mathcal{H}_{<k}, \mathcal{H}_{>k}, \mathcal{H}_{=k}$ are defined similarly. For width, use the notations $\mathcal{W}_{\leq k}, \mathcal{W}_{\geq k}, \mathcal{W}_{<k}, \mathcal{W}_{>k}$. The class of r-total (resp., r-connected, r-disconnected) regular posets is denoted by Λ (resp., $\mathcal{C}$, $\mathcal{D}$).

2.10 Characterizable Classes

The characterizability results of this section are proved by showing that a given class of posets is $\forall MSO_{\min}$ -definable, or that its complement is $\exists MSO_{\min}$ -definable. It may happen that the standard definition of a class is not suitable for such a definition, but it can be replaced by an equivalent definition which is suitable. As a first example of characterizability, a basic postulate characterizing revision operators obtained from r-total regular posets is given.

Theorem 2.10.1. *The class of Λ-revision operators is characterized by the basic postulate*

$$(\forall x(\neg(K(x) \wedge \phi(x))) \wedge \exists x \phi(x)) \rightarrow \exists! x((K * \phi)(x)). \tag{2.6}$$

Proof Let $*_R$ be a revision operator generated by an r-total regular poset R. If φ is satisfiable and inconsistent with K then it has a unique minimal satisfying truth assignment and so its revision is a singleton. Conversely, if R is a non-r-total regular poset then it has an incomparable non-minimal pair (a, b). If the revising formula ϕ has the two corresponding truth assignments as its models, then the models of the revision by ϕ will consist of these two assignments. $\Box$

The standard definition of totality is that any two elements are comparable. With the modification to consider only non-minimal elements, this becomes the statement that any two non-minimal elements are comparable. This is not suitable for construction of a postulate, because min-formulas cannot directly express comparability. An equivalent formulation in the language of min-formulas is that every nonempty set of non-minimal elements has a unique minimal element, expressed by (Equation 2.6).

The following results summarize several characterizable classes as well as demonstrate some strategies for constructing a characterization.

Theorem 2.10.2. *1. For every k, the class of $\mathcal{H}_{\leq k}$-revision operators is characterizable.*

2. For every k, the class of $\mathcal{H}_{\geq k}$-revision operators is characterizable.

3. For every k, the class of $\mathcal{W}_{\leq k}$-revision operators is characterizable.

Proof

Part 1: The following results shows that $\mathcal{H}_{>k}$ is $\exists MSO_{\min}$ -definable.

A regular poset has r-height greater than k iff there exists a chain of $k+1$ non-minimal elements. This is equivalent to the following: there are sets $A_1, \ldots, A_{k+1}$ such that

- the A_is are disjoint from the set of minimal elements,
- there are $k+1$ elements $a_1, \ldots, a_{k+1}$ such that $A_i = \{a_1, \ldots, a_i\}$ for every $i \leq k+1$, and

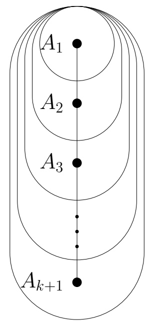

Figure 4: Chain of Subsets for Bounded Height Characterization

- $\min A_i = \{a_i\}$.

This construction is pictured in Figure 4. From this criteria, a $\exists MSO_{\min}$ sentence which expresses height greater than k can directly be constructed.

Part 2: Similarly, $\mathcal{H}_{<k}$ is $\exists MSO_{\min}$ -definable. A regular poset has height less than k iff there are sets $A_1, \ldots, A_{k-1}$ such that the A_is form a partition of the set of non-minimal elements, and each A_i is an antichain. This definition starts with existential second-order monadic quantifiers, thus the second-order quantifier structure is of the right form. The standard description of a set A_i being an antichain is that there are no comparable pairs in A_i. However, min-definability cannot directly express comparability. On the other hand, a min-definition can be given by the observation that A_i is an antichain iff it is equal to the set of its minimal elements, i.e., $A_i = \min A_i$.

Part 3: Show that $\mathcal{W}_{>k}$ is $\exists MSO_{\min}$ -definable. A regular poset has width greater than k iff there is an antichain of size at least $k+1$. This can be expressed similarly to the previous cases. $\square$

One case where a replacement of the standard definition by a $\forall MSO_{\min}$ -sentence cannot be found is having width at least k: the classes $\mathcal{W}_{\geq k}$ turn out to not be $\forall MSO_{\min}$ -definable, even though they are first-order definable. Connectedness and disconnectedness are also not $\forall MSO_{\min}$ -definable. These results are presented in the next section.

On the other hand, connected posets of bounded height turn out to be $\forall MSO_{\min}$ -definable. This is an example of an $\forall MSO_{\min}$ -definable class which is *not* first-order.

Theorem 2.10.3. *For every k, the class of $\mathcal{C} \cap \mathcal{H}_{\leq k}$-revision operators is characterizable.*

Proof It is sufficient to show that $\mathcal{H}_{=k} \cap \mathcal{D}$ is $\exists MSO_{\min}$ -definable. A regular poset R of r-height k is r-disconnected iff there is a partition of the non-minimal elements into $2k$ antichains $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ such that

- the A_is are all non-empty,
- some B_i is non-empty, and
- every union $A_i \cup B_j$ is also an antichain.

If R has r-height k and is disconnected, then k antichains in an r-height k component, plus at most k antichains for the other components satisfy these conditions. Conversely, assume that R has height k and is connected. Then there is a comparable pair (a, b) such that $a \in A_i$ and $b \in B_j$ for some i, j, contradicting the last condition.$\square$

2.11 Non-characterizable Classes

The negative results are based on Theorem 2.7.3 and Corollary 2.8.8, by constructing winning strategies for the Duplicator. First consider the class of $\mathcal{W}_{\geq 2}$-revision operators, i.e., the class of revision operators obtained from regular posets which are *not* r-total. As non r-total regular posets form a simple and natural first-order definable class, the non-characterizability of the corresponding class of revision operators might be considered somewhat surprising.

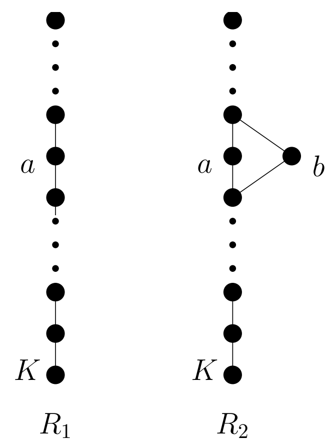

Figure 5: Logical Structures for Nonlinear Non-characterizability

Theorem 2.11.1. *The class of $\mathcal{W}_{\geq 2}$-revision operators is not characterizable.*

Proof Given ℓ and q, a winning strategy needs to be described for Duplicator in the $(\mathcal{R}, \ell, q)$ - $\exists MSO_{\min}$ game for the class of regular posets not in $\mathcal{W}_{\geq 2}$, i.e., for the class Λ of r-total regular posets.

The poset $R_1 = (X_1, \leq_1)$ picked in step 1. will be a chain on N elements, for some power of 2 to be determined later. Assume that Spoiler picks ℓ subsets $A_1, \ldots, A_\ell$. For any element $a \in X_1$ associate a bit-vector with components indicating which subsets A_i a belongs to, and for every Boolean combination ν of the A_is, whether a is minimal among elements belonging to ν. These bit-vectors form a coloring of X_1 with $L = 2^{\ell+2^{2^\ell}}$ colors. As there is at most one element which is minimal in a Boolean combination, there are at least $N - 2^{2^\ell}$ elements which are never minimal. If $N - 2^{2^\ell} > L$ then there are at least two never-minimal elements of the same color. Duplicator then forms R_2 by picking such an element a, splitting it into two incomparable elements a and b, and deleting another element c from the same color class. (See Figure 5 for illustration.) The sets B_i are the same as the A_is, with the exception of b replacing c. The monadic structures R_1^* and R_2^* are isomorphic and thus Duplicator can win the first-order game in Step 5.

The non-characterizability of $\mathcal{W}_{\geq k}$-revision operators follows similarly. In Theorem 2.10.3 the class of revision operators generated by connected regular posets of bounded height is characterizable. On the other hand, the full class of revision operators generated by connected regular posets turns out to be non-characterizable. This shows another limitation of $\forall MSO_{\min}$-definability, as connected posets are $\forall MSO$-definable. Finally, the class of revision operators generated by disconnected regular posets is not even $\forall MSO$-definable.

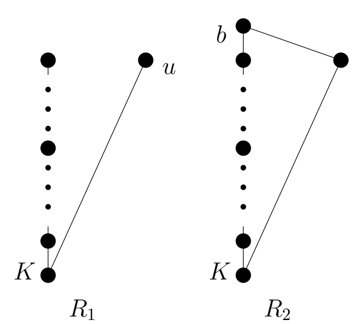

Figure 6: Logical Structures for Connected Non-characterizability

Theorem 2.11.2. *The classes of $\mathcal{C}$-revision operators and $\mathcal{D}$-revision operators are not characterizable.*

Proof The argument is similar to the previous one and it is illustrated by the Figure 6. Duplicator picks a poset R_1 consisting of a long chain and a single element u. After Spoiler picks the subsets $A_1, \ldots, A_\ell$, Duplicator considers the coloring described above, and finds a color class containing never-minimal elements. Duplicator then builds the poset R_2 by picking an element b from this color class and putting it above the chain and u. Then it follows as above that Duplicator can win in Step 5. of the game. □

2.12 Summary

The results on classes of revision operators are summarized in following Venn diagram. Characterizable classes (denoted by $\forall MSO_{\min}$ in Figure 7) turn out be a proper subset of universal monadic second-order definable classes, incomparable with first-order definable classes.

As noted, $\mathcal{C} \cap \mathcal{H}_{\leq k}$ is not first-order definable, but is $\forall MSO_{\min}$ -definable. This demonstrates that $\forall MSO_{\min}$ can define some $\forall MSO_{\min}$ classes which are not first-order definable ∀MSO sentences. On the other hand, $\forall MSO_{\min}$ cannot express the class of revision operators on connected posets. Furthermore, the restriction of bounded height restricts the class enough to permit it to be expressed in $\forall MSO_{\min}$.

Characterizable	**Non-characterizable**
1. Height $\leq k$, height $\geq k$	4. Width $\geq k$
2. Width $\leq k$	5. Connected
3. Connected and height $\leq k$	6. Disconnected

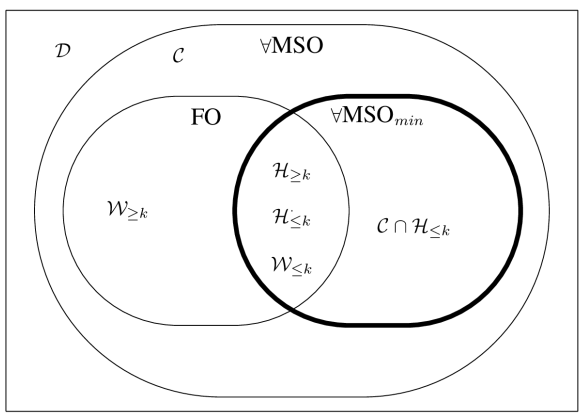

Figure 7: Summary of Characterizable and Noncharacterizable Revision Operators

CHAPTER 3

CHARACTERIZABILITY IN HORN BELIEF REVISION

3.1 Horn Belief Revision

As discussed in Chapter 2, the standard setup for the AGM framework is full propositional logic. However, recent work has considered similar questions for other logics such as the Horn fragment of propositional logic [1, 6, 7, 10, 12, 13, 20, 31, 41, 42]. Although most of the results on Horn belief change are about Horn belief contraction, Horn belief revision was considered by Delgrande and Peppas [11] and Zhang et al. [43]. Meanwhile Horn belief merging was addressed in Haret et al. [24]. The recent interest in the area may be due to the fact that Horn formulas have attractive computational properties.

A clause is a disjunction of literals. A clause is *Horn* if it contains at most one unnegated literal. A *Horn formula* is a conjunction of Horn clauses. The Horn fragment of propositional logic differs in several respects from standard logic. The most noticeable distinction is that the Horn fragment is not closed under negation. Thus, while a formula φ is in the Horn fragment, $\neg\varphi$ may not be. On the surface, it may be easy to underestimate the far-reaching consequences of this feature on Horn belief revision. After all only slight alteration is needed to the AGM postulates AGM4 and AGM8 in order to avoid negation.

H1 $H * \varphi$ is logically closed.

H2 $\varphi \in H * \varphi$

H3 $H * \varphi \subseteq H + \varphi$

H4 If $H * \varphi$ is consistent, then $H * \varphi = H + \varphi$.

H5 $H * \varphi$ is inconsistent only if φ is inconsistent.

H6 If $\mu = \varphi$, then $H * \mu = H * \varphi$.

H7 $H * (\mu \wedge \varphi) \subseteq (H * \mu) + \varphi$.

H8 If $H * \mu$ is consistent, then $(H * \mu) + \varphi \subseteq H * (\mu \wedge \varphi)$.

Nevertheless, Horn belief change turns out to be quite different from general belief change.

If the goal is to obtain a representation result analogous to that of Katsuno-Mendelzon for Horn belief revision, another modification must be made to the mapping of truth assignments. In [27] total preorders with an arbitrary mapping of truth assignments were used to generate revision operators by minimization. In the case of Horn revision, the mapping of truth assignments must be further restricted such that minimal sets model Horn formulas. This property is referred to as Horn compliance.

Delgrande and Peppas gave a characterization of a class of Horn belief operators. They characterized Horn belief revision operators obtained from Horn compliant faithful total preorders by minimization, showing that a Horn belief revision operator belongs to this class iff it satisfies the Horn AGM postulates and the acyclicity postulate scheme. The acyclicity scheme has a postulate for every $n \geq 3$, expressing the non-existence of a certain cyclic substructure.

As noted in [11], in the context of Katsuno-Mendelzon style belief revision, the limited expressibility of the Horn language fails to preclude revision operators generated by relations which are not total preorders. Moreover, they prove there exists revision operators generated by a psuedo-order which cannot be generated with a total preorder. In order to limit the class of revision operators generated by minimization to those defined by Horn complaint faithful total preorders, acyclicity is introduced.

As infinite postulate characterizations are unusual in belief revision, it is natural to ask whether it is necessary to use infinitely many postulates in Theorem 1.2.3. In this chapter, the following theorem will be presented as an answer to that question.

Theorem 3.8.1. *The class of Horn belief revision operators obtained from Horn compliant faithful rankings using minimization cannot be characterized by a finite set of postulates.*

In order to prove this, the framework developed in chapter 2 has to be modified for the present application. To accommodate Horn formulas, quantifiers of universal monadic second order logic must be restricted to quantifying over *closed* subsets of the ground set consisting of truth assignments. Also, the natural class of preference structures to consider (implicit in [11] and formalized here) goes beyond total preorders or even partial preorders, and contains structures with cyclic substructures as well. This observation is due to [11], and the structures used in this proof generalize an example presented in that paper.

A motivation to develop methods for proving non-characterizability is that the study of belief change for logics other than full propositional logic is “uncharted territory”, where it is not clear what kind of characterizations can be expected. Another candidate is the class of Horn belief revision operators with *strictly Horn compliant* faithful rankings, introduced by Zhang et al. [43]. Characterizability of this class of revision operators is an open problem, and approaching it from the point of view of non-characterizability might be useful. An understanding of the properties of strictly Horn compliant faithful rankings might be helpful for the study of their characterizability. Therefore some remarks are included on their properties and connections to classes of efficiently computable Horn belief revision operators introduced in [11].

This chapter is structured as follows: Sections 3.2-3.6 develop the framework for proving non-characterizability. Section 3.7 describes the structures used in the non-characterizability proof. Section 3.8 contains the proof of Theorem 3.8.1 and the statement of another non-characterizability result. The final section contains remarks on strict Horn compliance.

3.2 Specific Concepts

Many of the definitions used in Chapter 2 can be easily modified for the Horn case, because the definitions were not structure dependent. Some examples would be minimal sets and faithful structures. These concepts will be reused in this chapter even though the underlying structure may vary from those in Chapter 2.

The *weight* of a truth assignment is the number of its ones. The *intersection* of two truth assignments is the truth assignment formed by taking componentwise $\wedge$'s, e.g., $(1,0,1) \cap (0,1,1) = (0,0,1)$. A Boolean function is a *Horn function* if it is represented by a Horn formula. It is a basic fact that a Boolean function f is Horn iff $|f|$ is closed under intersection [25, 34]. In what follows, sets of truth assignments closed under intersection are referred to as *closed.*

3.3 Pseudo-orders and Horn revision by minimization

A *pseudo-order* $R = (V, \leq)$ is a total binary relation over a finite ground set V. Thus, in particular, pseudo-orders are reflexive. It is convenient to think of a pseudo-order as a directed graph, containing edges (u, v) such that $v < u$. (Thus edges between two vertices can go one way or both ways.) For a subset $S \subseteq V$, note that $\min_{\leq} S$ may be empty. This happens, for example, if S can be covered with a set of directed cycles of edges (u, v) such $v < u$.

The notion of a pseudo-order is used in [11] informally, and the definition above is one of the possible formalizations. Another possible formalization would be to require reflexivity only, i.e., to allow for vertices having no directed edges between them. The material presented here would work with modification for this interpretation of pseudo-order also. The use of the term "order" in this general context is explained by the possibility of formulating minimality as above.

Definition 3.3.1. *(Faithful structure) A* faithful structure F *for a Horn knowledge base* H_n *is a pseudo-order over* T_n*, such that* $\min T_n = |H_n|$*, and if* $u \in \min T_n$ *and* $v \notin \min T_n$ *then* $u < v$*.*

Definition 3.3.2. *(Horn compliance) A faithful structure is Horn compliant if for every Horn formula* φ *it holds that* $\min |\varphi|$ *is closed.*

In a Horn compliant faithful structure there are two relations: the preference relation $\leq$ of the underlying pseudo-order, and the componentwise partial ordering on truth assignments. The latter is only used implicitly when referring to closed sets. The notion of a Horn compliant faithful structure generalizes that of a *Horn compliant faithful total preorder* used by Kastuno-Mendelzon to an even broader class of structures then Definition 2.3.2.

Definition 3.3.3. *(Horn revision by minimization) The* revision operator $*_F$ for H, determined by a Horn compliant faithful structure F for H, by minimization *is*

$$H *_F \varphi = \langle \min |\varphi| \rangle.$$

The assumption of Horn compliance guarantees that $H *_F \varphi$ is well-defined. Horn revision operators defined by some Horn compliant faithful structure are called *pseudo-order based*.

3.4 Horn postulates and characterizability

As in Chapter 2 in order to be able to prove non-characterizability, one needs a formal definition of postulates and characterizability. These definitions will follow closely the form presented previously. In the following material the definitions will be illustrated using the acyclicity scheme from Theorem 1.2.3. Therefore, the following is a restatement of that scheme in a manner more convenient for the present work. Here indices are meant cyclically, i.e., $n + 1 = 1$.

Definition 3.4.1. *(Acyclicity) The acyclicity postulate $Acyc_n$ for $n \geq 3$ is the following: if $(H * \varphi_i) \wedge \varphi_{i+1}$ is satisfiable for $i = 1, \ldots, n$ then $(H * \varphi_1) \wedge \varphi_n$ is also satisfiable.*

The difference between the definition of a postulate used in this chapter and the general definition from Chapter 2 is that only conjunctions are allowed as arguments of the belief revision operator, instead of arbitrary Boolean combinations, because the class of Horn formulas is not closed under negation. As discussed in Chapter 2, while this definition seems to be a natural one in the present context, others could be considered as well. For example, one could use a language which included predicates or functions on truth assignments, such as a componentwise partial ordering. A framework allowing the acyclicity scheme to be considered a single postulate would allow the inclusion of natural numbers and a variable number of formulas; this seems to be hard to deal with and it is perhaps less natural in view of the types of postulates used in belief revision.

Definition 3.4.2. *(Horn postulate) A* Horn postulate $\mathcal{P}$ *is a first-order sentence with unary predicate symbols* $H, \varphi_1, \ldots, \varphi_\ell$ *and* $H * \mu_1, \ldots, H * \mu_m$*, where* $\mu_1, \ldots, \mu_m$ *are conjunctions of* $\varphi_1, \ldots, \varphi_\ell$*.*

A Horn revision operator satisfies a postulate for a Horn knowledge base H *if the postulate holds for all Horn revision formulas* $\varphi_1, \ldots, \varphi_\ell$*, with the variables ranging over the set of closed sets of truth assignments.*

As an example, the acyclicity postulates can be rewritten in this form as follows:

$$\left(\bigwedge_{i=1}^{n} \exists\, x((H * \varphi_i(x)) \wedge \varphi_{i+1}(x)) \right) \rightarrow \exists\, x((H * \varphi_1(x)) \wedge \varphi_n(x)).$$

Theorem 1.2.3 gives a postulate characterization of Horn belief revision operators obtained from Horn compliant faithful structures. The framework to be developed applies to a generalization of this setup.

Definition 3.4.3. *($\mathcal{F}$-revision operator) Let* $\mathcal{F}$ *be a family of faithful structures. Let* H *be a Horn knowledge base and* $*$ *be a Horn revision operator for* H*. Then* $*$ *is an* $\mathcal{F}$*-revision* *operator iff there is a faithful structure $F \in \mathcal{F}$ for H such that $* = *_F$, i.e., F represents $*$ using minimization.*

Definition 3.4.4. *(Characterization, characterizability) Let $\mathcal{F}$ be a family of faithful structures. A finite set of Horn postulates $\mathcal{P}$* characterizes *$\mathcal{F}$-revision operators if for every Horn knowledge base H and every Horn revision operator $*$ for H the following holds: $*$ satisfies the postulates in $\mathcal{P}$ iff $*$ is an $\mathcal{F}$-revision operator. The family of $\mathcal{F}$-revision operators is* characterizable *if there is a finite set of postulates characterizing $\mathcal{F}$-revision operators.*

The class considered in [11] is the following. Let $\mathcal{T}$ be the class of faithful structures where the underlying pseudo-order is a total preorder. Note that even if $\mathcal{P}$ characterizes $\mathcal{F}$-revision operators, it may happen that an $\mathcal{F}$-revision operator $*$ can also be represented by a faithful structure $F' \notin \mathcal{F}$. For example, for $\mathcal{T}$, Figure 2 in [11] gives an example of a revision operator generated by a Horn compliant faithful structure based on a pseudo-order which is not a total preorder, such that the same revision operator can also be generated by a Horn compliant faithful ranking. The following concept is useful to deal with this phenomenon.

Definition 3.4.5. *For a family $\mathcal{F}$ of faithful structures let*

$$\widetilde{\mathcal{F}} = \{F : \ F \textit{ is a faithful structure such that } *_F \textit{ is an } \mathcal{F}\textit{-revision operator}\}.$$

Thus $F \in \widetilde{\mathcal{F}}$ if there is an $F' \in \mathcal{F}$ such that $*_F = *_{F'}$

3.5 $H_{\min}$-formulas and translation

Again similarly to Chapter 2, Horn postulates are translated into sentences over an extension of the first-order language of pseudo-orders. The language of pseudo-orders contains the binary relation symbol $\leq$ and equality.

The translated sentences also contain additional unary predicate symbols $A_1, \ldots, A_\ell$, corresponding to Horn formulas $\varphi_1, \ldots, \varphi_\ell$ occurring in the postulates. In other words, the predicates $A_1, \ldots, A_\ell$ range over closed subsets of the ground set T_n.

Definition 3.5.1. *(Hat) Given a conjunction μ of $\varphi_1, \ldots, \varphi_\ell$, denote by $\hat{\mu}$ the first-order formula obtained by replacing the φ's with A's.*

For instance, for $\mu(x) = \varphi_1(x)$ one has $\hat{\mu}(x) = A_1(x)$, and for $\mu(x) = \varphi_1(x) \wedge \varphi_2(x)$ one has $\hat{\mu}(x) = A_1(x) \wedge A_2(x)$.

As previously in Chapter 2, a predicate is defined for minimal sets and an entire pseudo-order

.

$$\min\nolimits^{\nu}_{\leq}(x) \;\equiv\; \nu(x) \wedge \forall y(\nu(y) \rightarrow \neg(y < x)).$$

Minimal elements in the pseudo-order are defined by

$$\min(x) \;\equiv\; \forall y(\neg(y < x)).$$

While in Definition 2.7.1 ν could range over all Boolean combinations, in this context ν is a conjunction of closed sets, i.e. the unary predicates $A_1, \ldots, A_\ell$.

Definition 3.5.2. *($H_{\min}$-formula) A $H_{\min}$-formula over the unary predicate symbols $A_1, \ldots, A_\ell$ is a first-order formula built from the A_is and formulas of the form $\min^{\nu}_{\leq}(x)$, where the ν's are arbitrary conjunctions of the A_is.*

With these definitions, translation of a Horn postulate can be defined as follows.

Definition 3.5.3. *(Translation) The* translation $\tau(P)$ *of a Horn postulate $\mathcal{P}$ is the $H_{\min}$-sentence obtained from $\mathcal{P}$ by replacing*

1. *every occurrence of $K(x)$ with* $\min(x)$
2. *every occurrence $\mu_i(x)$ with its "hat" version*
3. *every occurrence of $K * \mu_i$ with* $\min^{\hat{\mu_i}}(x)$.

As before, Part 2 in the definition is redundant as the definition for φ_i is a special case of the definition for μ_i.

As an example of translation of a Horn postulate consider Definition 3.4.1.

Example 3.5.4. *(Translation of the acyclicity postulates) Applying Definition 3.5.3 yields the $H_{\min}$-sentence*

$$\left(\bigwedge_{i=1}^{n} \exists\, x(\min^{A_i}(x) \wedge A_{i+1}(x))\right) \rightarrow \exists\, x(\min^{A_1}(x) \wedge A_n(x)),$$

where, again, indices are meant cyclically.

As in Proposition 3.5.5, this is a syntactic transformation which goes from postulates to $H_{\min}$-sentences.

Proposition 3.5.5. *The mapping τ is a bijection between Horn postulates containing revising formulas $\varphi_1, \ldots, \varphi_\ell$ and $H_{\min}$-sentences over unary predicates $A_1, \ldots, A_\ell$.*

In order to interpret $H_{\min}$-formulas, the following definition will be needed.

Definition 3.5.6. *(ℓ-extension) Let $F = (X, \leq)$ be a faithful structure. An ℓ-*extension *of F is a structure*

$$F' = (X, \leq, A_1, \ldots, A_\ell),$$

where $A_1, \ldots, A_\ell$ are unary relations and every A_i is a closed set of truth assignments.

Given Horn formulas $H, \varphi_1, \ldots, \varphi_\ell$ and a faithful structure F for H, the definition of the $(\varphi_1, \ldots, \varphi_\ell)$-extension of F is standard, obtained by interpreting the unary predicate symbols $A_1, \ldots, A_\ell$ by $A_i(a) = \varphi_i(a)$. Again, the following proposition is a direct consequence of the definitions.

Proposition 3.5.7. *Let H be a Horn knowledge base, $F = (X, \leq)$ be a faithful structure for H and let $*_F$ be the Horn revision operator determined by F using minimization. Let $\varphi_1, \ldots, \varphi_\ell$ be Horn formulas and $\mathcal{P}$ be a postulate. Then $\mathcal{P}$ is satisfied by $*_F$ for $\varphi_1, \ldots, \varphi_\ell$ iff the $(\varphi_1, \ldots, \varphi_\ell)$-extension of F satisfies $\tau(P)$.*

3.6 $\forall MSO_{H_{\min}}$ -definability and games

The following modifications are made to the material presented in Sections 2.7 and 2.8 in order to develop a game for $\forall MSO_{H_{\min}}$ -definability. Section 2.7 constructs definability for a general belief revision operator. A brief introduction of the relevant material of finite model theory is presented in Section 2.8 as well as the a game to determine definability for a general belief revision operator.

The first modification of the section will be to the quantifiers used. Horn formulas correspond to closed subsets of truth assignments; therefore, quantification will only occur over closed subsets.

Definition 3.6.1. *(Closed set quantifier) The* closed-set *quantifiers* $\forall^c$ *and* $\exists^c$ *are generalized monadic second-order quantifiers interpreted in faithful structures, ranging over closed subsets of truth assignments.*

Definition 3.6.2. *($\forall MSO_{H_{\min}}$ sentence) A $\forall MSO_{H_{\min}}$ sentence is a second order sentence with universal closed-set quantifiers, of the form*

$$\Phi = \forall^c A_1, \ldots, A_\ell \ \Psi,$$

where Ψ is a $H_{\min}$-sentence.

The definition of $\exists MSO_{H_{\min}}$ sentences is analogous.

Definition 3.6.3. *($\forall MSO_{H_{\min}}$ -definability) A family $\mathcal{F}$ of faithful structures is $\forall MSO_{H_{\min}}$ -definable if there is a $\forall MSO_{H_{\min}}$ sentence Φ such that for every faithful structure F it holds that $F \in \mathcal{F}$ iff F satisfies Φ.*

The following theorem corresponds to Theorem 2.7.3 and establishes the link between characterizability and definability for Horn belief revision.

Theorem 3.6.4. *Let $\mathcal{F}$ be a family of faithful structures. The family of $\mathcal{F}$-revision operators is characterizable iff the family $\widetilde{\mathcal{F}}$ is $\forall MSO_{H_{\min}}$ -definable.*

As in Section 2.8 a version of the Ajtai-Fagin game is defined for faithful structures only. This game is called the Horn-KM game. For reasons similar to the general case, Definition 2.8.3 is modified for the current context.

Definition 3.6.5. *(ℓ-min-variant) Let $F = (X, \leq)$ be a faithful structure. Let $A_1, \ldots, A_\ell$ be unary predicate symbols. There are $L = 2^\ell$ conjunctions μ of $A_1, \ldots, A_\ell$. Let unary predicate symbols $M_1, \ldots, M_L$ represent them. An ℓ-min-variant of F is a structure*

$$F'' = (X, A_1, \ldots, A_\ell, M_1, \ldots, M_L),$$

where $F' = (X, \leq, A_1, \ldots, A_\ell)$ is an ℓ-extension of F, and $M_1, \ldots, M_L$ are the interpretations of the formulas $\min_{\leq}^{\nu}(x)$ in F', for conjunctions ν of the A_is.

Definition 3.6.6. *($(\mathcal{G}, \ell, q)$-$\exists MSO_{H_{\min}}$ game, or Horn-KM game) Given a class $\mathcal{G}$ of faithful structures and parameters ℓ and q, the $(\mathcal{G}, \ell, q)$-$\exists MSO_{H_{\min}}$ game is played by Spoiler and Duplicator as follows:*

1. *Duplicator picks a faithful structure $F_1 = (X_1, \leq_1)$ in $\mathcal{G}$,*
2. *Spoiler picks closed subsets $A_1, \ldots, A_\ell$ of X_1,*
3. *Duplicator picks a faithful structure $F_2 = (X_2, \leq_2) \notin \mathcal{G}$, and closed subsets $B_1, \ldots, B_\ell$ of X_2,*

4. *Form the* ℓ-min-*variant* F_1'' *of* F_1 *determined by the* ℓ*-extension* $F_1' = (X_1, \leq_1, A_1, \ldots, A_\ell)$, *and the* ℓ-min-*variant* F_2'' *of* F_2 *determined by the* ℓ*-extension* $F_2' = (X_2, \leq_2, B_1, \ldots, B_\ell)$,

5. *Spoiler and Duplicator play a q-round first-order Ehrenfeucht - Fraïssé game on* F_1'' *and* F_2''.

The following theorem shows how to use these games to prove undefinability (which, by Theorem 3.6.4, implies non-characterizability). This corresponds to Corollary 2.8.8 in Chapter 2. The proof closely parallels that of Theorem 2.8.5 and Theorem 2.8.7, and for this reason, has been omitted here.

Theorem 3.6.7. *Let* $\mathcal{G}$ *be a class of faithful structures. Then* $\mathcal{G}$ *is not* $\exists MSO_{H_{\min}}$ *-definable iff for every* ℓ *and* q*, Duplicator has a winning strategy in the* $(\mathcal{G}, \ell, q)$-$\exists MSO_{H_{\min}}$ *game.*

3.7 Ab wheels

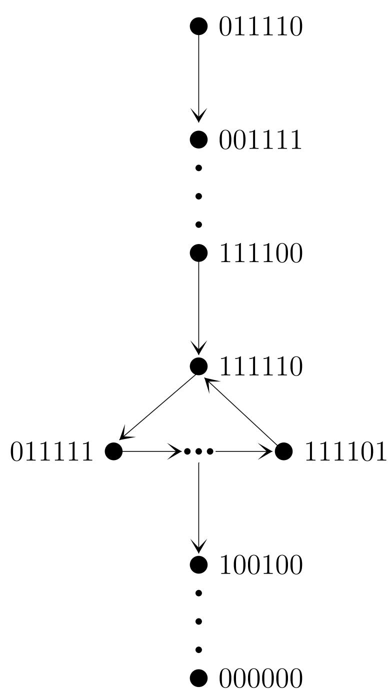

Figure 8: An ab wheel for n=6

After having developed the machinery to prove non-characterizability, this section describes the structures used in the proof of Theorem 3.8.1. These structures generalize the example of Figure 1 in [11].

Consider the knowledge base $H_n = \langle 0^n \rangle$. Let $C_n = \{a_1, \ldots, a_n\} \subset T_n$ be the set of truth assignments of weight $n-1$, where a_i is the truth assignment with a zero in the i'th position. Let $U_n = \{b_1, \ldots, b_n\}$ be the set of truth assignments of weight $n-2$ with two consecutive zeros. The truth assignment b_i has zeros in positions i and $i+1$. Indices are meant cyclically, i.e., for the case of 6 variables, $b_6 = 011110$. Thus $a_i \cap a_{i+1} = b_i$, and so $\{a_i, a_{i+1}, b_i\}$ is closed. Finally, let G_n be the directed graph on the vertex set C_n, containing one-way edges (a_i, a_{i+1}) (where addition is again meant cyclically) and the other edges both ways.

Definition 3.7.1. *(Ab wheel) An* ab wheel *W_n is a faithful structure for H_n with vertices T_n and $<$ corresponding to the following cases:*

1. *upper chain: truth assignments in U_n form a linear order and are greater than any other truth assignment,*
2. *cycle: truth assignments in C_n are below U_n and form the directed graph structure G_n as described above,*
3. *lower chain: all other truth assignments form a linear order and are smaller than any other truth assignment, with the all-zero truth assignment at the bottom.*

Note that

$$\min\{a_i, a_{i+1}, b_i\} = \{a_{i+1}\}. \tag{3.1}$$

Lemma 3.7.2. *Every closed subset of T_n has a unique minimum in W_n.*

Proof. Let S be a closed subset of T_n. The statement is clear if S is contained in the upper chain, or if it has an element in the lower chain. Otherwise assume that S contains $k \geq 1$ elements in the cycle, and possibly some elements in the upper chain.

If $k = 1$ then the statement is clear again. If $k = 2$ and the two elements in the cycle are consecutive, i.e., of the form a_i, a_{i+1}, then the unique minimal element of S is a_{i+1}. Other cases are not possible: if $k = 2$ and the two elements are non-consecutive, or if $k \geq 3$, the closure of S implies that it contains at least one element from the lower chain. $\square$

Lemma 3.7.3. *The following hold for the ab wheel W_n*

1. *it is a Horn compliant faithful structure for H_n,*
2. *$*_{W_n}$ is Horn revision operator which is not in $\widetilde{\mathcal{T}}$, i.e., it is not generated by any Horn compliant faithful ranking.*

Proof. For 1., only Horn compliance needs to be proved and it follows directly from Lemma 3.7.2. Part 2. follows from the facts that Horn compliant faithful rankings generate revision operators satisfying the acyclicity postulate scheme [11], and, on the other hand, $*_{W_n}$ falsifies $Acyc_n$. The latter claim follows from considering $\varphi_i = \langle\{a_i, a_{i+1}, b_i\}\rangle$. By (Equation 3.1) it holds that $(\min|\varphi_i|) \cap |\varphi_{i+1}| = \{a_{i+1}\}$, but

$$\min|\varphi_1| \cap |\varphi_n| = \{a_2\} \cap \{a_n, a_1, b_n\} = \emptyset.$$

$\square$

The following statement is not required for a proof, but does give other interesting properties of $*_{W_n}$.

Proposition 3.7.4. *The Horn revision operator* $*_{W_n}$

1. *satisfies the Horn AGM postulates,*

2. *satisfies the acyclicity postulates* $Acyc_\ell$ *for* $3 \leq \ell \leq n-1$.

3.8 Horn Belief Revision Results

Theorem 3.8.1. *The class of Horn belief revision operators obtained from Horn compliant faithful total preorders using minimization cannot be characterized by a finite set of postulates.*

Theorems 3.6.4 and 3.6.7 yield the following "characterization of characterizability".

Lemma 3.8.2. *Let* $\mathcal{F}$ *be a class of faithful structures and let* $\mathcal{G}$ *be the class of faithful structures not in* $\widetilde{\mathcal{F}}$. *The class of* $\mathcal{F}$*-revision operators is not characterizable iff for every* ℓ *and* q, *Duplicator has a winning strategy in the* $(\mathcal{G}, \ell, q)$-$\exists MSO_{\min}$ *game.*

Thus to show that the class of $\mathcal{T}$-revision operators is not characterizable, Duplicator needs a winning strategy in the $(\mathcal{G}, \ell, q)$-$\exists MSO_{\min}$ game for $\mathcal{G}$, the class of faithful structures not in $\widetilde{\mathcal{T}}$.

In the first round Duplicator picks the ab wheel W_n for $n = 2^\ell + 1$. Assume that Spoiler picks closed subsets $A_1, \ldots, A_\ell$ of T_n in the second round.

Let $I \subseteq \{1, \ldots, \ell\}$. Then $S_I = \cap_{i \in I} A_i$ is closed and has a unique minimum m_I. Let $a_j \in C_n$ be a truth assignment which is never minimal, i.e., $a_j \neq m_I$ for every I.

The Horn compliant faithful structure picked by Duplicator in the third round is a linear order where the upper and lower chains are the same as in W_n, and the cyclic structure G_n between them is replaced by the linear order

$$a_j > a_{j+1} > \ldots > a_n > a_1 > \ldots > a_{j-1}.$$

In other words, the cycle G_n is cut and it is made into a chain (referred to as the *middle* chain) by placing a_j on top. The closed sets $B_1, \ldots, B_\ell$ are the same as the ones selected by Spoiler in W_n. (Note that closedness is defined in terms of the truth assignments only, independently of the underlying pseudo-order.)

The claim is that the ℓ-min variants of the two structures are the same (as structures with unary relations over T_n), thus Duplicator wins the first-order game in the last part of the $Horn - KM$ game.

For this, one needs to show that for every $I \subseteq \{1, \ldots, \ell\}$ the set of minimal truth assignments in S_I and $S'_I = \cap_{i \in I} B_i$ are the same (in each case with respect to the corresponding pseudo-order). In both structures every closed subset has a unique minimum, it is sufficient to show that the minimal element m_I of S_I is also minimal in S'_I. This follows directly if m_I is in the upper or lower chain.

If m_I is in the cycle then there is two cases. If m_I is the only element of S_I in the cycle, then all other elements of S_I are in the upper chain. In the second structure m_I is in the middle chain and all the other elements of S'_I are in the upper chain, so m_I is indeed minimal in S'_I.

Otherwise, it must be the case that S_I has two elements in the cycle, m_I and its predecessor m'_I on the cycle, and all other elements of S_I are in the upper chain. The choice of a_j guarantees that $m_I \neq a_j$, thus m'_I is greater than m_I in the second structure, and all other elements of S'_I are in the upper chain. Thus m_I is again minimal in S'_I. This completes the proof of Theorem 3.8.1. □

The following formulates a non-characterizability result for another class of Horn revision operators as well. Let

$$\mathcal{B} = \{F : F \notin \widetilde{\mathcal{T}}\}$$

be the class of Horn compliant faithful structures not in $\widetilde{\mathcal{T}}$.

Theorem 3.8.3. *The class of $\mathcal{B}$-revision operators is not characterizable.*

The proof is similar to the proof of Theorem 3.8.1, with the Duplicator starting with a sufficiently large linear order with truth assignments of weight $n - 3$ and $n - 1$ at the bottom, and producing a faithful structure similar to an ab wheel in the third round.

3.9 Remarks on strict Horn compliance

Strictly Horn compliant faithful rankings are introduced by Zhang et al. [43]. In this section, some observations are made regarding properties of strictly Horn compliant faithful total preorders and their connections to classes studied in [11].

Horn compliance for a faithful total preorder is equivalent to the requirement that the intersection of two equivalent truth assignments is not above the two truth assignments [43]. A faithful total preorder is *strictly Horn compliant* if the intersection of two *arbitrary* truth assignments is not above both assignments. The following is a direct consequence of the definitions.

Proposition 3.9.1. *Let R be a total preorder on T_n with level sets $V_1, \ldots, V_r$. Then R is strictly Horn compliant iff $V_1 \cup \ldots \cup V_i$ is closed under intersection for every i, $1 \leq i \leq r$.*

In other words, strictly Horn compliant total preorders can be thought of as a sequence of Horn formulas $\varphi_0, \ldots, \varphi_r$, where $\varphi_0 = H$ is the knowledge base, φ_r is identically true, φ_i implies φ_{i+1} for every i, and the level of a truth assignment is determined by the first formula it satisfies. This also implies that strictly Horn compliant total preorders have a syntactic definition as well, using the completeness of unit resolution for Horn formulas, as each formula can be obtained from the previous one using a sequence of unit resolutions and weakenings which preserve the Horn property of clauses.

Strictly Horn compliant belief revision operators turn out to be of interest from the point of view of relating contractions and revisions for Horn logic [43]. Strictly Horn compliant belief revision operators are related to basic and canonical Horn belief revision operators introduced in [11].

A faithful total preorder is *basic* if truth assignments not in the knowledge base are ranked according to their weight, with lower weight truth assignments having lower rank. A *canonical* faithful ranking is specified by a Horn knowledge base H and a partition $(P_1, \ldots, P_r)$ of the variables. This determines a sequence $\varphi_0, \ldots, \varphi_{r+1}$ of Horn formulas, where $\varphi_0 = H$ and φ_i is obtained from H by adding the negations of every variable in $P_1 \cup \ldots P_i$ to every Horn clause of H. This, in turn, determines a total preorder on the truth assignments, where a truth assignment is on the i'th level if φ_i is the first formula it satisfies. There is an additional level added for truth assignments not satisfying φ_r.

It is clear, then, that both basic and canonical revisions are strongly Horn compliant. Proposition 3.9.1 implies that strictly Horn compliant revision operators, given their syntactic description, can be computed efficiently.

Thus strictly Horn compliant belief revision operators emerge as a natural class based on total preorders with Horn "fallback sets" [11] and good computational properties. Characterizability of Horn revision operators obtained from such rankings is an open problem. If the answer turns out to be negative then the framework of this papercould provide an approach to prove this.

CHAPTER 4

CONDITIONAL KNOWLEDGE BASES

4.1 Introduction

The book [28] introduces knowledge bases, composed of conditional statements each of which has an associated probability. The maximum entropy distribution is utilized as the method for inferring the probability of a condition outside the knowledge base with an unknown associated probability. In [29] and [30], an alternative technique for computing this method of inference was proposed. The approach consists of constructing a system of polynomials, one for each conditional in the knowledge base. In addition, given another conditional with an unknown probability, a new polynomial encoding the unknown probability can be appended to the polynomial system of the knowledge base. Using techniques from computational algebra, the unknown probability can be computed from this augmented system of polynomials.

There are several advantages to such an alternative framework. Most notably, this construction utilizes widely available and powerful techniques from computational algebraic geometry. Consequently, one could implement this method on almost any widely-used computer algebra system. In addition, this reformulation could offer further insight into the algebraic and geometric properties of conditional knowledge bases, providing mathematical insight into conditional knowledge base structure and its associated probability distribution. For these reasons, further investigation into these techniques would be of interest.

The purpose of this chapter is to investigate computational aspects of this technique. As part of this investigation, algebraic and geometric properties of the augmented polynomial systems of conditional knowledge bases will be explained. Experimental statistics will be presented to demonstrate the computational properties discussed. In conclusion, suggestions are made for further topics of interest and their relation to those discussed in this section.

4.2 Computational Aspects of Conditional Knowledge Bases

Section 4.3.1 provides tools to apply Theorem 4.3.10 for inference on conditional knowledge bases. In [30], several examples of these computations are given in which the Gröbner basis is satisfied by a finite set of points and only one root of the polynomial defining the elimination ideal is in the interval $(0, 1)$. The purpose of the following sections is to examine conditions under which these conditions may not hold.

In Section 4.4, the situation in which the solutions set to the Gröbner basis has high dimensional components will be considered. Algebraic background will be given relating sets of polynomials with computational results of elimination. The connection will then be used to examine several examples in which elimination does not output a univariate polynomial ideal.

Section 4.5 considers the case in which the elimination ideal is defined by a univariate polynomial, but that polynomial may have several roots on the interval $(0, 1)$. An example illustrates the phenomenon, and Theorem 4.3.6 is applied in an attempt to distinguish which roots are appropriate.

In [30], a method is proposed to simplify polynomials by removal of specific terms. Section 4.6 considers the effects of this simplification on the behavior described in Section 4.4 and Section 4.5. Examples are given to demonstrate situations which may occur.

Finally, Section 4.7 considers specific knowledge base forms with arbitrary formulas. Samples are taken of knowledge bases with and without the simplification mentioned above and the frequency of the behaviors from Section 4.4 and Section 4.5 are recorded. The computation time to complete elimination is also recorded.

4.3 Conditional Knowledge Bases

Let $\mathcal{L}$ be the language of propositional logic with finitely many variables, $a, b, c, \ldots$. Denote formulas of $\mathcal{L}$ as capital letters $A, B, C, \ldots$[1]. If $\mathcal{L}$ has n variables, denote the set of 2^n truth assignments as T_n. For each formula A, the truth assignments which satisfy A are denoted $|A|$. Given a probability distribution P which assigns probabilities to each element of $\omega \in T_n$, probabilities associated with formulas can be determined by

$$P(C) = \sum_{\omega \in |C|} P(\omega).$$

Moreover, conditional probabilities can be defined as follows

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[1] In order to remain consistent with the literature, the notation of this chapter differs from previous chapters.

$$P(B|A) = \frac{P(A \wedge B)}{P(A)} = \frac{\sum\limits_{\omega \in |A \wedge B|} P(\omega)}{\sum\limits_{\omega \in |A|} P(\omega)}, \text{ where } P(A) > 0. \tag{4.1}$$

Definition 4.3.1. *A conditional knowledge base is a subset of a language over conditional statements in which probabilities are rational,*

$$KB = \left\{ B_j | A_j \left[\frac{n_j}{m_j} \right] : A_j, B_j \in \mathcal{L}, m_j, n_j \in \mathbb{N},\ 0 \leq j \leq s \right\}.$$

Example 4.3.2. *Consider a situation in which there is knowledge of at least the three following events:*

A *is the event that an area has environmental pollution*

B *is the event that a particular species has not been sighted recently in the given area*

C *is the event that a particular species in this area is locally extinct*
with probabilities $P(B|A) = .15$ *and* $P(C|A) = .6$*. Using Definition 4.3.1, the knowledge base could be represented as*

$$\left\{ B|A \left[\frac{3}{20} \right], C|A \left[\frac{3}{5} \right] \right\}.$$

A given knowledge base may not contain enough information to uniquely determine a probability distribution over the set of truth assignments. In order to overcome this challenge, information theory has a well-established mathematical framework, the maximum entropy principle [26]. The maximum entropy principle employs the Shannon entropy measure to maximize entropy, i.e., maximize randomness in order to minimize bias. The maximum entropy principle integrates existing information, while minimizing unnecessary assumptions, to produce a probability distribution.

This distribution is acquired by maximizing the Shannon entropy measure

$$-\sum_{k=1}^{2^n} P_k(\omega_k) \ln P_k(\omega_k)$$

with the constraints

$$\sum_{k=1}^{2^n} P_k(\omega_k) = 1$$

and a linear constraint for each conditional in the knowledge base

$$0 = (1 - \frac{n_j}{m_j}) \sum_{\omega \models A_j \wedge B_j} P_k(\omega) - \frac{n_j}{m_j} \sum_{\omega \models A_j \wedge \neg B_j} P_k(\omega).$$

The result is a system of equations in 2^n number of variables, one for the probability of each truth assignment.

Definition 4.3.3. *Let*

$$f_j = \underbrace{(m_j - n_j) x_i^{m_j} \sum_{\omega \models A_j \wedge B_j} \prod_{\substack{i \neq j \\ \omega \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq j \\ \omega \models \neg A_i}} x_i^{n_i}}_{f_j^+} - \underbrace{n_j \sum_{\omega \models A_i \wedge \neg B_i} \prod_{\substack{i \neq j \\ \omega \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq j \\ \omega \models \neg A_i}} x_i^{n_i}}_{f_j^-}$$

for $1 \leq j \leq s$. The polynomial f_j can be decomposed into a positive and negative polynomials, f_j^+ and f_j^- respectively.

Given a knowledge base

$$\left\{B_1|A_1\left[\frac{n_1}{m_1}\right],\ldots,B_s|A_s\left[\frac{n_s}{m_s}\right]\right\}$$

denote the ideal in $\mathbb{C}[x_1,\ldots,x_s]$ *formed by these polynomials as* I_{KB}*.*

For each conditional $B_j|A_j$, the corresponding polynomial is a sum over the sets of truth assignments satisfying $A_j \wedge B_j$ and $A_j \wedge \neg B_j$. These sets of truth assignments are then compared to the truth assignments satisfying each of the other conditionals, $B_i|A_i$, in the knowledge base. Based on this comparison, a monomial, x_i^0, $x_i^{m_i}$, or $x_i^{n_i}$ is used to form a term in the j^{th} polynomial.

Example 4.3.4. *Let* $KB = \{b|c, c|a\}$ *with* $P(b|c) = \frac{1}{2}$ *and* $P(c|a) = \frac{1}{2}$ *over the three propositional variables* a, b, c*. The following truth assignments satisfy these formulas.*

$$|a| = \{111, 101, 110, 100\}$$

$$|b| = \{111, 011, 110, 010\}$$

$$|c| = \{111, 101, 011, 001\}$$

Then

$$f_1^+ = (2-1)x_1^2 \sum_{\omega \models A_1 \wedge B_1} \prod_{\substack{i \neq 1 \\ \omega \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ \omega \models \neg A_i}} x_i^{n_i} \;=\; x_1^2 \sum_{\{011,111\} \models b \wedge c} \prod_{\substack{i \neq 1 \\ \omega \models a \wedge c}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ \omega \models \neg a}} x_i^{n_i}$$

$$= x_1^2 \left[\prod_{\substack{i \neq 1 \\ 011 \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ 011 \models \neg A_i}} x_i^{n_i} + \prod_{\substack{i \neq j \\ 111 \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ 111 \models \neg A_i}} x_i^{n_i} \right]$$

$$= x_1^2 \left[\prod_{011 \not\models a \wedge c} x_2^0 \prod_{011 \models \neg a} x_2^1 + \prod_{111 \models a \wedge c} x_2^2 \prod_{111 \not\models \neg a} x_2^0 \right]$$

$$= x_1^2(x_2 + x_2^2),$$

$$f_1^- = \sum_{\omega \models A_1 \wedge \neg B_1} \prod_{\substack{i \neq 1 \\ \omega \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ \omega \models \neg A_i}} x_i^{n_i} \;=\; x_1^2 \sum_{\{101,001\} \models \neg b \wedge c} \prod_{\substack{i \neq 1 \\ \omega \models a \wedge c}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ \omega \models \neg a}} x_i^{n_i}$$

$$= \left[\prod_{\substack{i \neq 1 \\ 101 \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ 101 \models \neg A_i}} x_i^{n_i} + \prod_{\substack{i \neq j \\ 001 \models A_i \wedge B_i}} x_i^{m_i} \prod_{\substack{i \neq 1 \\ 001 \models \neg A_i}} x_i^{n_i} \right]$$

$$= \left[\prod_{101 \models a \wedge c} x_2^2 \prod_{101 \not\models \neg a} x_2^0 + \prod_{001 \not\models a \wedge c} x_2^0 \prod_{001 \models \neg a} x_2^1 \right]$$

$$= (x_2^2 + x_2).$$

This results in final polynomials

$$f_1 = f_1^+ - f_1^- = x_1^2(x_2 + x_2^2) - (x_2^2 + x_2)$$

and

$$f_2 = x_2^2(1 + x_1^2) - 2x_1.$$

One can utilize $|A_j|$ and $|B_j|$ to define conditional formulas constructing a formula which satisfies the given truth assignments. In the implementation, formulas are represented by their satisfying truth assignments. While representing formulas as sets of truth assignments may limit computational scalability, this formulation provides more flexibility and ease when producing formulas to input into conditionals. Moreover, the implementation of the algorithms utilized Python. Due to sets being a built-in datatype in Python, operations like set comparison and inclusion are already implemented saving time spent on implementation.

The maximum entropy principle does not guarantee that there is a probability distribution which can accommodate known probabilities of conditionals in the knowledge base. Therefore, criteria to discern whether there exists an applicable probability distribution is needed before applying the maximum entropy principle. If there is at least one probability distribution which satisfies the information in the knowledge base, the knowledge base is consistent.

Definition 4.3.5. *A conditional knowledge base,*

$$KB = \left\{ B_1|A_1 \left[\frac{n_1}{m_1}\right] \ldots B_s|A_s \left[\frac{n_s}{m_s}\right] \right\},$$

is consistent iff there exists a probability distribution, P, such that for all $B_j|A_j \in KB$

$$P(B_j|A_j) = \frac{n_j}{m_j}.$$

As part of [29] and [30], there is a method proposed to determine the consistency of probabilities assigned to elements in a knowledge base. Issues of consistency may be relevant as a preliminary test of a knowledge base. Consistency will be used later when discussing inference. In this context, consistency will be used as tool to evaluate the viability of several possible probabilities for a conditional which is not in the knowledge base. In Example 4.5.2, such a situation will occur, and this will offer a possible method to distinguish the appropriate value. The following theorem offers a computational method to assess consistency of a given knowledge base.

Theorem 4.3.6. *(Kern-Isberner et al.* [30]*) Given a conditional knowledge base, KB, composed of s conditionals with non-trivial probabilities, if the KB is consistent, then the polynomials defined by KB are not a generating set for* $\mathbb{C}[x_1, \ldots, x_s]$

Computationally, if

$$I_{KB} = (1),$$

then the original knowledge base was inconsistent. Given a polynomial system from a knowledge base, one common task is to infer the probability of a condition not in the knowledge base. For instance, a natural question in Example 4.3.2 may inquire about the probability that a species is extinct in a particular area, given pollution and no recent sightings. This would be equivalent to asking what is the probability of the conditional $C|A \wedge B$. Mathematically, this corresponds to constructing a new polynomial with a variable to represent the probability to be inferred.

Definition 4.3.7. *Let $B_{inf}|A_{inf}$ be a conditional probability to be inferred from a conditional knowledge base, KB. Similarly to Definition 4.3.3, a polynomial with a special variable, x_{inf}, to encode the probability of a condition $B_{inf}|A_{inf}$ can be defined as*

$$f_{inf} = \underbrace{x_{inf} \sum_{\omega \models A_{inf} \wedge B_{inf}} \prod_{\omega \models A_i \wedge B_i} x_i^{m_i} \prod_{\omega \models \neg A_i} x_i^{n_i}}_{f_{inf}^{+}} - \underbrace{\sum_{\omega \models A_{inf} \wedge \neg B_{inf}} \prod_{\omega \models A_i \wedge B_i} x_i^{m_i} \prod_{\omega \models \neg A_i} x_i^{n_i}}_{f_{inf}^{-}}.$$

where i indexes conditionals in KB.

Unlike the knowledge base polynomials, the inference polynomial does not contain information about a specific probability, but instead this information is encoded into the variable x_{inf}.

Example 4.3.8. *Consider inferring the probability of $a|(b \wedge c)$ with the knowledge base from Example 4.3.4. The corresponding inference polynomial would be*

$$f_{inf} = x_{inf}(x_1^2 x_2 + x_1^2 x_2^2) + x_1^2 x_2^2.$$

Definition 4.3.9. *Given a knowledge base of s conditional statements and a conditional $B_{inf}|A_{inf}$ for which the probability is to be inferred, the augmented polynomial system is formed by adding the inference polynomial for $B_{inf}|A_{inf}$ to the KB polynomials,*

$$\{f_1, \ldots, f_s\} \cup f_{inf}.$$

Denote the ideal of $\mathbb{C}[x_1, \ldots, x_s, x_{inf}]$ formed by the augmented polynomial system as I_{KB^*}.

The goal is to find the probability which has been encoded into the variable x_{inf} and satisfies the following theorem.

Theorem 4.3.10. *(Kern-Isberner et al.* [30]*) (Let* $KB = \{(B_1|A_1)\left[\frac{n_1}{m_1}\right], \ldots, (B_s|A_s)\left[\frac{n_s}{m_s}\right]\}$ *be consistent with nontrivial probabilities and* $B_{inf}|A_{inf}$ *a new conditional formula with a probability to be inferred from the* KB*. The probability* $\left[\frac{n_{inf}}{m_{inf}}\right]$ *under the maximum entropy distribution is a common root of*

$$I_{KB^*} \cap \mathbb{C}[x_{inf}].$$

This theorem leads naturally to computation of polynomials in the intersection of this ideal by means of elimination. Such calculations rely on the formation of a Gröbner basis. The following is a brief introduction to the topic.

4.3.1 Gröbner Basis and Elimination

Computationally, Gröbner bases are often used to represent the generating set of an ideal. A Gröbner basis preserves much of the vital algebraic and geometric information of the ideal. The following is intended to be a short introduction to Gröbner bases and their application in elimination theory. The background information presented in this section with more detail can be found in [8].

Definition 4.3.11. *A monomial order is a total well-ordered relation,* $\leq$*, on the set of monomials* x^α*,* $\alpha \in \mathbb{Z}^n_{>0}$ *such that if* $\alpha \leq \beta$*, then* $\alpha + \delta \leq \beta + \delta$*.*

With a fixed monomial ordering, leading terms of polynomials and the associated monomial ideals defined by the leading terms can be defined. Leading term ideals are often easier to work with and already contain important algebraic information.

Definition 4.3.12. *Given a monomial ordering the leading term of a nonzero polynomial f is the monomial of the maximal order in f and 0 if $f = 0$. Denote the leading term of f as $LT(f)$.*

Definition 4.3.13. *If I is an ideal in $\mathbb{C}[x_1, \ldots, x_n]$ then the leading term ideal,*

$$LT(I) = \{LT(f) : f \in I\}.$$

As a consequence of Hilbert's Basis theorem, every ideal in $\mathbb{C}[x_1, \ldots, x_s]$ is finitely generated. By a method analogous to Gaussian elimination, every ideal also has a generating set which is a Gröbner basis.

Definition 4.3.14. *A Gröbner basis for an ideal I is a set $\{g_1, \ldots, g_m\}$ such that*

$$I = (g_1, \ldots, g_m) \text{ and } LT(I) = (LT(g_1), \ldots, LT(g_m)).$$

Definition 4.3.15. *Let I be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Define the j^{th} elimination ideal with respect to $x_1 \leq \cdots \leq x_n$ as*

$$I_j = I \cap \mathbb{C}[x_{j+1}, \ldots, x_n].$$

Algorithms that depend on Gröbner basis calculations can be doubly exponential in time complexity. The computations in this chapter use Sage 6.4.1 and Singular 3.1.6 in which the implementation of elimination uses Gröbner bases as part of the computation [14, 21][1]. As a consequence, the degree of polynomials or the size of coefficients in intermediate steps may increase the computational time significantly [15].

With increases in computational power and development of new algorithms, the performance of techniques employing Gröbner bases has increased [15, 18, 35]. As existing software packages integrate these new algorithms, implementations of computational methods expressed here will become even more feasible. If algebraic limits can be set on degree of polynomials or size of coefficients, computational time may decrease.

On the other hand, from a design perspective, time is only one possible constraint. The reliability of software to produce a meaningful solution may outweigh a concern about computational time. In such a situation, the computational technique may only output relevant solutions for a specific set of inputs. Therefore, identifying the constraints on the input is essential to implementation.

In either case, a better understanding of the computational aspects and their algebraic consequences of the proposed approach would provide valuable information for practical situations. The remainder of this chapter is dedicated to investigating some of the computational features of this method. Whenever possible, algebraic explanation accompanies the observations of algebraic behavior.

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[1] Code will be posted to https://github.com/jonyaggie/thesis-code

According to the maximum entropy principle, the uniform distribution when applicable is the distribution which assumes the minimal amount of prior information. Therefore, to make comparison easier, all examples use probabilities from the uniform distribution unless otherwise stated. A variety of modern computer algebra systems can perform these calculations.

4.4 Algebraic Independence

In order to find the polynomial in the intersection, an elimination ideal is computed to eliminate $x_1, \ldots, x_n$, leaving a polynomial in x_{inf}. As with the previous chapter, the background concepts and propositions are from [8]. An affine variety is the zeros in $\mathbb{C}^n$ for a system of polynomials $\{f_1, \ldots, f_s\}$. If V is an affine variety, the coordinate ring, $\mathbb{C}[V]$, is formed from all the polynomial functions which vanish on V.

Definition 4.4.1. *The set of polynomials $f_1, \ldots, f_s \in \mathbb{C}[V]$ is algebraically independent over $\mathbb{C}$ if there is no nonzero polynomial, p, in s variables with coefficients $\mathbb{C}$ such that $p(f_1, \ldots, f_s) = 0$ in $\mathbb{C}[V]$.*

Proposition 4.4.2. *Let $V \subset \mathbb{C}^n$ be an affine variety. The maximal number of elements of $\mathbb{C}[V]$ which are algebraically independent over $\mathbb{C}$ is equal to the largest r such that there exists r variables for which the ideal formed from V intersected with $\mathbb{C}[x_1, \ldots, x_r]$ equals (0).*

In particular, f_{inf} may be algebraically independent and thus $I_{KB^*} \cap \mathbb{C}[x_{inf}] = \{0\}$. This could be problematic, because the inferred probability is encoded as a root of this intersection. Therefore, in such cases, one may not be able to ascertain a probability for C_{inf}.

Observation 4.4.3. *There exist knowledge bases for which f_{inf} is algebraically independent i.e. the elimination ideal, $I_{KB^*} \cap \mathbb{C}[x_{inf}] = (0)$.*

The following example proves the existence of such knowledge bases.

Example 4.4.4. *Consider another example, with knowledge base $B_1|A_1 = B|A[\frac{2}{3}]$ and $B_2|A_2 = C|A[\frac{1}{3}]$ with A, B, C the formulas which satisfy*

$$\begin{aligned} |A| &= \{(0,1,1),(0,1,0),(0,0,1)\} \\ |B| &= \{(1,1,0),(0,1,1),(1,0,0),(1,0,1),(0,0,0),(1,1,1),(0,1,0)\} \\ |C| &= \{(1,0,0),(0,0,1).\}. \end{aligned}$$

Applying Definition 4.3.3 and 4.3.7 yields the set

$$f_1 = 2x_1^3 - 2x_2^3$$

$$f_2 = 2x_2^3 - 2x_1^3$$

$$f_{inf} = x_2^3 x_{inf} + x_1^2 x_2 x_{inf} - x_1^2 x_2.$$

Using elimination on x_1, x_2 to find the intersection of this ideal and $\mathbb{C}[x_{inf}]$ yields the ideal (0).

A concern could be whether f_1 or f_2 being algebraically independent would have any effect on the intersection $I_{KB^*} \cap \mathbb{C}[x_{inf}]$. A geometric interpretation of elimination is in terms of projections. If the elimination ideal with respect to x_{inf} is a polynomial with a finite set of roots, then the solutions to the original system of polynomials only have a finite set of values for the x_{inf} component. Thus the algebraic independence of f_{inf} corresponds to an infinite set of values for x_{inf}.

It may occur that f_1 or f_2 are algebraically independent which has no effect on computing the elimination ideal with respect to f_{inf}. The following example illustrates a case in which f_1 and f_2 are each algebraically independent, but neither $\{f_{inf}\}$ or $\{f_1, f_2\}$ are.

Example 4.4.5. *Let A, B, and C be formulas satisfying*

$$\begin{aligned} |A| &= \{(1,1,0),(0,1,1),(1,0,0),(1,0,1),(0,1,0),(1,1,1)\} \\ |B| &= \{(1,0,0),(1,1,0),(0,1,0),(0,0,1),(1,0,1)\} \\ |C| &= \{(1,0,0),(1,1,0),(0,1,0),(0,0,0),(1,0,1)\} \end{aligned}$$

and the conditional knowledge base consist of the conditions $B_1|A_1 = B|A[\frac{2}{3}]$ and $B_2|A_2 = C|A[\frac{2}{3}]$. Consider inferring $C_{inf} = B|C$. The resulting polynomial system is

$$f_1 = 4x_1^3x_2^3 - 4$$

$$f_2 = 4x_1^3x_2^3 - 4$$

$$f_{inf} = 4x_1^3x_2^3x_{inf} - 4x_1^3x_2^3 + x_1^2x_2^2x_{inf}$$

The following are the elimination ideals for this system

$$I_{KB^*} \cap \mathbb{C}[x_1] = \{0\}$$

$$I_{KB^*} \cap \mathbb{C}[x_2] = \{0\}$$

$$I_{KB^*} \cap \mathbb{C}[x_{inf}] = \{65 * xi^3 - 192 * xi^2 + 192 * xi - 64\}.$$

In fact, if one considers eliminating less variables the intersection is no longer zero. For example, the intersection $I_{KB^*} \cap \mathbb{C}[x_1, x_2] = \{x_1^3 x_2^3 - 1\}$. Therefore, one is the largest number of algebraically independent polynomials in this system.

4.5 Multiple Roots

The purpose of this section is to consider the case when elimination results in a polynomial ideal of one variable. The primary concern in this situation is whether this polynomial can have multiple roots on the interval $(0, 1)$. Several examples of knowledge bases, their augmented polynomial systems, and the results of inference are presented.

Observation 4.5.1. *Given a conditional knowledge base, the elimination ideal for an augmented polynomial system for inferring the probability of $A_{inf}|B_{inf}$ may have multiple roots within $(0, 1)$. Moreover, there may be no method by which to determine the root which corresponds to the maximum entropy distribution.*

The following example proves this observation.

Example 4.5.2. *Let A, B, and C be formulas satisfying*

$$\begin{aligned} |A| &= \{(1,0,0),(1,1,0),(0,1,0),(0,0,1)\} \\ |B| &= \{(1,1,0),(0,1,1),(1,0,0),(1,0,1),(0,0,0),(1,1,1)\} \\ |C| &= \{(1,0,0),(0,1,1),(0,0,0),(0,0,1)\} \end{aligned}$$

and the conditional knowledge base consist of the conditions $B_1|A_1 = B|A[\frac{1}{2}]$ and $B_2|A_2 = C|A[\frac{1}{2}]$. Consider inferring $C_{inf} = B|C$.

From Definition 4.3.3 the following polynomials can be produced

$$f_1 = x_1^2 x_2^2 + x_1^2 - x_2^2 - 1$$

$$f_2 = x_1^2 x_2^2 - x_1^2 + x_2^2 - 1$$

$$f_{inf} = x_1^2 x_2^2 x_{inf} - x_1^2 x_2^2 + x_1^2 x_{inf} + 4x_1 x_2 x_{inf} - 2x_1 x_2$$

Eliminating x_1 and x_2 results in $\{32x_{inf}^3 - 48x_{inf}^2 + 22x_{inf} - 3\}$ which produces three real roots in (0,1) – $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$.

Unlike the Example 4.4.4 in the previous section, f_{inf} is not algebraically independent and the elimination ideal yields a degree three polynomial in x_{inf}. By Theorem 4.3.10, the probability of $b|c$ is a root of this polynomial. This polynomial has 3 real roots, $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$. While it is not discussed in detail in [30], ideally, the results of this calculation would be a polynomial with a unique root on the interval $(0, 1)$, yet this example has three such roots. How would one choose which root is appropriate?

A possible attempt to find the the correct root may appeal to Theorem 4.3.6. If adding C_{inf} to the knowledge base only was consistent for one of the three values produced, then there would be a method to identify the correct probability for C_{inf}. However, this is not the case which can be seen in the following example.

Example 4.5.3. *Let A, B, and C be formulas satisfying*

$$\begin{aligned}
|A| &= \{(0,1,1),(0,0,0)\} \\
|B| &= \{(1,1,0),(1,1,1),(0,0,0),(0,0,1),(1,0,1)\} \\
|C| &= \{(1,0,0),(1,1,0),(0,0,0),(0,0,1)\}
\end{aligned}$$

and the conditional knowledge base consist of the conditions $B_1|A_1 = B|A[\frac{1}{2}]$ *and* $B_2|A_2 = C|A[\frac{1}{2}]$. *Consider inferring* $C_{inf} = B|C$.

From Definition 4.3.3, the following polynomials can be produced

$$f_1 = x_1^2x_2^2 - 1$$

$$f_2 = x_1^2x_2^2 - 1$$

$$f_{inf} = x_1^2x_2^2x_{inf} - x_1^2x_2^2 + 3x_1x_2x_{inf} - 2x_1x_2.$$

Elimination produces two real roots in (0,1) – $\{\frac{1}{2}, \frac{3}{4}\}$.

Assigning $P(B|C) = \frac{1}{2}$ *and adding a polynomial* f_3 *to polynomial system of the knowledge base yields*

$$f_1 = x_1^2x_2^2x_3^2 - x_3$$

$$f_2 = x_1^2x_2^2x_3^2 - x_3$$

$$f_3 = x_1^2x_2^2x_3^2 + 2x_1x_2x_3^2 - 2x_1x_2.$$

On the other hand, using $P(B|C) = \frac{3}{4}$ *to generate* f_3 *results in*

$$f_1 = x_1^2 x_2^2 x_3^4 - x_3^3$$

$$f_2 = x_1^2 x_2^2 x_3^4 - x_3^3$$

$$f_3 = x_1^2 x_2^2 x_3^4 + 2x_1 x_2 x_3^4 - 6x_1 x_2.$$

Applying Theorem 4.3.6, neither of the Gröbner bases generated from these systems are trivial. Therefore, both roots are potential candidates for the probability of conditional $B|C$.

4.6 GCD Removal

One computational simplification proposed in [30] is to divide each f_i by $GCD(f_i^+, f_i^-)$ and similarly for f_{inf}. The goal is to remove solutions which may make computation easier without effecting the solutions containing valuable information about the knowledge base. Formally, in the remainder of this section, all polynomial systems will be composed of polynomials of the form

$$f_i^* = \frac{f_i^+ - f_i^-}{GCD(f_i^+, f_i^-)}.$$

The system of polynomials for a knowledge base or its augmented polynomial system would be composed the same as in previous sections but with this additional simplification. These systems will be called the GCD-simplified knowledge base polynomial systems and GCD-simplified augmented polynomial systems respectively.

Algebraically, the GCD-simplified systems cannot contain additional solutions. Each time a polynomial is divided, a solution to that polynomial is removed. Denote this solution as θ. If

θ was not a common solution to the system, the common solution set is unaffected. If θ was a common solution, then θ is no longer part of the common solutions. In either case, solutions are not *added* to the system. Consequently, the dimension and common roots of the system can never be adversely affected. Thus the worst case scenario would be that GCD-simplified systems would have all the same properties noted in the previous sections. Computationally, removal of terms may theoretically cause swell in size of the polynomials in the system.

The influence of removing the GCD has can be dramatic or minimal depending on the situation. In this section, examples which illustrate that the situations in Observation 4.4.3 and Observation 4.5.1 still can occur in GCD-simplified systems will be presented. The next section will then offer some experimental results to demonstrate the vastly different effect the technique may have on inference and computational time.

Observation 4.6.1. *Removal of the GCD from an augmented system of polynomials does not preclude the occurrence of the phenomena in Observation 4.4.3 and 4.5.1.*

The following examples prove the existence situation when GCD-simplified systems results in behaviors from Observation 4.4.3 and 4.5.1.

4.6.1 Multiple Roots

Example 4.6.2. *Repeating this process with the GCD removed from Example 4.5.2 results in the polynomial equations*

$$f_1 = x_1^2 - 1$$

$$f_2 = x_2^2 - 1$$

$$f_{inf} = x_1 x_2^2 x_{inf} - x_1 x_2^2 + x_1 x_{inf} + 4x_2 x_{inf} - 2x_2.$$

Finding the elimination ideal yields the polynomial $\{2x_{inf} - 1\}$ *which clearly has one real root –* $\frac{1}{2}$.

While this example did yield a positive result, there are cases in which removing the GCD does reduce the number of applicable roots.

Example 4.6.3. *Let A, B, and C be formulas satisfying*

$$\begin{aligned} |A| &= \{(0,1,1),(1,0,0),(0,0,1),(0,0,0),(1,1,1),(0,1,0)\} \\ |B| &= \{(0,1,1),(1,1,0),(0,0,1),(1,0,1),(0,0,0),(1,1,1)\} \\ |C| &= \{(0,1,1),(0,0,1),(1,0,1),(0,0,0),(1,1,1),(0,1,0)\} \end{aligned}$$

and the conditional knowledge base consist of the conditions $B_1|A_1 = B|A[\frac{2}{3}]$ *and* $B_2|A_2 = C|A[\frac{5}{6}]$. *Consider inferring* $C_{inf} = B|C$.

From Definition 4.3.3 the following polynomials can be produced

$$f_1 = 4x_1^3 x_2^6 - 2x_2^6 - 2$$

$$f_2 = 4x_1^3 x_2^6 + x_2^6 - 5$$

$$f_{inf} = 4x_1^3 x_2^6 x_{inf} - 4x_1^3 x_2^6 + x_1^2 x_2^5 x_{inf} - x_1^2 x_2^5 + x_2^6 x_{inf}.$$

Eliminating x_1 and x_2 results in $\{15624 * x_{inf}^6 - 74994 * x_{inf}^5 + 149985 * x_{inf}^4 - 159980 * x_{inf}^3 + 95985 * x_{inf}^2 - 30714 * x_{inf} + 4095\}$ *which produces two real roots in (0,1) –* $\{\frac{5}{6}, \frac{3}{4}\}$.

Removal of the GCD will only affect f_{inf}. Therefore, GCD-simplified augmented polynomials system would be

$$f_1 = 4x_1^3x_2^6 - 2x_2^6 - 2$$

$$f_2 = 4x_1^3x_2^6 + x_2^6 - 5$$

$$f_{inf} = 4x_1^3x_2 * x_{inf} - 4x_1^3x_2 + x_1^2x_{inf} - x_1^2 + x_2x_{inf}$$

which has the same elimination ideal with respect to x_i as the original polynomial system. Therefore, it also has the same two potential roots.

4.6.2 Algebraic Independence

Similar to the case of multiple roots, removing the GCD may have a positive influence on algebraic independence of f_{inf} by removing higher dimensional solutions. There is no guarantee that this will happen in all circumstances as illustrated by Example 4.6.4. Similar to Example 4.6.3, the removal of the GCD only effects f_{inf} and this does not change the polynomials generating the elimination ideal with respect to x_{inf}.

Example 4.6.4. *Let A, B, and C be formulas satisfying*

$$\begin{aligned} |A| &= \{(1,1,1),(0,0,1),(1,0,1)\} \\ |B| &= \{(0,1,1),(0,0,1),(1,0,1)\} \\ |C| &= \{(1,0,0),(0,1,1),(1,1,1),(0,0,0)\} \end{aligned}$$

and the conditional knowledge base consist of the conditions $B_1|A_1 = B|A[\frac{2}{3}]$ *and* $B_2|A_2 = C|A[\frac{1}{3}]$. *Consider inferring* $C_{inf} = B|C$.

From Definition 4.3.3 the following polynomials can be produced

$$f_1 = 2x_1^3 - 2x_2^3$$

$$f_2 = -2x_1^3 + 2x_2^3$$

$$f_{inf} = 3x_1^2x_2x_{inf} + x_2^3x_{inf} - x_1^2x_2$$

As noted above, the only GCD which can be removed is x_2 *in* f_{inf}. *However, removing this term does not change the fact that the elimination ideal with respect to* x_{inf} *is* (0).

4.7 Experimental Results

One noteworthy feature of this method of computing inferred probabilities is the influence of the knowledge base structure. For example, for three arbitrary formulas the knowledge bases $B|A, C|A$ and $B|A, A|C$ perform very differently when inferring the probability of $B|C$. As a initial step in analyzing this behavior, experiments were conducted looking at several knowledge base structures, some of which were known properties.

In [28], several properties of maximum entropy inference are proposed. These properties require specific structure of a knowledge base. The following experiments were conducted to investigate the relationship between knowledge base structure of these inference properties and a randomly selected knowledge base structure. Data was collected for each of the previously mentioned computational considerations – unique roots, ideal (0), and multiple roots. Three variable propositional formulas – A, B, C – were chosen at random and inserted into the given knowledge base structure. Probabilities for knowledge bases conditionals were determined by the uniform distribution. The augmented polynomial system was generated to infer the desired probability, and the inferred probability was calculated both with and without removal of the GCD of each polynomial. This process was repeated 1000 times for each knowledge base structure yielding the following results. The machine used for computation was running Red Hat 6.6 with a 2.6 GHz Intel Xeon E5-2670 processor and 126 GB RAM.

In the following, the knowledge base structure is represented so that the knowledge base is given in the numerator and the conditional to be inferred in the denominator. More formally, each KB structure is of the following form,

$$\frac{KB}{C_{inf}} = \frac{A_1|B_1, \ldots, A_j|B_j}{A_{inf}|B_{inf}}.$$

TABLE I: Knowledge Base Structure (Uniform)

		Without GCD Removed			**With GCD Removed**		
Property	Structure	Unique	(0)	Multiple	Unique	(0)	Multiple
Reasoning By Cases	$\frac{C\|(A\wedge B),C\|(A\wedge\neg B)}{B\|A}$	0%	100%	0%	100%	0%	0%
Transitive Chaining	$\frac{B\|A,C\|B}{C\|A}$	0%	100%	0%	57%	0%	43%
Cautious Monotonicity	$\frac{B\|A,C\|A}{C\|(A\wedge B)}$	50%	0%	50%	100%	0%	0%
Antecedent Conjunction	$\frac{C\|A,C\|B}{C\|(A\wedge B)}$	98%	0%	2%	100%	0%	0%
Conjunction Left	$\frac{B\|A,(B\wedge C)\|A}{C\|(A\wedge B)}$	100%	0%	0%	100%	0%	0%
KB form 1	$\frac{A\|B,(A\vee B)\|C}{B\|C}$	0%	100%	0%	68%	32%	0%

The data collected in Table I demonstrates how dramatically the ability to find a unique root can be depending on the knowledge base structure. For example, the forms for "Reasoning by Cases" and "Transitive Chaining" produced an elimination ideal of (0). While in both cases removing the GCD produced unique roots, the results for "Transitive Chaining" show that a bit over 40% of the samples resulted in multiple roots. However, it is also clear that removing the GCD cannot make the situation worse.

In addition to comparing the results of elimination, the computational time used to find the elimination ideal was recorded. In Table II, the mean and variance of computational time are displayed for each knowledge base form.

TABLE II: Computational Time (seconds)

	Without GCD Removed		**With GCD Removed**	
Property	Mean	Variance	Mean	Variance
Reasoning By Cases	6.320×10^{-4}	2.709×10^{-7}	5.502×10^{-4}	1.717×10^{-7}
Transitive Chaining	4.602×10^{-3}	6.605×10^{-5}	4.225×10^{-3}	7.316×10^{-5}
Cautious Monotonicity	6.020×10^{-4}	7.046×10^{-5}	4.008×10^{-3}	9.269×10^{-4}
Antecedent Conjunction	9.662×10^{-1}	3.471×10^{-1}	9.656×10^{-2}	3.4699×10^{-1}
Conjunction Left	3.005×10^{-4}	7.668×10^{-10}	3.001×10^{-4}	1.044×10^{-9}
KB form 1	1.106	1.776×10^{2}	2.434×10^{-1}	1.353

The measurements in Table II show that on average, removing the GCD has a mostly positive effect on the computational time. One exception is cautious monotonicity for which the mean and variance increase when the GCD is removed. In other cases, KB form 1 also shows the most significant improvement on average after removal of the GCD. In general, unlike the KB forms which adhere to a property, KB form 1 seems to be the most sensitive to changes in the formulas input.

Due to computational limitations, experiments were repeated for only a 100 samples. The probability distribution for the knowledge bases was produced by assigning $P(1) = \frac{1}{4}$ and $P(0) = \frac{3}{4}$ for each component of a truth assignment. For example, the truth assignment $(1, 0, 0)$ has the probability,

$$P(100) = P(1)P(0P(0) = \frac{1}{4}\frac{3}{4}\frac{3}{4} = \frac{9}{64}.$$

For several of the above properties, it was not possible to consistently produce enough results to record data. The following tables summarize the instances in which 100 samples were collected.

TABLE III: Knowledge Base Structure (Non-Uniform)

		Without GCD Removed			**With GCD Removed**		
Property	Structure	Unique	(0)	Multiple	Unique	(0)	Multiple
Reasoning By Cases	$\frac{C\mid(A\wedge B),C\mid(A\wedge\neg B)}{B\mid A}$	0%	100%	0%	10%	0%	90%
Cautious Monotonicity	$\frac{B\mid A,C\mid A}{C\mid(A\wedge B)}$	23%	0%	77%	78%	0%	22%
Conjunction Left	$\frac{B\mid A,(B\wedge C)\mid A}{C\mid(A\wedge B)}$	100%	0%	0%	100%	0%	0%

TABLE IV: Computational Time (seconds)

	Without GCD Removed		**With GCD Removed**	
Property	Mean	Variance	Mean	Variance
Reasoning By Cases	2.825	2.022×10^{2}	2.560	1.099×10^{2}
Cautious Monotonicity	3.223×10^{-4}	2.660×10^{-9}	3.158×10^{-3}	6.487×10^{-10}
Conjunction Left	3.052×10^{-4}	1.750×10^{-9}	3.011×10^{-4}	5.448×10^{-10}

CHAPTER 5

CONCLUSION

The thesis discussed two topics in knowledge representation: characterizability in belief revision and conditional knowledge bases. The results on both topics suggest several directions for further research. The thesis is concluded by very brief summaries of the results and a list of problems for further work.

5.1 Characterizability in Belief Revision

The framework established in Chapter 2 provides an approach to show that certain classes of belief revision operators are not characterizable by postulates. This framework is adapted in Chapter 3 to Horn belief revision. The approach merges techniques from finite model theory with theoretical models of belief revision. Several examples are given of classes of belief revision operators which are not characterizable. It is shown that the logical definability class corresponding to belief revision operators obtained by minimization from posets is a proper subset of universal monadic second-order logic and it is incomparable to first-order logic. For Horn belief revision, it is shown that the class of Horn belief operators obtained from Horn compliant total preorders cannot be characterized by finitely many postulates. Thus the characterization of this class given by Delgrande and Peppas, using infinitely many postulates, cannot be replaced by a finite characterization. It would be of interest to study the following problems related to this work.

Extending the framework to belief contraction. In general belief revision, the Harper and Levi identities allow for translation between revision and contraction, and revision by minimization has a natural counterpart for belief contraction. Using this definition for contraction by minimization, one could extend the current framework to consider characterizability of contraction operators generated by minimization.

Iterated belief revision. The current framework does not account for repeated revisions called iterated revision. This is a very important extension, which needs to be well understood for practical applications of belief revision. Darwiche and Pearl proposed postulates which have served as the foundation for many later works on iterated belief revision [9]. Developing a methodology to evaluate whether classes iterated revision operators are characterizable could be useful for the further development of iterated belief revision.

Strictly Horn compliant belief revision operators. While Section 3.9 presented some new information about strictly Horn compliant operators, their characterizability is still an open problem. The techniques developed in Chapter 3 may offer a way to prove their non-characterizability.

Reasoning about postulates. Reasoning about implications between postulates is an important part of the belief revision literature. This reasoning has not been formalized yet, and so basic questions about its hardness, such as the decidability of implication, appear to be open.

5.2 Conditional Knowledge Bases

The method introduced by Kern-Isberner et al. [30] offers a technique for maximum entropy inference on conditional knowledge bases. The experiments of Chapter 4 suggest that the method may be best applied to subclasses of this problem, in which the knowledge bases have specific structure, and consequently the polynomial systems produced can be handled efficiently. Alternatively, computational techniques which avoid elimination could be explored. The following are some possible future directions for research.

Structured subclasses. Identify structural properties of conditional knowledge bases where the computational algebra approach can infer the correct conditional probability and the computation time is feasible.

Zero dimensional components. In instances where the algebraic independence of f_{inf} cannot be eliminated by removing the GCD, isolated solutions to the augmented polynomial system may still exist. These isolated points may contain information about the value of the probability to be inferred. Further investigation into both theoretical and computational approaches to address this situation is needed.

Alternatives to elimination. Elimination is known to lead to certain computational challenges such as intermediate expression swell and large coefficients [15]. In order to give more flexibility tp this computational approach one could explore alternative symbolic methods or numerical methods to avoid elimination.

Risk and consequence of expression swell in GCD-simplified systems. Expression swell can occur when removing factors causes the introduction of additional terms. For example, consider the polynomial

$$f_i = f_i^+ - f_i^- = (a^3 + b^3) - (a^2 + b^2).$$

It holds that $GCD(f_i^+, f_i^-) = a + b$, and the removal of $a + b$ results in the polynomial $a^2 - 2ab + b^2 - a + b$ which is of lower degree, but more terms. Further information about frequency and conditions under which this may occur need to be identified. Moreover, the computational effects of such a situation would need to be assessed.

APPENDICES

Appendix A

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Appendix B

CODE FOR CHAPTER 4

The code used to produce examples in Chapter 4 can be found at:

https:github.comjonyaggiethesis-code

CITED LITERATURE

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VITA

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EDUCATION

Ph.D. in Mathematics, University of Illinois at Chicago, Anticipated: August 2016

Advisor: Dr. György Turán

Thesis Title: Topics in Knowledge Representation: Belief Revision and Conditional Knowledge Bases

M.A. in Mathematics, San Francisco State University, August 2010.

Advisor: Dr. Joseph Gubeladze

Thesis: *Variety of Finitary k-algebra Homomorphisms*

Thesis Topic: Implement an application to assist in classifying the structure of systems of multivariable polynomial equations generated by finitary k-algebra homomorphism.

B.A. in Mathematics, San Francisco State University, May 2008.

PUBLICATIONS

Characterizability in Belief Revision, Proceedings of the International Joint Conference in Artificial Intelligence 2015. Buenos Aires, Argentina. July, 2015.

Preliminary: A preliminary report of this paper was given at AAAI Spring Symposium, Logical Formalizations of Commonsense Reasoning at Stanford University, March 2015.

Description: Defines a two person game in which a winning strategy determines the characterizability of a model for revision with minimization and an associated fragment of universal monadic second-order logic.

György Turán, Jon Yaggie: *Non-characterizability of belief revision: an application of finite model theory.* (http://arxiv.org/abs/1403.6512) CoRR, April 2014.

Description: Demonstrates there are models of revision with minimization which are not characterizable within a standard framework.

WORK EXPERIENCE

Teaching Assistant, University of Illinois at Chicago, August 2011–July 2013, May 2014–present.

Mathematics Courses: Pre-calculus, First Semester College Algebra, Second Semester Calculus, Elementary Linear Algebra, Finite Mathematics for Business, Business Calculus

Computer Science Courses: Introduction to Computer Science(with Python), Data Structures(in C++)

Duties: Develop lessons to train students to think abstractly, problem solve, and promote numerical intuition. Assist students in implementing programming projects.

Research Assistant, University of Illinois at Chicago, August 2013–April 2014.

Duties: Assist in development and execution of professional development sessions in mathematical modeling and logic for secondary education professionals.

Coordinating Graduate Assistant, Valparaiso Experience in Research for Undergraduate Mathematicians (VERUM), June 2011–July 2011.

Duties: Coordination of group activities and meals. Serving as an intermediary between students and faculty. Facilitating undergraduate research in differential equations, natural language processing, and combinatorics.

Instructor, High Jump, September 2010–May 2013.

Courses: Math III

Duties: Planning and executing mathematics activities to enrich middle school student's mathematical knowledge and prepare them for more advanced classes. Example topics: algebraic exploration of limits and derivatives, creating art projects using mathematics, and creating spreadsheets and simple programs to explore numerical concepts

Math Circle Leader, Oakland/East Bay Math Circles, February 2010–May 2010.

Courses: Middle School Group

Duties: Planning and executing after school mathematics enrichment activities.

NSF GK-12 Fellow, Mission High School, July 2009–July 2010.

Duties: Collaborating with instructors to develop lesson plans with present research level mathematics accessible to high school students.

Graduate Teaching Assistant, San Francisco State University, September 2008–May 2009.

Courses: First and Second Semester College Algebra

Duties: Planning and executing classroom instruction, grading homework and exams, and offering referrals and guidance to struggling students.

Contract Web Developer, July 2001–August 2004.

Duties: Designing, implementing, and integrating payment modules into e-commerce solutions. Including retrieving, processing, and storing transactional data from relational databases as well as interacting with web APIs.

VOLUNTEER WORK AND OUTREACH

Peggy Notebaert Nature Museum, Data Analyst and DBA for frogsurvey.org June 2015–present

Duties: Evaluating current observational data, consulting on improving data collection methods, and proposing statistical methods for improved data. Consulting on database design, website development, and data visualization.

Expanding Your Horizons Chicago, Chicago, IL.

Board Member/Chair, AY 2014–present

Duties: Oversee general activity and budget. Coordinate and facilitate planning/execution of committee tasks. Direct leadership development and fundraising.

Volunteer Coordinator, 2013–14 Academic year

Duties: Recruiting, developing training materials, and supervising event volunteer positions

Organizing Committee member AY 2012–13 Academic year

Duties: Collaborate with committee members to establish a community and academic network.

Conference Support, Vienna Summer of Logic, July 2014, Vienna, Austria.

Duties: Providing technical support to conference participants with equipment such as computers, projectors, and portable devices.

Mission High School Math Circle Co-organizer, September 2009–May 2010, San Francisco, CA.

Duties: Coordinate activities, supervise circle leaders, and provide written reports on events.

Teaching English as a Second Language, May 1998–August 2000, Novosibirsk, Russia.

Duties: Design, adapt, and execute educational program to prepare computer scientists for short-term American positions.

PRESENTATIONS AND POSTERS

Characterizability in Belief Revision, International Joint Conference in Artificial Intelligence 2015. Buenos Aires, Argentina. July, 2015.

Characterizability in Belief Revision, CommonSense Reasoning 2015. Palo Alto, CA. March, 2015.

Providing a Training Ground for Future STEM Role Models, Chicago Symposium Series, Chicago, IL. January, 2015.

Non-characterization Results in Belief Revision, Workshop on Logic and Games. Vienna, Austria. July, 2014.

Non-characterization Results in Belief Revision, Student Poster Session, Doctoral Consortium KR2014. Vienna, Austria, July 2014.

Computing Affine k-algebra Homomorphisms, AMS Joint Meetings. Boston, MA, January 2012.

The Variety of Finitary Algebra Homomorphisms, Student Poster Session, MAA Northern California, Nevada, and Hawaii Section Meeting, San Francisco. CA, February 2010.

Pick's Theorem: A Math Circle Lesson, Student Poster Session, Bay Area Circle for Teachers Winter Workshop. San Francisco, CA, January 2010.

Student to Student Connections: Graduate Students in Math Circles, SIGMAA-MCST Special Session. AMS Joint Meetings, San Francisco, CA, January 2010.

WORKSHOPS AND CONFERENCES

International Joint Conference on Artificial Intelligence, Buenos Aires, Argentina, July 2015.

Strategies for Increasing Access for Women and Girls of Color in Science, San Francisco State University, San Franisco, CA, November 2014.

European Summer School in Language, Logic, and Information 2014, University of Tübingen, Germany, August 2014.

Vienna Summer of Logic, Technical University of Vienna, Austria, July 2014.

IEEE Conference on Computational Complexity, Stanford University, Palo Alto, CA, June 2013

Symposium on Theory of Computing, Palo Alto, CA, June 2013

Innovations in Theoretical Computer Science, University of California, Berkeley, CA, January 2013.

European Summer School in Language, Logic, and Information 2012, University of Opole, Poland, August 2012.

Toric Varieties Workshop, Mathematical Science Research Institute Summer Graduate Workshops, June 2009.

Computational Theory of Real Reductive Groups Workshop and Conference, University of Utah, July 2009.

SCHOLARSHIPS AND AWARDS

Chancellor's Student Service and Leadership Award, University of Illinois at Chicago, Spring 2015.

University Provost Award to attend the Vienna Summer of Logic conferences, University of Illinois at Chicago, Fall 2013.

Graduate Student Council Travel Award, University of Illinois at Chicago, Fall 2011.

College of Liberal Arts and Sciences PhD Student Travel Award, University of Illinois at Chicago, Fall 2011

Teaching Assistantship, University of Illinois at Chicago, 2011–Present.

$(CM)^2$ Fellowship, NSF GK–12 funded fellowship, San Francisco State University, September 2009–June 2010.

Teaching Assistantship, San Francisco State University, August 2008–June 2009.