

**Model Theory and Extremal Combinatorics:  
Structure, Enumeration, and 0-1 Laws**

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THESIS

Submitted as partial fulfillment of the requirements  
for the degree of Doctor of Philosophy in Mathematics  
in the Graduate College of the  
University of Illinois at Chicago, 2016

Chicago, Illinois

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To my mom and dad, whose love and support made this possible.

## Acknowledgements

I would first like to thank my advisor Dave Marker for being a consistent source of guidance and encouragement throughout my graduate career. I would like to also thank him for giving me travel support on numerous occasions and for bringing me with him to spend the semester at MSRI in Spring 2014. I would like to thank Dhruv Mubayi, my co-advisor, for teaching me many techniques in combinatorics and for collaborating with me on Chapters 4 and 5 of this thesis. I would like to thank Maryanthe Malliaris, whose undergraduate logic course at the University of Chicago made me want to be a model theorist. I would also like to thank her for collaborating with me on Chapter 8 of this thesis and in general for being a mentor and role model. Thank you to the rest of my committee members, John Baldwin and Sergei Starchenko.

Thank you to Isaac Golbring, whose class on continuous logic sparked my interest in the Urysohn sphere, which then led me to the questions addressed in Chapter 4. Thanks to Andrew Suk whose class on discrete geometry brought many interesting problems to my attention and who pointed me towards a paper on hereditary graph properties which informed my approach to Chapter 6. Thanks to Joszi Balogh, Hong Liu, Michelle Delcourt, Maryam Sharifzadeh, Sarka Perickova, and Adam Wagner for teaching me about the hypergraph containers theorem and its applications. This was key to the successful completion of Chapter 6.

I would like to thank the participants in the finite structures seminar which ran at Berkeley in the Spring of 2014 during the MSRI semester in model theory, including Cameron Hill, Alex Kruckman, Ward Henson, Carol Wood, Vince Guingoa, Gwyneth Harrison-Shermoen, Will Johnson, and Darío García. I learned a great deal during this semester, and this seminar in particular had a big impact on the direction of my research. I have benefited from conversations with many other model theorists, particularly those with interests relating to finite combinatorics. A few of these are Nate Ackerman, Cameron Freer, Rehana Patel, Vera Koponen, Ove Ahlman, Sergei Starchenko, and Artem Chernikov. Thanks to Lynn Scow who was always a generous source of knowledge and support during her time as a postdoc at UIC. Thank you to everyone who has been a part of the logic group at UIC during my time in graduate school for creating a wonderful environment to learn

and research in. Last but not least, I would like to thank Gabriel Conant for his support both personally and mathematically.

## Contribution of Authors

Chapter 1 is a list of notations. Chapter 2 contains background and history on the combinatorics questions addressed in this thesis. Chapter 3 gives a history of the model theoretic motivation for considering the combinatorics problems addressed in this thesis. Chapter 4 contains a joint paper with D. Mubayi which has been published on arXiv [82] and which has been reviewed and accepted at the Journal of Symbolic Logic, but has yet to appear. Chapter 5 contains joint work with D. Mubayi. This work has not yet been published, but we plan to publish this work in the future in a journal. Chapters 6 contains independent, unpublished work by myself. I plan to publish it in the future in a journal. Chapter 7 contains an explanation of how the results of Chapter 6 apply to various combinatorial results. These include previously known results by other authors ([20, 24, 45, 61, 62, 66, 73]) as well as previously known results due to myself and D. Mubayi (Chapter 4, [82], Chapter 5). Section 3 of Chapter 7 contains some new results which were originally proved by myself and D. Mubayi in unpublished work. Specifically, Corollaries 7.3.11 and 7.3.16 were originally proved by myself and D. Mubayi using techniques specific to multigraphs. The proofs appearing in this thesis rely on combining certain results from Chapter 5 with the machinery of Chapter 6. The machinery of Chapter 6 is a generalization of the specific techniques used originally by myself and D. Mubayi. This generalization was developed independently by me. Chapter 8 contains a paper which is joint work of myself and M. Malliaris. It has been published on arXiv [76] and is under review at a journal, but has not yet been accepted for publication. Chapter 8 reproves a theorem of Chudnovsky, Kim, Oum, and Seymour from [39].

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## Summary

This thesis investigates connections between model theory and extremal combinatorics. These two fields seem quite far apart in their subject matter and approach to mathematics. Model theory largely deals with infinite objects and uses very general tools, while extremal combinatorics largely deals with finite objects and uses tools tailored to specific structures. In spite of these differences, many important problems in both fields can be seen as versions of the same thematic question: how does infinite structure arise as a “limit” of finite structure? Existing work suggests that bridges between the two fields can be built by focusing on problems which address this question. Such bridges often lead to important theorems. One example is Ramsey’s theorem, which was first proved to answer a question in logic [91]. A question about logical zero-one laws motivated the important results of Kolaitis, Prömel, and Rothschild in [66]. More recent work [38, 74, 75, 96] demonstrates that dividing lines from infinite model theory can retain significance in the finite setting, reflected through strong versions of Ramsey’s theorem and Szemerédi’s regularity lemma. This thesis furthers this type of investigation in two ways: first by proving new theorems in extremal combinatorics which are directly related to model theoretic questions, and second, by applying the model theoretic perspective to existing theorems in combinatorics which address the shared thematic question. The motivation for this work is that the model theoretic viewpoint often formalizes intuitive similarities among results in different settings. We believe this type of organization could be especially useful in finite combinatorics, where the search for examples and counterexamples is often ad hoc, and where results about specific structures are hard to transfer to new settings. Further, the model theoretic perspective naturally leads to problems which are interesting purely from the perspective of combinatorics, and vice versa. We now give a brief summary of the contents of each chapter.

Chapter 2 contains background and motivation from the perspective of extremal combinatorics for the work contained in Chapters 4, 5, and 6. Chapter 3 contains background and motivation from the perspective of model theory for the work contained in Chapters 4, 5, and 6.

Chapter 4 gives an analysis, from the point of view of extremal combinatorics, of the family of finite metric spaces with underlying set  $[n]$  and distance set  $[r]$ , where  $r \geq 3$  is a fixed integer and  $n \rightarrow$



$\infty$ . Specifically, it provides asymptotic enumerations of the metric spaces and proves approximate structure results about them. In the case when  $r$  is even, exact structural and enumeration results are obtained, which imply the existence of a zero-one law. In this case we give an axiomatization of the almost sure theory and show it differs from the theory of the Fraïssé limit of finite metric spaces with distances in  $[r]$ . The work in this chapter is joint with D. Mubayi.

Chapter 5 investigates certain families of multigraphs from the perspective of extremal combinatorics. An  $(n, s, q)$ -graph is an  $n$ -vertex multigraph where every set of  $s$  vertices spans at most  $q$  edges. In Chapter 5, we study the problem of determining the maximum product of the edge multiplicities in  $(n, s, q)$ -graphs. We prove sharp results if the congruence class of  $q$  modulo  $\binom{s}{2}$  is at most  $s - 2$  and at least  $\binom{s}{2} - s/2$ . One of the smallest case that falls outside this range is  $(s, q) = (4, 15)$ , and here we prove that the maximum product of the edge multiplicities is  $2^{\gamma n^2 + O(n)}$ , where, assuming Schanuel's Conjecture,  $2^\gamma$  is transcendental. This could indicate the difficulty of solving the problem in full generality. We also prove a variety of other results for  $(n, s, q)$ -graphs, including stability theorems for the maximum product of edge multiplicities. Many of these results can be seen as extending work by Bondy-Tuza and Füredi-Kündgen about sums of edge multiplicities to the product setting. These results will also be used in Chapter 7 to prove enumeration theorems for  $(n, s, q)$ -graphs. The work in this chapter is joint with D. Mubayi.

Chapter 6 uses model theory to unify under a general framework, a collection of approximate enumeration and structure theorems from extremal combinatorics. Given a finite relational language  $\mathcal{L}$ , a hereditary  $\mathcal{L}$ -property is a class of  $\mathcal{L}$ -structures which is closed under isomorphism and has the model theoretic hereditary property. Given a hereditary  $\mathcal{L}$ -property  $\mathcal{P}$ , let  $\mathcal{P}_n$  denote the set of elements in  $\mathcal{P}$  with underlying set  $[n]$ . There is a long line of research in extremal combinatorics investigating asymptotic structure and enumeration results for  $\mathcal{P}_n$  in the setting of graphs. For instance in [3, 27], an asymptotic formula for  $|\mathcal{P}_n|$  is determined in terms of a parameter related to  $\mathcal{P}$  called the *coloring number* of  $\mathcal{P}$ . In Chapter 6, we define a generalization of the coloring number to the setting of hereditary  $\mathcal{L}$ -properties for arbitrary finite relational  $\mathcal{L}$ . We use this parameter to prove an enumeration theorem generalizing [3, 27]. We also generalize other notions from extremal combinatorics, such as extremal graphs and graph stability theorems. Our techniques include a new application of the hypergraph containers theorem to the setting of  $\mathcal{L}$ -structures, and a generalization of the graph removal lemma to the setting  $\mathcal{L}$ -structures which was proved by Aroskar and Cummings in [6]. This chapter generalizes a pattern of proof appearing in many papers from extremal combinatorics including [14, 18, 22, 73, 81, 92].

In Chapter 7 we demonstrate how the main theorems in Chapter 6 reduce to known enumeration and/or structure theorems in the settings of graphs, hypergraphs, colored hypergraphs, directed graphs, metric spaces, and multigraphs. In particular, we consider results from the following papers: [66] for graphs; [20], [24], [45], and [62] for hypergraphs; [61] for colored hypergraphs; [39] for digraphs; Chapter 4 of this thesis for metric spaces; and Chapter 5 of this thesis for multigraphs.

In Chapter 8, we consider a theorem in graph theory from the perspective of model theory. Chudnovsky, Kim, Oum, and Seymour established in [39] that any prime graph contains one of a short list of induced prime subgraphs. In Chapter 8, we reprove their theorem using many of the same ideas, but with the key model-theoretic ingredient of first determining the so-called amount of stability of the graph. This approach changes the applicable Ramsey theorem, improves the bounds and offers a different structural perspective on the graphs in question. Complementing this, we give an infinitary proof which implies the finite result. The work in this chapter is joint with M. Malliaris.

## Index of Notations

In this chapter we give notation and definitions in list form for easy reference.

**Sets and tuples.** Suppose  $X, V$  are sets,  $s \geq 2, n \geq 1$  are integers, and  $\bar{v} = (v_1, \dots, v_s) \in V^s$ .

- $\binom{X}{s} = \{Y \subseteq X : |Y| = s\}$ .
- $[n] = \{1, \dots, n\}$ .
- An *equipartition* of  $X$  is a partition  $U_1, \dots, U_m$  of  $X$  such that for all  $i \neq j$ ,  $\|U_i\| - \|U_j\| \leq 1$ .
- $X^{\neq} = \{(x_1, \dots, x_s) \in X^s : x_i \neq x_j \text{ for each } i \neq j\}$ .
- $\cup \bar{v} = \{v_1, \dots, v_s\}$ .
- $|\bar{v}| = s$  is the length of  $v$ .
- An *enumeration* of  $X$  is a tuple  $\bar{x} = (x_1, \dots, x_{|X|})$  such that  $\cup \bar{x} = X$ .
- $\mathcal{P}(X) = \{Y : Y \subseteq X\}$  is the power set of  $X$ . We will also denote this by  $2^X$ .

### Basic Combinatorial Structures.

- A *graph* a pair  $G = (V, E)$  where  $V$  is a set of vertices and  $E \subseteq \binom{V}{2}$  is a set of edges.
- Given an integer  $k \geq 2$ , a *k-uniform hypergraph* is a pair  $(V, E)$  where  $V$  is a set of vertices and  $E \subseteq \binom{V}{k}$  is a set of edges.
- An *multigraph* is a pair  $(V, w)$  where  $V$  is a set of vertices and  $w : \binom{V}{2} \rightarrow \mathbb{N}$  is a function.
- Given  $s \geq 2$  and  $q \geq 0$ , an  $(s, q)$ -graph is a multigraph  $G = (V, w)$  such that for all  $X \in \binom{V}{s}$ ,  $\sum_{xy \in \binom{X}{2}} w(xy) \leq q$ . An  $(n, s, q)$ -graph is an  $(s, q)$ -graph with vertex set of size  $n$ .
- Given  $r \geq 2$ , an *r-graph* is a pair  $(V, w)$  where  $V$  is a set of vertices and  $w : \binom{V}{2} \rightarrow 2^{[r]}$  is a function.
- A *directed graph* or *digraph* is a pair  $(V, E)$  where  $V$  is a set of vertices and  $E \subseteq V^2$  is a set of *directed edges*.
- An *oriented graph* is a pair  $(V, E)$  where  $V$  is a set of vertices and  $E \subseteq V^2$  is a set of directed edges with the property that that for each  $xy \in \binom{V}{2}$  at most one of  $(x, y)$  or  $(y, x)$  are in  $E$ .
- Given a set  $C$  and  $k \geq 2$ , a *C-colored k-uniform hypergraph* or  $(k, C)$ -graph, is a pair  $(V, H)$  where  $V$  is a set of vertices and  $H : \binom{V}{k} \rightarrow C$  is a function.

**Graphs.** Suppose  $G = (V, E)$  and  $H = (V', E')$ , are graphs,  $\mathcal{H}$  is a collection of finite graphs,  $\{x, y\} \in \binom{V}{2}$ ,  $X \subseteq V$ , and  $s \geq 2$ ,  $n \geq 1$  are integers.

- The *size* of  $G$  is  $|V|$ .
- $G$  is *complete* if  $E = \binom{V}{2}$ .
- $G$  is *independent* if  $E = \emptyset$ .
- $G[X]$  is the graph  $(X, \binom{X}{2} \cap E)$ .
- $X$  is *independent in  $G$*  if  $G[X]$  is independent.
- $X$  is *complete in  $G$*  if  $G[X]$  is complete.
- $K_s$  is the complete graph on  $s$  vertices.
- $xy = \{x, y\}$ .
- $V(G)$  is the vertex set of  $G$ .
- $E(G)$  is the edge set of  $G$ .
- $e(G) = |E(G)|$ .
- $N(x) = \{w : xw \in E(G)\}$  is the *neighborhood of  $x$  in  $G$* .
- $d(x) = |N(x)|$  is the *degree of  $x$  in  $G$* .
- $G$  is  *$s$ -colorable* if there is a partition  $U_1, \dots, U_s$  of  $V$  such that for all  $xy \in E$ , there is  $i \neq j$  such that  $x \in U_i$  and  $y \in U_j$ .
- $G$  is  *$s$ -partite* if and only if  $G$  is  $s$ -colorable.
- $H$  is a *non-induced subgraph* of  $G$ , written  $H \subseteq G$ , if  $V' \subseteq V$  and  $E' \subseteq E$ .
- $H$  is an *induced subgraph* of  $G$ , written  $H \subseteq_{ind} G$ , if  $V' \subseteq V$  and  $E' = E \cap \binom{V'}{2}$ .
- $H$  is *isomorphic* to  $G$ , written  $H \cong G$ , if there is a bijection  $f : V \rightarrow V'$  such that  $xy \in E$  if and only if  $f(x)f(y) \in E'$ .
- $G$  *omits  $H$*  as a (induced) subgraph if there is no (induced) subgraph of  $G$  isomorphic to  $H$ .
- $G$  is  *$H$ -free* if  $G$  omits  $H$  as a subgraph (note: this will mean something else in Chapter 6).
- $Forb(H)$  is the class of all finite  $H$ -free graphs.
- $Forb_{ind}(H)$  is the class of all finite graphs omitting  $H$  as an induced subgraph.
- $Forb(n, H)$  is the set of graphs in  $Forb(H)$  with vertex set  $[n]$ .
- $Forb_{ind}(n, H)$  is set of graphs in  $Forb_{ind}(H)$  with vertex set  $[n]$ .
- $G$  is  *$\mathcal{H}$ -free* if  $G$  is  $H'$ -free for all  $H' \in \mathcal{H}$ .
- $Forb(\mathcal{H})$  is the class of all finite  $\mathcal{H}$ -free graphs.
- $Forb_{ind}(\mathcal{H})$  is the class of all finite graphs omitting every  $H' \in \mathcal{H}$  as an induced subgraph.
- $Forb(n, \mathcal{H})$  is the set of graphs in  $Forb(\mathcal{H})$  with vertex set  $[n]$ .

- $Forb_{ind}(n, \mathcal{H})$  is set of graphs in  $Forb_{ind}(\mathcal{H})$  with vertex set  $[n]$ .
- $\mathcal{H}$  is a *hereditary graph property* if it is closed under isomorphism and induced subgraphs.
- $\mathcal{H}(n)$  is the collection of all elements in  $\mathcal{H}$  of size  $n$ .
- $\mathcal{H}_n$  is the set of all elements in  $\mathcal{H}$  with vertex set  $[n]$ .

**Hypergraphs.** Suppose  $k \geq 2$  and  $G = (V, E)$  is a  $k$ -uniform hypergraph and  $X \subseteq V$ .

- The *size* of  $G$  is  $|V|$ .
- $G$  is *complete* if  $E = \binom{V}{k}$ .
- $G$  is *independent* if  $E = \emptyset$ .
- $G[X]$  is the graph  $(X, \binom{X}{k} \cap E)$ .
- $X$  is *independent in  $G$*  if  $G[X]$  is independent.
- $X$  is *complete in  $G$*  if  $G[X]$  is complete.
- $V(G)$  is the vertex set of  $G$ .
- $E(G)$  is the edge set of  $G$ .
- If  $V$  is finite, the *average degree of  $G$*  is  $d = \frac{\binom{|V|}{k}|E|}{|V|}$ .

**Languages.**

- $\mathcal{L}_G = \{R\}$  is the *language of graphs* consisting of a single binary relation symbol  $R(x, y)$ .
- $\mathcal{L}_{hg}^k = \{R(x_1, \dots, x_k)\}$  is the *language of  $k$ -uniform hypergraphs* consisting of a single  $k$ -ary relation symbol  $R(x_1, \dots, x_k)$ .
- Given  $r \geq 2$ ,  $\mathcal{L}_r = \{R_1(x, y), \dots, R_r(x, y)\}$  consists of  $r$  binary relation symbols.
- Given  $q \geq 1$ ,  $\mathcal{L}_{mg}^q = \{R_0(x, y), \dots, R_q(x, y)\}$  is the *language of multigraphs of multiplicity at most  $q$*  consisting of  $q + 1$  binary relation symbols.

**Chapter 6 Notation.** This notation appears in Chapter 6. We include a list here for easy reference.

Suppose  $\mathcal{L} = \{R_1, \dots, R_t\}$  is a finite relational language,  $M$  is a finite  $\mathcal{L}$ -structure. As usual  $dom(M)$  denotes the domain of  $M$ . Let  $\mathcal{H}$  be a class of finite  $\mathcal{L}$ -structures,  $N$  an  $\mathcal{L}$ -structure,  $A \subseteq dom(M)$ ,  $\ell \in \mathbb{N}$ ,  $\bar{b} = (b_1, \dots, b_\ell) \in dom(M)^\ell$ , and  $C$  a set of constant symbols.

- $ar(R_i)$  is the arity of  $R_i$ .
- $r_{\mathcal{L}} = \max\{ar(R_i) : 1 \leq i \leq t\}$ .
- The *size* of  $M$  is  $|dom(M)|$ .
- $N \subseteq_{\mathcal{L}} M$  means  $N$  is a model theoretic substructure of  $M$ .
- $N \cong_{\mathcal{L}} M$  means  $M$  and  $N$  are isomorphic as  $\mathcal{L}$ -structures.

- $M$  is  $N$ -free if for all  $M' \subseteq_{\mathcal{L}} M$ ,  $M' \not\cong_{\mathcal{L}} N$ .
- $M$  is  $\mathcal{H}$ -free if it is  $H$ -free for all  $H$  in  $\mathcal{H}$ .
- $qftp_{\mathcal{L}}^M(\bar{a})$  is the quantifier-free type of  $\bar{a}$  in  $M$ .
- A type  $p(x_1, \dots, x_\ell)$  is *proper* if it contain  $x_i \neq x_j$  for each  $i \neq j$ .
- If  $\mathcal{L}' \subseteq \mathcal{L}$ ,  $M \upharpoonright_{\mathcal{L}'}$  is the reduct of  $M$  to  $\mathcal{L}'$ .
- $M[A]$  is the  $\mathcal{L}$ -structure induced on  $A$  by  $M$ .
- $C_A = \{c_a : a \in A\}$  is a new set of constant symbols, one for each element of  $A$ .
- $c_{\bar{b}} = (c_{b_1}, \dots, c_{b_\ell})$ .
- The *diagram of  $M$*  is  $Diag(M) = \{\phi(c_{\bar{a}}) : \phi(\bar{x}) \text{ is quantifier-free and } M \models \phi(\bar{a})\}$ .
- The *diagram of  $A$  in  $M$*  is  $Diag^M(A) = \{\phi(c_{\bar{a}}) : \phi(\bar{x}) \text{ is quantifier-free, } \cup \bar{a} \subseteq A, \text{ and } M \models \phi(\bar{a})\}$ .
- $S_\ell(\mathcal{L})$  is the set of complete, proper, quantifier-free types in  $\ell$  free variables.
- $S_\ell(\mathcal{H})$  is the set of complete, proper, quantifier-free types in  $\ell$  free variables which are realized in some  $H \in \mathcal{H}$ .
- $S_\ell(C) = \{p(\bar{c}) : p(\bar{x}) \in S_\ell(\mathcal{L}) \text{ and } \bar{c} \in C^\ell\}$ .
- $S_\ell(C, \mathcal{H}) = \{p(\bar{c}) : p(\bar{x}) \in S_\ell(\mathcal{H}) \text{ and } \bar{c} \in C^\ell\}$ .
- $\mathcal{H}(n)$  is the class of structures in  $\mathcal{H}$  of size  $n$ .
- $\mathcal{H}_n$  is the set of structures in  $\mathcal{H}$  with underlying set  $[n]$ .
- $\mathcal{H}$  is *trivial* if  $\mathcal{H}(n) = \emptyset$  for all sufficiently large  $n$ . Otherwise,  $\mathcal{H}$  is *non-trivial*.

## A Combinatorial Introduction

In this chapter, we provide the specific background and the general historical context needed to understand the combinatorial aspects of Chapters 4, 5, 6, and 7. Chapter 8 requires slightly different background and motivation, which is included in the introduction there. Our goals for this section are to show the following.

- Theorems which concern the number and structure of hereditary properties of graphs, and extensions of such results to other combinatorial structures, has been a major line of research within the field of extremal combinatorics.
- Precise structure and enumeration theorems in such settings are often proved by way of approximate structure and enumeration theorems.
- Major tools for proving approximate structure and enumeration theorems include Szemerédi's regularity lemma, the hypergraph containers theorem, graph removal lemmas, and supersaturation results.
- A general strategy has emerged for how to prove approximate structure and enumeration theorems using these tools.
- Whether this strategy can be made formal is an interesting question from the perspective of extremal combinatorics.

### 2.1. Hereditary Properties of Combinatorial Structures

In this section we show that theorems which concern the number and structure of hereditary properties of graphs, and extensions of such results to other combinatorial structures, is a major line of research within the field of extremal combinatorics.

**2.1.1. Hereditary Graph Properties.** A nonempty class of graphs  $\mathcal{P}$  is called a *hereditary graph property* if it is closed under isomorphism and induced subgraphs. Given a hereditary graph property  $\mathcal{P}$ , let  $\mathcal{P}_n$  denote the set of elements of  $\mathcal{P}$  with vertex set  $[n]$ . The *speed* of  $\mathcal{P}$  is the function which sends  $n \mapsto |\mathcal{P}_n|$ . There has been extensive investigation into the speed and asymptotic structure of hereditary properties of graphs, including [2, 3, 11, 12, 13, 26, 27, 93]. Independent

work of Alekseev in [3] and by Bollobás-Thomason in [27] focuses on the fastest growing  $\mathcal{P}$ , that is, where  $|\mathcal{P}_n|$  grows exponentially in  $n^2$ . For such  $\mathcal{P}$ , they show there is a discrete set of possibilities for the speed of  $\mathcal{P}$ . Moreover, they determine the speed of  $\mathcal{P}$  in terms of an integer-valued parameter called the *coloring number of  $\mathcal{P}$*  (see [27] for the precise definitions).

**THEOREM 2.1.1 (Alekseev, Bollobás-Thomason [3, 27]).** *Suppose  $r \in \mathbb{N}$  and  $\mathcal{P}$  is a hereditary graph property with coloring number  $r$ . Then*

$$|\mathcal{P}_n| = 2^{(1-\frac{1}{r}+o(1))\frac{n^2}{2}}.$$

Examples of hereditary graph properties include  $Forb(H)$  and  $Forb_{ind}(H)$  for any set of finite graphs  $H$ . Theorem 2.1.1 can be seen as a generalization of the many enumeration results about graph properties of these forms, for instance those appearing in [60, 66, 87, 88, 90].

**2.1.2. Hereditary Properties of Other Combinatorial Structures.** There are many results which extend the investigation of hereditary graph properties to other combinatorial structures. For instance, [9] studies hereditary properties of tournaments, [8] studies hereditary properties of oriented graphs and posets, and [45, 61, 65] study hereditary properties of  $k$ -uniform hypergraphs. In [61] and [45], a version of the coloring number for graphs properties is defined for  $k$ -uniform hypergraph properties and a corresponding version of Theorem 2.1.1 is proved. In fact, [61] proves these results for a more general class of structures, namely colored  $k$ -uniform hypergraphs. The results in [19, 20, 89, 83] investigate asymptotic enumeration and structure results for specific classes of  $H$ -free hypergraphs, which are examples of hereditary properties of hypergraphs. Similarly, the results in [73] concern specific classes of  $H$ -free digraphs, which are examples of hereditary properties of digraphs. The results in [22, 82] for metric spaces are similar in flavor, although they have not been studied explicitly as instances of hereditary properties. Thus, extending the investigation of hereditary graph properties to other combinatorial structures has been an active area of research for many years within extremal combinatorics. Our work in Chapters 4 and 5 continue this line of research in the setting of discrete metric spaces and of multigraphs.

## 2.2. A Pattern

We showed in the previous section that there has been many years of research which have successively extended and generalized results about hereditary graph properties to other combinatorial structures.



In this section, we demonstrate that as a result, a pattern of proof has emerged for how to prove approximate structure and enumeration theorems in these settings.

**2.2.1. Tools and Techniques.** In this subsection we give a brief outline of some of the tools and techniques which appear in the proofs we are interested in. We include this information here to facilitate our outline of the proof strategies in the next subsection. In most cases, we will merely provide an example of the tool we are introducing, usually in the setting of graphs.

### Extremal Structures.

Given graphs  $H$  and  $G$ , we say that  $G$  is  $H$ -free if it does not contain a copy of  $H$  as a subgraph (not necessarily induced). A major theme in extremal combinatorics is the investigation of  $H$ -free graphs, beginning with Turán's Theorem [101]. The *extremal number* of  $H$ ,  $\text{ex}(n, H)$ , is the maximum number of edges in any  $H$ -free graph with  $n$  vertices, and an  $H$ -free graph with  $n$  vertices is called *extremal* if it has  $\text{ex}(n, H)$  edges. The Turán graph,  $T_s(n)$ , is the complete  $s$ -partite graph on  $n$  vertices with no two part sizes differing by more than one. Let  $t_s(n)$  be the number of edges in  $T_s(n)$ .

**THEOREM 2.2.1. [Turán's theorem [101]]** *For  $n > s \geq 2$ , we have  $\text{ex}(n, K_s) = t_{s-1}(n)$  and equality holds only for  $T_{s-1}(n)$ .*

There exist in the literature many notions of extremal structures in other settings, see for example [73] for digraphs, [29, 53] for multigraphs, and [24] for hypergraphs. We will consider versions of extremal structures in the setting of metric spaces in Chapter 4 and in the setting of multigraphs in Chapter 5.

### Stability Theorems.

In [47] and [97], Erdős and Simonovits proved the following stability theorem, which intuitively says that if a  $K_s$ -free graph has “almost” the maximal number of possible edges, then it looks “almost” like an extremal  $K_s$ -free graph.

**THEOREM 2.2.2 (Erdős-Simonovits [46, 97]).** *For all  $\delta > 0$  and  $s \geq 2$ , there is an  $\epsilon > 0$  such that if  $H$  is a  $K_s$ -free graph with  $n$  vertices and  $t_{s-1}(n) - \epsilon n^2$  edges, then  $H$  can be made to be isomorphic to  $T_{s-1}(n)$  by adding or removing at most  $\delta n^2$  edges.*

In fact [97] contains versions of this theorem for  $F$ -free graphs for any finite graph  $F$  with at least one edge. Stability theorems have been proven for other types of structures as well, including directed

graphs in [73] and hypergraphs [62, 85]. We will prove stability theorems in the setting of metric spaces in Chapter 4 and for multigraphs in Chapter 5.

### Supersaturation.

The following supersaturation result follows from more general theorems for hypergraphs proved by Erdős and Simonovits in [50].

**THEOREM 2.2.3 (Erdős-Simonovits [50]).** *Suppose  $H$  is a graph with  $t$  vertices. Then for every  $c > 0$  there is  $c' > 0$  such that any graph on  $n$  vertices with at least  $ex(n, H) + cn^2$  edges contains at least  $c'n^t$  copies of  $H$ .*

Supersaturation theorems have also appeared for other combinatorial structures. Examples can be found for instance in [73] for digraphs and in [22] for metric spaces.

### Szemerédi's Regularity Lemma.

In this section we discuss Szemerédi's regularity lemma and one of its corollaries, the triangle removal lemma. We will not directly use these results, but we include them for reference as we will be using more general versions of them. Specifically, we will use a multicolor version of the regularity lemma in Chapter 4, a multicolor version of the triangle removal lemma in Chapter 5, and general version of the triangle removal lemma for finite relational languages (see [6]) in Chapter 6. The following definitions and results are quoted from [67]. We direct the reader there for a more extensive survey.

**DEFINITION 2.2.4.** Fix a finite graph  $G$  and disjoint subsets  $X, Y \subseteq V(G)$ .

- (1) Set  $e(X, Y) = |\{xy \in E(X, Y)\}|$ .
- (2) The *density* of  $(X, Y)$  in  $G$  is  $d(X, Y) = \frac{e(X, Y)}{|X||Y|}$ .
- (3)  $(X, Y)$  is  $\epsilon$ -*regular* for  $G$  if for all  $X' \subseteq X$  and  $Y' \subseteq Y$  with  $|X'| \geq \epsilon|X|$  and  $|Y'| \geq \epsilon|Y|$ ,

$$|d(X, Y) - d(X', Y')| \leq \epsilon.$$

We now state Szemerédi's Regularity Lemma.

**THEOREM 2.2.5 (Szemerédi [99]).** *For every  $\epsilon > 0$  and positive integer  $m$ , there are integers  $M = M(m, \epsilon)$  and  $N = N(m, \epsilon)$  such that if  $G$  is a finite graph with at least  $N(m, \epsilon)$  vertices, then there is a partition  $V_0, V_1, \dots, V_k$  of  $V(G)$  with the following properties.*

- $m \leq k \leq M$ .
- $|V_0| \leq \epsilon n$ .

- $|V_1| = \dots = |V_k|$ .
- All but at most  $\epsilon k^2$  pairs  $(V_i, V_j)$  where  $1 \leq i \neq j \leq k$  are  $\epsilon$ -regular.

Given two graphs  $G = (V, E)$  and  $G' = (V, E')$  and  $\delta > 0$ , we say  $G$  and  $G'$  are  $\delta$ -close if

$$|\{xy \in \binom{V}{2} : xy \in E \setminus E' \text{ or } xy \in E' \setminus E\}| \leq \delta n^2.$$

In other words,  $G$  is  $\delta$ -close to  $G'$  if we can obtain  $G'$  from  $G$  by adding or removing at most  $\delta n^2$  edges. We now state the triangle removal lemma.

**THEOREM 2.2.6 (Erdős-Frankl-Rödl [48]).** *Fix  $s \geq 3$ . For all  $0 < \epsilon < 1$ , there is  $0 < \delta < 1$  and  $M$  such that for all  $n > M$  the following holds. If  $|\{X \in \binom{V(G)}{s} : G[X] \cong K_s\}| \leq \delta \binom{n}{s}$ , then  $G$  is  $\epsilon$ -close to a  $K_s$ -free graph with the same vertex set.*

### Hypergraph Containers Theorem.

In this section we state a version of the hypergraph containers theorem, Theorem 2.2.7 below. The hypergraph containers theorem was independently developed by Balogh-Morris-Samotij in [18] and by Saxton-Thomason in [92]. Theorem 2.2.7 will be used in Chapter 6.

We begin with some definitions. Suppose  $\ell > 1$  is an integer and  $H$  is an  $\ell$ -uniform hypergraph with average degree  $d$  and vertex set of size  $n$ . Fix  $\tau > 0$ . Then for every  $\sigma \subseteq V(H)$ , set

$$d(\sigma) = |\{e \in E(H) : \sigma \subseteq e\}|.$$

Given  $v \in V(H)$  and  $j \in [\ell]$ , set

$$d^{(j)}(v) = \max\{d(\sigma) : v \in \sigma \subseteq V(H), |\sigma| = j\}.$$

If  $d > 0$ , then for each  $j \in [\ell]$ , define  $\delta_j = \delta_j(\tau)$  to satisfy the equation

$$\delta_j \tau^{j-1} n d = \sum_{v \in V(H)} d^{(j)}(v)$$

and set

$$\delta(H, \tau) = 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j.$$

If  $d = 0$ , set  $\delta(H, \tau) = 0$ . The following is a simplified version of Corollary 3.6 from [92] and is the version of the hypergraph contains theorem which we use in Chapter 6. Recall that given a  $k$ -uniform hypergraph  $G = (V, E)$ ,  $e(G)$  is the number of edges in  $G$  and given a set  $X \subseteq V$ ,  $G[X]$  is the hypergraph  $(X, E \cap \binom{V}{k})$ .

**THEOREM 2.2.7 (Saxton-Thomason [92]).** *Let  $H$  be an  $\ell$ -uniform hypergraph with a vertex set  $V$  of size  $n$ . Suppose  $0 < \epsilon, \tau < \frac{1}{2}$  and  $\tau$  satisfies  $\delta(H, \tau) \leq \epsilon/12\ell!$ . Then there exists a constant  $c = c(\ell)$  and a collection  $\mathcal{C} \subseteq \mathcal{P}(V)$  such that the following hold.*

- (i) *For every independent set  $I$  in  $H$ , there exists  $C \in \mathcal{C}$  such that  $I \subseteq C$ .*
- (ii) *For all  $C \in \mathcal{C}$ , we have  $e(H[C]) \leq \epsilon e(G)$ , and*
- (iii)  *$\log |\mathcal{C}| \leq c \log(1/\epsilon)n\tau \log(1/\tau)$ .*

**2.2.2. Combining the Tools: Patterns and a Question.** In this subsection we summarize general proof strategies which have been developed for proving approximate structure and enumeration theorems. These strategies involve various combinations of the tools in Subsection 2.2.1. Our focus will be on the strategies which make appearances later in this thesis. We will give citations where the reader can find examples of these strategies, however we have not attempted to give an exhaustive list of the many papers which have used these techniques.

**Regularity.** Szemerédi’s Regularity Lemma and its corollaries are often combined with stability theorems to prove approximate structure and/or enumeration theorems.

Stability Theorem + Regularity and Corollaries  $\Rightarrow$  Approximate structure and/or enumeration.

Examples of papers which use this strategy include [4, 10, 88] in the setting of graphs, [5] in the setting of oriented graphs, and [19, 20, 85] in the setting of hypergraphs. We will use a version of this strategy to prove approximate structure and enumeration theorems in the setting of discrete metric spaces in Chapter 4. The regularity lemma is also often used without a stability theorem to prove enumeration results. Examples of this include [45, 61, 65, 83].

**Containers.** Many new structure and enumeration results have been proved using the hypergraph containers theorem along with a stability and/or supersaturation theorem. In particular, the hypergraph containers theorem can be combined with a supersaturation result to prove approximate enumeration theorems.

Supersaturation + Containers  $\Rightarrow$  Approximate enumeration.

Papers which use this strategy to produce enumeration theorems include the two original papers on the hypergraph containers theorem [18, 92] for  $H$ -free graphs, [73] for digraphs, [22] for metric spaces, [21, 80, 81] for other specific families of graphs, and [17] for sum-free sets of integers. The containers theorem can also be combined with stability theorems (and other tools including graph

removal lemmas and supersaturation results) to produce structure results.

Containers

+

Stability *and* Graph Removal and/or Supersaturation

↓

Approximate Structure and/or Enumeration

Papers which employ this strategy to prove structure and enumeration theorems include [14, 16, 63] for graphs, [73] for directed graphs, and [15] for hypergraphs, vector spaces over finite fields, and permutations.

In this section we have shown there are many structure and enumeration results for combinatorial structures which employ similar tools and similar proof strategies. This pattern naturally raises the following question.

QUESTION 2.2.8. *Is there a way to view these results (and their proofs) as examples of a general theorem (and its proof)?*

Chapter 6 shows the answer is yes (for many cases), by formalizing the pattern of proof described in this section. We will use generalizations of some of the tools and techniques from Subsection 2.2.1. The precise statements of these theorems require preliminaries, so we leave them to Chapter 6

### 2.3. Conclusion

The study of hereditary properties of combinatorial structures has been a continual source of interesting research within the field of extremal combinatorics. Chapters 4 and 5 of this thesis contribute new examples to this line of research in the settings of discrete metric spaces and multigraphs.

Out of the many results in this line of research has emerged an pattern of proof for approximate structure and enumeration results. Chapter 6 formalizes this pattern, proving for instance, a general theorem in the setting of arbitrary finite relational languages which encompasses many known approximate enumeration theorems.

## A Logical Introduction

In this chapter, we explain the model theoretic motivation behind the results in Chapters 4, 5, 6 and 7. Chapter 8 does not fit directly into this narrative, and contains its own introduction. Our goal in this chapter is to demonstrate that the combinatorial results described in Chapter 2 are of direct importance to interesting model theoretic questions.

### 3.1. Logical 0-1 Laws.

In this section we give background on logical 0-1 laws.

DEFINITION 3.1.1. Suppose  $\mathcal{L}$  is a finite first-order language and  $F = \bigcup_{n \in \mathbb{N}} F(n)$ , where for each  $n$ ,  $F(n)$  is a set of  $\mathcal{L}$ -structures with underlying set  $[n]$ .

- (1) For each  $\mathcal{L}$ -sentence  $\psi$ , let  $\mu_n(\psi)$  to be the proportion of elements in  $F(n)$  which satisfy  $\psi$ , that is,

$$\mu_n(\psi) = \frac{|\{G \in F(n) : G \models \psi\}|}{|F(n)|}.$$

- (2) When it exists, the *asymptotic probability* of  $\psi$  is  $\mu(\psi) = \lim_{n \rightarrow \infty} \mu_n(\psi)$ .  
 (3)  $F$  has a *labeled first-order limit law* if for each  $\mathcal{L}$ -sentence  $\psi$ ,  $\mu(\psi)$  exists.  
 (4)  $F$  has a *labeled first-order 0-1 law* if for each  $\mathcal{L}$ -sentence  $\psi$ ,  $\mu(\psi) \in \{0, 1\}$ .  
 (5) The *almost sure theory* of  $F$  is  $T_{as}(F) = \{\psi : \psi \text{ is a first-order } \mathcal{L}\text{-sentences and } \mu(\psi) = 1\}$ .

While there are other kinds of logical 0-1 laws (e.g. monadic second-order), in this thesis, we will be restricting our attention to the first-order, labelled cases as described in Definition 3.1.1. Unless otherwise stated, 0-1 law (limit law) will always mean first-order labeled 0-1 law (first-order labelled limit law). Similarly, all languages and theories are first-order. The following is a standard fact (see for instance [103]).

REMARK 3.1.2. *Suppose  $F$  is as in Definition 3.1.1. If  $F$  has a 0-1 law, then  $T_{as}(F)$  is a complete, consistent theory.*

We will be interested in the case when  $F$  has a 0-1 law and  $T_{as}(F)$  has infinite models. When this happens, we can think of the models of  $T_{as}(F)$  as infinite structures whose first-order properties arise as the probabilistic “limit” of the finite structures in  $F$ . We now state some basic questions about 0-1 laws which form part of the motivation for this thesis. Assume  $\mathcal{L}$  is a finite language and  $F = \bigcup_{n \in \mathbb{N}} F(n)$ , where for each  $n$ ,  $F(n)$  is a set of  $\mathcal{L}$ -structures with underlying set  $[n]$ .

QUESTION 3.1.3. *How can we tell if  $F$  has a 0-1 law?*

QUESTION 3.1.4. *If  $F$  has a 0-1 law, what model theoretic properties must  $T_{as}(F)$  have?*

QUESTION 3.1.5. *If  $F$  has a 0-1 law, how does  $T_{as}(F)$  compare to theories which can be built as “limits” of  $F$  using other methods (e.g. Fraïssé limits, ultraproducts)?*

While progress has been made on all these questions in various settings, there is one class of examples which are not well understood from a model theoretic perspective, and which are the focus of this thesis. We explain some of this progress and introduce this class of examples in the next section.

### 3.2. Nontrivial, Fast-growing Families

In this section we define nontrivial, fast-growing families of first-order structures and give a list of our main examples, Example 3.2.5. We will show that while there exist general sufficient conditions for when a family of first-order structures has a 0-1 law, these do not apply to the families in Example 3.2.5. Further, we show these families generate interesting model theoretic questions.

**3.2.1. Definition and Main Examples.** In this subsection we define non-trivial, fast growing families and give our main list of examples in Example 3.2.5. Given a finite first-order language  $\mathcal{L}$  and a class of  $\mathcal{L}$ -structures  $\mathcal{K}$ , we say an  $\mathcal{L}$ -structure  $F$  is a *minimal forbidden configuration for  $\mathcal{K}$*  if for all  $A$  in  $\mathcal{K}$ ,  $F \not\subseteq_{\mathcal{L}} A$  and for all  $F' \subsetneq_{\mathcal{L}} F$ , there is  $A$  in  $\mathcal{K}$  such that  $F' \subseteq_{\mathcal{L}} A$ . Recall that if  $\mathcal{L}' \subseteq \mathcal{L}$  and  $M$  is an  $\mathcal{L}$ -structure, then  $M \upharpoonright_{\mathcal{L}'}$  denotes the reduct of  $M$  to  $\mathcal{L}'$ .

DEFINITION 3.2.1. Let  $\mathcal{L}$  be a finite relational language and let  $r \geq 2$  be the maximum arity of its relation symbols. For each  $n$ , suppose  $K(n)$  is a set of  $\mathcal{L}$ -structures with domain  $[n]$  and let  $K = \bigcup_{n \in \mathbb{N}} K(n)$ .

- (a) Given  $1 \leq \ell \leq r$ , let  $\mathcal{L}(\ell) = \{R \in \mathcal{L} : \text{the arity of } R \text{ is at most } \ell\}$ .
- (b)  $K \upharpoonright_{\ell} = \{G \upharpoonright_{\mathcal{L}(\ell)} \text{ such that } G \in K\}$  is the *reduct of  $K$  to  $\mathcal{L}(\ell)$* .

(c)  $K$  is *trivial* if the following holds for each  $1 \leq \ell \leq r$ . If  $F$  is a minimal forbidden configuration for  $K \upharpoonright_\ell$ , then  $|\text{dom}(F)| \leq \ell$ . Otherwise,  $K$  is *non-trivial*.

(d)  $K$  is *fast-growing* if  $|K(n)| \geq \Omega(n^r)$ .

The above definition of triviality in part (c) of Definition 3.2.1 comes from a definition of Koponen in [70], where it is called *randomness*. We have decided to change the name here to avoid confusion with terminology in Chapter 6.

OBSERVATION 3.2.2. *In the notation of Definition 3.2.1, if all the relation symbols in  $\mathcal{L}$  have the same arity  $r$ , then  $K$  is trivial if and only if all its minimal forbidden configurations have size at most  $r$ .*

In all of the examples we will consider,  $\mathcal{L}$  and  $K$  will be as in Observation 3.2.2. The following are examples of  $K$  which are *not* non-trivial, fast-growing families.

EXAMPLE 3.2.3. Suppose for each  $n$ ,  $K(n)$  is the set of graphs with vertex set  $[n]$  and  $K = \bigcup_{n \in \mathbb{N}} K(n)$ . Then  $K$  is trivial because all its minimal forbidden configurations have size at most 2.

EXAMPLE 3.2.4. Suppose  $k \in \mathbb{N}$  is a fixed positive integer. For each  $n$ , let  $K(n)$  be the set of graphs with vertex set  $[n]$  which have VC-dimension at most  $k$ , and let  $K = \bigcup_{n \in \mathbb{N}} K(n)$ . Clearly  $K$  is nontrivial. However, Theorem 2 in [4] implies there is an  $\epsilon = \epsilon(k) > 0$  such that  $|K(n)| \leq 2^{n^{2-\epsilon}}$ . Thus  $K$  is not fast-growing.

We now give the main list of examples which form the focus of this thesis.

EXAMPLE 3.2.5. In the following list,  $K = \bigcup_{n \in \mathbb{N}} K(n)$  are non-trivial fast growing families. See the citations listed for enumeration theorems showing the families are fast-growing.

- (1)  $K(n)$  is the set of  $K_s$ -free graphs with vertex set  $[n]$ , where  $s \geq 3$  is an integer. See [66].
- (2)  $K(n)$  is the set of  $H$ -free graphs with vertex set  $[n]$ , where  $H$  is a fixed edge-critical graph of chromatic number at least 3 ( $H$  is *edge-critical* if there is an edge in  $H$  such that removing the edge results in a graph of strictly smaller chromatic number than  $H$ ). See [89, 60].
- (3)  $K(n)$  is the set of directed graphs with vertex set  $[n]$  omitting transitive tournaments of size  $k$ , where  $k \geq 3$  is a fixed integer. See [73].
- (4)  $K(n)$  is the set of triangle-free 3-uniform hypergraphs with vertex set  $[n]$ . See [20].
- (5)  $K(n)$  is the set of Fano-free 3-uniform hypergraphs with vertex set  $[n]$ . See [85].



- (6)  $K(n)$  is the set of metric spaces with underlying set  $[n]$  and distances in  $[r]$ , where  $r \geq 2$  is a fixed even integer. See [82] or Chapter 4 of this thesis.

We will see later in this chapter that for all the  $K$  in Example 3.2.5,  $K$  has a 0-1 law. These are the 0-1 laws this thesis is focused on understanding from a more general perspective. We would like to point out that the multigraphs investigated in Chapter 5 are also fast-growing non-trivial families.

**3.2.2. Existing General Theorems Do Not Cover Example 3.2.5.** In this section we show that while there exist some general sufficient conditions for when a family has a logical 0-1 law, they do not apply to the families in Example 3.2.5. We begin with a bit of history.

The first major theorem concerning logical 0-1 laws is the following result, which was proved independently by Fagin in [51] and by Glebskiĭ et. al in [55].

**THEOREM 3.2.6 (Fagin [51], Glebskiĭ et. al [55]).** *Suppose  $\mathcal{L}$  is a finite language and for each  $n$ ,  $F(n)$  is the set of all  $\mathcal{L}$ -structures with underlying set  $[n]$ . Then  $F = \bigcup_{n \in \mathbb{N}} F(n)$  has a 0-1 law and  $T_{as}(F)$  has infinite models.*

Thus, for every finite first-order language  $\mathcal{L}$ , the family of all  $\mathcal{L}$ -structures with universe  $[n]$  for  $n \in \mathbb{N}$  has a 0-1 law. The same arguments used in [51] and [55] also show that the family of all finite graphs has a 0-1 law (see [103] for the full argument). However, for nontrivial families of first-order structures, questions about 0-1 laws cannot be answered using the techniques from Theorem 3.2.6 exclusively.

The most general sufficient conditions for the existence of 0-1 laws consider families where the number of structures of size  $n$  does not grow too quickly as  $n \rightarrow \infty$ . For these families, algebraic techniques are often used. The first results of this type were introduced by Compton in [40, 41]. Further results of this flavor appear in [23, 36, 40, 41] and provide a large number of examples of logical limit laws. Other results employing algebraic techniques to prove logical limit laws for relatively slow growing families include [58] for minor-closed classes of graphs and [69] for graphs of bounded degree.

Fast-growing families grow too quickly for the types of algebraic arguments used in the Compton-style results of the previous paragraph. In particular, for any  $F = \bigcup_{n \in \mathbb{N}} F(n)$  from Example 3.2.5,  $|F(n)|$  grows at least exponentially in  $n^2$ . The results in [23, 36, 40, 41] all require growth rates which are subexponential in  $n^2$ , and therefore tell us nothing about the families from Example 3.2.5.

In [68], Koponen proves a general theorem which shows that for any  $\ell \geq 2$ , the family of  $\ell$ -colorable structures in any finite relational language has a 0-1 law. We will discuss in Section 3.3 how this result contributes to some of the 0-1 laws in Example 3.2.5. However, the results in [68] alone do not tell us any of these families have 0-1 laws.

We have shown that while there has been progress in finding sufficient conditions for when a family of structures in a finite relational language has a 0-1 law, none of these results apply directly to the set of examples in Example 3.2.5.

### 3.3. Fast Growing Families and Extremal Combinatorics

In this section we show that the 0-1 laws for the structures in Example 3.2.5 rely on precise structure and enumeration results from extremal combinatorics. The outline of how these results work is the following.

Precise Asymptotic Structure and Enumeration + Model Theoretic Arguments  $\Rightarrow$  New 0-1 Law.

For example, let  $s \geq 3$  and consider the family  $Forb(K_s) = \bigcup_{n \in \mathbb{N}} Forb(n, K_s)$ , where for each  $n$ ,  $Forb(n, K_s)$  is the set of  $K_s$ -free graphs with vertex set  $[n]$  (this is (1) in Example 3.2.5). Let  $Col_{s-1}(n)$  be the set of  $(s-1)$ -colorable graphs with vertex set  $[n]$  and let  $Col_{s-1} = \bigcup_{n \in \mathbb{N}} Col_{s-1}(n)$ . Recall for all  $n$ ,  $Col_{s-1}(n) \subseteq Forb(n, K_s)$ . The argument proving  $Forb(K_s)$  has a 0-1 law in [66] proceeds as follows.

Almost all elements in  $Forb(n, K_s)$  are in  $Col_{s-1}(n)$  (Precise Asymptotic Structure and Enumeration)

+

$Col_{s-1}$  has a 0-1 law (Model Theoretic Arguments)

$\Downarrow$

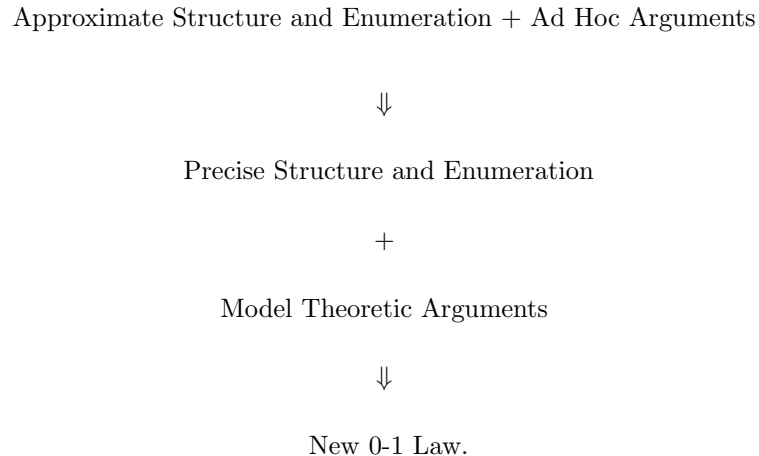
$Forb(n, K_s)$  has a 0-1 law and  $T_{as}(Forb(K_s)) = T_{as}(Col_{s-1})$  (New 0-1-Law).

The 0-1 law for number (2) of Example 3.2.5 is proved by Patel in [84] by combining model theoretic arguments from [66] with combinatorial results from [89, 60]. In [68], Koponen proves a general theorem for  $\ell$ -colorable structures in an arbitrary finite relational language. This result can be combined (following the recipe above) with the combinatorial results in [73], [20], and [85] to yield

new 0-1 laws for the families of structures in numbers (3), (4), and (5) of Example 3.2.5 respectively. In Chapter 4 we prove the existence of a new 0-1 law for certain discrete metric spaces (number (6) of Example 3.2.5) using this strategy of first proving precise structure and enumeration theorems. Thus we have shown all the 0-1 laws for Example 3.2.5 rely crucially on precise structure and enumeration results from extremal combinatorics. There exists almost no work on understanding these structure and enumeration results from a model theoretic perspective. This is a main goal of this thesis.

### 3.4. The Role of Approximate Structure

We have established in the previous section that precise structure and enumeration results play key roles in proving 0-1 laws for many nontrivial, fast-growing families. In this section we show that approximate structure and enumeration theorems play key roles in proving these precise results. For example, the precise structure and enumeration results in [20, 73, 85] and Chapter 4 (used for numbers (3)-(6) of Example 3.2.5 respectively) all explicitly use approximate structure and enumeration theorems as preliminary steps. Further, the precise structure and enumeration results required for the 0-1 laws in parts (1) and (2) of Example 3.2.5 can also be proved using this strategy. Thus, the following is a more complete outline of how the 0-1 laws in Example 3.2.5 can be proven.



This shows that Question 2.2.8 from Chapter 2, and Chapter 6 which answers it, contribute to our understanding the logical 0-1 laws in Example 3.2.5 from a model theoretic point of view.

**3.4.1. Model Theoretic Motivation.** In this section we discuss some model theoretic questions related to the 0-1 laws in Example 3.2.5.

Questions about how finite structure interacts with the infinite structure are recurring themes within model theory. One such question concerns the relationship between pseudofinite structures and

Fraïssé limits, two types of infinite structures which are “limits” of finite structures. Specifically, a well-known open question of Cherlin [17] asks: is the Fraïssé limit of triangle-free graphs, called the Henson graph, also a limit in the pseudofinite sense?

The investigation of  $K_s$ -free graphs by Kolaitis, Prömel, and Rothschild in [66] can be seen as addressing the “almost sure” version of Cherlin’s question: is the Henson graph also the limit of triangle-free graphs in the almost sure sense? The authors show the answer is no, proving that triangle-free graphs have a zero-one law, with almost sure theory differing from the theory of the Henson graph,  $T_{hg}$ . They in fact show much more, proving analogous results for  $K_{l+1}$ -free graphs for all  $l \geq 2$ . Later investigation demonstrated that for triangle-free graphs, the almost sure theory and  $T_{hg}$  fall into different classes of model theoretic complexity. In particular, the almost sure theory is simple, while  $T_{hg}$  has  $\text{SOP}_3$  (which implies it is non-simple, see [95]). This is a pattern which repeats in many examples. For instance, we will show in Chapter 4 that an analogous phenomenon occurs in the case of discrete metric spaces (i.e. number (6) in Example 3.2.5).

In [1], Ahlman proves a general theorem (Theorem 5.7 there) which implies the almost sure theories arising from all the examples in Example 3.2.5 are supersimple with  $SU$ -rank 1. The application of this theorem to these examples relies crucially on having an explicit understanding their asymptotic structure, which comes from extremal combinatorics.

We end with a conjecture which arises naturally from what is known about Example 3.2.5. Given a finite relational language, a class  $\mathcal{K}$  of  $\mathcal{L}$ -structures is a *hereditary  $\mathcal{L}$ -property* if it is closed under isomorphism and has the model-theoretic hereditary property. Given  $\mathcal{K}$  a hereditary  $\mathcal{L}$ -property, let  $\mathcal{K}_n$  be the set of elements from  $\mathcal{K}$  with universe  $[n]$ . Observe that for each  $K$  in Example 3.2.5, there is an appropriate finite relational language  $\mathcal{L}$  and a hereditary  $\mathcal{L}$ -property  $\mathcal{K}$  such that for all  $n \in \mathbb{N}$ ,  $K(n) = \mathcal{K}_n$ . Thus hereditary  $\mathcal{L}$ -properties are an appropriate setting in which to generalize results and questions related to Example 3.2.5.

**CONJECTURE 3.4.1.** *Suppose  $\mathcal{L}$  is a finite relational language and  $\mathcal{K}$  is a hereditary  $\mathcal{L}$ -property such that  $K := \bigcup_{n \in \mathbb{N}} \mathcal{K}_n$  is fast-growing and non-trivial. If  $K$  has a 0-1 law, then  $T_{as}(K)$  is supersimple with  $SU$ -rank 1.*

We believe that understanding the above will rely on a general model theoretic understanding of the asymptotic structure results which lead to the 0-1 laws in Example 3.2.5. This is the model theoretic motivation for Chapter 6, which focuses on understanding asymptotic enumeration and structure results in the setting of hereditary  $\mathcal{L}$ -properties.

### 3.5. Conclusion

In sum, 0-1 laws for the non-trivial, fast-growing families in Example 3.2.5 generate interesting model theoretic questions and are not completely understood from a model theoretic perspective. These 0-1 laws rely on precise structure and enumeration results, which in turn rely on approximate structure and enumeration results. Chapter 6 of this thesis addresses approximate structure and enumeration results from a model theoretic perspective, and thus can be seen as contributing to our understanding of these 0-1 laws. Chapter 4 of this thesis proves a new 0-1 law for certain families of discrete metric spaces. Chapter 5 contains combinatorial results for certain families of multigraphs, which are key ingredients in forthcoming work by the author and D. Mubayi which proves precise asymptotic structure results as well as new new 0-1 laws for these families. The results in Chapters 4 and 5 contributed directly to the development of the results in Chapter 6, and may serve useful in generalizing further aspects of the arguments used to prove the 0-1 laws for the examples in Example 3.2.5.

## Discrete Metric Spaces

### 4.1. Introduction

The work in this chapter is joint with D. Mubayi and appears in [82]. Given integers  $n, r \geq 3$ , define  $M_r(n)$  to be the set of all metric spaces with underlying set  $[n] := \{1, \dots, n\}$  and distances in  $\{1, \dots, r\}$ . The goal of this chapter is to investigate the approximate structure of most elements of  $M_r(n)$  for fixed  $r$  and large  $n$ , and in the case when  $r$  is even, to prove that  $M_r(n)$  has a labeled first-order 0-1 law.

**4.1.1. Background.** Given a set  $X$ , let  $\binom{X}{2} = \{Y \subseteq X : |Y| = 2\}$  and  $2^X = \{Y : Y \subseteq X\}$ . An  $r$ -graph  $G$  is a pair  $(V, c)$ , where  $V$  is a (vertex) set, and  $c : \binom{V}{2} \rightarrow 2^{[r]}$ ; we call  $G$  a *simple complete  $r$ -graph* if  $|c(xy)| = 1$  for all  $xy \in \binom{V}{2}$ . Elements of  $M_r(n)$  are naturally viewed as simple complete  $r$ -graphs by assigning edge colors corresponding to distances. Given a set  $\mathcal{H}$  of  $r$ -graphs, let  $\text{Forb}_n^r(\mathcal{H})$  be the set of simple complete  $r$ -graphs with vertex set  $[n]$  which contain no element of  $\mathcal{H}$  as a substructure. By taking  $\mathcal{H}$  to be the set of simple complete  $r$ -graphs on three vertices which contain violations of the triangle inequality, we see that  $M_r(n) = \text{Forb}_n^r(\mathcal{H})$ . In this way, we can view  $M_r(n)$  as an edge-colored analogue of  $\text{Forb}_n(\mathcal{H})$ . This analogy suggests that one could prove similar results as in [49] and [66] about  $M_r(n)$ . In this chapter we show that this is indeed the case, utilizing techniques from graph theory to describe the approximate structure of most elements of  $M_r(n)$  for large  $n$ .

We may view elements of  $M_r(n)$  as first-order structures in the language  $\mathcal{L}_r$  consisting of  $r$  binary predicates, one for each edge color. In this setting, as a corollary of our structural results, we are able to prove in the case when  $r$  is even, that there is a labeled first-order 0-1 law for  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  and to give an axiomatization of its almost sure theory. We consider only  $r \geq 3$  for the following reason. There is no way to violate the triangle inequality using distances in  $\{1, 2\}$ , so  $M_2(n)$  consists of the set of *all* simple complete 2-graphs. This means that given a pair  $x, y$  of distinct elements of  $[n]$ , the distance between  $x$  and  $y$  is equal to 1 in exactly half of the elements of  $M_2(n)$ . For each  $G \in M_2(n)$ , associate a graph  $\mathcal{G}$  with vertex set  $[n]$  such that for each  $x, y \in [n]$ , there is an

edge between  $x$  and  $y$  in  $\mathcal{G}$  if and only if the distance between  $x$  and  $y$  is equal to 1 in  $G$ . Under this association, we see that  $M_2(n)$  behaves exactly like the random graph  $G(n, 1/2)$ , the structural properties of which have been studied extensively (see [25]), and which is known to have a labeled first-order 0-1 law [51, 55].

The proofs of our main results will rely on a stability theorem which is proved using a multi-color version of the Szemerédi regularity lemma [7]. While our proof techniques bear some resemblance to the classical results in [48, 49, 64], we need several new ideas that are motivated by work on weighted Turán-type problems [54]. The results in this chapter also add to existing results that study metric spaces as combinatorial objects [37, 71, 78, 79]. In particular, [79] and [71] address questions similar to ours in the continuous setting. In [79], Mascioni investigates the following problem. Given an integer  $n$  and a fixed set  $X$  of  $n$  points, if we assign i.i.d. uniform real numbers in  $[0, 1]$  to the elements of  $\binom{X}{2}$ , what is the probability we get a metric space? It is shown in [79] that this probability  $p$  satisfies

$$(1) \quad \left(\frac{1}{2}\right)^{\binom{n}{2}} \leq p \leq \left(\frac{1}{2}\right)^{\lfloor n/2 \rfloor} \left(\frac{2}{3}\right)^{\lfloor n/2 \rfloor (\lfloor n/2 \rfloor - 2)},$$

where the lower bound is obtained by noting that any assignment of distances from  $[\frac{1}{2}, 1]$  yields a metric space. In more recent work, Kozma, Meyerovitch, Peled, and Samotij [71] identify the set of metric spaces on  $[n]$  having all distances in  $[0, 1]$  with elements in the cube  $[0, 1]^{\binom{n}{2}}$ . Let  $M_n$  be the subset of  $[0, 1]^{\binom{n}{2}}$  which corresponds to the set of metric spaces on  $[n]$ . Then [71] shows that there are constants  $c$ ,  $c'$ , and  $C$  such that  $c > 0$  and

$$(2) \quad \frac{1}{2} + \frac{c'}{\sqrt{n}} \leq (\text{vol} M_n)^{1/\binom{n}{2}} \leq \frac{1}{2} + \frac{C}{n^c}.$$

They also prove that with high probability, all distances are between  $1/2 - n^{-c}$  and 1. The upper bound in (2) implies that  $p^{1/\binom{n}{2}}$  approaches  $1/2$  as  $n \rightarrow \infty$ , where  $p = \text{vol} M_n$  is the probability from (1). Given a fixed even  $r \geq 4$ , our results about  $M_r(n)$  can be translated into results about metric spaces on  $[n]$  with all distances in  $\{\frac{1}{r}, \dots, \frac{r-1}{r}, 1\}$ . In this setting, our Theorem 4.1.2 says that almost all such metric spaces (as  $n \rightarrow \infty$ ) have all of their distances in  $[\frac{1}{2}, 1]$ , therefore capturing a similar phenomenon as the results of [71] (for odd  $r$  the situation is slightly more complicated). If it were possible to generalize our results to the setting where  $r \rightarrow \infty$  and  $n$  is fixed, then they could apply to the continuous setting.

While existing results on logical 0-1 laws do not apply directly to  $M_r$ , a result of [68] does imply that a subfamily  $C_r$  of  $M_r$ , (which will be defined later) has a labeled first-order 0-1 law. Our results

will show that when  $r$  is even, almost all elements of  $M_r$  are in  $C_r$ , which will yield that  $M_r$  has a labeled first-order 0-1 law. Therefore, this chapter provides the combinatorial argument required to reduce the existence of a labeled first-order 0-1 law for  $M_r$  to the existence of one for  $C_r$ , while the fact that  $C_r$  has a labeled first-order 0-1 law follows from known results, and is in fact very easy to prove directly. Part of the motivation for this work is the idea that having more examples of logical limit laws in languages other than that of graphs, and seeing the techniques used to prove them, will improve our general understanding of when a family of finite structures has a logical limit law.

**4.1.2. Results.** In this section we state the results in this chapter. First we give some necessary definitions and notation. Given positive integers  $r, s$  and a set  $X$ , set  $[r] = \{1, \dots, r\}$ ,  $\binom{X}{s} = \{Y \subseteq X : |Y| = s\}$ , and  $2^X = \{Y : Y \subseteq X\}$ . Recall that an  $r$ -graph  $G$  is a pair  $(V, c)$ , where  $V$  is a set, and  $c : \binom{V}{2} \rightarrow 2^{[r]}$ . We call  $V$  the *vertex set* of  $G$  and  $c$  the *coloring* of  $G$ . In the case when  $|c(e)| \leq 1$  for every  $e \in \binom{V}{2}$ , we say that  $(V, c)$  is *simple*, and when  $c(e) \neq \emptyset$  for each  $e \in \binom{V}{2}$ , we say  $G$  is *complete*. Given integers  $r, n \geq 3$ , we consider  $M_r(n)$  as the set of simple complete  $r$ -graphs  $([n], c)$  satisfying the triangle inequality, i.e., for every three pairwise distinct elements  $x, y, z$  of  $[n]$ ,

$$\text{if } \{i\} = c(xy), \{j\} = c(yz), \text{ and } \{k\} = c(xz), \text{ then we have } i \leq j + k.$$

Given a set  $X$  and  $\{x, y\} \in \binom{X}{2}$ , we will write  $xy$  to mean  $\{x, y\}$ . Given integers  $i < j$ , set  $[i, j] = \{i, i + 1, \dots, j\}$ .

**DEFINITION 4.1.1.** For an even integer  $r \geq 4$  and any integer  $n$ , let  $C_r(n)$  be the set of all simple complete  $r$ -graphs  $G = ([n], c)$  such that  $c(e) \subset [\frac{r}{2}, r]$  for all  $e \in \binom{[n]}{2}$ .

When  $r$  is even, there is no way to violate the triangle inequality using distances in  $[\frac{r}{2}, r]$ , so  $C_r(n) \subset M_r(n)$ . The strongest structural result we will prove (Theorem 4.1.2 below) says that when  $r \geq 4$  is even, almost all elements in  $M_r(n)$  are in  $C_r(n)$ .

**THEOREM 4.1.2.** *Let  $r \geq 4$  be an even integer. Then there is  $\beta > 0$  and  $M > 0$  such that for all  $n \geq M$ ,*

$$|C_r(n)| \geq |M_r(n)|(1 - 2^{-\beta n}).$$

When  $r$  is even,  $|C_r(n)| = \left(\frac{r}{2} + 1\right) \binom{n}{2}$ . Therefore Theorem 4.1.2 yields that when  $r$  is even,

$$\left(\frac{r}{2} + 1\right) \binom{n}{2} \leq |M_r(n)| \leq \left(\frac{1}{1 - 2^{-\beta n}}\right) \left(\frac{r}{2} + 1\right) \binom{n}{2}$$

for some positive  $\beta$  and sufficiently large  $n$ . We obtain the following Corollary.



COROLLARY 4.1.3. *Let  $r \geq 4$  be an even integer. Then  $|M_r(n)| = \left(\frac{r}{2} + 1\right) \binom{n}{2}^{+o(1)}$ .*

When  $r$  is odd, we still obtain a result on the approximate structure of most elements of  $M_r(n)$  (Theorem 4.1.5 below), however the situation in this case is more complicated.

DEFINITION 4.1.4. Let  $r \geq 3$  be an odd integer. Define  $C_r(n)$  to be the set of simple complete  $r$ -graphs  $G = ([n], c)$  such that there is a partition  $V_1 \cup \dots \cup V_t$  of  $[n]$  and for every  $xy \in \binom{[n]}{2}$ ,

$$c(xy) \subset \begin{cases} \left[\frac{r-1}{2}, r-1\right] & \text{if } xy \in \binom{V_i}{2} \text{ for some } i \in [t] \\ \left[\frac{r+1}{2}, r\right] & \text{if } x \in V_i, y \in V_j \text{ for some } i \neq j \in [t]. \end{cases}$$

It is easy to see that for  $r$  odd,  $C_r(n) \subset M_r(n)$ . Given  $\delta > 0$ , two  $r$ -graphs  $G = (V, c)$  and  $G' = (V, c')$  with the same vertex set  $V$  are  $\delta$ -close if  $|\{e \in \binom{V}{2} : c(e) \neq c'(e)\}| \leq \delta|V|^2$ . Set

$$C_r^\delta(n) = \{G \in M_r(n) : \text{there is } G' \in C_r(n) \text{ such that } G \text{ and } G' \text{ are } \delta\text{-close}\}.$$

We now state our structure theorem which holds for all  $r \geq 3$ . Informally, it states that most members of  $M_r(n)$  are in  $C_r^\delta(n)$  for small  $\delta$  and  $n$  large enough depending on  $\delta$ .

THEOREM 4.1.5. *Let  $r \geq 3$  be an integer. Then for all  $\delta > 0$ , there exists an  $M$  and  $\beta > 0$  such that  $n > M$  implies*

$$\frac{|M_r(n) \setminus C_r^\delta(n)|}{|M_r(n)|} \leq \frac{|M_r(n) \setminus C_r^\delta(n)|}{\lceil \frac{r+1}{2} \rceil \binom{n}{2}} \leq 2^{-\beta n^2}.$$

We will prove the following enumeration theorem as a corollary of Theorem 4.1.5 in Section 4.3.

COROLLARY 4.1.6. *Let  $r \geq 3$  be an integer. Then  $|M_r(n)| = \lceil \frac{r+1}{2} \rceil \binom{n}{2}^{+o(n^2)}$ .*

Given  $r \geq 3$ , define  $\mathcal{L}_r = \{R_1, \dots, R_r\}$  where each  $R_i$  is a binary relation symbol. Given  $n \in \mathbb{N}$ , we can naturally consider elements  $G = ([n], c) \in M_r(n)$  as  $\mathcal{L}_r$ -structures by interpreting for each  $(x, y) \in [n]^2$ ,  $R_i^G(x, y) \Leftrightarrow xy \in \binom{[n]}{2}$  and  $c(xy) = \{i\}$ . We will prove as a consequence of Theorem 4.1.2 that, when  $r$  is even,  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  has a labeled first-order 0-1 law. In fact, we will define a set  $T_r$  of  $\mathcal{L}_r$ -sentences such that the following holds (see Section 4.2 for the definition of  $T_r$ ).

THEOREM 4.1.7. *Let  $r \geq 4$  be an even integer and consider the elements of  $M_r(n)$  and  $C_r(n)$  as  $\mathcal{L}_r$ -structures. Then  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  and  $C_r = \bigcup_{n \in \mathbb{N}} C_r(n)$  have labeled first-order 0-1 laws. Moreover they have the same almost sure theory, and this theory is axiomatized by  $T_r$ .*

When  $r$  is odd, the error term in Corollary 4.1.6 cannot be improved from  $o(n^2)$  to  $O(n)$ , and moreover, Theorem 4.1.2 does not hold (see Section 7 for a detailed discussion). This leads us to make the following conjecture.

**CONJECTURE 4.1.8.** *Let  $r \geq 3$  be an odd integer and consider elements of  $M_r(n)$  as  $\mathcal{L}_r$ -structures as in Theorem 4.1.7. Then  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  has a labeled first-order limit law, but does not have a labeled first-order 0-1 law.*

**4.1.3. Notation and outline.** Throughout the chapter, we will omit floors and ceilings where they are unimportant to the argument. Let  $r \geq 3$  be an integer and let  $G$  be an  $r$ -graph. We will write  $V(G)$  to denote the vertex set of  $G$  and  $c^G$  to denote its coloring. For simplicity of notation we set  $E(G) = \binom{V(G)}{2}$ , and for subsets  $X, Y \subseteq V(G)$ , set  $E(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$ , and  $E(X) = E(X, X)$ . Given a simple complete  $r$ -graph  $G$ , we define  $d^G : E(G) \rightarrow [r]$  to be the function sending  $xy \in E(G)$  to the unique  $i \in [r]$  such that  $c^G(xy) = \{i\}$ . We will sometimes also wish to discuss graphs, meaning a set equipped with a single binary, symmetric, irreflexive relation. In order to avoid confusion, these graphs will be denoted by  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  is the vertex set of  $\mathcal{G}$  and  $\mathcal{E} \subseteq \binom{\mathcal{V}}{2}$  is the edge set of  $\mathcal{G}$ . Given a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  and  $v \in \mathcal{V}$ , we will write  $\mathcal{DEG}(v) = |\{u : uv \in \mathcal{E}\}|$ .

By a *violating triple* we will mean a tuple  $(i, j, k) \in \mathbb{N}^3$  such that  $|i - j| \leq k \leq i + j$  is false. By a *violating triangle*, we will mean an  $r$ -graph  $H$  such that  $V(H) = \{x, y, z\}$ , and for some violating triple  $(i, j, k)$ ,  $i \in c^H(xy)$ ,  $j \in c^H(yz)$ , and  $k \in c^H(xz)$ . Given two  $r$ -graphs  $H$  and  $G$ , we say  $H$  is a *sub- $r$ -graph* of  $G$  if  $V(H) \subseteq V(G)$  and for all  $xy \in \binom{V(H)}{2}$ ,  $c^H(xy) \subseteq c^G(xy)$ . We say  $G$  *omits*  $H$  if  $H$  is not a sub- $r$ -graph of  $G$ . Define a *metric  $r$ -graph* to be an  $r$ -graph  $G = (V, c)$  which omits all violating triangles. We will say an  $r$ -graph *contains a violating triangle* if it contains a violating triangle as a sub- $r$ -graph. Given two finite  $r$ -graphs  $G$  and  $G'$  with  $V(G) = V(G')$ , set

$$\Delta(G, G') = \{xy \in E(G) : c^G(xy) \neq c^{G'}(xy)\}.$$

In this notation, given  $\delta > 0$ ,  $G$  and  $G'$  are  $\delta$ -close if  $|\Delta(G, G')| \leq \delta|V(G)|^2$ . Given a set of finite  $r$ -graphs  $S$  and a finite  $r$ -graph  $G$ , say that  $G$  is  $\delta$ -close to  $S$  if  $G$  is  $\delta$ -close to some element of  $S$ . Given  $r \geq 3$ , set

$$m(r) = \left\lceil \frac{r+1}{2} \right\rceil.$$

A subset  $A \subseteq [r]$  is called a *metric set* if  $A^3$  contains no violating triples. Note that when  $r$  is even,  $\lceil \frac{r}{2}, r \rceil$  is a metric subset of  $[r]$  of size  $m(r)$ . When  $r$  is odd, both  $\lceil \frac{r-1}{2}, r-1 \rceil$  and  $\lceil \frac{r+1}{2}, r \rceil$

are metric subsets of  $[r]$  of size  $m(r)$ . As remarked earlier, any  $r$ -graph meeting the requirements in the definition of  $C_r(n)$  is already in  $M_r(n)$ . In particular,  $C_r(n)$  contains all simple complete metric  $r$ -graphs with distances in  $[m(r), r]$ , therefore  $|C_r(n)| \geq m(r)^{\binom{n}{2}}$ . These observations yield the following fact we will use throughout the chapter.

REMARK 4.1.9. *Let  $n, r \geq 3$  be integers. Then*

$$|M_r(n)| \geq |C_r(n)| \geq m(r)^{\binom{n}{2}},$$

*and if  $r$  is even, then  $|C_r(n)| = m(r)^{\binom{n}{2}}$ .*

We now give an outline of the chapter. In Section 4.2 we introduce the notion of a labeled first-order 0-1 law, and prove as a consequence of Theorem 4.1.2 that Theorem 4.1.7 is true, i.e. when  $r \geq 4$  is an even integer,  $M_r$  has a labeled first-order 0-1 law in the language consisting of  $r$  binary predicates. In Section 4.3 we prove Corollary 4.1.6, which provides an asymptotic enumeration of  $M_r(n)$  as a consequence of Theorem 4.1.5. In Section 4.4 we provide preliminaries and notation regarding a multi-color version of Szemerédi's regularity lemma, then we prove Theorem 4.4.13, which is a stability result needed to prove Theorem 4.1.5. In Section 4.5 we prove Theorem 4.1.5, and in Section 4.6 we prove Theorem 4.1.2. Finally, in Section 4.7, we explain why Corollary 4.1.3 and Theorem 4.1.2 do not hold when  $r$  is odd, then discuss open questions concerning  $M_r(n)$  when  $r$  is odd.

## 4.2. Proof of logical 0-1 law

In this section we assume Theorem 4.1.2 and prove Theorem 4.1.7, which says that for even integers  $r \geq 4$ , the family  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  has a labeled first-order 0-1 law in the language  $\mathcal{L}_r$  consisting of  $r$  binary relation symbols. The outline of the argument is as follows. Theorem 4.1.2 allows us to reduce Theorem 4.1.7 to showing the existence of a labeled first-order 0-1 for the subfamily  $C_r = \bigcup_{n \in \mathbb{N}} C_r(n)$ . The existence of a labeled first-order 0-1 law for  $C_r$  follows from a standard argument. In particular, it follows from a theorem in [68] which generalizes the method in [51]. We assume familiarity with the theory of Fraïssé limits. For background on Fraïssé limits, see Chapter 7 of [59]. For background on 0-1 laws, see Chapter 3.

Fix an even integer  $r \geq 4$  for the rest of the section. Define  $\mathcal{L}_r = \{R_1(x, y), \dots, R_r(x, y)\}$ , where each  $R_i(x, y)$  is a binary relation symbol. Given an  $r$ -graph  $G$ , make  $G$  into an  $\mathcal{L}_r$ -structure by

interpreting for all  $(x, y) \in V(G)^2$ ,

$$R_i^G(x, y) \Leftrightarrow xy \in E(G) \text{ and } i \in c^G(xy).$$

From here on, all  $r$ -graphs will be considered as  $\mathcal{L}_r$ -structures in this way. We now prove that as a consequence of Theorem 4.1.2,  $M_r$  has a labeled first-order 0-1 law if and only if  $C_r$  does.

LEMMA 4.2.1. *For all  $\mathcal{L}_r$ -sentences  $\psi$ , if  $\mu^{C_r}(\psi)$  exists, then  $\mu^{M_r}(\psi)$  exists, and moreover,  $\mu^{C_r}(\psi) = \mu^{M_r}(\psi)$ .*

PROOF. Assume  $\mu^{C_r}(\psi)$  exists. For all  $n$ ,

$$(3) \quad \mu_n^{M_r}(\psi) = \frac{|\{G \in M_r(n) \setminus C_r(n) : G \models \psi\}|}{|M_r(n)|} + \frac{|\{G \in C_r(n) : G \models \psi\}|}{|M_r(n)|}.$$

By Theorem 4.1.2, there is  $\beta > 0$  such that for sufficiently large  $n$ ,

$$|M_r(n) \setminus C_r(n)| \leq 2^{-\beta n} |M_r(n)| \text{ and } |C_r(n)| \leq |M_r(n)| \leq (1 + 2^{-\beta n}) |C_r(n)|,$$

where the second inequality is because for all  $n$ ,  $C_r(n) \subseteq M_r(n)$ . Thus for sufficiently large  $n$ ,

$$\frac{|\{G \in C_r(n) : G \models \psi\}|}{|C_r(n)|(1 + 2^{-\beta n})} \leq \frac{|\{G \in C_r(n) : G \models \psi\}|}{|M_r(n)|} \leq \frac{|\{G \in C_r(n) : G \models \psi\}|}{|C_r(n)|}.$$

and

$$\frac{|\{G \in M_r(n) \setminus C_r(n) : G \models \psi\}|}{|M_r(n)|} \leq 2^{-\beta n}.$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{|\{G \in M_r(n) \setminus C_r(n) : G \models \psi\}|}{|M_r(n)|} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{|\{G \in C_r(n) : G \models \psi\}|}{|M_r(n)|} = \lim_{n \rightarrow \infty} \frac{|\{G \in C_r(n) : G \models \psi\}|}{|C_r(n)|} = \mu^{C_r}(\psi).$$

Combining these with (3) yields that  $\mu^{M_r}(\psi) = \mu^{C_r}(\psi)$ .  $\square$

Towards stating the definition of  $T_r$  from Theorem 4.1.7, we now fix some notation. Fix an integer  $k \geq 2$ . Given  $A \in M_r(k)$ , write  $x_1 \dots x_k \equiv A$  as short hand for the  $\mathcal{L}_r$ -formula which says that sending  $x_i \mapsto i$  makes  $x_1 \dots x_k$  isomorphic to  $A$ . Explicitly we mean the formula  $\psi(x_1, \dots, x_k)$  given by

$$\bigwedge_{1 \leq i < j \leq k} \left( R_{d^A(i,j)}(x_i, x_j) \wedge \bigwedge_{s \neq d^A(i,j)} \neg R_s(x_i, x_j) \right).$$

Given  $A \in M_r(k)$  and  $A' \in M_r(k+1)$ , write  $A \prec A'$  to denote that for all  $ij \in \binom{[k]}{2}$ ,  $d^A(ij) = d^{A'}(ij)$ .

Given such a pair  $A \prec A'$ , let  $\sigma_{A'/A}$  be the following sentence:

$$\forall x_1 \dots \forall x_k ((x_1 \dots x_k \equiv A) \rightarrow \exists y (x_1 \dots x_k y \equiv A')).$$

Sentences of this form are called *extension axioms*. Let  $T$  be a set of  $\mathcal{L}_r$ -sentences axiomatizing an infinite metric space with distances all in  $[\frac{r}{2}, r]$ ,

$$T' = \bigcup_{k \in \mathbb{N}} \{\sigma_{A'/A} : A \in C_r(k), A' \in C_r(k+1), A \prec A'\}, \text{ and}$$

$$T_r = T \cup T'.$$

Given an  $\mathcal{L}_r$ -structure  $M$ , let  $Th(M)$  denote the set of  $\mathcal{L}_r$ -sentences true in  $M$ . Given a set of  $\mathcal{L}_r$ -sentences  $\Gamma$ , we will write  $M \models \Gamma$  to denote that  $M \models \phi$  for all  $\phi \in \Gamma$ .

**Proof of Theorem 4.1.7.** Lemma 4.2.1 implies that if  $C_r$  has a labeled first-order 0-1 law then so does  $M_r$ , and moreover  $T_{as}^{C_r} = T_{as}^{M_r}$ . Therefore, it suffices to show  $C_r$  has a labeled first-order 0-1 law and to show  $T_r$  axiomatizes  $T_{as}^{C_r}$ . Let  $\mathcal{C}_r$  be the class of  $\mathcal{L}_r$ -structures obtained by closing  $C_r$  under isomorphism. Given  $M \in \mathcal{C}_r$ , we will write  $V(M)$  to denote the underlying set of  $M$ . That  $\mathcal{C}_r$  is a Fraïssé class is straightforward to see. For the sake of completeness we verify that  $\mathcal{C}_r$  has the amalgamation property. Given  $X, Y \in \mathcal{C}_r$ , an isometry  $f : X \rightarrow Y$  is an injective map from  $V(X)$  into  $V(Y)$  such that for all  $xy \in E(X)$ ,  $d^X(x, y) = d^Y(f(x), f(y))$ . Suppose  $A, B, C \in \mathcal{C}_r$  and  $f : C \rightarrow A$ ,  $g : C \rightarrow B$  are isometries. Without loss of generality, assume that  $f$  and  $g$  are inclusion maps and  $V(A) \cap V(B) = V(C)$ . To verify the amalgamation property, we want to find  $D \in \mathcal{C}_r$  and isometries  $h : A \rightarrow D$  and  $s : B \rightarrow D$  such that for all  $c \in V(C)$ ,  $s(c) = h(c)$ . We do this by setting  $V(D) = V(A) \cup V(B)$  and for  $xy \in \binom{V(D)}{2}$ , setting

$$(4) \quad d^D(x, y) = \begin{cases} d^A(x, y) & \text{if } xy \in E(A), \\ d^B(x, y) & \text{if } xy \in E(B) \setminus E(A), \\ r & \text{if } x \in (V(A) \setminus V(C)), y \in (V(B) \setminus V(C)). \end{cases}$$

$D$  is a simple complete  $r$ -graph with  $d^D(x, y) \in [\frac{r}{2}, r]$  for all  $xy \in E(D)$ , so  $D \in \mathcal{C}_r$ . Define  $h : A \rightarrow D$  and  $s : B \rightarrow D$  to be the inclusion maps. Then for all  $c \in V(C)$ ,  $h(c) = s(c) = c$ , as desired, and  $\mathcal{C}_r$  has the amalgamation property. Note that we could have chosen any color in  $[\frac{r}{2}, r]$  to assign the

edges in the third case of (62), as there are no forbidden configurations in  $\mathcal{C}_r$ . We leave the rest of the verification that  $\mathcal{C}_r$  is a Fraïssé class to the reader.

Let  $FL(\mathcal{C}_r)$  be the Fraïssé limit of  $\mathcal{C}_r$  and make  $FL(\mathcal{C}_r)$  into an  $\mathcal{L}_r$ -structure by interpreting, for each  $(x, y) \in FL(\mathcal{C}_r)^2$ ,  $R_i(x, y)$  if and only if  $d^{C_r}(x, y) = i$ . It is a standard exercise to see that  $FL(\mathcal{C}_r) \models T_r$  and further that  $T_r$  axiomatizes  $\text{Th}(FL(\mathcal{C}_r))$ . Therefore  $T_r$  is a complete, consistent  $\mathcal{L}_r$ -theory, so to show  $\mathcal{C}_r$  has a labeled first-order 0-1 law, it suffices to show that for each  $\psi \in T_r$ ,  $\mu^{C_r}(\psi) = 1$ . For  $\psi \in T$ , this is obvious. Because there are no forbidden configurations in  $\mathcal{C}_r$ , a straightforward counting argument shows that for  $\psi \in T'$ ,  $\mu^{C_r}(\neg\psi) = 0$ , and therefore  $\mu^{C_r}(\psi) = 1$ . An example of such an argument applied to graphs is the proof of Lemma 2.4.3 of [77]. The proof in our case is only slightly more complicated, so we omit it. We also point out that this fact (that for all  $\psi \in T'$ ,  $\mu^{C_r}(\psi) = 1$ ) follows directly from a much more general result, Theorem 3.15 of [68]. Because this theorem is much more powerful than what our example requires, we leave it to the interested reader to verify it applies to  $\mathcal{C}_r$  and  $\psi \in T'$ .  $\square$

We end this section by showing that while there is a Fraïssé limit naturally associated to  $M_r$ , its theory is very different from the almost sure theory we obtain from  $M_r$ . Let  $\mathcal{M}_r$  be the class of finite metric spaces obtained by closing  $M_r$  under isomorphism, that is,  $\mathcal{M}_r$  is the class of all finite metric spaces with distances all in  $[r]$ . It is well known that  $\mathcal{M}_r$  is a Fraïssé class. For instance, this is a simple case of general results contained in [43], which tell us when, given  $S \subseteq \mathbb{R}$ , the class of finite metric spaces with distances all in  $S$  forms a Fraïssé class. For completeness we verify the amalgamation property for our case, that is, when  $S = [r]$ .

Suppose  $A, B, C \in \mathcal{M}_r$  and  $f : C \rightarrow A, g : C \rightarrow B$  are isometries. Without loss of generality, assume that  $f$  and  $g$  are inclusion maps and  $V(A) \cap V(B) = V(C)$ . To verify the amalgamation property, we want to find  $D \in \mathcal{M}_r$  and isometries  $h : A \rightarrow D$  and  $s : B \rightarrow D$  such that for all  $c \in V(C)$ ,  $s(c) = h(c)$ . Given  $s, t \in [r]$ , let  $t \dot{+} s = \min\{r, t + s\}$ . Set  $V(D) = V(A) \cup V(B)$  and for  $xy \in \binom{V(D)}{2}$ , set

$$(5) \quad d^D(x, y) = \begin{cases} d^A(x, y) & \text{if } xy \in E(A), \\ d^B(x, y) & \text{if } xy \in E(B) \setminus E(A), \\ \max\{d^A(x, c) \dot{+} d^B(c, y) : c \in V(C)\} & \text{if } x \in (V(A) \setminus V(C)), y \in (V(B) \setminus V(C)). \end{cases}$$

We leave it to the reader to verify that the assigned distances do not violate the triangle inequality, and therefore, that  $D$  is in  $\mathcal{M}_r$ . Define  $h : A \rightarrow D$  and  $s : B \rightarrow D$  to be the inclusion maps. Then

for all  $c \in V(C)$ ,  $h(c) = s(c) = c$ , as desired, and  $\mathcal{M}_r$  has the amalgamation property. Note that unlike in the proof of the amalgamation property for  $\mathcal{C}_r$ , the distance in the third line of (5) must be chosen carefully, as there are many forbidden configurations in  $\mathcal{M}_r$ .

Let  $FL(\mathcal{M}_r)$  be the Fraïssé limit of  $\mathcal{M}_r$ . It is a standard exercise that the theory of  $FL(\mathcal{M}_r)$  is axiomatized by the axioms for an infinite metric space with distances all in  $[r]$  and the collection of all extension axioms of the form  $\sigma_{A'/A}$  for some  $A \in M_r(k)$ ,  $A' \in M_r(k+1)$  with  $A \prec A'$ , and  $k \geq 0$ . We can see now that  $\text{Th}(FL(\mathcal{M}_r))$  and  $\text{Th}(FL(\mathcal{C}_r))$  are different. For instance, let  $\psi$  be the sentence

$$\exists x \exists y R_1(x, y).$$

Then  $\psi \in \text{Th}(FL(\mathcal{M}_r))$ , while clearly  $\text{Th}(FL(\mathcal{C}_r)) \models \neg\psi$ . Model theoretically,  $\text{Th}(FL(\mathcal{C}_r))$  is simple (in the sense of Definition 7.2.1 in [100]). This can be seen by adapting the argument used to prove the theory of the random graph is simple, as  $\mathcal{C}_r$  is just an edge-colored version of the random graph (see Corollary 7.3.14 in [100] for a proof that the theory of the random graph is simple). On the other hand, a straightforward adjustment of the construction in Theorem 5.5(b) of [42] shows that  $\text{Th}(FL(\mathcal{M}_r))$  has the *r-strong order property* ( $\text{SOP}_r$ ), a measure of the complexity of a first-order theory defined in [95]. It is shown in [95] that for all  $n \geq 3$ , a theory with  $\text{SOP}_n$  is not simple. In sum, when  $r \geq 4$  is even, we have a family of labeled finite structures,  $M_r$ , associated to two theories which differ in model theoretic complexity:

- $\text{Th}(FL(\mathcal{M}_r))$  where  $\mathcal{M}_r$  is obtained by closing  $M_r$  under isomorphism. This theory has  $\text{SOP}_r$  (and therefore is not simple).
- $T_{as}^{M_r} = T_{as}^{C_r} = \text{Th}(FL(\mathcal{C}))$ , where  $C_r \subseteq M_r$  is a special subfamily, and  $\mathcal{C}_r$  is obtained by closing  $C_r$  under isomorphism. This theory is simple.

### 4.3. Asymptotic Enumeration

In this section we assume Theorem 4.1.5 and prove Corollary 4.1.6, which asymptotically enumerates  $M_r(n)$  for all  $r \geq 3$ . Recall that for all integers  $r \geq 3$ ,  $m(r) = \lceil \frac{r+1}{2} \rceil$ .

**Proof of Corollary 4.1.6.** Fix an integer  $r \geq 3$ . All logs will be base  $m(r)$  unless otherwise stated. Remark 4.1.9 implies that  $|M_r(n)| \geq m(r)^{\binom{n}{2}}$ , so it suffices to show that for all  $0 < \gamma < 1$ , there is  $M$  such that  $n > M$  implies  $|M_r(n)| < m(r)^{\binom{n}{2} + \gamma n^2}$ .

Fix  $0 < \gamma < 1$ . Let  $H(x) = -x \log_2 x - (1-x) \log_2(1-x)$ . Recall that  $\binom{n}{xn} \leq 2^{H(x)n}$  for all  $n \in \mathbb{N}$  and  $0 < x \leq \frac{1}{2}$  (see for instance page 427 of [52]). Basic calculus shows  $H(x) \rightarrow 0$  as  $x \rightarrow 0$ . Choose  $\delta > 0$  small enough so that

$$(H(\delta) + \delta) \log 2 + \delta \log r < \frac{\gamma}{4}.$$

Theorem 4.1.5 implies there exists a  $\beta = \beta(\delta) > 0$  and  $M_1 = M_1(\delta)$  such that  $n > M_1$  implies

$$|M_r(n) \setminus C_r^\delta(n)| \leq 2^{-\beta n^2} m(r) \binom{n}{2}.$$

Choose  $M > M_1$  large enough so that  $n > M$  implies  $\frac{\gamma}{4}n^2 + n \log n < \frac{\gamma}{2}n^2$  and  $\frac{\gamma}{2}n^2 + \log 2 \leq \gamma n^2$ .

We now assume  $n > M$  and bound the size of  $C_r^\delta(n)$ . All elements  $G \in C_r^\delta(n)$  can be constructed as follows:

- Choose an element of  $G' \in C_r(n)$ . There are  $|C_r(n)|$  ways to do this. If  $r$  is even, then  $|C_r(n)| = m(r) \binom{n}{2}$ . If  $r$  is odd, we must find an upper bound for  $|C_r(n)|$ . When  $r$  is odd, we can construct any element of  $C_r(n)$  by first choosing a partition of  $[n]$ , then assigning a color to each edge in a way compatible with the partition. There are at most  $n^n m(r) \binom{n}{2}$  ways to do this.
- Choose at most  $\delta n^2$  edges to be in  $\Delta(G, G')$ . There are at most  $\binom{n^2}{\delta n^2} 2^{\delta n^2} \leq 2^{(H(\delta) + \delta)n^2}$  ways to do this.
- Assign a color to each edge in  $\Delta(G, G')$ . There are at most  $r^{\delta n^2}$  ways to do this.

Thus

$$|C_r^\delta(n)| \leq n^n m(r) \binom{n}{2} 2^{(H(\delta) + \delta)n^2} r^{\delta n^2} = m(r) \binom{n}{2} + n^2 ((H(\delta) + \delta) \log 2 + \delta \log r) + n \log n.$$

By our assumptions on  $\delta$  and  $M$ , this is at most  $m(r) \binom{n}{2} + \frac{\gamma}{4}n^2 + n \log n < m(r) \binom{n}{2} + \frac{\gamma}{2}n^2$ . Therefore, since  $M_r(n) = (M_r(n) \setminus C_r^\delta(n)) \cup C_r^\delta(n)$  we have

$$|M_r(n)| \leq m(r) \binom{n}{2} - n^2 \beta \log 2 + m(r) \binom{n}{2} + \frac{\gamma}{2}n^2 \leq 2m(r) \binom{n}{2} + \frac{\gamma}{2}n^2 = m(r) \binom{n}{2} + \frac{\gamma}{2}n^2 + \log 2 \leq m(r) \binom{n}{2} + \gamma n^2,$$

where the last inequality is by the choice of  $M$ . □

#### 4.4. Stability Theorem

In this section we prove a stability theorem which implies that for all integers  $r \geq 3$ , for large enough  $n$ , if  $G \in M_r(n)$  has close to the maximal number of different distances occurring between its vertices, then it is structurally close to an element of  $C_r(n)$ . This is a crucial step in the proofs of Theorems 4.1.2 and 4.1.5. Before proceeding further, we require some definitions and notation.



**4.4.1. Regularity Lemmas and Preliminaries.** In this section we state a version of Szemerédi's Regularity Lemma which applies to  $r$ -graphs. We will also prove easy consequences of this for our situation.

DEFINITION 4.4.1. Let  $r \geq 3$  be an integer. Fix a finite  $r$ -graph  $G$  and disjoint subsets  $X, Y \subseteq V(G)$ .

- (1) We call a partition  $\mathcal{A} = \{A_1, \dots, A_m\}$  of  $V(G)$  an *equipartition of order  $m$*  if  $||A_i| - |A_j|| \leq 1$  for all  $i \neq j$ . A *refinement* of  $\mathcal{A}$  is a partition  $\mathcal{B} = \{B_1, \dots, B_k\}$  such that for each  $i \in [k]$ , there is  $j \in [m]$  such that  $B_i \subseteq A_j$ .
- (2) For  $l \in [r]$ , set

$$e_l^G(X, Y) := |\{xy \in E(X, Y) : l \in c^G(xy)\}| \text{ and}$$

$$\rho_l^G(X, Y) := \frac{e_l(X, Y)}{|X||Y|}.$$

- (3) The *density vector* of  $(X, Y)$  in  $G$  is  $(\rho_1^G, \dots, \rho_r^G)$  where  $\rho_i^G = \rho_i^G(X, Y)$ .
- (4)  $(X, Y)$  is  $\epsilon$ -regular for  $G$  if for all  $X' \subseteq X$  and  $Y' \subseteq Y$  with  $|X'| \geq \epsilon|X|$  and  $|Y'| \geq \epsilon|Y|$ , for all  $l \in [r]$ ,

$$|\rho_l^G(X, Y) - \rho_l^G(X', Y')| \leq \epsilon.$$

- (5) A partition  $\mathcal{B} = \{B_1, \dots, B_k\}$  of  $V(G)$  is called  $\epsilon$ -regular for  $G$  if it is an equipartition of  $V(G)$ , and for all but at most  $\epsilon k^2$  of the pairs  $ij \in \binom{[k]}{2}$ ,  $(B_i, B_j)$  is  $\epsilon$ -regular for  $G$ .

We now state the multi-color version of the Szemerédi Regularity Lemma and one of its corollaries we will use in this chapter. Both results appear in [7].

THEOREM 4.4.2. (*Regularity Lemma*) Fix an integer  $r \geq 2$ . For every  $\epsilon > 0$  and positive integer  $m$ , there is an integer  $CM = CM(m, \epsilon)$  such that if  $G$  is a finite  $r$ -graph with at least  $CM$  vertices, and  $\mathcal{A}$  is an equipartition of  $G$  of order  $m$ , then there  $k$  such that  $m \leq k \leq CM$  and a refinement  $\mathcal{B}$  of  $\mathcal{A}$  of order  $k$  which is  $\epsilon$ -regular for  $G$ .

THEOREM 4.4.3. (*Embedding Lemma*) Fix an integer  $r \geq 2$ . For every  $0 < d < 1$  and  $k \in \mathbb{N} \setminus \{0\}$ , there is a positive  $\gamma = \gamma_{el}(d, k) \leq d$  and  $\delta = \delta_{el}(d, k)$  such that the following holds. Suppose that  $H$  and  $G$  are  $r$ -graphs and  $V(H) = \{v_1, \dots, v_k\}$ . Suppose  $V_1, \dots, V_k$  are pairwise disjoint subsets of  $V(G)$  such that for every  $ij \in \binom{[k]}{2}$ ,  $(V_i, V_j)$  is  $\gamma$ -regular for  $G$ , and for each  $l \in [r]$ ,  $l \in c^H(v_i v_j)$  implies  $\rho_l^G(V_i, V_j) \geq d$ . Then there are at least  $\delta \prod_{i=1}^k |V_i|$   $k$ -tuples  $(w_1, \dots, w_k) \in V_1 \times \dots \times V_k$  such that for each  $ij \in \binom{[k]}{2}$ ,  $c^H(v_i v_j) \subseteq c^G(w_i w_j)$ .

We will apply these theorems to what are called *reduced  $r$ -graphs*, which we define below. Recall that a metric  $r$ -graph is an  $r$ -graph with no violating triangles.

DEFINITION 4.4.4. Let  $r \geq 2$  be an integer,  $G$  a finite  $r$ -graph, and  $0 < \eta \leq d \leq 1$ .

- (1) Suppose  $\mathcal{P} = \{V_1, \dots, V_t\}$  is an  $\eta$ -regular partition for  $G$ . Let  $R(G, \mathcal{P}, d)$  be the  $r$ -graph  $R$  with vertex set  $[t]$  such that  $s \in c^R(ij)$  if and only if  $(V_i, V_j)$  is  $\eta$ -regular for  $G$  and  $\rho_s(V_i, V_j) \geq d$ . We say  $R$  is a *reduced  $r$ -graph* obtained from  $G$  with parameters  $\eta$  and  $d$ .
- (2) Let  $\tilde{M}_r(t)$  be the set of metric  $r$ -graphs on  $[t]$  and set

$Q_{\eta,d,t}(G) = \{R(G, \mathcal{P}, d) : \mathcal{P} \text{ is an } \eta\text{-regular equipartition for } G \text{ and } \mathcal{P} \text{ has order } t\}$ , and

$$Q_{\eta,d}(G) = \bigcup_{t=\frac{1}{\eta}}^{CM(\frac{1}{\eta}, \eta)} Q_{\eta,d,t}(G).$$

We emphasize that the difference between  $\tilde{M}_r(t)$  and  $M_r(t)$  is that  $r$ -graphs in  $\tilde{M}_r(t)$  need not be simple and need not be complete. The following two lemmas will be needed.

LEMMA 4.4.5. Let  $r \geq 2$  be an integer,  $0 < d < 1$ ,  $0 < \gamma \leq \gamma_{el}(d, 3)$ , and  $\delta \leq \delta_{el}(d, 3)$ . Let  $(i, j, k) \in [r]^3$  be a violating triple. Suppose  $G \in M_r(n)$  and  $V_1, V_2, V_3 \subseteq V(G)$  are pairwise disjoint and pairwise  $\gamma$ -regular for  $G$  with  $\delta|V_1||V_2||V_3| \geq 1$ . If  $\{X, Y, Z\} = \{V_1, V_2, V_3\}$ , then

$$(6) \quad \min\{\rho_i^G(X, Y), \rho_j^G(Y, Z), \rho_k^G(X, Z)\} < d.$$

PROOF. Suppose for contradiction that  $\{X, Y, Z\} = \{V_1, V_2, V_3\}$  and (6) fails. By Theorem 4.4.3 there exists at least  $\delta|V_1||V_2||V_3| \geq 1$  tuples  $(x, y, z) \in X \times Y \times Z$  such that  $i \in c^G(xy)$ ,  $j \in c^G(yz)$  and  $k \in c^G(xz)$ . But now  $\{x, y, z\}$  is a violating triangle in  $G$ , a contradiction.  $\square$

LEMMA 4.4.6. Let  $0 < d < 1$  and  $0 < \eta \leq \gamma_{el}(d, 3)$ . There is an  $M$  such that  $n > M$  implies that for all  $G \in M_r(n)$ ,  $\emptyset \neq Q_{\eta,d}(G) \subseteq \bigcup_{t=\frac{1}{\eta}}^{CM(\frac{1}{\eta}, \eta)} \tilde{M}_r(t)$ . In other words, any reduced  $r$ -graph obtained from  $G$  with parameters  $d$  and  $\eta$  omits all violating triangles.

PROOF. Let  $M = \frac{2CM(\frac{1}{\eta}, \eta)}{\delta_{el}(d, 3)^{\frac{3}{2}}}$ . Suppose  $n > M$  and  $G \in M_r(n)$ . As  $n > CM(\frac{1}{\eta}, \eta)$ , there is  $t$  with  $\frac{1}{\eta} \leq t \leq CM(\frac{1}{\eta}, \eta)$  and  $\mathcal{P} = \{V_1, \dots, V_t\}$  an  $\eta$ -regular partition for  $G$ . Therefore  $Q_{\eta,d,t}(G) \neq \emptyset$ , so  $Q_{\eta,d}(G) \neq \emptyset$ . Let  $R = R(G, \mathcal{P}, d) \in Q_{\eta,d,t}(G)$ . We will show that  $R \in \tilde{M}_r(t)$ . Note that for all  $V_i, V_j, V_k \in \mathcal{P}$ ,

$$\delta_{el}(d, 3)|V_i||V_j||V_k| \geq \delta_{el}(d, 3) \left(\frac{n}{t} - 1\right)^3 > \delta_{el}(d, 3) \left(\frac{n}{2t}\right)^3 \geq \delta_{el}(d, 3) \frac{n^3}{8CM(\frac{1}{\eta}, \eta)^3} > 1,$$

by assumption on  $M$ . Thus by Lemma 4.4.5,  $R$  contains no violating triangle, so  $R \in \tilde{M}_r(t)$ .  $\square$

We spend the rest of this section stating various definitions and facts we will need for our proofs.

We will work with the following subset  $\tilde{C}_r(n) \subseteq \tilde{M}_r(n)$  which is an analogue of  $C_r(n) \subseteq M_r(n)$ .

DEFINITION 4.4.7. Let  $r \geq 3$  be an integer. Set  $\tilde{C}_r(t)$  to be the set of complete  $r$ -graphs  $R$  with  $V(R) = [t]$  such that

- (i) if  $r$  is even, then for all  $xy \in E(R)$ ,  $c^R(xy) = \lfloor \frac{r}{2}, r \rfloor$ .
- (ii) if  $r$  is odd, then there is a partition  $[t] = V_1 \cup \dots \cup V_s$  such that for all  $xy \in \binom{[t]}{2}$ ,

$$c^R(xy) = \begin{cases} \lfloor \frac{r-1}{2}, r-1 \rfloor & \text{if } xy \in \binom{V_i}{2} \text{ for some } i \in [s] \\ \lfloor \frac{r+1}{2}, r \rfloor & \text{if } xy \in E(V_i, V_j) \text{ for some } i \neq j \in [s]. \end{cases}$$

Note that elements of  $\tilde{C}_r(t)$  contain no violating triangles, so  $\tilde{C}_r(t) \subseteq \tilde{M}_r(t)$ . Further, note that when  $r$  is even,  $\tilde{C}_r(t)$  consists of a single  $r$ -graph. The following weight function defined on metric  $r$ -graphs is crucial to our proof.

DEFINITION 4.4.8. Let  $t \geq 2$  and  $r \geq 3$  be integers and let  $R \in \tilde{M}_r(t)$ . For  $ij \in \binom{[t]}{2}$ , set

$$f^R(i, j) = \max\{|c^R(ij)|, 1\} \quad \text{and} \quad W(R) = \prod_{ij \in \binom{[t]}{2}} f^R(i, j).$$

Note that for integers  $r \geq 3$  and  $t \geq 2$ , any  $r$ -graph  $R$  with  $t$  vertices has  $W(R) \leq r^{\binom{t}{2}}$ . Recall that when  $r$  is even  $m(r) = \lfloor \frac{r}{2}, r \rfloor$  and when  $r$  is odd,  $m(r) = \lfloor \frac{r-1}{2}, r-1 \rfloor = \lfloor \frac{r+1}{2}, r \rfloor$ , so for any integers  $r \geq 3$  and  $t \geq 2$ , for all  $R \in \tilde{C}_r(t)$  and  $ij \in \binom{[t]}{2}$ ,  $f^R(i, j) = m(r)$ , and thus  $W(R) = m(r)^{\binom{t}{2}}$ .

We now state a lemma which restricts how many colors we can assign to the edges of a triangle  $\{i, j, k\}$  in an  $r$ -graph without creating a violating triangle. The proof of this lemma is elementary but somewhat tedious, and for this reason is relegated to the Appendix.

LEMMA 4.4.9. Fix an integer  $r \geq 3$ . Let  $A$ ,  $B$ , and  $C$  be nonempty subsets of  $[r]$  such that  $|A| \geq |B| \geq |C|$ ,  $|A| > m(r)$ , and  $|B| \geq m(r)$ . Set  $x = |A| - m(r)$  and  $y = |B| - m(r)$ , and suppose

$$|C| \geq \begin{cases} \max\{m(r) - x - y, 1\} & \text{if } r \text{ is even} \\ \max\{m(r) - x - y + 2, 1\} & \text{if } r \text{ is odd.} \end{cases}$$

Then there is a violating triple  $(a, b, c) \in A \times B \times C$ .

A straightforward consequence of this is that  $m(r)$  is the largest size of a metric subset of  $[r]$ . Another important consequence is the following.

**COROLLARY 4.4.10.** *Let  $r, t \geq 3$  be integers and let  $R \in \tilde{M}_r(t)$ . Suppose  $uv, vw, uw \in E(R)$ , and  $f^R(u, v) \geq f^R(v, w) > m(r)$ . Then  $f^R(u, w) < m(r)$  and  $\max\{f^R(u, v)f^R(u, w), f^R(v, w)f^R(u, w)\} \leq m(r)^2 - 1$ .*

**PROOF.** For  $xy \in \binom{[t]}{2}$ , set  $f(x, y) = f^R(x, y)$ . Given  $A, B, C \subseteq [r]$  and  $x, y \in [r]$ , write  $P(A, B, C, x, y)$  if  $A, B, C, x, y$  satisfy the hypotheses of Lemma 4.4.9, that is, if the following hold.

- $|A| \geq |B| \geq |C|$ ,
- $x = |A| - m(r)$  and  $y = |B| - m(r)$ ,
- $|A| > m(r)$  and  $|B| \geq m(r)$ .

Set  $A = c^R(uv)$ ,  $B = c^R(vw)$ ,  $C = c^R(uw)$ ,  $x = |A| - m(r)$ , and  $y = |B| - m(r)$ . We show  $P(A, B, C, x, y)$  holds. By definition of  $x$  and  $y$ , the second bullet holds. Since  $|A| = f(u, v)$ ,  $|B| = f(v, w)$ , and  $|C| = f(u, w)$ , our assumptions imply  $|A| \geq |B| > m(r)$ , so the last bullet holds. We now show the first bullet holds, that is, that  $|A| \geq |B| \geq |C|$ . Suppose for a contradiction that  $|C| > |B|$ . Let  $z = |C| - m(r)$  and note our assumptions imply that either  $P(A, C, B, x, z)$  or  $P(C, A, B, z, x)$  holds. In either case,  $|B| > m(r) \geq m(r) - x - z + 2$  implies by Lemma 4.4.9 that there is a violating triple  $(a, b, c) \in A \times B \times C$ . Now  $\{u, v, w\}$  is a violating triangle in  $R$ , a contradiction. Thus  $|A| \geq |B| \geq |C|$ .

Consequently,  $P(A, B, C, x, y)$  holds, so if  $|C| \geq m(r) - x - y + 2$  were true, Lemma 4.4.9 would imply that there is a violating triple  $(a, b, c) \in A \times B \times C$ , making  $\{u, v, w\}$  a violating triangle in  $R$ , a contradiction. Therefore, we must have  $|C| < m(r) - x - y + 2$ . Our assumptions imply that  $x, y \geq 1$ , so in fact,  $|C| < m(r)$ . Further, we have shown that

$$|B||C| = f(v, w)f(u, w) \leq (m(r)+y)(m(r)-x-y+1) \leq (m(r)+y)(m(r)-y) = m(r)^2 - y^2 \leq m(r)^2 - 1,$$

and

$$|A||C| = f(u, v)f(u, w) \leq (m(r)+x)(m(r)-x-y+1) \leq (m(r)+x)(m(r)-x) = m(r)^2 - x^2 \leq m(r)^2 - 1,$$

as desired. □

**4.4.2. Two Lemmas.** In this section, we prove two lemmas toward our stability result. The first lemma bounds the size of  $W(R)$  for  $R \in \tilde{M}_r(t)$ . We will frequently use the following inequality

which holds for all integers  $r \geq 3$ :

$$(7) \quad m(r)^2 - 1 \geq r.$$

LEMMA 4.4.11. *Let  $t, r \geq 3$  be integers and  $R \in \tilde{M}_r(t)$ . Let  $a_R = |\{ij \in E(R) : f^R(i, j) > m(r)\}|$ .*

*Then*

$$W(R) \leq m(r)^{\binom{t}{2} + t + 5} \left( \frac{m(r)^2 - 1}{m(r)^2} \right)^{a_R}.$$

PROOF. Fix an integer  $r \geq 3$ . Given an integer  $t$  and  $R \in \tilde{M}_r(t)$ , set  $g(R) = m(r)^{\binom{t}{2} + t + 5} \left( \frac{m(r)^2 - 1}{m(r)^2} \right)^{a_R}$ .

We proceed by induction on  $t$ . Assume  $t = 3$  and fix  $R \in \tilde{M}_r(t)$ . In this case  $a_R \leq 3$ , so  $g(R) \geq m(r)^5(m(r)^2 - 1)^3$ . It is straightforward to verify that  $r^3 \leq m(r)^5$ , as  $r \geq 3$ . Therefore,

$$W(R) \leq r^3 \leq m(r)^5(m(r)^2 - 1)^3 \leq g(R).$$

Assume now that  $t > 3$  and the claim holds for all  $t'$  with  $3 \leq t' < t$ . Fix  $R \in \tilde{M}_r(t)$ , set  $a = a_R$ , and for  $xy \in \binom{[t]}{2}$ , set  $f(x, y) = f^R(x, y)$ . If  $a = 0$  then  $W(R) \leq m(r)^{\binom{t}{2}} \leq g(R)$  trivially. So assume  $a > 0$ .

Choose  $uv \in E(R)$  such that  $f(u, v)$  is maximum, and note that  $a > 0$  implies  $f(u, v) > m(r)$ . Define  $R'$  to be the  $r$ -graph with  $V(R') = [t] \setminus \{u, v\}$  and for each  $xy \in E(R')$ ,  $c^{R'} = c^R|_{V(R')}$ . Let  $a' = a_{R'}$ ,

$$Y = \{z \in V(R') : \max\{f(u, z), f(v, z)\} > m(r)\},$$

and set  $s = |Y|$ . For all  $z \in Y$ , because  $\max\{f(u, z), f(v, z)\} > m(r)$  and  $f(u, v) > m(r)$ , Corollary 4.4.10 implies  $\min\{f(u, z), f(v, z)\} < m(r)$  and  $f(u, z)f(v, z) \leq m(r)^2 - 1$ . By the definition of  $Y$ , for all  $z \in V(R') \setminus Y$ ,  $\max\{f(u, z), f(v, z)\} \leq m(r)$ , so  $f(u, z)f(v, z) \leq m(r)^2$ . Combining these facts we have

$$\begin{aligned} W(R) &= W(R')f(u, v) \left( \prod_{z \in Y} f(u, z)f(z, v) \right) \left( \prod_{z \notin Y} f(u, z)f(z, v) \right) \\ &\leq W(R')f(u, v)(m(r)^2 - 1)^s m(r)^{2(t-2-s)} \leq W(R')r(m(r)^2 - 1)^s m(r)^{2(t-2-s)}. \end{aligned}$$

Using (7), we can upper bound this by

$$W(R')(m(r)^2 - 1)^{s+1} m(r)^{2(t-2-s)} = W(R') \left( \frac{m(r)^2 - 1}{m(r)^2} \right)^{s+1} m(r)^{2t-2}.$$

By the induction hypothesis, this is at most

$$m(r)^{\binom{t-2}{2}+t-2+5} \left( \frac{m(r)^2-1}{m(r)^2} \right)^{a'} \left( \frac{m(r)^2-1}{m(r)^2} \right)^{s+1} m(r)^{2t-2} = m(r)^{\binom{t}{2}+t+4} \left( \frac{m(r)^2-1}{m(r)^2} \right)^{a'+s+1}.$$

Note that  $a = a' + |\{zu : z \in Y \text{ and } f(u, z) > m(r)\} \cup \{vz : z \in Y \text{ and } f(v, z) > m(r)\} \cup \{uv\}|$ .

Because for each  $z \in Y$  exactly one of  $f(u, z)$  or  $f(v, z)$  is strictly greater than  $m(r)$ , this shows

$a = a' + s + 1$ . Therefore,

$$W(R) \leq m(r)^{\binom{t}{2}+t+4} \left( \frac{m(r)^2-1}{m(r)^2} \right)^a < g(R).$$

This completes the proof.  $\square$

We now fix some notation. Suppose  $r \geq 3$  is an integer,  $0 < \epsilon < 1$ ,  $R$  is an  $r$ -graph,  $u \in V(R)$ , and  $t = |V(R)|$ . For  $i \in [r]$ , set

$$\Gamma_i^R(u) = \{v \in V(R) : f^R(u, v) = i\},$$

$$\mu_i^R(u) = |\Gamma_i^R(u)|, \text{ and}$$

$$V_0^R(\epsilon) = \{v \in V(R) : \mu_{m(r)}^R(v) < (1 - \sqrt{\epsilon})(t - 1)\}.$$

We now prove the second lemma.

LEMMA 4.4.12. *For every integer  $r \geq 3$  there are  $C_1, C_2, C_3$ , depending only on  $r$  such that for every  $0 < \epsilon < 1$ , there is  $M$  such that if  $t > M$  the following holds. Suppose  $R \in \tilde{M}_r(t)$  with  $W(R) > m(r)^{(1-\epsilon)\binom{t}{2}}$ . Let  $a_R = |\{ij \in E(R) : f^R(i, j) > m(r)\}|$  and  $b_R = |\{ij \in E(R) : f^R(i, j) < m(r)\}|$ .*

*Then*

$$(1) \ a_R \leq C_1 \epsilon t^2,$$

$$(2) \ b_R \leq C_2 \epsilon t^2, \text{ and}$$

$$(3) \ |V_0^R(\epsilon)| \leq \sqrt{\epsilon} C_3 t.$$

PROOF. Let  $r, t \geq 3$  be integers. Fix  $\epsilon > 0$  and suppose  $R \in \tilde{M}_r(t)$  is such that  $W(R) > m(r)^{(1-\epsilon)\binom{t}{2}}$ .

Set  $a = a_R$  and  $b = b_R$ . All logs in this proof are base  $m(r)$ . Our assumptions and Lemma 4.4.11

imply  $m(r)^{(1-\epsilon)\binom{t}{2}} < W(R) \leq m(r)^{\binom{t}{2}+t+5} \left( \frac{m(r)^2-1}{m(r)^2} \right)^a$ . Consequently,

$$(8) \quad \left( \frac{m(r)^2}{m(r)^2-1} \right)^a < m(r)^{\epsilon \binom{t}{2} + t + 5}.$$

Suppose  $M_1$  is large enough so that  $t > M_1$  implies  $t(1 - \frac{\epsilon}{2}) + 5 < \frac{\epsilon t^2}{4}$ , and assume  $t > M_1$ . Taking log of both sides of (8) we obtain

$$a \log \left( \frac{m(r)^2}{m(r)^2 - 1} \right) \leq \epsilon \binom{t}{2} + t + 5 < \frac{\epsilon}{2} t^2 + \frac{\epsilon}{4} t^2 = \frac{3\epsilon t^2}{4},$$

where the last inequality is by assumption on  $M_1$ . Therefore  $a \leq C_1 \epsilon t^2$ , for appropriate choice of  $C_1 = C_1(r)$ . This proves (1). For (2), note that by the definitions of  $W(R)$ ,  $a$ , and  $b$  we have

$$W(R) \leq (m(r) - 1)^b r^a m(r)^{\binom{t}{2} - a - b}.$$

Thus our assumptions and part (1) imply that,

$$m(r)^{(1-\epsilon)\binom{t}{2}} < (m(r)-1)^b r^{C_1 \epsilon t^2} m(r)^{\binom{t}{2} - a - b} \leq (m(r)-1)^b r^{C_1 \epsilon t^2} m(r)^{\binom{t}{2} - b} = \left( \frac{m(r) - 1}{m(r)} \right)^b r^{C_1 \epsilon t^2} m(r)^{\binom{t}{2}}.$$

Consequently,

$$\left( \frac{m(r)}{m(r) - 1} \right)^b < m(r)^{\epsilon \binom{t}{2}} r^{C_1 \epsilon t^2}.$$

Taking log of both sides, we obtain

$$b \log \left( \frac{m(r)}{m(r) - 1} \right) < \epsilon \binom{t}{2} + C_1 \epsilon t^2 \log r < \left( \frac{1}{2} + C_1 \log r \right) \epsilon t^2,$$

from which (2) follows directly for an appropriate choice of  $C_2 = C_2(r)$ . For (3), parts (1) and (2) yield

$$|\{ij \in E(R) : f(i, j) = m(r)\}| = \binom{t}{2} - a - b \geq \binom{t}{2} - (C_1 + C_2) \epsilon t^2 = \left( \frac{1}{2} - (C_1 + C_2) \epsilon \right) t^2 - \frac{t}{2}.$$

Setting  $m = |V_0^R(\epsilon)| = |\{u \in V(R) : \mu_{m(r)}^R(u) < (1 - \sqrt{\epsilon})(t - 1)\}|$ , it is clear that

$$\sum_{v \in V(R)} \mu_{m(r)}^R(v) \leq m(1 - \sqrt{\epsilon})(t - 1) + (t - m)(t - 1) = t^2 - t - \sqrt{\epsilon} m t + \sqrt{\epsilon} m.$$

On the other hand, let  $\mathcal{G}$  be the graph with vertex set  $\mathcal{V} = [t]$  and edge set  $\mathcal{E} = \{ij \in \binom{\mathcal{V}}{2} : f^R(ij) = m(r)\}$ . Then

$$\sum_{v \in V(R)} \mu_{m(r)}^R(v) = \sum_{v \in \mathcal{V}} \mathcal{D}\mathcal{E}\mathcal{G}(v) = 2|\mathcal{E}| \geq 2 \left( \left( \frac{1}{2} - \epsilon(C_1 + C_2) \right) t^2 - \frac{t}{2} \right) = (1 - 2\epsilon(C_1 + C_2))t^2 - t.$$

Consequently  $(1 - 2\epsilon(C_1 + C_2))t^2 - t \leq t^2 - t - \sqrt{\epsilon} m t + \sqrt{\epsilon} m$ . Simplifying this we obtain

$$m \leq \frac{2\epsilon(C_1 + C_2)t^2}{\sqrt{\epsilon}(t - 1)} = 2\sqrt{\epsilon}(C_1 + C_2) \frac{t^2}{t - 1}.$$

Set  $C_3 = 3(C_1 + C_2)$ . It is now clear that there is  $M_2$  such that if  $t > M_2$ , then  $m \leq \sqrt{\epsilon}C_3t$ , so (3) holds. Therefore if  $t > M = \max\{M_1, M_2\}$ , (1), (2), and (3) hold.  $\square$

**4.4.3. Proof of the Stability Result.** In this section we will prove our stability result below.

**THEOREM 4.4.13.** *Fix an integer  $r \geq 3$ . For all  $\delta > 0$  there is  $0 < \epsilon < 1$  and  $M$  such that for all  $t > M$  the following holds. If  $R \in \tilde{M}_r(t)$  and  $W(R) > m(r)^{(1-\epsilon)\binom{t}{2}}$ , then  $R$  is  $\delta$ -close to  $\tilde{C}_r(t)$ .*

The following is a consequence of Lemma 4.4.9, so its proof appears in the appendix along with the proof of Lemma 4.4.9.

**LEMMA 4.4.14.** *Suppose  $r \geq 3$  is an integer and  $A, B, C \subseteq [r]$  are such that  $|A| = |B| = |C| = m(r)$  and there is no violating triple  $(a, b, c) \in A \times B \times C$ . Then one of the following holds:*

- (1)  $r$  is even and  $A = B = C = [m(r) - 1, r]$ .
- (2)  $r$  is odd and for some relabeling  $\{A, B, C\} = \{D, E, F\}$  one of the following holds:
  - (a)  $D = F = E = [m(r) - 1, r - 1]$ .
  - (b)  $D = F = [m(r), r]$ ,  $E \subseteq [m(r) - 1, \dots, r]$ .

An immediate corollary of this is the following.

**COROLLARY 4.4.15.** *Suppose  $r, t \geq 3$  are integers,  $R \in \tilde{M}_r(t)$ , and  $xy, yz, xz \in \binom{[t]}{2}$  are such that  $f^R(x, y) = f^R(y, z) = f^R(x, z) = m(r)$ . Then one of the following holds:*

- (1)  $r$  is even and  $c^R(xy) = c^R(yz) = c^R(xz) = [m(r) - 1, r]$ .
- (2)  $r$  is odd and for some relabeling  $\{x, y, z\} = \{u, v, z\}$  one of the following holds:
  - (a)  $c^R(uv) = c^R(uw) = c^R(vw) = [m(r) - 1, r - 1]$ .
  - (b)  $c^R(uv) = c^R(uw) = [m(r), r]$ ,  $c^R(vw) \subseteq [m(r) - 1, r]$ .

**PROOF.**  $R \in \tilde{M}_r(t)$  implies there is no violating triple  $(a, b, c) \in c^R(uv) \times c^R(uw) \times c^R(vw)$ . Thus the corollary follows immediately by applying Lemma 4.4.14 to  $A = c^R(uv)$ ,  $B = c^R(uw)$  and  $C = c^R(vw)$ .  $\square$

We will use the following consequence of Corollary 4.4.15.

**LEMMA 4.4.16.** *For all integers  $r \geq 3$  and  $0 < \epsilon < 1$ , there is  $M$  such that  $t > M$  and  $R \in \tilde{M}_r(t)$  implies the following. Let  $V = [t]$ . Then one of the following holds.*



- (i)  $r$  is even and for all  $xy \in \binom{V}{2} \setminus E(V, V_0^R(\epsilon))$ ,  $f^R(x, y) = m(r)$  implies  $c^R(xy) = [m(r) - 1, r]$ .
- (ii)  $r$  is odd and for all  $xy \in \binom{V}{2} \setminus E(V, V_0^R(\epsilon))$ ,  $f^R(x, y) = m(r)$  implies both of the following:
- (a) either  $r \in c^R(xy)$  or  $c^R(xy) = [m(r) - 1, r - 1]$ .
- (b) either  $m(r) - 1 \in c^R(xy)$  or  $c^R(xy) = [m(r), r]$ .

PROOF. Fix an integer  $r \geq 3$  and  $0 < \epsilon < 1$ . Choose  $M$  large enough so that  $t > M$  implies  $t - 2 - 2\sqrt{\epsilon}(t - 1) \geq 1$  and fix  $R \in \tilde{M}_r(t)$ . Suppose  $xy \in \binom{V}{2} \setminus E(V_0^R(\epsilon), V)$  and  $f^R(x, y) = m(r)$ . Since  $x, y \notin V_0^R(\epsilon)$ ,  $\min\{\mu_{m(r)}^R(x), \mu_{m(r)}^R(y)\} \geq (1 - \sqrt{\epsilon})(t - 1)$ . Therefore

$$|V \cap \Gamma_{m(r)}^R(x) \cap \Gamma_{m(r)}^R(y)| \geq t - 2 - 2\sqrt{\epsilon}(t - 1) \geq 1,$$

where the last inequality holds by our assumption on  $M$ . Thus there is  $z \in V \setminus \{x, y\}$  such that  $f^R(x, y) = f^R(y, z) = f^R(x, z) = m(r)$ . If  $r$  is even, part (1) of Corollary 4.4.15 implies  $c^R(xy) = [m(r) - 1, r]$ , so (i) holds. If  $r$  is odd, part (2) of Corollary 4.4.15 implies  $c^R(xy) \subseteq [m(r) - 1, r]$ . Recall that since  $r$  is odd,  $|[m(r) - 1, r]| = m(r) + 1$ . Therefore, since  $|c^R(xy)| = f^R(x, y) = m(r)$  and  $c^R(xy) \subseteq [m(r) - 1, r]$ ,  $m(r) - 1 \notin c^R(xy)$  implies  $c^R(xy) = [m(r), r]$  (so (i) holds), and  $r \notin c^R(xy)$  implies  $c^R(xy) = [m(r) - 1, r - 1]$  (so (ii) holds).  $\square$

We now fix some notation. Suppose  $r \geq 3$  is an integer,  $R$  is an  $r$ -graph, and  $u \in V(R)$ . For  $i \in [r]$ , set

$$N_i^R(u) = \{v \in V(R) : i \in c^R(uv)\} \text{ and}$$

$$\deg_i^R(u) = |N_i^R(u)|.$$

**Proof of Theorem 4.4.13.** Let  $r \geq 3$  be an integer, and fix  $\delta > 0$ . Let  $C_1, C_2, C_3$  be as in Lemma 4.4.12. We will consider the cases when  $r$  is even and odd separately.

Case 1:  $r$  is even. Fix  $0 < \epsilon < 1$  small enough so that  $\max\{\sqrt{\epsilon}C_3, (C_1 + C_2)\epsilon\} < \frac{\delta}{2}$ . Apply Lemma 4.4.12 to  $\epsilon$  to obtain  $M_1$ , and apply Lemma 4.4.16 to  $\epsilon$  to obtain  $M_2$ . Set  $M = \max\{M_1, M_2\}$ . Fix  $t > M$  and  $R \in \tilde{M}_r(t)$  such that  $W(R) \geq m(r)^{(1-\epsilon)\binom{t}{2}}$ . Set  $V = [t]$ . Let  $R'$  be the unique element of  $\tilde{C}_r(t)$ , that is,  $R'$  is the complete  $r$ -graph with vertex set  $V$  such that for all  $xy \in \binom{V}{2}$ ,  $c^{R'}(xy) = [m(r) - 1, r]$ . We show  $|\Delta(R, R')| \leq \delta t^2$ .

Let  $V_0 = V_0^R(\epsilon)$  and  $V_1 = V \setminus V_0$ . Define  $A = E(V_0, V) \cup \{xy \in \binom{V}{2} : f^R(x, y) \neq m(r)\}$ . Suppose  $xy \in \binom{V}{2} \setminus A$ . Then  $xy \in \binom{V}{2} \setminus E(V, V_0)$  and  $f^R(x, y) = m(r)$ , so Lemma 4.4.16 (i) implies

$c^R(xy) = [m(r) - 1, r]$ . Thus  $c^R(xy) = c^{R'}(xy)$  and  $xy \notin \Delta(R, R')$ . We have shown  $\Delta(R, R') \subseteq A$ , and consequently  $|\Delta(R', R)| \leq |A|$ .

We now bound  $|A|$ . The definition of  $A$  and parts (1), (2), and (3) of Lemma 4.4.12 imply

$$|A| \leq |V||V_0| + a_R + b_R \leq (\sqrt{\epsilon}C_3 + (C_1 + C_2)\epsilon)t^2.$$

By assumption on  $\epsilon$ ,  $(\sqrt{\epsilon}C_3 + (C_1 + C_2)\epsilon)t^2 < (\frac{\delta}{2} + \frac{\delta}{2})t^2 = \delta t^2$ , and consequently,  $|\Delta(R, R')| \leq \delta t^2$  as desired.

Case 2:  $r$  is odd. Fix  $0 < \epsilon < 1$  small enough so that  $\max\{\sqrt{\epsilon}C_3, (C_1 + C_2)\epsilon, 2\sqrt{\epsilon}\} < \frac{\delta}{5}$ . Apply Lemma 4.4.12 to  $\epsilon$  to obtain  $M_1$  and apply Lemma 4.4.16 to  $\epsilon$  to obtain  $M_2$ . Choose  $M > \max\{M_1, M_2\}$  large enough so that  $t > M$  implies  $\frac{2}{\sqrt{\epsilon}t} < \frac{\delta}{5}$ ,  $\sqrt{\epsilon}t^2 + t \leq 2\sqrt{\epsilon}t^2$ , and  $\frac{t^2}{t-1} < 2t$ . Fix  $t > M$  and  $R \in \tilde{M}_r(t)$  such that  $W(R) \geq m(r)^{(1-\epsilon)\binom{t}{2}}$  and set  $V = [t]$ . We construct an element  $R' \in \tilde{C}_r(t)$ , then show  $|\Delta(R, R')| \leq \delta t^2$ . First we choose integers  $k \geq 1$ ,  $\ell \in \{0, \dots, k\}$ , and a partition  $V_0, V_1, \dots, V_\ell, \dots, V_k$  of  $V$  with the following properties:

- $|V_0| \leq \sqrt{\epsilon}C_3t$ ,
- If  $0 < l$ , then for each  $1 \leq i \leq l$ , there is  $u_i \in V$  and  $B_i \subseteq V$  such that  $V_i = (N_{m(r)-1}^R(u_i) \cap B_i) \cup \{u_i\}$ ,
- If  $l < k$ , then  $V_{l+1}, \dots, V_k$  are singletons.

Step 1: Let  $V_0 = V_0^R(\epsilon)$ . Note that part (3) of Lemma 4.4.12 implies  $|V_0| \leq \sqrt{\epsilon}C_3t$ . Define  $B_1 = V \setminus V_0$ . If there exists  $u \in B_1$  such that  $|N_{m(r)-1}^R(u) \cap B_1| \geq \sqrt{\epsilon}(t-1)$ , then choose  $u_1$  to be any  $u \in B_1$  with  $|N_{m(r)-1}^R(u) \cap B_1|$  maximal, and set  $V_1 = (N_{m(r)-1}^R(u_1) \cap B_1) \cup \{u_1\}$ . If  $V \setminus (V_0 \cup V_1) = \emptyset$ , set  $k = l = 1$  and end the construction. If not, go to step 2. If no  $u$  exists in  $B_1$  such that  $|N_{m(r)-1}^R(u) \cap B_1| \geq \sqrt{\epsilon}(t-1)$ , then put each element of  $B_1$  into its own part and end the construction. This means we set  $l = 0$ ,  $k = t - |V_0|$ , and let  $V_1, \dots, V_k$  partition  $B_1$  into singletons.

Step  $i+1$ : Suppose  $i \geq 1$  and we have chosen  $V_i, B_i$ , and  $u_i$  such that  $V_i = (N_{m(r)-1}^R(u_i) \cap B_i) \cup \{u_i\}$  and  $V \setminus \bigcup_{j=0}^i V_j \neq \emptyset$ . Set  $B_{i+1} = V \setminus \bigcup_{j=0}^i V_j$ . If there exists  $u \in B_{i+1}$  such that  $|N_{m(r)-1}^R(u) \cap B_{i+1}| \geq \sqrt{\epsilon}(t-1)$ , choose  $u_{i+1}$  to be any  $u \in B_{i+1}$  with  $|N_{m(r)-1}^R(u) \cap B_{i+1}|$  maximal, and set  $V_{i+1} = (N_{m(r)-1}^R(u_{i+1}) \cap B_{i+1}) \cup \{u_{i+1}\}$ . If  $V \setminus \bigcup_{j=0}^{i+1} V_j = \emptyset$ , set  $k = l = i+1$  and end the construction. Otherwise go to step  $i+2$ . If no  $u$  exists in  $B_{i+1}$  such that  $|N_{m(r)-1}^R(u) \cap B_{i+1}| \geq \sqrt{\epsilon}(t-1)$ , then put each element of  $B_{i+1}$  into its own part and end the construction. This means we set  $l = i$ ,  $k = t - |\bigcup_{j=0}^i V_j|$ , and let  $V_{i+1}, \dots, V_k$  partition  $B_{i+1}$  into singletons. This completes the

construction of the partition  $V_0, V_1, \dots, V_l, \dots, V_k$ . Given  $xy \in \binom{V}{2}$ , define

$$c^{R'}(xy) = \begin{cases} [m(r) - 1, r - 1] & \text{if } xy \in \binom{V_i}{2} \text{ some } 0 \leq i \leq l \\ [m(r), r] & \text{otherwise.} \end{cases}$$

This completes our construction of  $R'$ . We now bound  $|\Delta(R, R')|$ . Set

$$A = E(V_0, V) \cup \left\{ xy \in \binom{V}{2} : f^R(x, y) \neq m(r) \right\} \cup \bigcup_{i=1}^l E(\{u_i\}, V) \cup E(V_i, V \setminus \Gamma_{m(r)}^R(u_i)).$$

We first bound  $|A|$ , then  $|\Delta(R, R') \setminus A|$ . By parts (1), (2), and (3) of Lemma 4.4.12,

$$\left| E(V_0, V) \cup \left\{ xy \in \binom{V}{2} : f^R(x, y) \neq m(r) \right\} \right| \leq |V||V_0| + a_R + b_R \leq C_3\sqrt{\epsilon}t^2 + C_1\epsilon t^2 + C_2\epsilon t^2.$$

By construction, for each  $1 \leq i \leq l$ ,  $|V_i| \geq \sqrt{\epsilon}(t-1)$ , therefore  $l \leq \frac{t}{\sqrt{\epsilon}(t-1)}$ . Thus

$$\left| \bigcup_{i=1}^l E(\{u_i\}, V) \right| \leq lt \leq \frac{t^2}{\sqrt{\epsilon}(t-1)} \leq \frac{2t}{\sqrt{\epsilon}},$$

where the last inequality is by assumption on  $M$ . By construction, for each  $1 \leq i \leq l$ ,  $u_i \notin V_0$  implies  $|V \setminus \Gamma_{m(r)}^R(u_i)| \leq \sqrt{\epsilon}(t-1) + 1$ . Therefore

$$\left| \bigcup_{i=1}^l E(V_i, V \setminus \Gamma_{m(r)}^R(u_i)) \right| \leq \sum_{i=1}^l |V_i| |V \setminus \Gamma_{m(r)}^R(u_i)| \leq (\sqrt{\epsilon}(t-1) + 1) \sum_{i=1}^l |V_i| \leq (\sqrt{\epsilon}(t-1) + 1)t \leq 2\sqrt{\epsilon}t^2,$$

where the last inequality is by assumption on  $M$ . Combining all of this yields that

$$|A| \leq \left( \sqrt{\epsilon}C_3 + (C_1 + C_2)\epsilon + \frac{2}{\sqrt{\epsilon}t} + 2\sqrt{\epsilon} \right) t^2.$$

We now bound  $|\Delta(R, R') \setminus A|$ . An edge  $xy \in \Delta(R, R') \setminus A$  is contained in one of the following:

- $X = \bigcup_{l+1 \leq i < j \leq k} \{xy \in E(V_i, V_j) \setminus A : c^R(xy) \neq [m(r), r]\}$ .
- For some  $1 \leq i \leq l$ ,  $Y_i = \{xy \in E(V_i) \setminus A : c^R(xy) \neq [m(r) - 1, r - 1]\}$ .
- For some  $1 \leq i \leq \ell$  and  $i < j \leq k$ ,  $Z_{ij} = \{xy \in E(V_i, V_j) \setminus A : c^R(xy) \neq [m(r), r]\}$ .

We now bound  $|X|$ . Define  $\mathcal{G}$  to be the graph with vertex set  $\mathcal{V} = \bigcup_{j=l+1}^k V_j$  and edge set

$$\mathcal{E} = \left\{ xy \in \binom{\mathcal{V}}{2} : m(r) - 1 \in c^R(xy) \right\}.$$

By definition of  $X$ , for all  $xy \in X$  we have  $xy \in \binom{V}{2} \setminus E(V_0, V)$ ,  $f^R(x, y) = m(r)$ , and  $c^R(xy) \neq [m(r), r]$ , so Lemma 4.4.16 (ii)(b) implies  $m(r) - 1 \in c^R(xy)$ , and therefore  $X \subseteq \mathcal{E}$ . By construction,

for all  $u \in \mathcal{V}$ ,  $\mathcal{DEG}(v) = |N_{m(r)-1}^R(u) \cap \mathcal{V}| < \sqrt{\epsilon}(t-1)$ , thus

$$|X| \leq |\mathcal{E}| = \frac{1}{2} \sum_{v \in \mathcal{V}} \mathcal{DEG}(v) < \frac{\sqrt{\epsilon}t^2}{2}.$$

We now show each  $Y_i$  is empty. If  $l = 0$  this is vacuous, so assume  $l \geq 1$ . Suppose for a contradiction that for some  $1 \leq i \leq l$ ,  $Y_i \neq \emptyset$ . Then there is  $xy \in E(V_i)$  such that  $f^R(x, y) = m(r)$  and  $c^R(xy) \neq [m(r) - 1, r - 1]$ . By Lemma 4.4.16 (ii)(a),  $r \in c^R(xy)$ . But by construction,  $m(r) - 1 \in c^R(u_i x) \cap c^R(u_i y)$ . Now  $(r, m(r) - 1, m(r) - 1) \in c^R(xy) \times c^R(u_i x) \times c^R(u_i y)$  is a violating triple, making  $\{x, y, u_i\}$  a violating triangle, a contradiction.

We now show each  $Z_{ij}$  is empty. Suppose for a contradiction that for some  $1 \leq i \leq \ell$  and some  $i < j \leq k$ , there is  $xy \in Z_{ij}$ , say with  $x \in V_i$ ,  $y \in V_j$ . Then  $f^R(x, y) = f^R(u_i, y) = m(r)$  and  $c^R(xy) \neq [m(r), r]$ . By Lemma 4.4.16 (ii)(b),  $m(r) - 1 \in c^R(xy)$ , and by construction  $m(r) - 1 \in c^R(xu_i)$ . Also by construction,  $m(r) - 1 \notin c^R(u_i y)$ , so Lemma 4.4.16 (ii)(b) implies  $c^R(u_i y) = [m(r), r]$ . But now  $(r, m(r) - 1, m(r) - 1) \in c^R(u_i y) \times c^R(u_i x) \times c^R(xy)$  is a violating triple, making  $\{u_i, x, y\}$  a violating triangle, a contradiction.

Combining all of this yields that  $|\Delta(R, R') \setminus A| < \frac{\sqrt{\epsilon}t^2}{2}$ , so

$$|\Delta(R, R')| \leq (\sqrt{\epsilon}C_3 + (C_1 + C_2)\epsilon + \frac{2}{\sqrt{\epsilon}t} + 2\sqrt{\epsilon} + \frac{\sqrt{\epsilon}}{2})t^2.$$

By our assumptions on  $\epsilon$  and because  $\frac{2}{\sqrt{\epsilon}t} < \frac{\delta}{5}$ ,  $(\sqrt{\epsilon}C_3 + (C_1 + C_2)\epsilon + \frac{2}{\sqrt{\epsilon}t} + 2\sqrt{\epsilon} + \frac{\sqrt{\epsilon}}{2})t^2 < 5\frac{\delta}{5}t^2 = \delta t^2$ , and  $|\Delta(R, R')| < \delta t^2$  as desired.  $\square$

#### 4.5. Proof of Theorem 4.1.5

In this section we prove Theorem 4.1.5, which says that for all integers  $r \geq 3$  and all  $\delta > 0$ , almost all elements of  $M_r(n)$  are  $\delta$ -close to  $C_r(n)$ . We begin with some key definitions. For  $n, r, s \geq 3$  integers, and  $\delta, \eta, d, \epsilon \geq 0$ , set

$$\tilde{C}_r^\delta(s) = \{R \in \tilde{M}_r(s) : R \text{ is } \delta\text{-close to } \tilde{C}_r(s)\},$$

$$D_r(n, \delta, \eta, d) = \{G \in M_r(n) : Q_{\eta, d}(G) \neq \emptyset \text{ and for all } R \in Q_{\eta, d}(G), R \in \tilde{C}_r^\delta(t) \text{ where } t = |V(R)|\},$$

$$\tilde{E}_r(s, \epsilon) = \{R \in \tilde{M}_r(s) : W(R) \geq m(r)^{(1-\epsilon)\binom{s}{2}}\}, \text{ and}$$

$$E_r(n, \epsilon, \eta, d) = \{G \in M_r(n) : \text{for all } R \in Q_{\eta, d}(G), R \in \tilde{E}_r(t, \epsilon) \text{ where } t = |V(R)|\},$$

and recall that  $C_r^\delta(n) = \{G \in M_r(n) : G \text{ is } \delta\text{-close to } C_r(n)\}$ . Theorem 4.1.5 follows from two lemmas that we now prove. The first lemma below informally states that  $r$ -graphs in  $M_r(n)$  with

reduced  $r$ -graphs close to  $\tilde{C}_r(t)$  are themselves close to  $C_r(n)$ . We will use throughout the following observations.

FACT 4.5.1. *Suppose  $3 \leq t \leq n$ . Then  $\binom{\lceil n/t \rceil}{2} \leq \binom{n/t+1}{2} \leq \frac{n^2}{t^2}$  and  $\lceil n/t \rceil^2 \leq (n/t+1)^2 \leq 4n^2/t^2$ .*

LEMMA 4.5.2. *Let  $r, n \geq 3$  be integers. For all  $\delta > 0$ , there is  $d_0$  such that for all positive  $d \leq d_0$  and  $\eta \leq \gamma_{el}(d, 3)$ ,*

$$D_r(n, \delta/8, \eta, d) \subseteq C_r^\delta(n).$$

PROOF. Fix  $\delta > 0$  and set  $d_0 = \frac{\delta}{2(2r+5)}$ . Fix  $d \leq d_0$  and  $\eta \leq \gamma_{el}(d, 3)$ , and suppose  $G \in D_r(n, \delta/8, \eta, d)$ . Then by definition of  $D_r(n, \delta/8, \eta, d)$ ,  $G \in M_r(n)$  and there is  $R \in Q_{\eta, d}(G)$  which is  $\frac{\delta}{8}$ -close to  $\tilde{C}_r(t)$  where  $t = |V(R)|$ . Let  $R' \in \tilde{C}_r(t)$  be such that  $R$  is  $\frac{\delta}{8}$ -close to  $R'$ . We will build an element  $G' \in C_r(n)$  such that  $G$  is  $\delta$ -close to  $G'$ .

Let  $\mathcal{P} = \{V_1, \dots, V_t\}$  be an  $\eta$ -regular partition for  $G$  such that  $R = R(G, \mathcal{P}, d)$ . Define

$$A = \Delta(R, R') \cup \left\{ ij \in \binom{[t]}{2} : (V_i, V_j) \text{ is not } \eta\text{-regular for } G \right\}.$$

Note that  $|A| \leq \frac{\delta}{8}t^2 + \eta t^2$ . Define  $G'$  by  $V(G') = V(G) = [n]$  and for  $xy \in \binom{[n]}{2}$ ,

$$d^{G'}(x, y) = \begin{cases} r-1 & \text{if } xy \in E(V_i) \text{ for some } i \in [t] \\ r-1 & \text{if } xy \in E(V_i, V_j) \text{ for some } ij \in \binom{[t]}{2} \text{ such that either } ij \in A \text{ or } d^G(x, y) \notin c^{R'}(ij) \\ d^G(x, y) & \text{if } xy \in E(V_i, V_j) \text{ for some } ij \in \binom{[t]}{2} \setminus A \text{ and } d^G(x, y) \in c^{R'}(ij). \end{cases}$$

Set

$$U_r = \begin{cases} [m(r), r] & \text{if } r \text{ is odd} \\ [m(r)-1, r] & \text{if } r \text{ is even} \end{cases} \quad \text{and} \quad L_r = \begin{cases} [m(r)-1, r-1] & \text{if } r \text{ is odd} \\ [m(r)-1, r] & \text{if } r \text{ is even.} \end{cases}$$

Note that  $r-1 \in U_r \cap L_r$ . By the definition of  $\tilde{C}_r(t)$ , there is a partition  $\tilde{W}_1, \dots, \tilde{W}_s$  of  $[t]$  such that for all  $ij \in \binom{[t]}{2}$ ,

$$c^{R'}(ij) = \begin{cases} L_r & \text{if } ij \in E(\tilde{W}_u) \text{ some } u \in [s] \\ U_r & \text{if } ij \in E(\tilde{W}_u, \tilde{W}_v) \text{ some } uv \in \binom{[s]}{2}. \end{cases}$$

Define a new partition  $W_1, \dots, W_s$  of  $[n]$  by setting  $W_u = \bigcup_{i \in \bar{W}_u} V_i$  for each  $u \in [s]$ . Then by construction, for all  $xy \in \binom{[n]}{2}$ ,

$$d^{G'}(x, y) \in \begin{cases} L_r & \text{if } xy \in E(W_u) \text{ some } u \in [s] \\ U_r & \text{if } xy \in E(W_u, W_v) \text{ some } uv \in \binom{[s]}{2}. \end{cases}$$

Therefore,  $G' \in C_r(n)$  by definition. We now show  $|\Delta(G, G')| \leq \delta n^2$ . Recall that by definition of  $\mathcal{Q}_{\eta, d}(G)$  and  $\eta, 3 \leq \frac{1}{\eta} \leq t \leq n$ . Edges  $xy \in \Delta(G, G')$  fall into the following categories:

- $xy \in E(V_i)$  for some  $i \in [t]$ . There are at most  $t \binom{\lceil n/t \rceil}{2} \leq t \frac{n^2}{t^2} = \frac{n^2}{t} \leq \eta n^2$  such edges.
- $xy \in E(V_i, V_j)$  for some  $ij \in A$ . The number of such edges is at most

$$|A| \binom{\lceil n/t \rceil}{2} \leq |A| \frac{4n^2}{t^2} \leq \left( \frac{\delta}{8} t^2 + \eta t^2 \right) \frac{4n^2}{t^2} = \left( \frac{\delta}{2} + 4\eta \right) n^2.$$

- $xy \in E(V_i, V_j)$  for some  $ij \in \binom{[t]}{2} \setminus A$  such that  $d^G(x, y) \notin c^{R'}(ij)$ . This means  $(V_i, V_j)$  is  $\eta$ -regular for  $G$  and  $c^{R'}(ij) = c^R(ij)$ . Because  $R = R(G, \mathcal{P}, d)$ , for each  $l \in [r] \setminus c^R(ij)$  we have that  $e_l^G(V_i, V_j) \leq d|V_i||V_j|$ . Therefore there are at most  $dr \binom{t}{2} \left[ \frac{n}{t} \right]^2 \leq dr \frac{t^2}{2} \frac{4n^2}{t^2} = 2drn^2$  such edges.

Combining these bounds with the fact that  $\eta \leq d \leq d_0 = \frac{\delta}{2(2r+5)}$  yields

$$|\Delta(G, G')| \leq n^2 \left( 5\eta + \frac{\delta}{2} + 2dr \right) \leq n^2 \left( 5d_0 + \frac{\delta}{2} + 2d_0r \right) = n^2 \left( \frac{\delta}{2} + d_0(2r+5) \right) = \delta n^2.$$

□

We now prove the second lemma. Informally, it says that most graphs in  $M_r(n)$  have all their reduced graphs  $R$  with  $W(R)$  quite large.

LEMMA 4.5.3. *For all  $\epsilon > 0$ , there is  $\beta = \beta(\epsilon)$  and  $d_0 = d_0(\epsilon) > 0$ , such that for all positive  $d \leq d_0$  and  $\eta \leq \gamma_{el}(d, 3)$ , there is  $M$  such that  $n \geq M$  implies*

$$(9) \quad \frac{|M_r(n) \setminus E_r(n, \epsilon, \eta, d)|}{m(r) \binom{n}{2}} \leq 2^{-\beta n^2}.$$

PROOF. All logs in this proof are base 2. Fix  $\epsilon > 0$  and set  $\beta = \frac{\epsilon \log m(r)}{8}$ . Define

$$F(x) = 5x \log r + 4r(H(x) + x) - 2\beta,$$

and choose  $d_0 < \frac{1}{3}$  small enough so that  $F(d_0) < -\beta$ . Recall that for  $0 \leq y \leq x \leq \frac{1}{2}$ ,  $H(y) \leq H(x)$ , so for any  $0 \leq y \leq x \leq d_0$ ,  $F(y) \leq F(x)$ . Fix  $d \leq d_0$  and  $\eta \leq \gamma_{el}(d, 3) \leq d$ . Set  $N = CM(\frac{1}{\eta}, \eta)$  and define

$$\begin{aligned}
C &= \log(N - \frac{1}{\eta} + 1) + \log(2^r) \binom{N}{2} + (H(\eta) + \eta)N^2, \\
C' &= \log N + \frac{\log m(r)}{2} + N^2 \log r, \text{ and} \\
C'' &= 5\eta \log r + 4r(H(d) + d) - 4\beta.
\end{aligned}$$

For any integer  $n$ , define

$$\begin{aligned}
S(n) &= n \log(N) + (H(\eta) + \eta)N^2 + (5\eta \log r)n^2 + 4r(H(d) + d)n^2 + nN^2 \log r, \\
S'(n) &= S(n) + \log(N - \frac{1}{\eta} + 1) + \binom{N}{2} \log(2^r), \text{ and} \\
S''(n) &= S'(n) - 4\beta n^2 + \frac{\log m(r)}{2}n.
\end{aligned}$$

Notice that  $S''(n) = C + C'n + C''n^2$  and  $C'' \leq F(d) - 2\beta$ . Choose  $M \geq N$  large enough so that  $n > M$  implies  $S''(n) < (C'' + 2\beta)n^2 \leq F(d)n^2$ . We show  $n > M$  implies (9) holds. Fix  $n > M$ . Our assumptions on  $d \leq d_0$  and  $M$  imply  $S''(n) < F(d)n^2 \leq F(d_0)n^2 < -\beta n^2$ , so it suffices to show

$$(10) \quad \frac{|M_r(n) \setminus E_r(n, \epsilon, \eta, d)|}{m(r) \binom{n}{2}} \leq 2^{S''(n)}.$$

By definition of  $E(n, \epsilon, \eta, d)$ , we have  $G \in M_r(n) \setminus E_r(n, \epsilon, \eta, d)$  if and only if there is  $\frac{1}{\eta} \leq t \leq N$  and  $R \in \tilde{M}_r(t)$  such that  $R \in Q_{\eta, d}(G)$  and  $W(R) < m(r)^{(1-\epsilon)\binom{t}{2}}$ . We give an upper bound for the number of such  $G$ .

Fix some  $\frac{1}{\eta} \leq t \leq N$  and  $R \in \tilde{M}_r(t)$  such that  $W(R) < m(r)^{(1-\epsilon)\binom{t}{2}}$ . All  $G \in M_r(n)$  such that  $R \in Q_{\eta, d}(G)$  can be constructed as follows:

- Choose an equipartition of  $[n]$  into  $t$  pieces  $V_1, \dots, V_t$ . There are at most  $t^n \leq N^n$  such partitions. Note that for each  $i \in [t]$ ,  $|V_i| \leq \lceil n/t \rceil$ .
- Choose  $J \subseteq \binom{[t]}{2}$  to be the set of  $ij$  such that  $(V_i, V_j)$  is not  $\eta$ -regular for  $G$ . There are at most  $\binom{\binom{t}{2}}{\eta \binom{t}{2}} 2^{\eta \binom{t}{2}} \leq 2^{H(\eta)t^2 + \eta t^2} \leq 2^{(H(\eta) + \eta)N^2}$  ways to do this.
- Choose  $d^G(x, y)$  for each  $xy \in E(V_i)$  and  $i \in [t]$ . There are at most  $r^{t \binom{\lceil n/t \rceil}{2}} \leq r^{t \frac{n^2}{t^2}} = r^{\frac{n^2}{t}} \leq r^{\eta n^2}$  ways to do this.
- Choose  $d^G(x, y)$  for each  $xy \in E(V_i, V_j)$  where  $ij \in J$ . The number of ways to do this is at most  $(r^{\lceil n/t \rceil^2})^{\eta t^2} \leq r^{\frac{4n^2}{t^2} \eta t^2} = r^{4\eta n^2}$ .
- Choose  $d^G(x, y)$  for each  $xy \in E(V_i, V_j)$  where  $ij \in I = \binom{[t]}{2} \setminus J$ . For each  $ij \in I$ ,  $(V_i, V_j)$  is  $\eta$ -regular, so the colors for edges in  $E(V_i, V_j)$  can be chosen as follows:
  - (a) For each  $s \notin c^R(ij)$ , choose a subset of  $E(V_i, V_j)$  of size at most  $d|V_i||V_j|$  to have color  $s$ . The number of ways to do this is at most

$$\left( \binom{\lceil n/t \rceil^2}{d \lceil n/t \rceil^2} 2^{d \lceil n/t \rceil^2} \right)^r \leq 2^{r(H(d)\lceil n/t \rceil^2 + d \lceil n/t \rceil^2)} \leq 2^{\frac{4rn^2}{t^2}(H(d) + d)}.$$

- (b) Assign colors from  $c^R(ij)$  to the rest of the edges in  $E(V_i, V_j)$ . The number of ways to do this is at most  $f^R(i, j)^{\lceil n/t \rceil^2} \leq f^R(i, j)^{\frac{n^2}{t^2} + \frac{2n}{t} + 1} \leq f^R(i, j)^{\frac{n^2}{t^2}} r^{\frac{2n}{t} + 1} \leq f^R(i, j)^{\frac{n^2}{t^2}} r^n$  ways to do this.

Therefore, the total number of ways to choose  $d^G(x, y)$  for  $xy \in E(V_i, V_j)$  where  $ij \in I$  is at most

$$\prod_{ij \in I} 2^{(H(d)+d)\frac{4rn^2}{t^2}} f^R(i, j)^{\frac{n^2}{t^2}} r^n \leq 2^{4r(H(d)+d)n^2} r^{nN^2} \left( \prod_{ij \in I} f^R(i, j)^{\frac{n^2}{t^2}} \right) \leq 2^{4r(H(d)+d)n^2} r^{nN^2} W(R)^{\frac{n^2}{t^2}}.$$

By the assumption,  $W(R) < m(r)^{(1-\epsilon)\binom{t}{2}}$ . Therefore

$$W(R)^{\frac{n^2}{t^2}} < m(r)^{(1-\epsilon)\binom{t}{2}\frac{n^2}{t^2}} < m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)}.$$

Combining the above yields that the number of  $G \in M_r(n)$  with  $R \in Q_{\eta, d}(G)$  is at most

$$N^n 2^{(H(\eta)+\eta)N^2} r^{5\eta n^2} 2^{4r(H(d)+d)n^2} r^{nN^2} m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)} = 2^{S(n)} m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)}.$$

The number of  $R \in \tilde{M}_r(t)$  with  $\frac{1}{\eta} \leq t \leq N$  is at most  $(N - \frac{1}{\eta} + 1)|\tilde{M}_r(N)|$ , so

$$\begin{aligned} |M_r(n) \setminus E_r(n, \epsilon, \eta, d)| &< (N - \frac{1}{\eta} + 1)|\tilde{M}_r(N)| 2^{S(n)} m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)} \\ &< (N - \frac{1}{\eta} + 1)(2^r)^{\binom{N}{2}} 2^{S(n)} m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)} = 2^{S'(n)} m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)}. \end{aligned}$$

Thus

$$\frac{|M_r(n) \setminus E_r(n, \epsilon, \eta, d)|}{m(r)^{\binom{n}{2}}} < \frac{2^{S'(n)} m(r)^{(1-\epsilon)\left(\binom{n}{2} + \frac{n}{2}\right)}}{m(r)^{\binom{n}{2}}} = 2^{S''(n)}.$$

We have shown that  $n > M$  implies (10) holds, so we are done.  $\square$

**Proof of Theorem 4.1.5.** Fix  $\delta > 0$ . Apply Theorem 4.4.13 to  $\frac{\delta}{8}$  to obtain  $\epsilon$  and  $M_{4.4.13}$ . Apply Lemma 4.5.2 to  $\delta$  to obtain  $(d_0)_{4.5.2}$ . Apply Lemma 4.5.3 to  $\epsilon$  to obtain  $\beta$  and  $(d_0)_{4.5.3}$ . Let  $d_0 = \min\{(d_0)_{4.5.2}, (d_0)_{4.5.3}\}$ . Apply Lemma 4.5.3 to  $d = d_0 \leq (d_0)_{4.5.3}$  and  $\eta = \min\{\gamma_{el}(d, 3), \frac{1}{M_{4.4.13}}\}$  to obtain  $M_{4.5.3}$ . Set  $M = \max\{CM(\eta, \frac{1}{\eta}), M_{4.5.3}\}$  and fix  $n > M$ . Lemma 4.5.3 implies

$$(11) \quad \frac{|M_r(n) \setminus E_r(n, \epsilon, \eta, d)|}{m(r)^{\binom{n}{2}}} \leq 2^{-\beta n^2}.$$

We now show  $E_r(n, \epsilon, \eta, d) \subseteq D_r(n, \delta/8, \eta, d)$ . Suppose  $G \in E_r(n, \epsilon, \eta, d)$ . We need to show that  $Q_{\eta, d}(G) \neq \emptyset$  and for all  $R \in Q_{\eta, d}(G)$ ,  $R \in \tilde{C}_r^{\delta/8}(t)$  where  $t = |V(R)|$ . As  $n > CM(\eta, \frac{1}{\eta})$ , by Theorem 4.4.2, we have  $Q_{\eta, d}(G) \neq \emptyset$ . Suppose  $R \in Q_{\eta, d}(G)$  and set  $t = |V(R)|$ . By definition of  $E_r(n, \epsilon, \eta, d)$ ,  $R \in \tilde{E}_r(t, \epsilon)$ . Theorem 4.4.13 and our assumptions on  $\eta$  imply that  $R \in \tilde{C}_r^{\delta/8}(t)$ , so  $E_r(n, \epsilon, \eta, d) \subseteq D_r(n, \delta/8, \eta, d)$ . Lemma 4.5.2 implies  $D_r(n, \delta/8, \eta, d) \subseteq C_r^\delta(n)$ . Combining these



inclusions with (11) we have that

$$\frac{|M_r(n) \setminus C_r^\delta(n)|}{m(r) \binom{n}{2}} \leq 2^{-\beta n^2}.$$

By Remark 4.1.9,  $|M_r(n)| \geq m(r) \binom{n}{2}$ , so

$$\frac{|M_r(n) \setminus C_r^\delta(n)|}{|M_r(n)|} \leq \frac{|M_r(n) \setminus C_r^\delta(n)|}{m(r) \binom{n}{2}} \leq 2^{-\beta n^2},$$

which completes our proof of Theorem 4.1.5.  $\square$

#### 4.6. Proof of Theorem 4.1.2

In this section we prove Theorem 4.1.2, which says that for all even integers  $r \geq 4$ , almost all  $G$  in  $M_r(n)$  are in  $C_r(n)$ . The outline of the proof is as follows. Given  $\epsilon > 0$  and integers  $r, n \geq 3$ , define

$$A_r(n, \epsilon) = \{G \in M_r(n) : \exists x \in [n] \text{ such that for some } l \in [m(r) - 2], |N_l^G(x)| \geq \epsilon n\},$$

$$A'_r(n, \epsilon) = \{G \in M_r(n) \setminus A_r(n, \epsilon) : \exists xy \in \binom{[n]}{2} \text{ with } d^G(x, y) \in [m(r) - 2]\}.$$

For all  $\epsilon > 0$ ,  $n \in \mathbb{N}$ , and even integers  $r \geq 4$ , we have that  $M_r(n) = C_r(n) \cup A_r(n, \epsilon) \cup A'_r(n, \epsilon)$ , and thus  $M_r(n) \setminus C_r(n) = A_r(n, \epsilon) \cup A'_r(n, \epsilon)$ . We will show that when  $r$  is even, there are  $\epsilon > 0$  and  $\beta > 0$  such that for large  $n$ ,  $|A_r(n, \epsilon) \cup A'_r(n, \epsilon)| \leq 2^{-\beta n} |M_r(n)|$ , from which Theorem 4.1.2 will follow. We do this in two lemmas, one for each of the sets  $A_r$  and  $A'_r$  defined above. The first lemma will apply to all  $r \geq 3$ , while the second will apply only to even  $r \geq 4$ .

LEMMA 4.6.1. *For all integers  $r \geq 3$  and all  $\epsilon > 0$  there is  $\beta > 0$  and  $M$  such that  $n > M$  implies*

$$(12) \quad |A_r(n, \epsilon)| \leq 2^{-\beta n^2} |C_r(n)|.$$

PROOF. Let  $r \geq 3$  be an integer and fix  $\epsilon > 0$ . By Remark 4.1.9, it suffices to find  $\beta > 0$  and  $M$  such that  $n > M$  implies

$$|A_r(n, \epsilon)| \leq 2^{-\beta n^2} m(r) \binom{n}{2}.$$

Choose  $T > 0$  large enough so that  $\frac{\epsilon^2 T^2}{64} - \frac{\epsilon T}{8} \geq 1$ , then choose  $0 < \delta < \min\{\frac{1}{T}, \frac{\epsilon^2}{128}\}$ . Apply Theorem 4.4.13 to  $\delta$  to obtain  $\epsilon_{4.4.13}$  and  $M_{4.4.13}$ . Apply Lemma 4.5.3 to  $\epsilon_{4.4.13}$  to obtain  $d_0$  and  $\beta > 0$ . Choose  $d \leq d_0$  and  $\eta < \min\{\delta, \gamma_{el}(d, 3), \frac{\epsilon}{2}, d, \frac{1}{M_{4.4.13}}\}$ . Apply Lemma 4.5.3 to this  $d$  and  $\eta$  to

obtain  $M_{4.5.3}$ . Choose  $M \geq \max\{M_{4.5.3}, CM(\frac{1}{\eta}, \eta)\}$ . Lemma 4.5.3 implies that for all  $n > M$ ,

$$\frac{|M_r(n) \setminus E_r(n, \epsilon_{4.4.13}, \eta, d)|}{m(r)^{\binom{n}{2}}} \leq 2^{-\beta n^2}.$$

Therefore, it suffices to prove that  $n > M$  implies that  $A_r(n, \epsilon) \subseteq M_r(n) \setminus E_r(n, \epsilon_{4.4.13}, \eta, d)$ . Fix  $n > M$  and suppose for a contradiction that there is some  $G \in A_r(n, \epsilon) \cap E_r(n, \epsilon_{4.4.13}, \eta, d)$ . Since  $G \in A_r(n, \epsilon)$ , there is  $x \in [n]$  and  $l \in [m(r) - 2]$  such that  $|N_l^G(x)| \geq \epsilon n$ . Because  $n > CM(\frac{1}{\eta}, \eta)$ , by Theorem 4.4.2, there is  $R \in Q_{\eta, d}(G)$ . Also,  $G \in E_r(n, \epsilon_{4.4.13}, \eta, d)$  implies that  $W(R) \geq m(r)^{(1-\epsilon_{4.4.13})\binom{t}{2}}$  where  $t = |V(R)|$ . Then  $t \geq \frac{1}{\eta} > M_{4.4.13}$  implies that there is  $R' \in \tilde{C}_r(t)$  such that  $|\Delta(R, R')| \leq \delta t^2$ .

Let  $\mathcal{P} = \{V_1, \dots, V_t\}$  be an  $\eta$ -regular partition for  $G$  such that  $R = R(G, \mathcal{P}, d)$ , and define  $\Sigma = \{i \in [t] : |N_i^G(x) \cap V_i| \geq \frac{\epsilon}{2}|V_i|\}$ . We have that

$$(13) \quad \epsilon n \leq |N_l^G(x)| = \sum_{i \in \Sigma} |N_l^G(x) \cap V_i| + \sum_{i \notin \Sigma} |N_l^G(x) \cap V_i| \leq |\Sigma| \left\lceil \frac{n}{t} \right\rceil + (t - |\Sigma|) \frac{\epsilon}{2} \left\lceil \frac{n}{t} \right\rceil.$$

Note that  $n \geq 3$  implies  $\lceil n/t \rceil \leq \frac{4n}{3t}$ , so

$$|\Sigma| \left\lceil \frac{n}{t} \right\rceil + (t - |\Sigma|) \frac{\epsilon}{2} \left\lceil \frac{n}{t} \right\rceil = |\Sigma| \left(1 - \frac{\epsilon}{2}\right) \left\lceil \frac{n}{t} \right\rceil + \frac{\epsilon t}{2} \left\lceil \frac{n}{t} \right\rceil \leq |\Sigma| \left(1 - \frac{\epsilon}{2}\right) \frac{4n}{3t} + \frac{\epsilon t}{2} \frac{4n}{3t} = |\Sigma| \left(1 - \frac{\epsilon}{2}\right) \frac{4n}{3t} + \frac{2\epsilon n}{3}.$$

Combining this with (29), yields that  $\epsilon n \leq |\Sigma| \left(1 - \frac{\epsilon}{2}\right) \frac{4n}{3t} + \frac{2\epsilon n}{3}$ . Rearranging this, we obtain that  $|\Sigma| \geq (\epsilon n/3) / ((4n/3t)(1 - \frac{\epsilon}{2})) = \frac{\epsilon t}{4(1 - \frac{\epsilon}{2})} \geq \frac{\epsilon t}{4}$ . Set

$$I = \{ij \in E(\Sigma) : (V_i, V_j) \text{ is } \eta\text{-regular for } G \text{ and } c^R(ij) = c^{R'}(ij)\}.$$

Applying that  $\mathcal{P}$  is an  $\eta$ -regular partition for  $G$ , that  $|\Delta(R, R')| \leq \delta t^2$ , and that  $\frac{\epsilon t}{4} \leq |\Sigma|$  yields

$$(14) \quad |I| \geq \left(\frac{\epsilon t}{4}\right) - \eta t^2 - \delta t^2 = t^2 \left(\frac{\epsilon^2}{32} - \eta - \delta\right) - \frac{\epsilon t}{8} \geq t^2 \left(\frac{\epsilon^2}{32} - 2\delta\right) - \frac{\epsilon t}{8},$$

where the last inequality is because  $\eta < \delta$ . By our assumptions,  $t \geq \frac{1}{\delta} \geq T$  and  $\delta < \frac{\epsilon^2}{128}$ . These facts imply the right hand side of (14) is at least  $\frac{\epsilon^2 T}{64} - \frac{\epsilon T}{8} \geq 1$ . Thus  $I \neq \emptyset$ .

Take  $ij \in I$  and let  $W_i = N_l^G(x) \cap V_i$  and  $W_j = N_l^G(x) \cap V_j$ . Since  $\eta < \frac{\epsilon}{2}$  and  $(V_i, V_j)$  is  $\eta$ -regular for  $G$ , we have  $\rho_{r-1}^G(W_i, W_j) \geq \rho_{r-1}^G(V_i, V_j) - \eta$ . Because  $c^R(ij) = c^{R'}(ij)$ , we have that  $r-1 \in c^R(ij)$ . Therefore, by definition of  $R$ ,  $\rho_{r-1}^G(V_i, V_j) \geq d$ , so  $\rho_{r-1}^G(W_i, W_j) \geq d - \eta > 0$ , where the last inequality is by the assumption on  $\eta$ . Therefore, there is  $(x_i, x_j) \in W_i \times W_j$  such that  $d^G(x_i, x_j) = r-1$ . But now  $d^G(x, x_i) = l$ ,  $d^G(x, x_j) = l$ , and  $d^G(x_i, x_j) = r-1$  implies that  $\{x, x_i, x_j\}$  is a violating

triangle in  $G$ , a contradiction. This finishes the proof that  $A_r(n, \epsilon) \subseteq M_r(n) \setminus E_r(n, \epsilon_{4.4.13}, \eta, d)$ , so we are done.  $\square$

LEMMA 4.6.2. *Let  $r \geq 4$  be an even integer. There are  $\epsilon, \beta > 0$  and  $N$  such that  $n > N$  implies*

$$(15) \quad |A'_r(n, \epsilon)| \leq 2^{N^2 - \beta n} |C_r(n)|.$$

PROOF. All logs are base 2. Set  $\beta = \frac{1}{2}(\log m(r)^2 - \log(m(r)^2 - 2))$  and choose  $\epsilon > 0$  small enough so that

$$(16) \quad 2r(H(\epsilon) + \epsilon) - 2\beta < -\frac{3\beta}{2}.$$

Given an integer  $k$ , set

$$F(k) = \log \binom{k}{2} + \log(m(r) - 2) - 2\log(m(r)^2 - 2) + 2rk(H(\epsilon) + \epsilon) \text{ and}$$

$$F'(k) = F(k) + 3\log m(r).$$

By Corollary 4.1.6, there is  $n_0$  such that  $n > n_0$  implies

$$(17) \quad |M_r(n)| \leq 2^{(n-1)^2 - \beta n} m(r) \binom{n}{2} = 2^{(n-1)^2 - \beta n} |C_r(n)|.$$

By (16) and definition of  $F'(n)$ , there is  $n_1$  such that  $n > n_1$  implies

$$(18) \quad F'(n) - 2\beta n + 5 < -\beta n.$$

Apply Lemma 4.6.1 to  $\epsilon$  to obtain  $M_{4.6.1}$  and  $\beta_{4.6.1}$ . Choose  $N > \max\{M_{4.6.1}, n_0, n_1\}$  large enough so  $\beta_{4.6.1}(N-2)^2 > 1$ . We show by induction that for all  $n \geq N$ , (15) holds. We begin with the base cases  $n = N$  and  $n = N + 1$ . Combining (17) with the fact that for all  $n$ ,  $A'_r(n, \epsilon) \subseteq M_r(n)$  yields

$$|A'_r(N, \epsilon)| \leq |M_r(N)| \leq 2^{(N-1)^2 - \beta N} |C_r(N)| < 2^{N^2 - \beta N} |C_r(N)| \text{ and}$$

$$|A'_r(N+1, \epsilon)| \leq |M_r(N+1)| \leq 2^{N^2 - \beta(N+1)} |C_r(N+1)|.$$

Therefore (15) holds for  $n = N$  and  $n = N + 1$ . Suppose now  $n \geq N + 2$  and (15) holds for all  $m$  such that  $N \leq m \leq n - 1$ . We show it holds for  $n$ . We can construct any element  $G$  of  $A'_r(n, \epsilon)$  as follows.

- Choose a pair of elements  $xy \in \binom{[n]}{2}$ . There are  $\binom{n}{2}$  ways to do this.
- Choose  $d^G(x, y) \in [m(r) - 2]$ . There are  $m(r) - 2$  ways to do this.

- Put a structure on  $[n] \setminus \{x, y\}$ . There are at most  $|M_r(n-2)|$  ways to do this.
- For each  $l \in [m(r)-2]$ , choose  $N_l(x)$  and  $N_l(y)$ . Since  $G$  is not in  $A_r(n, \epsilon)$ , for each  $l \in [m(r)-2]$ ,  $\max\{|N_l(x)|, |N_l(y)|\} \leq \epsilon n$ . Therefore, there are at most  $\binom{n}{\epsilon n} 2^{\epsilon n} 2^{(m(r)-2)} \leq 2^{2rn(H(\epsilon)+\epsilon)}$  ways to do this.
- For each  $z \in [n] \setminus (\{x, y\} \cup \bigcup_{l=1}^{m(r)-2} N_l(x) \cup N_l(y))$ , choose  $d^G(x, z)$  and  $d^G(y, z)$ . Note that  $(d^G(x, z), d^G(y, z))$  must be chosen from  $[m(r)-1, r] \times [m(r)-1, r] \setminus \{(m(r)-1, r), (r, m(r)-1)\}$ , so there are at most  $m(r)^2 - 2$  choices.

Combining all of this we obtain that  $|A'_r(n, \epsilon)|$  is at most

$$(19) \quad \binom{n}{2} (m(r)-2) 2^{2rn(H(\epsilon)+\epsilon)} (m(r)^2 - 2)^{n-2} |M_r(n-2)| = 2^{F(n)} (m(r)^2 - 2)^n |M_r(n-2)|.$$

Because  $M_r(n-2) = C_r(n-2) \cup A_r(n-2, \epsilon) \cup A'_r(n-2, \epsilon)$ ,

$$|M_r(n-2)| = |C_r(n-2)| + |A_r(n-2, \epsilon)| + |A'_r(n-2, \epsilon)|.$$

Lemma 4.6.1 implies  $|A_r(n-2, \epsilon)| \leq |C_r(n-2)| 2^{-\beta_{4.6.1}(n-2)^2}$ , and our induction hypothesis implies  $|A'_r(n-2, \epsilon)| \leq |C_r(n-2)| 2^{N^2 - \beta(n-2)}$ . Remark 4.1.9 implies  $|C_r(n)| = m(r)^{2n-3} |C_r(n-2)|$ .

Combining these facts with (19), we obtain that

$$(20) \quad \begin{aligned} |A'_r(n, \epsilon)| &\leq 2^{F(n)} (m(r)^2 - 2)^n (1 + 2^{-\beta_{4.6.1}(n-2)^2} + 2^{N^2 - \beta(n-2)}) |C_r(n-2)| \\ &= 2^{F(n)} (m(r)^2 - 2)^n m(r)^{-2n+3} (1 + 2^{-\beta_{4.6.1}(n-2)^2} + 2^{N^2 - \beta(n-2)}) |C_r(n)| \\ &= 2^{F'(n) - 2\beta n} (1 + 2^{-\beta_{4.6.1}(n-2)^2} + 2^{N^2 - \beta(n-2)}) |C_r(n)|. \end{aligned}$$

By assumption on  $N$ ,  $-\beta_{4.6.1}(n-2)^2 < -1$ , so we have that

$$1 + 2^{-\beta_{4.6.1}(n-2)^2} + 2^{N^2 - \beta(n-2)} \leq 2 + 2^{N^2 - \beta(n-2)} \leq \begin{cases} 4 & \text{if } N^2 - \beta(n-2) \leq 1, \\ 2(2^{N^2 - \beta(n-2)}) & \text{if } N^2 - \beta(n-2) > 1. \end{cases}$$

Combining this with (20) yields that

$$|A'_r(n, \epsilon)| \leq \begin{cases} 2^{F'(n) - 2\beta n + 2} |C_r(n)| & \text{if } N^2 - \beta(n-2) \leq 1 \text{ and} \\ 2^{F'(n) - 3\beta n + N^2 + 5} |C_r(n)| & \text{if } N^2 - \beta(n-2) > 1. \end{cases}$$

In both cases we have  $|A'_r(n, \epsilon)| \leq 2^{N^2 + F'(n) - 2\beta n + 5} |C_r(n)|$ , so by (18),  $|A'_r(n, \epsilon)| \leq 2^{N^2 - \beta n} |C_r(n)|$ .

This completes the induction.  $\square$

**Proof of Theorem 4.1.2.** Fix  $r \geq 4$  an even integer. Apply Lemma 4.6.2 to obtain  $\epsilon_{4.6.2}$ ,  $\beta_{4.6.2}$  and  $N_{4.6.2}$ . Apply Lemma 4.6.1 to  $\epsilon_{4.6.2}$  to obtain  $\beta_{4.6.1}$  and  $M_{4.6.1}$ . Set  $\epsilon = \epsilon_{4.6.2}$  and  $\beta = \frac{1}{2}\beta_{4.6.2}$ . Let  $M'$  be large enough so that  $n > M'$  implies  $2^{-\beta_{4.6.1}n^2} + 2^{N_{4.6.2}^2 - \beta_{4.6.2}n} < 2^{-\beta n}$ . Set  $M = \max\{M_{4.6.1}, N_{4.6.2}, M'\}$ . For all  $n$ , by definition,  $M_r(n) \setminus C_r(n) = A_r(n, \epsilon) \cup A'_r(n, \epsilon)$ . Therefore, when  $n > M$  our assumptions imply

$$|M_r(n) \setminus C_r(n)| = |A_r(n, \epsilon)| + |A'_r(n, \epsilon)| \leq (2^{-\beta_{4.6.1}n^2} + 2^{N_{4.6.2}^2 - \beta_{4.6.2}n})|C_r(n)| < 2^{-\beta n}|C_r(n)|.$$

Rearranging yields that  $|C_r(n)| \geq |M_r(n)|(1 - 2^{-\beta n})$ , as desired.  $\square$

#### 4.7. Concluding remarks

- When  $r$  is odd, the error term in Corollary 4.1.6 cannot be strengthened from  $o(n^2)$  to  $o(1)$  (or even to  $O(n)$ ), as in Corollary 4.1.3. This can be seen by constructing a large collection of elements of  $M_r(n)$ , which will show that  $|M_r(n)|$  is at least  $m(r) \binom{n}{2}^{\Omega(n \log_{m(r)}(n))}$ . Fix  $n$  a sufficiently large integer. Define a *matching* to be a set  $S \subseteq \binom{[n]}{2}$  such that no two elements of  $S$  have nonempty intersection. Given a matching  $S$ , define  $A(S)$  to be the set of simple complete  $r$ -graphs  $G$  such that for each  $xy \in S$ ,  $d^G(x, y) = m(r) - 1$  and for each  $xy \in \binom{[n]}{2} \setminus S$ ,  $d^G(x, y) \in [m(r), r]$ . One can easily verify that for any matching  $S$ , no element of  $A(S)$  contains a violating triangle, so  $A(S) \subseteq M_r(n)$ , and that given another matching  $S' \neq S$ ,  $A(S) \cap A(S') = \emptyset$ . Further, it is clear that  $|A(S)| = m(r) \binom{n}{2}^{-|S|}$  and  $|S| \leq \frac{n}{2}$ , so  $|A(S)| \geq m(r) \binom{n}{2}^{-\frac{n}{2}}$ . Finally, note that there are at least  $(\frac{n}{2})!$  distinct matchings on  $[n]$ . This and Stirling's approximation yields that

$$|M_r(n)| \geq \left(\frac{n}{2}\right)! m(r) \binom{n}{2}^{-\frac{n}{2}} = m(r) \binom{n}{2}^{\Omega(n \log_{m(r)} n)}.$$

Combining this with Theorem 4.1.6, the best bounds we have obtained for  $|M_r(n)|$  are

$$m(r) \binom{n}{2}^{\Omega(n \log n)} \leq |M_r(n)| \leq m(r) \binom{n}{2}^{o(n^2)}.$$

We conjecture that in fact,  $|M_r(n)| = m(r) \binom{n}{2}^{\Theta(n \log n)}$ .

- It is impossible to extend Theorem 4.1.2 to the case when  $r$  is odd. Indeed, one can show that

$$|C_r(n)| \leq (1 - r^{-66r^2})|M_r(n)|.$$

The proof of this (see the appendix of [82]) in fact shows that there is an  $\mathcal{L}_r$ -sentence  $\psi$  such that for all  $n$ ,  $C_r(n) \subseteq \{G \in M_r(n) : G \models \neg\psi\}$ , and

$$(21) \quad |C_r(n)| \leq r^{65r^2} |\{G \in M_r(n) : G \models \psi\}|.$$

Suppose we knew that for some  $\alpha > 0$ ,  $|C_r(n)| \geq \alpha|M_r(n)|$  for all sufficiently large  $n$ . Then since for all  $G \in C_r(n)$ ,  $G \models \neg\psi$  we would know that

$$|\{G \in M_r(n) : G \models \neg\psi\}| \geq \alpha|M_r(n)|$$

Dividing both sides of this by  $|M_r(n)|$  gives us that  $\mu^{M_r}(\neg\psi) \geq \alpha$ , and therefore  $\mu^{M_r}(\psi) \leq 1 - \alpha$ . By dividing the quantities in (21) by  $|M_r(n)|$ , we obtain that  $|C_r(n)|/|M_r(n)| \leq \mu^{M_r}(\psi)r^{65r^2}$ , and therefore  $\alpha/r^{65r^2} \leq \mu^{M_r}(\psi)$ . Combining these inequalities, we would have that

$$0 < \frac{\alpha}{r^{65r^2}} \leq \mu^{M_r}(\psi) \leq 1 - \alpha < 1,$$

that is,  $\mu^{M_r}(\psi) \notin \{0, 1\}$ . Therefore, if we could show such an  $\alpha$  existed, we would know that  $M_r$  had no labeled first-order 0-1 law. However, we do not know that such an  $\alpha$  exists. In fact it seems likely to the authors that instead,  $\lim_{n \rightarrow \infty} |C_r(n)|/|M_r(n)| = 0$ .

#### 4.8. Appendix

**Proof of Lemma 4.4.9.** Given an integer  $r \geq 3$ , subsets  $A, B, C \subseteq [r]$ , and integers  $x, y$ , write  $H_r(A, B, C, x, y)$  to mean  $A, B, C, x, y$  satisfy the hypotheses of the lemma for  $r$ . We show by induction on  $r$  that for all  $r \geq 3$ ,  $A, B, C \subseteq [r]$ , and  $x, y \in \mathbb{N}$ ,  $H_r(A, B, C, x, y)$  implies  $A \times B \times C$  contains a violating triple.

Case  $r = 3$ : Fix  $A, B, C \subseteq [3]$ , and integers  $x, y$  such that  $H_3(A, B, C, x, y)$ . As  $m(3) = 2$  and  $3 - m(3) = 1$ , we have  $|A| = 3$ ,  $x = 1$ ,  $|B| \geq 2$ ,  $0 \leq y \leq 1$ , and  $|C| \geq \max\{2 - 1 - y + 2, 1\} = \max\{3 - y, 1\}$ . If  $y = 0$ , then  $|B| = 2$  and  $|C| \geq 3 - y = 3$ , contradicting that  $|B| \geq |C|$ . Therefore,  $y = 1$ ,  $|B| = 3$ , and  $|C| \geq 2$ . This implies that  $A = B = [3]$  and  $C \cap \{1, 3\} \neq \emptyset$ , so either  $(3, 1, 1)$  or  $(1, 1, 3)$  is in  $A \times B \times C$ , and we are done.

Case  $r > 3$ : Let  $r > 3$  and suppose by induction that the claim holds for all  $3 \leq r' < r$ . Fix  $A, B, C \subseteq [r]$  and integers  $x, y$  such that  $H_r(A, B, C, x, y)$ . Notice this implies  $x \geq y \geq 0$  and  $x \geq 1$ .

Suppose  $A, B, C \subseteq [r-1]$ . Then

$$|A| = m(r) + x = \begin{cases} m(r-1) + x + 1 & \text{if } r \text{ is even} \\ m(r-1) + x & \text{if } r \text{ is odd,} \end{cases}$$

$$|B| = m(r) + y = \begin{cases} m(r-1) + y + 1 & \text{if } r \text{ is even} \\ m(r-1) + y & \text{if } r \text{ is odd} \end{cases}$$

and

$$|C| \geq \begin{cases} \max\{m(r) - x - y, 1\} = \max\{m(r-1) - (x+1) - (y+1) + 3, 1\} & \text{if } r \text{ is even} \\ \max\{m(r) - x - y + 2, 1\} = \max\{m(r-1) - x - y + 2, 1\} & \text{if } r \text{ is odd.} \end{cases}$$

Thus,  $H_{r-1}(A, B, C, x, y)$  holds when  $r$  is odd, and  $H_{r-1}(A, B, C, x+1, y+1)$  holds when  $r$  is even, so we are done by the induction hypothesis. Assume now one of  $A, B$ , or  $C$  contains  $r$ . Let  $a = \min A$ ,  $b = \min B$ ,  $c = \min C$ ,  $a' = \max A$ ,  $b' = \max B$ , and  $c' = \max C$ . Our assumptions imply that

$$(22) \quad a \leq r - |A| + 1 = r - (m(r) + x) + 1 = \begin{cases} m(r) - 1 - x & \text{if } r \text{ is even} \\ m(r) - x & \text{if } r \text{ is odd,} \end{cases}$$

and

$$b \leq r - |B| + 1 \leq r - (m(r) + y) + 1 = \begin{cases} m(r) - 1 - y & \text{if } r \text{ is even} \\ m(r) - y & \text{if } r \text{ is odd.} \end{cases}$$

Thus

$$a + b \leq \begin{cases} m(r) - 1 - x + m(r) - 1 - y = r - x - y & \text{if } r \text{ is even} \\ m(r) - x + m(r) - y = r - x - y + 1 & \text{if } r \text{ is odd.} \end{cases}$$

If

$$c' > \begin{cases} r - x - y & \text{if } r \text{ is even} \\ r - x - y + 1 & \text{if } r \text{ is odd,} \end{cases}$$

then  $(a, b, c')$  is a violating triple and we are done. So assume

$$(23) \quad c' \leq \begin{cases} r - x - y & \text{if } r \text{ is even} \\ r - x - y + 1 & \text{if } r \text{ is odd.} \end{cases}$$

Note that

$$c \leq \begin{cases} r - x - y - |C| + 1 \leq r - x - y - (m(r) - x - y) + 1 = m(r) - 1 & \text{if } r \text{ is even} \\ r - x - y + 1 - |C| + 1 \leq r - x - y + 1 - (m(r) - x - y + 2) + 1 = m(r) - 1 & \text{if } r \text{ is odd.} \end{cases}$$

Therefore

$$c + a \leq \begin{cases} m(r) - 1 + m(r) - 1 - x = r - x & \text{if } r \text{ is even} \\ m(r) - 1 + m(r) - x = r - x & \text{if } r \text{ is odd.} \end{cases}$$

If  $b' > r - x$ , then  $(a, b', c)$  is a violating triple and we are done. So assume  $b' \leq r - x$ . Because  $x \geq 1$ , this implies  $r \notin B$ . Further,

$$(24) \quad b \leq \begin{cases} r - x - (m(r) + y) + 1 = m(r) - x - y - 1 & \text{if } r \text{ is even} \\ r - x - (m(r) + y) + 1 = m(r) - x - y & \text{if } r \text{ is odd.} \end{cases}$$

Suppose  $r \notin C$ . Then we must have that  $a' = r \in A$ . Therefore,

$$a' - b \geq \begin{cases} r - (m(r) - x - y - 1) = m(r) + x + y - 1 & \text{if } r \text{ is even} \\ r - (m(r) - x - y) = m(r) + x + y - 1 & \text{if } r \text{ is odd.} \end{cases}$$

We now have  $c \leq m(r) - 1 < m(r) + x + y - 1 \leq a' - b$ , so  $(a', b, c)$  is a violating triple, and we are done.

Suppose now  $c' = r \in C$ . By (23), this implies that  $r$  is odd,  $x = 1$  and  $y = 0$ . By (24),  $b \leq m(r) - 1$ .

Therefore,

$$c' - b \geq r - (m(r) - 1) = m(r) > m(r) - x,$$

so by (22),  $(a, b, c')$  is a violating triple. This completes the induction.  $\square$

**Proof of Lemma 4.4.14.** We proceed by induction on  $r \geq 3$ . The base case  $r = 3$  can easily be verified. Suppose now the claim holds for all  $3 \leq r' < r$ . Set  $A' = A \cap [r - 1]$ ,  $B' = B \cap [r - 1]$ , and  $C' = C \cap [r - 1]$ .

Suppose that  $r$  is odd. If  $A, B, C \subseteq [r - 1]$ , then because  $|A| = |B| = |C| = m(r) = m(r - 1)$ , the induction hypothesis implies that  $A = B = C = [m(r - 1) - 1, r - 1] = [m(r) - 1, r - 1]$ , i.e. case (2)(a) holds. Suppose now at least one of  $A, B$ , or  $C$  contain  $r$ . By relabeling if necessary, we may assume  $r \in A$ . Let  $a' = r \in A$ ,  $b = \min B$  and  $c = \min C$ . Then  $b \leq r - |B| + 1 = m(r)$ . Therefore  $c \geq a' - b \geq r - m(r) = m(r) - 1$ , so  $C \subseteq [m(r) - 1, r]$ . Similarly,  $c \leq r - |C| + 1 = m(r)$ , so  $b \geq a' - c \geq r - m(r) = m(r) - 1$  implies  $B \subseteq [m(r) - 1, r]$ . If  $b = c = m(r) - 1$ , then  $(a', b, c)$  is



a violating triple, a contradiction. Thus as most one of  $b$  or  $c$  is  $m(r) - 1$ . Therefore, by relabeling if necessary, we may assume  $B \subseteq [m(r), r]$  and  $C \subseteq [m(r) - 1, r]$ . Recall that  $|[m(r) - 1, r]| = m(r) + 1 = |B| + 1$ , so this implies that  $B = [m(r), r]$ . Let  $a = \min A$ . Then  $r \in B$  and  $c \leq m(r)$  implies  $a \geq r - m(r) = m(r) - 1$ , so  $A \subseteq [m(r) - 1, r]$ . If  $C = [m(r), r]$ , then we are done. If  $C \neq [m(r), r]$ , then  $c < m(r)$  implies  $(m(r) - 1, r, c)$  is a violating triple, so  $m(r) - 1 \notin A$ . Thus  $A \subseteq [m(r), r]$  and  $|A| = |[m(r), r]|$  implies  $A = [m(r), r]$  and we are done.

Suppose now that  $r$  is even. Note that  $\min\{|A'|, |B'|, |C'|\} \geq m(r) - 1$ . If two elements of the set  $\{|A'|, |B'|, |C'|\}$  are strictly greater than  $m(r) - 1 = m(r - 1)$ , then Lemma 4.4.9 implies there is a violating triple in  $A' \times B' \times C'$ , a contradiction. Therefore by relabeling if necessary, we may assume  $|A'| = |B'| = m(r) - 1$ , so  $r \in A \cap B$ . Let  $a = \min A$ ,  $b = \min B$ ,  $c = \min C$  and note that  $\max\{a, b, c\} \leq r - m(r) + 1 = m(r) - 1$ . Now  $(a, r, c)$  and  $(r, b, c)$  cannot be violating triples, so

$$a \geq r - c \geq r - (m(r) - 1) = m(r) - 1,$$

$$c \geq r - b \geq r - (m(r) - 1) = m(r) - 1 \text{ and}$$

$$b \geq r - c \geq r - (m(r) - 1) = m(r) - 1.$$

Thus,  $A, B, C \subseteq [m(r) - 1, r]$ . Since  $|A| = |B| = |C| = |[m(r) - 1, r]|$ , this implies  $A = B = C = [m(r) - 1, r]$ .  $\square$

**4.8.1. Proof that when  $r$  is odd,  $C_r(n)$  is not almost all of  $M_r(n)$ .** Fix  $r \geq 3$  an odd integer for the rest of this section. In this section we show that it is not the case that almost all elements of  $M_r(n)$  are in  $C_r(n)$  by constructing, for each integer  $n \geq 4$ , a map  $f : C_r(n) \rightarrow M_r(n) \setminus C_r(n)$  which is at most  $r^{65r^2}$ -to-1. This will imply that for all  $n \geq 4$ ,

$$(25) \quad |C_r(n)| \leq (1 - r^{-66r^2})|M_r(n)|.$$

We start with some preliminary definitions. Given an integer  $n$  and  $X, Y$  disjoint subsets of  $[n]$ , set  $X <_* Y$  if and only if

$$(i) \quad |X| < |Y| \text{ or}$$

$$(ii) \quad |X| = |Y| \text{ and } \min X < \min Y.$$

DEFINITION 4.8.1. Fix an integer  $n \geq 3$  and  $G \in M_r(n)$ .

- (1) A set  $X \subseteq [n]$  is a *component* of  $G$  if for all  $xy \in \binom{X}{2}$ , there is a sequence  $(z_1, \dots, z_k)$  of distinct elements of  $X$  such that  $x = z_1$ ,  $y = z_k$ , and for each  $1 \leq i \leq k-1$ ,  $d^G(z_i, z_{i+1}) = m(r) - 1$ .
- (2) A *component decomposition* of  $G$  is a partition  $X_1, \dots, X_l$  of  $[n]$  such that each  $X_i$  is a component of  $G$ . Note that there is a unique component decomposition of  $G$ , up to relabeling.
- (3) If  $X_1, \dots, X_l$  is the component decomposition of  $G$  and  $X_1 <_* \dots <_* X_l$ , we say  $X_1, \dots, X_l$  is the *canonically ordered component decomposition* (c.o.c.d.) of  $G$ .
- (4) A component  $X$  of  $G$  is *large* if  $|X| \geq 2r$ . Otherwise it is *small*.
- (5) Suppose  $X_1, \dots, X_l$  is the c.o.c.d. of  $G$ . The *minimal large component* of  $G$  is

$$ML(G) = \begin{cases} \emptyset & \text{if } \max\{|X_1|, \dots, |X_l|\} < 2r, \\ X_i \text{ where } i = \min\{j \in [l] : |X_j| \geq 2r\} & \text{otherwise.} \end{cases}$$

- (6)  $H$  is the simple complete  $r$ -graph with vertex set  $[4]$  such that  $d^H(1, 3) = d^H(2, 4) = r - 1$ ,  $d^H(1, 4) = r$ , and  $d^H(1, 2) = d^G(2, 3) = d^H(3, 4) = m(r) - 1$ .
- (7) A *bad cycle in  $G$*  is a sequence  $(z_1, \dots, z_k)$  of distinct elements of  $[n]$  such that for each  $1 \leq i \leq k-1$ ,  $d^G(z_i, z_{i+1}) = m(r) - 1$  and  $d^G(z_1, z_k) = r$ . Say  $G$  *contains a bad cycle* if there are  $z_1, \dots, z_k \in [n]$  such that  $(z_1, \dots, z_k)$  is a bad cycle in  $G$ .

LEMMA 4.8.2.  $H \in M_r(4)$ , and for any integers  $n \geq k \geq 4$ , if  $G \in M_r(n)$  contains a bad cycle, then  $G \in M_r(n) \setminus C_r(n)$ . In particular, if  $G \in M_r(n)$  and  $G$  contains a copy of  $H$ , then  $G \notin C_r(n)$ .

PROOF. That  $H$  contains no violating triangles and is therefore in  $M_r(4)$  can be checked easily. Suppose now  $n \geq k \geq 4$ ,  $G \in M_r(n)$ , and  $(y_1, \dots, y_k)$  is a bad cycle in  $G$ . Suppose for a contradiction that  $G \in C_r(n)$ . Then there is a partition  $\mathcal{P} = \{V_1, \dots, V_t\}$  of  $[n]$  such that for all  $xy \in \binom{[n]}{2}$ ,

$$d^G(x, y) \in \begin{cases} [m(r), r] & \text{if } xy \in E(V_i, V_j) \text{ some } 1 \leq i < j \leq t, \\ [m(r) - 1, r - 1] & \text{if } xy \in \binom{V_i}{2} \text{ some } 1 \leq i \leq t. \end{cases}$$

Note that for all  $xy \in \binom{[n]}{2}$ , if  $x$  and  $y$  are in the same component of  $G$ , then they are in the same element of  $\mathcal{P}$ . Fix  $1 \leq i \leq t$  such that  $y_1 \in V_i$ . Then  $d^G(y_1, y_2) = \dots = d^G(y_{k-1}, y_k) = m(r) - 1$  implies  $y_k$  is in the same component of  $G$  as  $y_1$ , so  $y_k \in V_i$ . This implies  $d^G(y_1, y_k) \in [m(r) - 1, r - 1]$ . Because  $(y_1, \dots, y_k)$  is a bad cycle in  $G$ , by definition,  $d^G(y_1, y_k) = r$ , a contradiction. Since

$H$  contains a bad cycle, it follows immediately that if  $G \in M_r(n)$  contains a copy of  $H$ , then  $G \notin C_r(n)$ .  $\square$

Suppose  $n$  is an integer and  $G \in M_r(n)$ . Given  $X \subseteq [n]$ , let  $G[X]$  denote the simple complete  $r$ -graph with vertex set  $X$  such that for all  $xy \in E(X)$ ,  $d^{G[X]}(x, y) = d^G(x, y)$ . Set

$$D_1(n) = \{G \in C_r(n) : \text{the c.o.c.d. of } G \text{ has at least 4 small components}\},$$

$$D_2(n) = \{G \in C_r(n) \setminus D_1(n) : \text{if } \{y_1, \dots, y_4\} \text{ are the least four elements of } ML(G),$$

then  $G[ML(G) \setminus \{y_1, \dots, y_4\}]$  has at most 3 large components\},

$$D_3(n) = C_r(n) \setminus (D_1(n) \cup D_2(n)).$$

We are now ready to define our map  $f$ .

DEFINITION 4.8.3. Given  $n \geq 4$  and  $G \in C_r(n)$ , define  $f(G)$  to be the simple complete  $r$ -graph with vertex set  $[n]$  satisfying the following, where  $Y_1, \dots, Y_u$  denotes the c.o.c.d. of  $G$ .

- (1) If  $G \in D_1(n)$ , set  $Y = \bigcup_{i=1}^4 Y_i$ , and for each  $i \in [4]$ , set  $y_i = \min Y_i$ . Given  $xy \in \binom{[n]}{2}$ , set

$$d^{f(G)}(x, y) = \begin{cases} d^H(i, j) & \text{if } xy = y_i y_j \in \binom{\{y_1, \dots, y_4\}}{2}, \\ r - 1 & \text{if } xy \in \binom{Y}{2} \setminus \binom{\{y_1, \dots, y_4\}}{2}, \\ d^G(x, y) & \text{otherwise.} \end{cases}$$

- (2) If  $G \in D_2(n)$ , let  $s \in [4]$  be such that  $Y_s = ML(G)$  and let  $y_1 < y_2 < y_3 < y_4$  be the least four elements of  $Y_s$ . Set  $Y = \bigcup_{i=1}^{s-1} Y_i$  and  $Y'_s = Y_s \setminus \{y_1, \dots, y_4\}$ . Given  $xy \in \binom{[n]}{2}$ , set

$$d^{f(G)}(x, y) = \begin{cases} d^H(i, j) & \text{if } xy = y_i y_j \in \binom{\{y_1, \dots, y_4\}}{2}, \\ r & \text{if } xy \in \binom{Y}{2} \cup E(Y, \{y_1, \dots, y_4\}) \\ d^G(x, y) + 1 & \text{if } xy = y_i z \text{ for some } y_i \in \{y_1, \dots, y_4\} \text{ and } z \in Y'_s, \\ d^G(x, y) & \text{otherwise.} \end{cases}$$

Note that any small component of  $f(G)$  is either a singleton coming from  $Y$ , or is a small component of  $f(G)[Y'_s]$ . If  $X$  is a small component of  $f(G)[Y'_s]$ , then since  $X$  and  $\{y_1, \dots, y_4\}$  were in the same component of  $G$ , there must be  $x \in X$  and  $y \in \{y_1, \dots, y_4\}$  such that  $d^G(x, y) = m(r) - 1$ , and thus,  $d^{f(G)}(x, y) = m(r)$ . In particular, if  $X = \{x\}$  is a singleton, then for some  $y \in \{y_1, \dots, y_4\}$ ,  $d^{f(G)}(x, y) = m(r)$ . On the other hand,

if  $X = \{x\}$  is a singleton coming from  $Y$ , then by construction, for all  $y \in \{y_1, \dots, y_4\}$ ,  $d^{f(G)}(x, y) = r$ .

- (3) If  $G \in D_3(n)$ , let  $s \in [4]$  be such that let  $Y_s = ML(G)$  and let  $y_1 < y_2 < y_3 < y_4$  be the least four elements of  $Y_s$ . Set  $Y = \bigcup_{i=1}^{s-1} Y_i$  and  $Y'_s = Y_s \setminus \{y_1, \dots, y_4\}$ . Let  $Z^1, \dots, Z^k$  be the large components of  $G[Y'_s]$  listed so that  $Z^1 <_* \dots <_* Z^k$ . Note that for each  $i, j \in \binom{[k]}{2}$  and  $xy \in E(Z^i, Z^j)$ , because  $Z^i$  and  $Z^j$  are different components in  $G[Y'_s]$ ,  $d^G(x, y) \neq m(r) - 1$ . Since  $Z^i$  and  $Z^j$  are contained in the same component of  $G$ , we know  $d^G(x, y) \in [m(r) - 1, r - 1]$ . Therefore we must have  $d^G(x, y) \in [m(r), r - 1]$ . Enumerate each  $Z^i = \{z_1^i, \dots, z_{|Z^i|}^i\}$  in increasing order.

We inductively build a sequence  $i_1, \dots, i_k$  with the following properties:

- (i) For each  $1 \leq j \leq k$ ,  $i_j \in [2r]$ .
- (ii) For each  $2 \leq j \leq k - 1$ ,  $d^G(z_{i_{j-1}}^{j-1}, z_{i_j}^j) = |i_j - i_{j+1}| \in [r]$ .

Set  $i_1 = i_2 = 1$ ,  $i_3 = d^G(z_{i_1}^1, z_{i_2}^2) + 1$ . Notice  $1 \leq i_3 \leq (r - 1) + 1 = r$ , so  $i_3 \in [2r]$ , and by construction,  $i_3 - i_2 = d^G(z_{i_1}^1, z_{i_2}^2) \in [r]$ , so (i) and (ii) are satisfied for  $j = 1, 2$ . Suppose we've defined  $i_1, \dots, i_j$  for  $2 \leq j < k$  such that (i) and (ii) hold for  $j - 1$ . Set

$$i_{j+1} = \begin{cases} i_j + d^G(z_{i_{j-1}}^{j-1}, z_{i_j}^j) & \text{if } i_j \leq r, \\ i_j - d^G(z_{i_{j-1}}^{j-1}, z_{i_j}^j) & \text{if } i_j > r. \end{cases}$$

By the induction hypothesis,  $i_j \in [2r]$ , so by the above definition, if  $i_j \leq r$ , then  $i_{j+1} \in [2, 2r]$  and if  $i_j > r$ , then  $i_{j+1} \in [1, 2r - 1]$ . In either case,  $i_{j+1} \in [2r]$  so (i) is satisfied for  $j + 1$ . We also have that (ii) is satisfied by  $j + 1$  since by definition,

$$|i_j - i_{j+1}| = d^G(z_{i_{j-1}}^{j-1}, z_{i_j}^j) \in [r].$$

This completes the construction of  $i_1, \dots, i_k$ . Given  $xy \in \binom{[n]}{2}$ , set

$$d^{f(G)}(x, y) = \begin{cases} r & \text{if } xy \in \binom{Y}{2} \cup E(Y, \{y_1, \dots, y_4\}) \cup \{z_{i_1}^1 z_{i_k}^k\}, \\ d^H(i, j) & \text{if } xy = y_i y_j \in \binom{\{y_1, \dots, y_4\}}{2}, \\ m(r) - 1 & \text{if } xy = z_{i_j}^j z_{i_{j+1}}^{j+1} \in \{z_{i_1}^1 z_{i_2}^2, \dots, z_{i_{k-1}}^{k-1} z_{i_k}^k\}, \\ d^G(x, y) + 1 & \text{if } xy = y_i z \text{ for some } y_i \in \{y_1, \dots, y_4\} \text{ and } z \in Y'_s, \\ d^G(x, y) & \text{otherwise.} \end{cases}$$

Note that the same remarks as above for the case when  $G \in D_2(n)$  apply here. That is, if  $X = \{x\}$  is a singleton component of  $f(G)$ , then either  $d^{f(G)}(x, y) = r$  for all  $y \in \{y_1, \dots, y_4\}$  in which case  $x$

is an element in a small component of  $G$ , or there is  $y \in \{y_1, \dots, y_4\}$  such that  $d^{f(G)}(x, y) = m(r)$ , in which case  $x$  is an element of  $ML(G) \setminus \{y_1, \dots, y_4\}$ .

LEMMA 4.8.4. *Let  $n \geq 4$  be an integer and  $G \in C_r(n)$ . Then  $f(G) \in M_r(n) \setminus C_r(n)$ .*

PROOF. By definition,  $f(G)$  must contain a copy of  $H$ , so  $f(G)$  is not in  $C_r(n)$  by Lemma 4.8.2. We now show  $f(G) \in M_r(n)$ . We leave the verification of the case when  $G \in D_1(n)$  to the reader, since it requires only the simplest types of arguments which we show below for the other cases. So assume  $G \in D_2(n) \cup D_3(n)$ . Let  $Y_1, \dots, Y_u$  be the c.o.c.d. of  $G$ , let  $s$  be such that  $Y_s = ML(G)$ , and let  $y_1 < \dots < y_4$  be the least elements of  $Y_s$ . Set  $Y = \bigcup_{i=1}^{s-1} Y_i$  and  $Y'_s = Y_s \setminus \{y_1, \dots, y_4\}$ . It suffices to show that if  $x, y, z \in [n]$  are pairwise distinct and  $E(\{x, y, z\}) \cap \Delta(G, f(G)) \neq \emptyset$ , then  $\{x, y, z\}$  is not a violating triangle in  $f(G)$ , or equivalently,  $(d^{f(G)}(x, y), d^{f(G)}(y, z), d^{f(G)}(x, z))$  is not a violating triple. We consider only the cases where  $\{x, y, z\} \subseteq Y_s$ , as the rest of the cases are similar to these or trivial.

Fix  $x, y, z \in [n]$  pairwise distinct such that  $E(\{x, y, z\}) \cap \Delta(G, f(G)) \neq \emptyset$ ,  $\{x, y, z\} \subseteq Y_s$ . If  $\{x, y, z\} \subseteq \{y_1, \dots, y_4\}$ , let  $i, j, k \in [4]$  be such that  $x = y_i$ ,  $y = y_j$ ,  $z = y_k$ . Then by definition of  $f(G)$ ,  $\{x, y, z\}$  is a violating triangle in  $f(G)$  if and only if  $\{i, j, k\}$  is a violating triangle in  $H$ . Since, by Lemma 4.8.2,  $H$  contains no violating triangles, we are done. If  $x, y \in \{y_1, \dots, y_4\}$  and  $z \in Y'_s$  or if  $x, y \in Y'_s$  and  $z \in \{y_1, \dots, y_4\}$ , then by definition,

$$d^{f(G)}(x, z) = d^G(x, z) + 1, \quad d^{f(G)}(y, z) = d^G(y, z) + 1, \quad \text{and} \quad d^{f(G)}(x, y) \in [m(r) - 1, r].$$

Because  $x, y, z$  were in the same component of  $G$ ,  $d^G(x, z), d^G(y, z) \in [m(r) - 1, r - 1]$ . Therefore

$$(d^{f(G)}(x, z), d^{f(G)}(y, z), d^{f(G)}(x, y)) \in [m(r), r] \times [m(r), r] \times [m(r) - 1, r],$$

which contains no violating triples. Up to relabeling, this leaves us with the case where  $\{x, y, z\} \subseteq Y'_s$ . This case is vacuous when  $G \in D_2(n)$ , because for  $G \in D_2(n)$ ,  $E(Y'_s) \cap \Delta(G, f(G)) = \emptyset$ . So we are left with the case when  $G \in D_3(n)$  and  $\{x, y, z\} \subseteq Y'_s$ .

Let  $Z^1 <_* \dots <_* Z^k$  be the c.o.c.d. of  $G[Y'_s]$ , and for  $1 \leq j \leq k$ , let  $z_{i_j}^j \in Z^j$  be as in the definition of  $f(G)$ . We must have  $E(\{x, y, z\}) \cap \{z_{i_1}^1 z_{i_2}^2, \dots, z_{i_{k-1}}^{k-1} z_{i_k}^k, z_{i_1}^1 z_{i_k}^k\} \neq \emptyset$  since otherwise  $E(\{x, y, z\}) \cap \Delta(G, f(G)) = \emptyset$ . Assume that  $xy \in \{z_{i_1}^1 z_{i_2}^2, \dots, z_{i_{k-1}}^{k-1} z_{i_k}^k, z_{i_1}^1 z_{i_k}^k\}$ , and note this implies  $d^{f(G)}(x, y) \in \{r, m(r) - 1\}$ .

If  $z \in \{z_{i_1}^1, \dots, z_{i_k}^k\}$ , and  $xz, yz \notin \{z_{i_1}^1 z_{i_2}^2, \dots, z_{i_{k-1}}^{k-1} z_{i_k}^k, z_{i_1}^1 z_{i_k}^k\}$ , then by definition of  $f$ ,

$$d^{f(G)}(x, z) = d^G(x, z) \quad \text{and} \quad d^{f(G)}(y, z) = d^G(y, z).$$

Because  $z$  is the same component of  $G$  as  $x$  and  $y$ ,  $d^G(x, z), d^G(y, z) \in [m(r) - 1, r - 1]$ . Because  $z$  is in a different component of  $G[Y'_s]$  than both  $x$  and  $y$ ,  $d^G(x, z), d^G(y, z) \neq m(r) - 1$ . Therefore

$$d^G(x, z), d^G(y, z) \in [m(r), r - 1], \text{ so}$$

$$(d^{f(G)}(x, z), d^{f(G)}(y, z), d^{f(G)}(x, y)) \in [m(r), r - 1] \times [m(r), r - 1] \times \{m(r) - 1, r\},$$

which contains no violating triples. If  $z \in \{z_{i_1}^1, \dots, z_{i_k}^k\}$  and  $xz \in \{z_{i_1}^1 z_{i_2}^2, \dots, z_{i_{k-1}}^{k-1} z_{i_k}^k, z_{i_1}^1 z_{i_k}^k\}$ , then since  $k \geq 4$ , this implies  $yz \notin \{z_{i_1}^1 z_{i_2}^2, \dots, z_{i_{k-1}}^{k-1} z_{i_k}^k, z_{i_1}^1 z_{i_k}^k\}$ . By definition,  $d^{f(G)}(x, z) \in \{m(r) - 1, r\}$ , and as above, because  $y$  and  $z$  are in the same component of  $G$  but different components of  $G[Y'_s]$ ,  $d^{f(G)}(y, z) = d^G(y, z) \in [m(r), r - 1]$ . Therefore

$$(d^{f(G)}(x, z), d^{f(G)}(y, z), d^{f(G)}(x, y)) \in \{m(r) - 1, r\} \times [m(r), r - 1] \times \{m(r) - 1, r\},$$

which contains no violating triples. Up to relabeling we have now covered the cases where  $z \in \{z_{i_1}^1, \dots, z_{i_k}^k\}$ , so assume  $z \in Y'_s \setminus \{z_{i_1}^1, \dots, z_{i_k}^k\}$ . Then by definition,

$$d^{f(G)}(x, z) = d^G(x, z), \quad d^{f(G)}(y, z) = d^G(y, z) \in [m(r) - 1, r - 1].$$

If  $z$  is in the same component of  $G[Y'_s]$  as  $x$ , then  $y$  and  $z$  are in the same component of  $G$  but different components of  $G[Y'_s]$ , so  $d^G(y, z) \neq m(r) - 1$ . Therefore we have that

$$(d^{f(G)}(x, z), d^{f(G)}(y, z), d^{f(G)}(x, y)) \in [m(r) - 1, r - 1] \times [m(r), r - 1] \times \{m(r) - 1, r\},$$

which contains no violating triples. A similar argument covers the case where  $z$  is instead in the same component of  $G[Y'_s]$  as  $y$ . If  $z$  is in a different component of  $G[Y'_s]$  than  $x$  and  $y$ , then

$$d^G(x, z), d^G(y, z) \neq m(r) - 1 \text{ so}$$

$$(d^{f(G)}(x, z), d^{f(G)}(y, z), d^{f(G)}(x, y)) \in [m(r), r - 1] \times [m(r), r - 1] \times \{m(r) - 1, r\},$$

which contains no violating triples. This completes the proof.  $\square$

We will use the following lemmas. Given  $K \subseteq M_r(n)$ , set  $f^{-1}(K) = \{G \in C_r(n) : f(G) \in K\}$ .

LEMMA 4.8.5. *Let  $n \geq 4$  be an integer. For all  $G \in f(D_1(n))$ , there is  $E \subseteq \binom{[n]}{2}$  such that  $|E| \leq \binom{4+8r}{2}$  and for all  $G' \in f^{-1}(G) \cap D_1(n)$ ,  $\Delta(G, G') \subseteq E$ .*

PROOF. Suppose  $G \in f(D_1(n))$  and  $X_1, \dots, X_l$  is the c.o.c.d. of  $G$ . Suppose  $X_1, \dots, X_s$  enumerate the components which are singletons and  $X_w$  is the unique component such that  $G[X_w]$  contains a copy of  $H$ . Then by definition of  $f$ , for any  $G' \in C_r(n)$ ,  $f(G') = G$  implies

$$\Delta(G, G') \subseteq E(X_w \cup \bigcup_{i=1}^s X_i).$$

If  $s \leq 8r$ , then set  $E = E(X_w \cup \bigcup_{i=1}^s X_i)$ . Since in this case,

$$|E(X_w \cup \bigcup_{i=1}^s X_i)| \leq \binom{4+s}{2} \leq \binom{4+8r}{2},$$

we are done. Assume now  $s > 8r$ . Let  $G' \in f^{-1}(G) \cap D_1(n)$  and let  $Y_1, \dots, Y_u$  be the c.o.c.d. of  $G'$ . For  $i \in [4]$ , let  $\{y_i\} = \min Y_i$  and  $Y = \bigcup_{i=1}^4 Y_i$ . By definition of  $f$ ,  $\Delta(G, G') \subseteq \binom{Y}{2}$ . Note that for each  $1 \leq j \leq 4$ ,  $Y_j$  has size at most  $2r - 1$ , so  $|Y| \leq 4(2r - 1) < 8r$ . Since  $s > 8r$ , there is some  $1 \leq i \leq s$  such that  $X_i \cap Y = \emptyset$ . Combining this with the fact that  $\Delta(G, G') \subseteq \binom{Y}{2}$ , yields that

$$X_i \in \{Y_5, \dots, Y_u\},$$

say  $X_i = Y_k$ , some  $5 \leq k \leq u$ . Then  $|Y_k| = 1$  and  $Y_k >_* Y_4 >_* \dots >_* Y_1$  implies by definition of  $<_*$  that  $|Y_4| = |Y_3| = |Y_2| = |Y_1| = 1$ . Therefore  $Y = \{y_1, \dots, y_4\} = X_w$ , and  $\Delta(G', G) \subseteq \binom{X_w}{2}$ . Setting  $E = \binom{X_w}{2}$  we are done, as  $|X_w| = 4$ .

□

LEMMA 4.8.6. *Let  $n \geq 4$  be an integer. For all  $G \in f(D_2(n))$ , there are  $G_1, \dots, G_8 \in D_2(n)$  and  $E \subseteq \binom{[n]}{2}$  such that  $f(G_1) = \dots = f(G_8) = G$ ,  $|E| \leq \binom{4+6r}{2}$ , and for all  $G' \in f^{-1}(G) \cap D_2(n)$ , there is  $1 \leq t \leq 8$  such that  $\Delta(G_t, G') \subseteq E$ .*

PROOF. Suppose  $G \in f(D_2(n))$  and  $X_1, \dots, X_l$  is the c.o.c.d. of  $G$ . Let  $t$  be such that  $ML(G) = X_t$ . By definition of  $f$ , there is a unique index  $1 \leq w \leq l$  such that  $G[X_w]$  consists of a copy of  $H$ . There is also be a unique (possibly empty) sequence  $1 \leq i_1 < \dots < i_v < w$  with the following properties:

- For each  $1 \leq j \leq v$ ,  $X_{i_j} = \{x_{i_j}\}$  is a singleton, and
- For each  $1 \leq j \leq v$ , for each  $y \in X_w$ ,  $d^G(x_{i_j}, y) = r$ , and
- For all  $j \notin \{i_1, \dots, i_v\}$ , if  $X_j = \{x_j\}$  is a singleton, then for some  $y \in X_w$ ,  $d^G(x, y) = m(r)$ .

Suppose  $G' \in f^{-1}(G) \cap D_2(n)$ . Suppose  $Y_1, \dots, Y_u$  is the c.o.c.d. of  $G'$  and  $s$  is such that  $ML(G') = Y_s$ . By definition of  $f$  on  $D_2(n)$ , we must have that  $X_w$  consists of the least 4 elements of  $Y_s$ . By the discussion following the definition of  $f$  on  $D_2(n)$ ,

$$\bigcup_{j=1}^v X_{i_j} = \bigcup_{i=1}^{s-1} Y_i,$$

and the small components of  $G'[Y_s \setminus X_w]$  are exactly the elements of  $\{X_1, \dots, X_{t-1}\} \setminus \{X_{i_1}, \dots, X_{i_v}, X_w\}$ . Notice that by definition of  $D_2(n)$ ,  $s \leq 4$ , so

$$\left| \bigcup_{j=1}^v X_{i_j} \right| = \left| \bigcup_{i=1}^{s-1} Y_i \right| \leq 3(2r-1) < 6r.$$

By definition of  $f$ , we have that

$$(26) \quad \{X_t, \dots, X_l\} = \{Y_{s+1}, \dots, Y_u\} \cup \{\text{the large components of } G'[Y_s \setminus X_w]\}.$$

If  $X_i$  is a large component of  $G'[Y_s \setminus X_w]$ , then  $|X_i| \leq |Y_s \setminus X_w| < |Y_s| \leq |Y_{s+1}| \leq |Y_u|$ . So by definition of  $<_*$ ,

$$(27) \quad X_i <_* Y_{s+1} <_* \dots <_* Y_u.$$

By definition of  $D_2(n)$  there are at most 3 large components of  $G'[Y_s \setminus X_w]$ . Combining this with (26) and (27), we have that the large components of  $G'[Y_s \setminus X_w]$  are contained in  $\{X_t, X_{t+1}, X_{t+2}\}$  (where we let  $X_i = \emptyset$  if  $i > l$ ). In sum, for any  $G' \in f^{-1}(G) \cap D_2(n)$ , we have the following.

- (i)  $X_w$  consists of the least 4 elements of  $ML(G')$ ,
- (ii)  $\bigcup_{j=1}^v X_{i_j}$  is the union of the small components of  $G'$  and has size strictly less than  $6r$ ,
- (iii) The small components of  $G'[ML(G') \setminus X_w]$  are the elements of  $\{X_1, \dots, X_{t-1}\} \setminus \{X_{i_1}, \dots, X_{i_v}, X_w\}$ ,
- (iv) The set of large components of  $G'[ML(G') \setminus X_w]$  is some subset  $S$  of  $\{X_t, X_{t+1}, X_{t+2}\}$ .

Set  $E = E(X_w \cup \bigcup_{j=1}^v X_{i_j})$ , and given  $S \subseteq \{X_t, X_{t+1}, X_{t+2}\}$ , set

$$X_S = \left( \bigcup_{X_i \in S} X_i \right) \cup \left( \bigcup_{j \in [t-1] \setminus \{i_1, \dots, i_v\}} X_j \right).$$

Then (iii) and (iv) show that for all  $G' \in f^{-1}(G) \cap D_2(n)$ , there is  $S \subseteq \{X_t, X_{t+1}, X_{t+2}\}$  such that  $ML(G') = X_S$ . Moreover, given such a  $G'$  and  $S$ , by definition of  $f$  and (i)-(iv),

- $\Delta(G, G') \subseteq E \cup E(X_w, X_S)$  and
- For all  $xy \in E(X_w, X_S)$ ,  $d^{G'}(x, y) = d^G(x, y) - 1$ .



Therefore, for all other  $G'' \in f^{-1}(G) \cap D_2(n)$  such that  $ML(G'') = X_S$ , we have that for all  $xy \in E(X_w, X_S)$ ,  $d^{G''}(x, y) = d^G(x, y) - 1 = d^{G'}(x, y)$ , so  $\Delta(G', G'') \cap E(X_w, X_S) = \emptyset$ . This implies that

$$\Delta(G', G'') \subseteq (\Delta(G', G) \cup \Delta(G'', G)) \setminus E(X_w, X_S) \subseteq E.$$

We now define  $G_1, \dots, G_8$ . Let  $S_1, \dots, S_8$  enumerate the subsets of  $\{X_t, X_{t+1}, X_{t+2}\}$ . For each  $1 \leq i \leq 8$ , if there is  $G' \in f^{-1}(G) \cap D_2(n)$  such that  $ML(G') = X_{S_i}$ , choose  $G_i$  to be such a  $G'$ . If no such  $G'$  exists, choose  $G_i$  to be any element of  $D_2(n)$ . By what we've shown, for all  $G' \in f^{-1}(G) \cap D_2(n)$ , there is  $1 \leq i \leq 8$  such that  $ML(G') = X_{S_i}$ , and therefore  $\Delta(G', G_i) \subseteq E$ . By (ii),  $|\bigcup_{j=1}^v X_{i_j}| < 6r$ , so  $|X_w \cup \bigcup_{j=1}^v X_{i_j}| < 4 + 6r$  and  $|E| \leq \binom{4+6r}{2}$ . This completes the proof.  $\square$

LEMMA 4.8.7. *Let  $n \geq 4$  be an integer. For all  $G \in f(D_3(n))$ , there is  $G_1 \in f^{-1}(G) \cap D_3(n)$  and  $E \subseteq \binom{[n]}{2}$  such that  $|E| \leq \binom{4+6r}{2} + 2$ , and for all  $G' \in f^{-1}(G) \cap D_1(n)$ ,  $\Delta(G_1, G') \subseteq E$ .*

PROOF. Suppose  $G \in f(D_3(n))$  and  $X_1, \dots, X_l$  is the c.o.c.d. of  $G$ . By definition of  $f$ , there are exactly two indices  $1 \leq w < b \leq l$  such that  $G[X_w]$  consists of a copy of  $H$ , and such that there is a sequence  $(z^1, \dots, z^k)$  which is a bad cycle in  $G[X_b]$  of some length  $k \geq 4$ . Let  $B$  be the simple complete  $r$ -graph with vertex set  $X_b$  such that for all  $1 \leq i \leq k-1$ ,  $d^B(z^i, z^{i+1}) = d^B(z^1, z^k) = r-1$ , and for all other  $xy \in E(X_b)$ ,  $d^B(x, y) = d^G(x, y)$ . Then by definition of  $f$ ,  $B$  must have  $k$  components,  $Z^1, \dots, Z^k$  such that for each  $1 \leq i \leq k$ ,  $Z^i$  is a large component of  $B$  containing  $z^i$ . Moreover, we must have that either  $Z^1 <_* \dots <_* Z^k$  or  $Z^k <_* \dots <_* Z^1$ . Because  $(z^1, \dots, z^k)$  is a bad cycle if and only if  $(z^k, \dots, z^1)$  is a bad cycle, we can relabel  $(z^1, \dots, z^k)$  if necessary so that  $Z^1 <_* \dots <_* Z^k$ . There is also be a unique (possibly empty) sequence  $1 \leq i_1 < \dots < i_v < w$  with the following properties:

- For each  $1 \leq j \leq v$ ,  $X_{i_j} = \{x_{i_j}\}$  is a singleton, and
- For each  $1 \leq j \leq v$ , for each  $y \in X_w$ ,  $d^G(x_{i_j}, y) = r$ , and
- For all  $j \notin \{i_1, \dots, i_v\}$ , if  $X_j = \{x_j\}$  is a singleton, then for some  $y \in X_w$ ,  $d^G(x, y) = m(r)$ .

Suppose  $G' \in f^{-1}(G) \cap D_3(n)$  and  $Y_1, \dots, Y_u$  is the c.o.c.d. of  $G'$ . Let  $s$  be such that  $ML(G') = Y_s$ . The same arguments as in the case when  $G \in D_2(n)$  imply that  $X_w$  consists of the least 4 elements of  $Y_s$ ,

$$\bigcup_{j=1}^v X_{i_j} = \bigcup_{i=1}^{s-1} Y_i,$$

the small components of  $G'[Y_s \setminus X_w]$  are exactly the elements of  $\{X_1, \dots, X_{t-1}\} \setminus \{X_{i_1}, \dots, X_{i_v}, X_w\}$ , and  $|\bigcup_{j=1}^v X_{i_j}| < 6r$ . Further, by definition of  $f$  we must have that  $Z^1, \dots, Z^k$  are the large components of  $G'[Y_s \setminus X_w]$ . In sum, for any  $G' \in f^{-1}(G) \cap D_2(n)$ , we have the following.

- (i)  $X_w$  consists of the least 4 elements of  $ML(G')$ ,
- (ii)  $\bigcup_{j=1}^v X_{i_j}$  is the union of the small components of  $G'$  and has size strictly less than  $6r$ ,
- (iii)  $\{X_1, \dots, X_{t-1}\} \setminus \{X_{i_1}, \dots, X_{i_v}, X_w\}$  is the set of small components of  $G'[ML(G') \setminus X_w]$ ,
- (iv)  $Z^1, \dots, Z^k$  are the large components of  $G'[ML(G') \setminus X_w]$ , and  $Z^1 <_* \dots <_* Z^k$ .

Set  $X = \bigcup_{j \in [t-1] \setminus \{i_1, \dots, i_v\}} X_j$  and  $Z = \bigcup_{j=1}^k Z^j$ , and note (iii) and (iv) imply that  $ML(G') = X \cup Z$ .

Define

$$E_1 = E(X_w \cup \bigcup_{j=1}^v X_{i_j}) \text{ and } E_2 = E(X_w, X \cup Z).$$

Then for all  $G' \in f^{-1}(G) \cap D_3(n)$ , the definition of  $f$  and (i)-(iv) imply that  $\Delta(G, G') \subseteq E_1 \cup E_2 \cup \{z^1 z^2, z^2 z^3, \dots, z^1 z^k\}$  and for all  $xy \in E_2$ ,  $d^{G'}(x, y) = d^G(x, y) - 1$ . We now show that we can also recover the value of  $d^{G'}(z^{j-1}, z^j)$  for each  $2 \leq j \leq k-1$ . For each  $1 \leq j \leq k$ , let  $z_{j_1}^j, \dots, z_{j_1 z_{j_1}^j}^j$  enumerate the elements of  $Z^j$  in increasing order. Let  $s_1, \dots, s_k$  be the indices such that  $(z_{s_1}^1, \dots, z_{s_k}^k) = (z^1, \dots, z^k)$ . By definition of  $f$ , for each  $2 \leq i \leq k-1$ ,  $d^{G'}(z^{i-1}, z^i) = |s_{i+1} - s_i|$ . We have now shown that for all  $G', G'' \in f^{-1}(G) \cap D_3(n)$ ,

- $\Delta(G, G') \cup \Delta(G, G'') \subseteq E_1 \cup E_2 \cup \{z^1 z^2, z^2 z^3, \dots, z^1 z^k\}$ ,
- For all  $xy \in E_2$ ,  $d^{G'}(x, y) = d^G(x, y) - 1 = d^{G''}(x, y)$ , and
- For all  $z^i z^{i+1} \in \{z^1 z^2, \dots, z^{k-2} z^{k-1}\}$ ,  $d^{G'}(x, y) = |s_{i+1} - s_i| = d^{G''}(x, y)$ .

Therefore,

$$\Delta(G', G'') \subseteq (\Delta(G, G') \cup \Delta(G'', G)) \setminus (E_2 \cup \{z^1 z^2, \dots, z^{k-2} z^{k-1}\}) \subseteq E_1 \cup \{z^{k-1} z^k, z^1 z^k\}.$$

Set  $E = E_1 \cup \{z^{k-1} z^k, z^1 z^k\}$  and take  $G_1$  to be any element of  $f^{-1}(G) \cap D_3(n)$ . By (ii),  $|\bigcup_{j=1}^v X_{i_j}| < 6r$ , so  $|X_w \cup \bigcup_{j=1}^v X_{i_j}| < 4 + 6r$  and  $|E| \leq \binom{4+6r}{2} + 2$ . This completes the proof.  $\square$

We now prove that for all  $n \geq 4$ , (25) holds. Fix an integer  $n \geq 4$  and  $G \in f(C_r(n))$ . Define  $E_1, \dots, E_{10} \subseteq \binom{[n]}{2}$  and  $G_1, \dots, G_{10} \in C_r(n)$  as follows. If  $G \notin f(D_1(n))$ , set  $E_1 = \emptyset$  and  $G_1 = G$ . Otherwise, let  $G_1 = G$  and let  $E_1 \subseteq \binom{[n]}{2}$  be as in Lemma 4.8.5. If  $G \notin f(D_2(n))$ , let  $E_2 = \dots = E_9 = \emptyset$  and  $G_2 = \dots = G_9 = G$ . Otherwise let  $E \subseteq \binom{[n]}{2}$  and  $G_2, \dots, G_9 \in D_2(n)$  be as in Lemma 4.8.6, and set  $E_2 = \dots = E_9 = E$ . If  $G \notin f(D_3(n))$ , let  $E_{10} = \emptyset$  and  $G_{10} = G$ . Otherwise, let

$E_{10} \subseteq \binom{[n]}{2}$  and  $G_{10} \in D_3(n)$  be as in Lemma 4.8.7. Then Lemmas 4.8.5, 4.8.6, and 4.8.7 imply that

$$\begin{aligned} f^{-1}(G) \cap D_1(n) &\subseteq \{G' \in C_r(n) : \Delta(G_1, G') \subseteq E_1\}, \\ f^{-1}(G) \cap D_2(n) &\subseteq \bigcup_{i=2}^9 \{G' \in C_r(n) : \Delta(G_i, G') \subseteq E_i\}, \text{ and} \\ f^{-1}(G) \cap D_3(n) &\subseteq \{G' \in C_r(n) : \Delta(G_{10}, G') \subseteq E_{10}\}. \end{aligned}$$

Since  $C_r(n) = D_1(n) \cup D_2(n) \cup D_3(n)$ , we have that

$$(28) \quad f^{-1}(G) \subseteq \bigcup_{i=1}^{10} \{G' \in C_r(n) : \Delta(G_i, G') \subseteq E_i\}.$$

For each  $1 \leq i \leq 10$ , every element of  $\{G' \in C_r(n) : \Delta(G_i, G') \subseteq E_i\}$  can be constructed by starting with  $G_i$ , then changing the edges contained in  $E_i$ . There are at most  $r^{|E_i|}$  ways to do this, and for each  $i$ ,  $|E_i| \leq \binom{4+8r}{2} \leq 64r^2$ . Therefore, for each  $i$ ,  $|\{G' \in C_r(n) : \Delta(G_i, G') \subseteq E_i\}| \leq r^{64r^2}$ . Combining this with (28), we have that

$$|f^{-1}(G)| \leq 10r^{64r^2} \leq r^{65r^2}.$$

Since  $f(C_r(n)) \subseteq M_r(n) \setminus C_r(n)$ , this implies  $|M_r(n) \setminus C_r(n)| \geq |f(C_r(n))| \geq \frac{|C_r(n)|}{r^{65r^2}}$ . Rearranging this yields that

$$|C_r(n)| \leq \frac{r^{65r^2}}{r^{65r^2} + 1} |M_r(n)| = \left(1 - \frac{1}{r^{65r^2} + 1}\right) |M_r(n)| < (1 - r^{-66r^2}) |M_r(n)|,$$

as desired.

## Multigraphs

### 5.1. Introduction

The work in this chapter is joint with D. Mubayi. A *multigraph*  $G$  is a pair  $(V, w)$ , where  $V$  is a set of vertices and  $w : \binom{V}{2} \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$  is a function. Given integers  $s \geq 2$  and  $q \geq 0$ , a multigraph  $(V, w)$  is an  $(s, q)$ -*graph* if for every  $X \in \binom{V}{s}$ ,  $\sum_{xy \in \binom{X}{2}} w(xy) \leq q$ . An  $(n, s, q)$ -graph is an  $(s, q)$ -graph with  $n$  vertices. In this chapter, we will prove multigraph versions of Turán's Theorem (Theorem 2.2.1) and the Erdős-Simonovits Stability Theorem (Theorem 2.2.2). Our object of study will be the set of  $(n, s, q)$ -graphs with vertex set  $[n] := \{1, \dots, n\}$ , which we denote by  $F(n, s, q)$ .

DEFINITION 5.1.1. Given a multigraph  $G = (V, w)$ , define

$$S(G) = \sum_{xy \in \binom{V}{2}} w(xy), \quad P(G) = \prod_{xy \in \binom{V}{2}} w(xy),$$

$$\text{ex}_\Sigma(n, s, q) = \max\{S(G) : G \in F(n, s, q)\}, \quad \text{and} \quad \text{ex}_\Pi(n, s, q) = \max\{P(G) : G \in F(n, s, q)\}.$$

An  $(n, s, q)$ -graph  $G$  is *sum-extremal* (*product-extremal*) if  $S(G) = \text{ex}_\Sigma(n, s, q)$  ( $P(G) = \text{ex}_\Pi(n, s, q)$ ). Let  $\mathcal{S}(n, s, q)$  ( $\mathcal{P}(n, s, q)$ ) be the set of all sum-extremal (product-extremal)  $(n, s, q)$ -graphs with vertex set  $[n]$ .

Determining  $\text{ex}_\Sigma(n, s, q)$  and the structure of elements in  $\mathcal{S}(n, s, q)$  for a given  $s$  and  $q$  is a natural multigraph version of Theorem 2.2.1 which has been studied extensively in the literature. In [29], Bondy and Tuza determine the structure of multigraphs in  $\mathcal{S}(n, s, q)$  when  $n$  is large compared to  $s$  and  $q \equiv 0, -1 \pmod{\binom{s}{2}}$  and when  $s = 3$ . One interesting phenomenon discovered in [29] is that  $\mathcal{S}(n, s, q)$  has many non-isomorphic multigraphs when  $s = 3$ ,  $q \equiv 2 \pmod{\binom{s}{2}}$  and  $n$  is large. In [53], Füredi and Kündgen (among other things) determine the asymptotic value of  $\text{ex}_\Sigma(n, s, q)$  for all  $s, q$  with a  $O(n)$  error term, and the exact value is determined for many cases. The results in both [29] and [53] apply to the more general setting of integer-weighted graphs. Other special cases of these questions have appeared in [72]. Versions of Theorems 2.2.1 and 2.2.2 for families of multigraphs and digraphs with forbidden configurations have also appeared in a long and ongoing investigation

in [30, 31, 32, 33, 34, 35] (see [35] for a survey of these results). These papers also deal exclusively with questions related to *summing* the total number of edges in a multigraph or digraph.

This chapter investigates  $\text{ex}_{\Pi}(n, s, q)$  and  $\mathcal{P}(n, s, q)$  for various values of  $(s, q)$ . Questions about  $\text{ex}_{\Pi}(n, s, q)$  and  $\mathcal{P}(n, s, q)$  appear to be of independent interest, as they are natural “product versions” of the questions about extremal sums for  $(n, s, q)$ -graphs investigated in [29, 53]. Another reason to study these problems is that they can be used to prove asymptotic structure, enumeration, and 0-1 law results for  $(n, s, q)$ -graphs similar to those proved in [66] in the setting of  $K_s$ -free graphs. Indeed, the crucial parameter that appears in the expression for the number of  $(n, s, q)$ -graph is  $\text{ex}_{\Pi}(n, s, q)$ . This is developed in the multigraphs section of Chapter 7 and in forthcoming work by the author and D. Mubayi.

## 5.2. Main Results

Observe that if  $n \geq s \geq 2$ ,  $G = ([n], w) \in F(n, s, q)$  and  $q < \binom{s}{2}$ , then  $P(G) = 0$  because any  $s$ -set of vertices must contain a pair  $uv$  with  $w(uv) = 0$ . Therefore, throughout the chapter we assume  $q \geq \binom{s}{2} \geq 1$ . Given a multigraph  $G = (V, w)$  and  $xy \in \binom{V}{2}$ , we will refer to  $w$  as the *weight function* of  $G$  and  $w(xy)$  as the *multiplicity* of  $xy$ . The *multiplicity* of  $G$  is  $\mu(G) = \max\{w(xy) : xy \in \binom{V}{2}\}$ .

DEFINITION 5.2.1. Given integers  $s \geq 2$  and  $q \geq \binom{s}{2}$ , define the *asymptotic product density* and the *asymptotic sum density*, respectively, as the following limits (which both exist):

$$\text{ex}_{\Pi}(s, q) = \lim_{n \rightarrow \infty} \left( \text{ex}_{\Pi}(n, s, q) \right)^{\frac{1}{\binom{n}{2}}} \quad \text{and} \quad \text{ex}_{\Sigma}(s, q) = \lim_{n \rightarrow \infty} \frac{\text{ex}_{\Sigma}(n, s, q)}{\binom{n}{2}}.$$

Given two multigraphs  $G = (V, w)$  and  $G' = (V, w')$ , set  $\Delta(G, G') = \{xy \in \binom{V}{2} : w(xy) \neq w'(xy)\}$ . We say  $G$  and  $G'$  are  $\delta$ -close if  $|\Delta(G, G')| \leq \delta n^2$ , otherwise they are  $\delta$ -far. Given a multigraph  $G = (V, w)$  and a set  $X \subseteq V$ ,  $G[X]$  denotes the multigraph with vertex set  $X$  and weight function  $w \upharpoonright_{\binom{X}{2}}$ . Suppose that  $q \equiv b \pmod{\binom{s}{2}}$ . Our results fall into three cases depending on the value of  $b$ .

### 5.2.1. Case (i): $0 \leq b \leq s - 2$ .

DEFINITION 5.2.2. Given  $n \geq s \geq 1$  and  $a \geq 1$ , let  $\mathbb{U}_{s,a}(n)$  be the set of multigraphs  $G = ([n], w)$  such that there is a partition  $A_0, A_1, \dots, A_{\lfloor \frac{n}{s} \rfloor}$  of  $[n]$  for which the following holds.

- For each  $1 \leq i \leq \lfloor n/s \rfloor$ ,  $|A_i| = s$ , and  $|A_0| = n/s - \lfloor n/s \rfloor$ .
- For each  $0 \leq i \leq \lfloor n/s \rfloor$ , and  $G[A_i]$  is a star with  $|A_i| - 1$  edges of multiplicity  $a + 1$  and all other edges of multiplicity  $a$ .

- For all  $xy \notin \bigcup \binom{A_i}{2}$ ,  $w(xy) = a$ .

Let  $\mathbb{U}_a(n)$  be the unique element of  $\mathbb{U}_{1,a}(n)$ , i.e.  $\mathbb{U}_a(n) = ([n], w)$  where  $w(xy) = a$  for all  $xy \in \binom{[n]}{2}$ .

**THEOREM 5.2.3.** *Suppose  $n, s, q, a$  are integers satisfying  $n \geq s \geq 2$ ,  $a \geq 1$ , and  $q = a \binom{s}{2} + b$  for some  $0 \leq b \leq s - 2$ .*

- (Extremal) Then  $a \binom{n}{2} \leq \text{ex}_{\Pi}(n, s, q) \leq a \binom{n}{2} ((a+1)/a)^{\lfloor \frac{b}{b+1} n \rfloor}$  and thus  $\text{ex}_{\Pi}(s, q) = a$ . Further,
  - (a) If  $b = 0$ , then  $\mathcal{P}(n, s, q) = \{\mathbb{U}_a(n)\}$  and  $\text{ex}_{\Pi}(n, s, q) = a \binom{n}{2}$ .
  - (b) If  $b = s - 2$ , then  $\mathbb{U}_{s-1,a}(n) \subseteq \mathcal{P}(n, s, q)$  and  $\text{ex}_{\Pi}(n, s, q) = a \binom{n}{2} \left(\frac{a+1}{a}\right)^{\lfloor \frac{(s-2)n}{s-1} \rfloor}$ . Also,  $\mathcal{P}(n, 3, q) = \mathbb{U}_{2,a}(n)$ .
- (Stability) For all  $\delta > 0$ , there is  $\epsilon > 0$  and  $M$  such that for all  $n > M$  and  $G \in F(n, s, q)$ , if  $P(G) > \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$ , then  $G$  is  $\delta$ -close to  $\mathbb{U}_a(n)$ .

**5.2.2. Case (ii):**  $b = \binom{s}{2} - t$  and  $1 \leq t \leq \frac{s}{2}$ . Call a partition  $U_1, \dots, U_k$  of a finite set  $X$  an *equipartition* if  $||U_i| - |U_j|| \leq 1$  for all  $i \neq j$ .

**DEFINITION 5.2.4.** Given integers  $a \geq 2$  and  $n \geq s \geq 1$ , define  $\mathbb{T}_{s,a}(n)$  to be the set of multigraphs  $G = ([n], w)$  with the following property. There is an equipartition  $U_1, \dots, U_s$  of  $[n]$  such that

$$w(xy) = \begin{cases} a - 1 & \text{if } xy \in \binom{U_i}{2} \text{ for some } i \in [s]. \\ a & \text{if } (x, y) \in U_i \times U_j \text{ for some } i \neq j \in [s]. \end{cases}$$

We think of elements of  $\mathbb{T}_{s,a}(n)$  as multigraph analogues of Turán graphs. Recall that  $t_s(n)$  is the number of edges in  $T_s(n)$ .

**THEOREM 5.2.5.** *Let  $s, q, a, t$  be integers satisfying  $a \geq 2$ ,  $q = a \binom{s}{2} - t$  and either*

- (a)  $s \geq 2$  and  $t = 1$  or
- (b)  $s \geq 4$  and  $2 \leq t \leq \frac{s}{2}$ .

- (Extremal) Then for all  $n \geq s$ ,  $\mathbb{T}_{s-t,a}(n) \subseteq \mathcal{P}(n, s, q)$ ,  $\text{ex}_{\Pi}(n, s, q) = (a-1) \binom{n}{2} \left(\frac{a}{a-1}\right)^{t_{s-t}(n)}$ , and  $\text{ex}_{\Pi}(s, q) = (a-1) \left(\frac{a}{a-1}\right)^{\frac{s-t-1}{s-t}}$ . If (a) holds and  $n \geq s$  or (b) holds and  $n$  is sufficiently large, then  $\mathcal{P}(n, s, q) = \mathbb{T}_{s-t,a}(n)$ .
- (Stability) For all  $\delta > 0$ , there is  $M$  and  $\epsilon$  such that for all  $n > M$  and  $G \in F(n, s, q)$ , if  $P(G) > \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$  then  $G$  is  $\delta$ -close to an element of  $\mathbb{T}_{s-t,a}(n)$ .

**5.2.3. Case (iii):**  $(s, q) = (4, 15)$ . This is one of the first cases not covered by cases (i) and (ii). It is our most difficult result, and also perhaps our most interesting, as it gives an indication of the difficulty of determining  $\text{ex}_{\Pi}(n, s, q)$  in general.

DEFINITION 5.2.6. Given  $n$ , let  $W(n)$  be the set of multigraphs  $G = ([n], w)$  for which there is a partition  $L, R$  of  $[n]$  such that  $w(xy) = 1$  if  $xy \in \binom{L}{2}$ ,  $w(xy) = 2$  if  $xy \in \binom{R}{2}$ , and  $w(xy) = 3$  if  $(x, y) \in L \times R$ .

Notice  $W(n) \subseteq F(n, 4, 15)$  for all  $n \in \mathbb{N}$ . Straightforward calculus shows that for  $G \in W(n)$ , the product  $P(G)$  is maximized when  $|R| \approx \beta n$ , where  $\beta = \frac{\log 3}{2 \log 3 - \log 2}$ . Write  $\mathcal{P}(W(n))$  for the set of  $G \in W(n)$  with  $P(G) = \max\{P(G') : G' \in W(n)\}$ .

THEOREM 5.2.7. *There is  $M$  such that for all  $n \geq M$ ,  $\mathcal{P}(W(n)) \subseteq \mathcal{P}(n, 4, 15)$ . Consequently*

$$\text{ex}_{\Pi}(n, 4, 15) = 2^{\gamma n^2 + O(n)},$$

where  $\gamma = \beta^2/2 + \beta(1 - \beta) \log_2 3$  and  $\beta = \frac{\log 3}{2 \log 3 - \log 2}$ .

For reference,  $\beta \approx .73$  and  $2^\gamma \approx 1.49$ . Recall that Schanuel's Conjecture states the following: if  $z_1, \dots, z_n$  are complex numbers which are linearly independent over  $\mathbb{Q}$ , then  $\mathbb{Q}(z_1, \dots, z_n, e^{z_1}, \dots, e^{z_n})$  has transcendence degree at least  $n$  over  $\mathbb{Q}$ . As promised in the abstract, we now show that assuming Schanuel's Conjecture,  $2^\gamma$  is transcendental.

PROPOSITION 5.2.8. *Assuming Schanuel's Conjecture,  $2^\gamma$  is transcendental.*

PROOF. Assume Schanuel's Conjecture holds. It is well-known that Schanuel's conjecture implies  $\log 2$  and  $\log 3$  are algebraically independent over  $\mathbb{Q}$  (see for instance [102]). Observe  $\gamma = \frac{f(\log 2, \log 3)}{g(\log 2, \log 3)}$  where  $f(x, y) = xy^2/2 + y^2(y - x)$  and  $g(x, y) = x(2y - x)^2$ . Note the coefficient of  $x^3$  in  $f(x, y)$  is 0 while in  $g(x, y)$  it is 1. We now show  $\log 2, \log 3, \gamma \log 2$  are linearly independent over  $\mathbb{Q}$ . Suppose towards a contradiction that this is not the case. Then there are non-zero rationals  $p, q, r$  such that

$$p \log 2 + q \log 3 + r \gamma \log 2 = 0.$$

Replacing  $\gamma$  with  $\frac{f(\log 2, \log 3)}{g(\log 2, \log 3)}$ , this implies  $p \log 2 + q \log 3 + r \frac{f(\log 2, \log 3)}{g(\log 2, \log 3)} \log 2$ . By clearing the denominators of  $p, q, r$  and multiplying by  $g(\log 2, \log 3)$ , we obtain that there are non-zero integers  $a, b, c$  such that

$$(a \log 2 + b \log 3)g(\log 2, \log 3) + cf(\log 2, \log 3) \log 2 = 0.$$

Let  $p(x, y) = (ax+by)g(x, y) + cf(x, y)x$ . Then  $p(x, y)$  is a rational polynomial and  $p(\log 2, \log 3) = 0$ . Since the coefficient of  $x^3$  is 1 in  $g(x, y)$  and 0 in  $f(x, y)$ , the coefficient of  $x^4$  in  $p(x, y)$  is  $a \neq 0$ . Thus  $p(x, y)$  has at least one non-zero coefficient, contradicting that  $\log 2$  and  $\log 3$  are algebraically independent over  $\mathbb{Q}$ . Thus  $\log 2, \log 3, \gamma \log 2$  are linearly independent over  $\mathbb{Q}$ , so Schanuel's conjecture implies  $\mathbb{Q}(\log 2, \log 3, \gamma \log 2, 2^\gamma)$  has transcendence degree at least 3 over  $\mathbb{Q}$ . Suppose towards a contradiction that  $2^\gamma$  is not transcendental. Then  $\log 2, \log 3, \gamma \log 2$  must be algebraically independent over  $\mathbb{Q}$ . Let  $h(x, y, z) = zyg(x, y) - xf(x, y)$ . Then  $h(x, y, z)$  has non-zero coefficients and  $h(\log 2, \log 3, \gamma \log 2) = 0$  implies  $\log 2, \log 3, \gamma \log 2$  are algebraically dependent over  $\mathbb{Q}$ , a contraction. Thus  $2^\gamma$  is transcendental.  $\square$

We remark that the question of obtaining transcendental densities for natural extremal problems is an intriguing one, first explicitly posed by Fox (see [86]) in the context of Turán densities of hypergraphs. Pikhurko [86] showed the set of hypergraph Turán densities is uncountable, thereby proving the existence of transcendental ones (see also [57]). But to our knowledge, Theorem 5.2.7 is the first example of a fairly natural extremal problem whose answer is given (explicitly) by a transcendental number (modulo Schanuel's Conjecture of course).

### 5.3. Extremal Results: Cases (i) and (ii)

In this section we prove the extremal statements in Theorems 5.2.3 and 5.2.5. We begin with some preliminaries. The results in [53] imply that for all  $s$  and  $q$ ,  $\text{ex}_\Sigma(s, q)$  always exists. We now show this for  $\text{ex}_\Pi(s, q)$  for the values of  $s$  and  $q$  of interest to us in this chapter.

PROPOSITION 5.3.1. *For all  $n \geq s \geq 2$  and  $q \geq \binom{s}{2}$ ,  $\text{ex}_\Pi(s, q)$  exists and  $\text{ex}_\Pi(n, s, q) \geq \text{ex}_\Pi(s, q)^{\binom{n}{2}}$ .*

PROOF. For all  $n \geq s$ ,  $b_n := (\text{ex}_\Pi(n, s, q))^{\frac{1}{\binom{n}{2}}} \geq 1$  since  $q \geq \binom{s}{2}$  implies that  $\mathbb{U}_1(n) \in F(n, s, q)$ . We now show the  $b_n$  are non-increasing. For  $n > s$  and  $G \in F(n, s, q)$ ,

$$P(G) = \left( \prod_{i \in [n]} P(G[[n] \setminus \{i\}]) \right)^{1/(n-2)} \leq \left( \prod_{i \in [n]} b_{n-1}^{\binom{n-1}{2}} \right)^{1/(n-2)} = b_{n-1}^{n \binom{n-1}{2} / (n-2)} = b_{n-1}^{\binom{n}{2}}.$$

Therefore for all  $G \in F(n, s, q)$ ,  $P(G)^{1/\binom{n}{2}} \leq b_{n-1}$ , so  $b_n \leq b_{n-1}$  and  $\lim_{n \rightarrow \infty} b_n = \text{ex}_\Pi(s, q)$  exists. The inequality  $\text{ex}_\Pi(n, s, q) \geq \text{ex}_\Pi(s, q)^{\binom{n}{2}}$  follows because the  $b_n$  are non-increasing.  $\square$

Suppose  $s \geq 2$  and  $q \geq \binom{s}{2}$ . The AM-GM inequality implies that

$$(29) \quad \text{ex}_\Pi(s, q) = \lim_{n \rightarrow \infty} \text{ex}_\Pi(n, s, q)^{1/\binom{n}{2}} \leq \lim_{n \rightarrow \infty} \frac{\text{ex}_\Sigma(n, s, q)}{\binom{n}{2}} = \text{ex}_\Sigma(s, q).$$



The following lemma is an integer version of the AM-GM inequality.

LEMMA 5.3.2. *If  $\ell \geq 2$ ,  $k \in [\ell]$  and  $a, x_1, \dots, x_\ell$  are positive integers such that  $\sum_{i=1}^\ell x_i \leq a\ell - k$ , then  $\prod_{i=1}^\ell x_i \leq a^{\ell-k}(a-1)^k$ . Moreover, equality holds if and only if exactly  $k$  of the  $x_i$  are equal to  $a-1$  and the rest are equal to  $a$ .*

PROOF. If there are  $x_i$  and  $x_j$  with  $x_i < x_j - 1$ , then replacing  $x_i$  with  $x_i + 1$  and replacing  $x_j$  with  $x_j - 1$  increases the product and keeps the sum unchanged. So no two of the  $x_i$ 's differ by more than one when the product is maximized.  $\square$

COROLLARY 5.3.3. *Let  $n \geq s \geq 2$ ,  $a \geq 2$ , and  $(a-1)\binom{s}{2} \leq q < a\binom{s}{2}$ . Suppose  $G \in \mathcal{S}(n, s, q)$  has all edge multiplicities in  $\{a, a-1\}$  and contains exactly  $k$  edges of multiplicity  $a-1$ . Then for all other  $G' \in F(n, s, q)$ ,  $G' \in \mathcal{P}(n, s, q)$  if and only if  $G'$  has  $k$  edges of multiplicity  $a-1$  and all other edges of multiplicity  $a$ . Consequently,  $G \in \mathcal{P}(n, s, q) \subseteq \mathcal{S}(n, s, q)$ .*

PROOF. Fix  $G$  so that the hypotheses hold. Then  $S(G) = a\binom{n}{2} - k$  and  $P(G) = a^{\binom{n}{2}-k}(a-1)^k$ . Let  $G' = ([n], w)$  be another element of  $F(n, s, q)$ . Since  $G \in \mathcal{S}(n, s, q)$ , we have

$$S(G') \leq S(G) = a\binom{n}{2} - k.$$

By Lemma 5.3.2 with  $\ell = \binom{n}{2}$ ,  $P(G') \leq a^{\binom{n}{2}-k}(a-1)^k$  with equality if and only if  $\{w(xy) : xy \in \binom{[n]}{2}\}$  consists of  $k$  elements equal to  $a-1$  and the rest equal to  $a$ . This shows  $G' \in \mathcal{P}(n, s, q)$  if and only if  $G'$  has  $k$  edges of multiplicity  $a-1$  and the rest of multiplicity  $a$ . Consequently,  $G \in \mathcal{P}(n, s, q)$ . To show  $\mathcal{P}(n, s, q) \subseteq \mathcal{S}(n, s, q)$ , let  $G' \in \mathcal{P}(n, s, q)$ . Then by what we have shown,  $S(G') = a\binom{n}{2} - k = S(G)$ , so  $G \in \mathcal{S}(n, s, q)$  implies  $G' \in \mathcal{S}(n, s, q)$ .  $\square$

The following is a consequence of Theorem 5.2 in [29] (case  $b = 0$ ) and Theorems 8 and 9 in [53] (cases  $0 < b \leq s - 2$ ).

THEOREM 5.3.4 (Bondy-Tuza [29], Füredi-Kündgen [53]). *Let  $n \geq s \geq 2$ ,  $a \geq 1$ ,  $0 \leq b \leq s - 2$ , and  $q = a\binom{s}{2} + b$ . Then*

$$\text{ex}_\Sigma(n, s, q) \leq a\binom{n}{2} + \left\lfloor \frac{b}{b+1}n \right\rfloor.$$

*with equality holding when  $b = s - 2$  and when  $b = 0$ .*

**Proof of Theorem 5.2.3 (Extremal).** Since  $\mathbb{U}_a(n) \in F(n, s, q)$ ,  $a\binom{n}{2} \leq \text{ex}_\Pi(n, s, q)$ . On the other hand, let  $G \in F(n, s, q)$ . Theorem 5.3.4 implies that  $S(G) \leq a\binom{n}{2} + \lfloor \frac{b}{b+1}n \rfloor$ . This along with Lemma

5.3.2 implies that  $P(G) \leq a^{\binom{n}{2}}((a+1)/a)^{\lfloor \frac{b}{b+1}n \rfloor}$ . Thus  $a^{\binom{n}{2}} \leq \text{ex}_{\Pi}(n, s, q) \leq a^{\binom{n}{2}}((a+1)/a)^{\lfloor \frac{b}{b+1}n \rfloor}$ , which implies  $\text{ex}_{\Pi}(s, q) = a$ .

Case (a): If  $b = 0$ , then Theorem 5.3.4 implies  $\mathbb{U}_a(n) \in \mathcal{S}(n, s, q)$ . Because  $\mathbb{U}_a(n)$  has all edge multiplicities in  $\{a\}$ , Corollary 5.3.3 implies  $\mathbb{U}_a(n) \in \mathcal{P}(n, s, q)$  and moreover, every other element of  $\mathcal{P}(n, s, q)$  has all edges of multiplicity  $a$ . In other words,  $\{\mathbb{U}_a(n)\} = \mathcal{P}(n, s, q)$ , so  $\text{ex}_{\Pi}(n, s, q) = a^{\binom{n}{2}}$ .

Case (b): If  $b = s-2$ , then it is straightforward to check  $\mathbb{U}_{s-1,a}(n) \subseteq F(n, s, q)$ . Since  $S(G) = a^{\binom{n}{2}} + \lfloor \frac{s-2}{s-1}n \rfloor$  for all  $G \in \mathbb{U}_{s-1,a}(n)$ , Theorem 5.3.4 implies  $\mathbb{U}_{s-1,a}(n) \subseteq \mathcal{S}(n, s, q)$ . Because every element in  $\mathbb{U}_{s-1,a}(n)$  has all edge multiplicities in  $\{a+1, a\}$ , Corollary 5.3.3 implies  $\mathbb{U}_{s-1,a}(n) \subseteq \mathcal{P}(n, s, q)$  and every  $G' \in \mathcal{P}(n, s, q)$  contains exactly  $\lfloor \frac{s-2}{s-1}n \rfloor$  edges of multiplicity  $a+1$ , and all others of multiplicity  $a$ . Thus  $\text{ex}_{\Pi}(n, s, q) = a^{\binom{n}{2}}(\frac{a+1}{a})^{\lfloor \frac{s-2}{s-1}n \rfloor}$ . Suppose  $s = 3$ ,  $b = 1$ , and  $G' = ([n], w) \in \mathcal{P}(n, s, q)$ . If there are  $x, y \neq z \in [n]$  such that  $w(xy) = w(xz) = a+1$ , then because  $G'$  contains only edges of multiplicity  $a+1$  and  $a$ ,  $S(\{x, y, z\}) \geq 2(a+1) + a = 3a+2 > q$ , a contradiction. Thus the edges of multiplicity  $a+1$  form a matching of size  $\lfloor \frac{n}{2} \rfloor$  in  $G'$ , so  $G' \in \mathbb{U}_{s-1,a}(n)$ . This shows  $\mathbb{U}_{s-1,a}(n) = \mathcal{P}(n, s, q)$ .  $\square$

The following is a consequence of Theorem 5.2 of [29].

**THEOREM 5.3.5 (Bondy-Tuza [29]).** *Suppose  $n \geq s \geq 2$ ,  $a \geq 1$ , and  $q = a^{\binom{s}{2}} - 1$ . Then*

$$\text{ex}_{\Sigma}(n, s, q) = (a-1) \binom{n}{2} + t_{s-1}(n).$$

**Proof of Theorem 5.2.5(a) (Extremal).** Since  $\mathbb{T}_{s-1,a}(n) \subseteq F(n, s, q)$  and for all  $G \in \mathbb{T}_{s-1,a}(n)$ ,  $S(G) = (a-1) \binom{n}{2} + t_{s-1}(n)$ , Theorem 5.3.5 implies that  $\mathbb{T}_{s-1,a}(n) \subseteq \mathcal{S}(n, s, q)$ . Therefore Corollary 5.3.3 implies  $\mathbb{T}_{s-1,a}(n) \subseteq \mathcal{P}(n, s, q)$  and each  $G \in \mathcal{P}(n, s, q)$  has  $t_{s-1}(n)$  edges of multiplicity  $a$  and the rest of multiplicity  $a-1$ . Fix  $G = ([n], w) \in \mathcal{P}(n, s, q)$  and let  $G'$  be the graph with vertex set  $[n]$  and edge set  $E = \{xy \in \binom{[n]}{2} : w(xy) = a\}$ . Then  $G'$  is  $K_s$ -free and has  $t_{s-1}(n)$  edges, so by Turán's theorem,  $G' = T_{s-1}(n)$  and thus  $G \in \mathbb{T}_{s-1,a}(n)$ . So we have shown,  $\mathcal{P}(n, s, q) = \mathbb{T}_{s-1,a}(n)$ . Consequently,  $\text{ex}_{\Pi}(n, s, q) = (a-1) \binom{n}{2} (\frac{a}{a-1})^{t_{s-1}(n)}$  and  $\text{ex}_{\Pi}(s, q) = (a-1) (\frac{a}{a-1})^{\frac{s-2}{s-1}}$ .  $\square$

To prove Theorem 5.2.5(b) (Extremal), we will need the following theorem, as well as a few lemmas.

**THEOREM 5.3.6. [Dirac [44], Bondy-Tuza [29]]** *Let  $n \geq s \geq 4$ ,  $a \geq 1$ , and  $q = a^{\binom{s}{2}} - t$  for some  $2 \leq t \leq \frac{s}{2}$ . Then  $\text{ex}_{\Sigma}(n, s, q) = \text{ex}_{\Sigma}(n, s', q')$  where  $s' = s - t + 1$  and  $q' = a^{\binom{s'}{2}} - 1$ .*

PROOF. Let  $n \geq s \geq 4$  and  $2 \leq t \leq s/2$ . In [44], Dirac proved that  $\text{ex}_\Sigma(n, s, \binom{s}{2} - t) = t_{s-t}(n)$ .

This along with Lemma 5.1 in [29] implies that for all  $a \geq 1$ ,

$$\text{ex}_\Sigma(n, s, a \binom{s}{2} - t) = \text{ex}_\Sigma(n, s, \binom{s}{2} - t) + (a-1) \binom{n}{2} = t_{s-t}(n) + (a-1) \binom{n}{2} = \text{ex}_\Sigma(n, s', a \binom{s'}{2} - 1),$$

where the last equality is by Theorem 5.3.5 applied to  $s'$  and  $a \binom{s'}{2} - 1$ .  $\square$

LEMMA 5.3.7. *If  $s, q, a, t$  are integers satisfying case (b) of Theorem 5.2.5, and  $s' = s - t + 1$ ,  $q' = a \binom{s'}{2} - 1$ , then for all  $n \geq s$ ,  $\mathbb{T}_{s'-1}(n) \subseteq \mathcal{P}(n, s, q)$  and  $\text{ex}_\Pi(n, s, q) = \text{ex}_\Pi(n, s', q')$ .*

PROOF. Set  $s' = s - t + 1$  and  $q' = a \binom{s'}{2} - 1$ , and fix  $n \geq s$ . Fix  $G \in \mathbb{T}_{s'-1, a}(n)$ . It is straightforward to check that  $G \in F(n, s, q)$ . By Theorem 5.3.6,  $\text{ex}_\Sigma(n, s', q') = \text{ex}_\Sigma(n, s, q)$ . Since  $S(G) = (a-1) \binom{n}{2} + t_{s'-1}(n)$ , by Theorem 5.3.5 applied to  $s'$  and  $q'$ , we have that  $S(G) = \text{ex}_\Sigma(n, s', q') = \text{ex}_\Sigma(n, s, q)$ . This shows  $G \in \mathcal{S}(n, s, q)$ . By Corollary 5.3.3, since  $G$  has all edge multiplicities in  $\{a, a-1\}$ ,  $G \in \mathcal{P}(n, s, q)$ , so  $P(G) = \text{ex}_\Pi(n, s, q)$ . Since  $G \in \mathbb{T}_{s'-1, a}(n)$  and  $\mathbb{T}_{s'-1, a}(n) \subseteq \mathcal{P}(n, s', q')$  by Theorem 5.2.5(a) (Extremal),  $P(G) = \text{ex}_\Pi(n, s', q')$ . Thus  $\text{ex}_\Pi(n, s, q) = P(G) = \text{ex}_\Pi(n, s', q')$ .  $\square$

We now fix some notation. Given  $n \in \mathbb{N}$ ,  $z \in [n]$ ,  $Y \subseteq [n]$ , and  $G = ([n], w)$ , set

$$S(Y) = \sum_{xy \in \binom{Y}{2}} w(xy), \quad S_z(Y) = \sum_{y \in Y} w(yz), \quad P(Y) = \prod_{xy \in \binom{Y}{2}} w(xy), \quad \text{and} \quad P_z(Y) = \prod_{y \in Y} w(yz)$$

If  $X \subseteq [n]$  is disjoint from  $Y$ , set  $P(X, Y) = \prod_{x \in X, y \in Y} w(xy)$ .

CLAIM 5.3.8. *Suppose  $s, q, a, t$  are integers satisfying the hypotheses of case (b) of Theorem 5.2.5. Then for all  $n \geq 2s$  and  $s - t + 1 \leq y \leq s - 1$ ,*

$$\text{ex}_\Pi(n - y, s, q) \leq \text{ex}_\Pi(n, s, q) (a-1)^{-\binom{y}{2}} \left( a^{y-2} (a-1)^2 \right)^{-(n-y)} \left( \frac{a-1}{a} \right)^{\frac{n-y}{s-t}}.$$

PROOF. Set  $s' = s - t + 1$  and  $q' = a \binom{s'}{2} - 1$ . Fix  $n \geq s$  and  $s' \leq y \leq s - 1$ . Choose some  $H = ([n-y], w) \in \mathbb{T}_{s'-1, a}(n-y)$  and let  $U_1, \dots, U_{s'-1}$  be the partition of  $[n-y]$  corresponding to  $H$ . Observe that there is some  $i$  such that  $|U_i| \geq \frac{n-y}{s'-1}$ . Without loss of generality, assume  $|U_1| \geq \frac{n-y}{s'-1}$ . Assign the elements of  $Y' := [n] \setminus [n-y]$  to the  $U_i$  in as even a way as possible, to obtain an equipartition  $U'_1, \dots, U'_{s'-1}$  of  $[n]$  extending  $U_1, \dots, U_{s'-1}$ . Observe that because  $s' \leq |Y'| \leq s - 1$  and  $s' - 1 = s - t \geq s/2$ , for each  $i$ ,  $|U'_i \setminus U_i| \in \{1, 2\}$ , and there is at least one  $i$  such that  $|U'_i \setminus U_i| = 1$ . Since  $|U_1| \geq \frac{n-y}{s-t}$ , by redistributing  $Y'$  if necessary, we may assume that  $|U'_1 \setminus U_1| = 1$ . Define a new multigraph  $H' = ([n], w')$  so that  $w'(xy) = a - 1$  if  $xy \in \binom{U'_i}{2}$  for some  $i \in [s' - 1]$  and  $w'(xy) = a$  if  $(x, y) \in U'_i \times U'_j$  for some  $i \neq j$ . Note that by construction  $H' \in \mathbb{T}_{s'-1, a}(n)$  and  $H'[[n-y]] = H$ .

By Lemma 5.3.7, since  $n - y \geq s$ ,  $H \in \mathbb{T}_{s'-1, a}(n - y)$  and  $H' \in \mathbb{T}_{s'-1, a}(n)$  imply  $H \in \mathcal{P}(n - y, s, q)$  and  $H' \in \mathcal{P}(n, s, q)$ . These facts imply the following.

$$(30) \quad \text{ex}_{\Pi}(n, s, q) = P(H') = P(H)P(Y')P(Y', [n - y]) = \text{ex}_{\Pi}(n - y, s, q)P(Y')P(Y', [n - y]).$$

By definition of  $H'$ , if  $|U'_i \setminus U_i| = 2$ , then for all  $z \in U_i$ ,  $P_z(Y') = a^{y-2}(a - 1)^2$  and if  $|U'_i \setminus U_i| = 1$ , then for all  $z \in U_i$ ,  $P_z(Y') = a^{y-1}(a - 1)$ . Since  $|U'_1 \setminus U_1| = 1$ , this implies

$$(31) \quad P(Y', [n - y]) \geq \left(a^{y-2}(a - 1)^2\right)^{n-y-|U_1|} \left(a^{y-1}(a - 1)\right)^{|U_1|} = \left(a^{y-2}(a - 1)^2\right)^{n-y} \left(\frac{a}{a - 1}\right)^{|U_1|}.$$

By construction,  $P(Y') \geq (a - 1)^{\binom{y}{2}}$ . Combining this with (30), (6.9.3), and the fact that  $|U_1| \geq \frac{n-y}{s-t}$ , we obtain

$$\text{ex}_{\Pi}(n, s, q) \geq \text{ex}_{\Pi}(n - y, s, q)(a - 1)^{\binom{y}{2}} \left(a^{y-2}(a - 1)^2\right)^{n-y} \left(\frac{a}{a - 1}\right)^{\frac{n-y}{s-t}}.$$

Rearranging this yields  $\text{ex}_{\Pi}(n - y, s, q) \leq \text{ex}_{\Pi}(n, s, q)(a - 1)^{-\binom{y}{2}} \left(a^{y-2}(a - 1)^2\right)^{-(n-y)} \left(\frac{a-1}{a}\right)^{\frac{n-y}{s-t}}$ .  $\square$

LEMMA 5.3.9. *Let  $n \geq s \geq 4$ ,  $a \geq 2$ , and  $q = a \binom{s}{2} - t$  for some  $2 \leq t \leq \frac{s}{2}$ . Suppose  $G \in F(n, s, q)$  and  $Y \in \binom{[n]}{s-t+1}$  satisfies  $S(Y) \geq a \binom{s-t+1}{2}$ . Then there is  $Y \subseteq Y' \subseteq [n]$  such that  $s - t + 1 \leq |Y'| \leq s - 1$  and for all  $z \in [n] \setminus Y'$ ,  $S_z(Y') \leq a|Y'| - 2$ , and consequently,  $P_z(Y') \leq a^{|Y'|-2}(a - 1)^2$ .*

PROOF. Suppose towards a contradiction that  $Y \in \binom{[n]}{s-t+1}$  satisfies  $S(Y) \geq a \binom{s-t+1}{2}$  but for all  $Y \subseteq Y' \subseteq [n]$  such that  $s - t + 1 \leq |Y'| \leq s - 1$ , there is  $z \in [n] \setminus Y'$  with  $S_z(Y') > a|Y'| - 2$ . Apply this fact with  $Y' = Y$  to choose  $z_1 \in [n] \setminus Y$  such that  $S_{z_1}(Y) > a|Y| - 2$ . Then inductively define a sequence  $z_2, \dots, z_{t-1}$  so that for each  $1 \leq i \leq t - 2$ ,  $S_{z_{i+1}}(Y \cup \{z_1, \dots, z_i\}) \geq a(s - t + 1 + i) - 1$  (to define  $z_{i+1}$ , apply the fact with  $Y' = Y \cup \{z_1, \dots, z_i\}$ ). Then  $|Y \cup \{z_1, \dots, z_{t-1}\}| = s$  and

$$\begin{aligned} S(Y \cup \{z_1, \dots, z_{t-1}\}) &\geq S(Y) + S_{z_1}(Y) + S_{z_2}(Y \cup \{z_1\}) + \dots + S_{z_{t-1}}(Y \cup \{z_1, \dots, z_{t-2}\}) \\ &\geq a \binom{s-t+1}{2} + a(s-t+1) - 1 + \dots + a(s-1) - 1 \\ &= a \binom{s}{2} - (t-1) > a \binom{s}{2} - t, \end{aligned}$$

contradicting that  $G \in F(n, s, q)$ . Therefore there is  $Y \subseteq Y' \subseteq [n]$  such that  $s - t + 1 \leq |Y'| \leq s - 1$  and for all  $z \in [n] \setminus Y'$ ,  $S_z(Y') \leq a|Y'| - 2$ . By Lemma 5.3.2, this implies  $P_z(Y') \leq a^{|Y'|-2}(a - 1)^2$ .  $\square$

Two multigraphs  $G = (V, w)$  and  $G' = (V', w')$  are *isomorphic* if there is a bijection  $f : V \rightarrow V'$  such that for all  $xy \in \binom{V}{2}$ ,  $w(xy) = w'(f(x)f(y))$ .

LEMMA 5.3.10. *Suppose  $s, q, a, t$  are integers satisfying the hypotheses of case (b) of Theorem 5.2.5. Then there are constants  $C > 1$  and  $0 < \alpha < 1$  such that for all  $n \geq 1$  the following holds. Suppose  $G \in F(n, s, q)$  and  $k(G)$  is the maximal number of pairwise disjoint elements of  $\{Y \in \binom{[n]}{s-t+1} : S(G[Y]) \geq a \binom{s-t+1}{2}\}$ . Then*

$$(32) \quad P(G) \leq C^{k(G)} \alpha^{k(G)n} \text{ex}_{\Pi}(n, s, q).$$

PROOF. Set  $\alpha = \left(\frac{a-1}{a}\right)^{\frac{1}{2t(s-t)}}$ . Choose  $C \geq q^{\binom{s-1}{2}}$  sufficiently large so that  $\text{ex}_{\Pi}(n, s, q) \leq C\alpha^{n^2}$  holds for all  $1 \leq n \leq s^3$ . We proceed by induction on  $n$ . If  $1 \leq n \leq s^3$  and  $G \in F(n, s, q)$ , then (32) is clearly true of  $k(G) = 0$ . If  $k(G) \geq 1$ , then by choice of  $C$  and since  $k(G) \leq n$  and  $\alpha < 1$ ,

$$P(G) \leq \text{ex}_{\Pi}(n, s, q) \leq C\alpha^{n^2} \leq C\alpha^{k(G)n} \leq C^{k(G)} \alpha^{k(G)n} \text{ex}_{\Pi}(n, s, q).$$

Now let  $n > s^3$  and suppose by induction (32) holds for all  $G' \in F(n', s, q)$  where  $1 \leq n' < n$ . If  $G \in F(n, s, q)$ , then (32) is clearly true if  $k(G) = 0$ . If  $k(G) > 0$ , let  $Y_1, \dots, Y_k$  be a maximal set of pairwise disjoint elements in  $\{Y \in \binom{[n]}{s-t+1} : S(G[Y]) \geq a \binom{s-t+1}{2}\}$ . Apply Lemma 5.3.9 to find  $Y'$  such that  $Y_1 \subseteq Y' \subseteq [n]$ ,  $s-t+1 \leq |Y'| \leq s-1$ , and for all  $z \in [n] \setminus Y'$ ,  $P_z(Y') \leq a^{|Y'|-2}(a-1)^2$ . Let  $|Y'| = y$ . Then note

$$(33) \quad P(Y', [n] \setminus Y') = \prod_{z \in [n] \setminus Y'} P_z(Y') \leq \left(a^{y-2}(a-1)^2\right)^{n-y}.$$

Observe that  $G[[n] \setminus Y']$  is isomorphic to some  $H \in F(n-y, s, q)$ . Since  $Y'$  can intersect at most  $t-2$  other  $Y_i$ , and since  $Y_1, \dots, Y_k$  was maximal, we must have  $k(H) + 1 \leq k(G) \leq k(H) + t - 1$ . By our induction hypothesis,

$$(34) \quad P([n] \setminus Y') = P(H) \leq C^{k(H)} \alpha^{k(H)(n-y)} \text{ex}_{\Pi}(n-y, s, q).$$

Since  $\mu(G) \leq q$  and  $y \leq s-1$ , and by our choice of  $C$ ,  $P(Y') \leq q^{\binom{y}{2}} \leq C$ . Combining this with (33), (34) and the fact that  $\mu(H) \leq \mu(G)$  we obtain that

$$\begin{aligned} P(G) &= P([n] \setminus Y') P(Y', [n] \setminus Y') P(Y') \leq C^{k(H)} \alpha^{k(H)(n-y)} \text{ex}_{\Pi}(n-y, s, q) \left(a^{y-2}(a-1)^2\right)^{n-y} C \\ &= C^{k(H)+1} \alpha^{k(H)(n-y)} \text{ex}_{\Pi}(n-y, s, q) \left(a^{y-2}(a-1)^2\right)^{n-y}. \end{aligned}$$

Plugging in the upper bound for  $\text{ex}_{\Pi}(n-y, s, q)$  from Claim 5.3.8 yields that  $P(G)$  is at most

$$(35) \quad C^{k(H)+1} \alpha^{k(H)(n-y)} \text{ex}_{\Pi}(n, s, q) (a-1)^{-\binom{y}{2}} \left(\frac{a-1}{a}\right)^{\frac{n-y}{s-t}} \leq C^{k(H)+1} \alpha^{k(H)(n-y)+2t(n-y)} \text{ex}_{\Pi}(n, s, q),$$

where the last inequality is because  $(a-1)^{-\binom{y}{2}} < 1$  and by definition of  $\alpha$ ,  $(\frac{a-1}{a})^{1/(s-t)} = \alpha^{2t}$ . We claim that the following holds.

$$(36) \quad k(H)(n-y) + 2t(n-y) \geq (k(H) + t - 1)n.$$

Rearranging this, we see (36) is equivalent to  $yk(H) \leq tn + n - 2ty$ . Since  $2 \leq t \leq s/2$  and  $y \leq s-1$ ,  $tn + n - 2ty \geq 3n - s(s-1)$ , so it suffices to show  $yk(H) \leq 3n - s(s-1)$ . By definition,  $k(H) \leq \frac{n-y}{s-t+1}$  so  $yk(H) \leq \frac{y(n-y)}{s-t+1}$ . Combining this with the facts that  $s-t+1 \leq y \leq s-1$  and  $s/2 < s-t+1$  yields

$$yk(H) \leq \frac{(s-1)(n-(s-t+1))}{s-t+1} = n \left( \frac{s-1}{s-t+1} \right) - s + 1 < 2n \left( \frac{s-1}{s} \right) - s + 1.$$

Thus it suffices to check  $2n(\frac{s-1}{s}) - s + 1 \leq 3n - s(s-1)$ . This is equivalent to  $(s-1)^2 \leq n(\frac{s+2}{s})$ , which holds because  $n \geq s^3$ . This finishes the verification of (36). Combining (67), (36), and the fact that  $k(H) + 1 \leq k(G) \leq k(H) + t - 1$  yields

$$P(G) \leq C^{k(H)+1} \alpha^{(k(H)+t-1)n} \text{ex}_{\Pi}(n, s, q) \leq C^{k(G)} \alpha^{k(G)n} \text{ex}_{\Pi}(n, s, q).$$

□

**Proof of Theorem 5.2.5(b)(Extremal).** Set  $s' = s - t + 1$  and  $q' = a\binom{s'}{2} - 1$ . Fix  $n \geq s$ . By Lemma 5.3.7 and definition of  $s'$ ,  $\mathbb{T}_{s-t,a}(n) = \mathbb{T}_{s'-1,a}(n) \subseteq \mathcal{P}(n, s, q)$  and

$$\text{ex}_{\Pi}(n, s, q) = \text{ex}_{\Pi}(n, s', q') = (a-1)^{\binom{n}{2}} \left( \frac{a}{a-1} \right)^{t_{s'-1}(n)},$$

where the last equality is by Theorem 5.2.5(a) (Extremal) applied to  $s'$  and  $q'$ . By definition, we have  $\text{ex}_{\Pi}(s, q) = (a-1) \left( \frac{a}{a-1} \right)^{1 - \frac{1}{s'-1}}$ . We have left to show that  $\mathcal{P}(n, s, q) \subseteq \mathbb{T}_{s'-1,a}(n)$  holds for large  $n$ . Assume  $n$  is sufficiently large and  $C$  and  $\alpha$  are as in Lemma 5.3.10. Note  $\text{ex}_{\Pi}(n, s, q) = \text{ex}_{\Pi}(n, s', q')$  implies  $\mathcal{P}(n, s, q) \cap F(n, s', q') \subseteq \mathcal{P}(n, s', q') = \mathbb{T}_{s'-1,a}(n)$ , where the equality is by Theorem 5.2.5 (a) (Extremal). So it suffices to show  $\mathcal{P}(n, s, q) \subseteq F(n, s', q')$ . Suppose towards a contradiction that there exists  $G = ([n], w) \in \mathcal{P}(n, s, q) \setminus F(n, s', q')$ . Then in the notation of Lemma 5.3.10,  $k(G) \geq 1$ . Combining this with Lemma 5.3.10, we have

$$P(G) \leq C^{k(G)} \alpha^{k(G)n} \text{ex}_{\Pi}(n, s, q) = \left( C \alpha^n \right)^{k(G)} \text{ex}_{\Pi}(n, s, q) < \text{ex}_{\Pi}(n, s, q),$$

where the last inequality is because  $n$  is large,  $\alpha < 1$ , and  $k(G) \geq 1$ . But now  $P(G) < \text{ex}_{\Pi}(n, s, q)$  contradicts that  $G \in \mathcal{P}(n, s, q)$ . □

### 5.4. Stability: Cases (i) and (ii)

In this section we prove the product-stability results for cases (i) and (ii). We will use the fact that for any  $(s, q)$ -graph  $G$ ,  $\mu(G) \leq q$ . If  $G = (V, w)$  and  $a \in \mathbb{N}$ , let  $E_a(G) = \{xy \in \binom{V}{2} : w(xy) = a\}$  and  $e_a(G) = |E_a(G)|$ . In the following notation,  $p$  stands for “plus” and  $m$  stands for “minus.”

$$p_a(G) = |\{xy \in \binom{V}{2} : w(xy) > a\}| \quad \text{and} \quad m_a(G) = |\{xy \in \binom{V}{2} : w(xy) < a\}|.$$

LEMMA 5.4.1. *Let  $s \geq 2$ ,  $q \geq \binom{s}{2}$  and  $a > 0$ . Suppose there exist  $0 < \alpha < 1$  and  $C > 1$  such that for all  $n \geq s$ , every  $G \in F(n, s, q)$  satisfies*

$$P(G) \leq \text{ex}_{\Pi}(n, s, q)q^{Cn}\alpha^{p_a(G)}.$$

*Then for all  $\delta > 0$  there are  $\epsilon, M > 0$  such that for all  $n > M$  the following holds. If  $G \in F(n, s, q)$  and  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$  then  $p_a(G) \leq \delta n^2$ .*

PROOF. Fix  $\delta > 0$ . Choose  $\epsilon > 0$  so that  $\frac{2\epsilon \log q}{\log(1/\alpha)} = \delta$ . Choose  $M \geq s$  sufficiently large so that  $n \geq M$  implies  $(\epsilon n^2 + Cn) \log q \leq 2\epsilon(\log q)n^2$ . Let  $n > M$  and  $G \in F(n, s, q)$  be such that  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$ . Our assumptions imply

$$\text{ex}_{\Pi}(n, s, q)^{1-\epsilon} \leq P(G) \leq \text{ex}_{\Pi}(n, s, q)q^{Cn}\alpha^{p_a(G)}.$$

Rearranging  $\text{ex}_{\Pi}(n, s, q)^{1-\epsilon} \leq \text{ex}_{\Pi}(n, s, q)q^{Cn}\alpha^{p_a(G)}$  yields  $\left(\frac{1}{\alpha}\right)^{p_a(G)} \leq \text{ex}_{\Pi}(n, s, q)^{\epsilon}q^{Cn} \leq q^{\epsilon n^2 + Cn}$ , where the second inequality is because  $\text{ex}_{\Pi}(n, s, q) \leq q^{n^2}$ . Taking logs of both sides, we obtain

$$p_a(G) \log(1/\alpha) \leq (\epsilon n^2 + Cn) \log q \leq 2\epsilon n^2 \log q,$$

where the second inequality is by assumption on  $n$ . Dividing both sides by  $\log(1/\alpha)$  and applying the definition of  $\epsilon$  yields  $p_a(G) \leq \frac{2\epsilon n^2 \log q}{\log(1/\alpha)} = \delta n^2$ .  $\square$

We now prove the key lemma for this section.

LEMMA 5.4.2. *Let  $s, q, b, a$  be integers satisfying  $s \geq 2$  and either*

$$(i) \ a \geq 1, \ 0 \leq b \leq s - 2, \ \text{and} \ q = a \binom{s}{2} + b \ \text{or}$$

$$(ii) \ a \geq 2, \ b = 0, \ \text{and} \ q = a \binom{s}{2} - 1.$$

*Then there exist  $0 < \alpha < 1$  and  $C > 1$  such that for all  $n \geq s$  and all  $G \in F(n, s, q)$ ,*

$$(37) \quad P(G) \leq \text{ex}_{\Pi}(n, s, q)q^{Cn}\alpha^{p_a(G)}.$$

PROOF. We prove this by induction on  $s \geq 2$ , and for each fixed  $s$ , by induction on  $n$ . Let  $s \geq 2$  and  $q, b, a$  be as in (i) or (ii) above. Set

$$\xi = \begin{cases} 0 & \text{if case (i) holds.} \\ 1 & \text{if case (ii) holds.} \end{cases}$$

Suppose first  $s = 2$ . Set  $\alpha = 1/2$  and  $C = 2$ . Since  $G$  is an  $(n, 2, a - \xi)$ -graph,  $p_a(G) = 0$ . Therefore for all  $n \geq 2$ ,

$$P(G) \leq \text{ex}_{\Pi}(n, s, q) \leq \overline{\text{ex}}_{\Pi}(n, s, q)q^{Cn} = \text{ex}_{\Pi}(n, s, q)q^{Cn}\alpha^{p_a(G)}.$$

Assume now  $s > 2$ . Let  $\mathcal{I}$  be the set of  $(s', q', b') \in \mathbb{N}^3$  such that  $2 \leq s' < s$  and  $s', q', b', a$  satisfy (i) or (ii). Observe that  $\mathcal{I}$  is finite. Suppose by induction on  $s$  that  $(s', q', b') \in \mathcal{I}$  implies there are  $0 < \alpha(s', q', b') < 1$  and  $C(s', q', b') > 1$  such that for all  $n' \geq s'$  and  $G' \in F(n', s', q')$ ,  $P(G') \leq \text{ex}_{\Pi}(n', s', q')q^{C(s', q', b')n'}\alpha(s', q', b')^{p_a(G')}$ . Set

$$\alpha = \max \left( \left\{ q^{-1}, \left( \frac{a^{s-2}(a-\xi) - 1}{a^{s-2}(a-\xi)} \right)^{\frac{1}{s-2}}, \left( \frac{a-1}{a} \right)^{\frac{1}{s-2}} \right\} \cup \left\{ \alpha(s', q', b') : (s', q', b') \in \mathcal{I} \right\} \right).$$

Observe  $0 < \alpha < 1$ . Choose  $C \geq \binom{s-1}{2}$  sufficiently large so that for all  $n \leq s$

$$(38) \quad q^{\binom{n}{2}} \leq q^{Cn}(a-\xi)^{\binom{n}{2}} \left( \frac{a}{a-\xi} \right)^{t_{s-1}(n)} \alpha^{\binom{n}{2}},$$

and so that for all  $(s', q', b') \in \mathcal{I}$ ,  $C(s', q', b') \leq C/2$  and  $\left( \frac{a+1}{a} \right)^{(s-3)/(s-2)} \leq q^{C/2}$ . Given  $G \in F(n, s, q)$ , set

$$\Theta(G) = \left\{ Y \subseteq \binom{[n]}{s-1} : S(Y) \geq a \binom{s-1}{2} + (1-\xi)b \right\},$$

and let  $A(n, s, q) = \{G \in F(n, s, q) : \Theta(G) \neq \emptyset\}$ . We show the following holds for all  $n \geq 1$  and  $G \in F(n, s, q)$  by induction on  $n$ .

$$(39) \quad P(G) \leq q^{Cn}(a-\xi)^{\binom{n}{2}} \left( \frac{a}{a-\xi} \right)^{t_{s-1}(n)} \alpha^{p_a(G)}.$$

This will finish the proof since  $(a-\xi)^{\binom{n}{2}} \left( \frac{a}{a-\xi} \right)^{t_{s-1}(n)} \leq \text{ex}_{\Pi}(n, s, q)$  (by Theorem 5.2.3 (Extremal) for case (i) and Theorem 5.2.5(a) (Extremal) for case (ii)). If  $n \leq s$  and  $G \in F(n, s, q)$ , then (39) holds because of (38) and the fact that  $P(G) \leq q^{\binom{n}{2}}$ . So assume  $n > s$ , and suppose by induction that (39) holds for all  $s \leq n' < n$  and  $G' \in F(n', s, q)$ . Let  $G = ([n], w) \in F(n, s, q)$ . Suppose first that  $G \in A(n, s, q)$ . Choose  $Y \in \Theta(G)$  and set  $R = [n] \setminus Y$ . Given  $z \in R$ , note that

$$a \binom{s-1}{2} + (1-\xi)b + S_z(Y) \leq S(Y) + S_z(Y) = S(Y \cup \{z\}) \leq a \binom{s}{2} + (1-\xi)b - \xi,$$



and therefore  $S_z(Y) \leq a(s-1) - \xi$ . Then for all  $z \in R$ , Lemma 5.3.2 implies  $P_z(Y) \leq a^{s-2}(a-\xi)$ , with equality only if  $\{w(yz) : y \in Y\}$  consists of  $s-1-\xi$  elements equal to  $a$  and  $\xi$  elements equal to  $a-1$ . Let  $R_1 = \{z \in R : \exists y \in Y, w(yz) > a\}$  and  $R_2 = R \setminus R_1$ . Then  $z \in R_1$  implies  $P_z(Y) < a^{s-2}(a-\xi)$ , so  $P_z(Y) \leq a^{s-2}(a-\xi) - 1$ . Let  $k = |R_1|$ . Observe that  $G[R]$  is isomorphic to an element of  $F(n', s, q)$ , where  $n' = n - |R| \geq 1$ . By induction (on  $n$ ) and these observations we have that the following holds, where  $p_a(R) = p_a(G[R])$ .

$$\begin{aligned} P(G) &= P(R)P(Y) \prod_{z \in R_1} P_z(Y) \prod_{z \in R_2} P_z(Y) \\ &\leq q^{C(n-s+1)}(a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)} q^{\binom{s-1}{2}} \left(a^{s-2}(a-\xi) - 1\right)^k \left(a^{s-2}(a-\xi)\right)^{n-s+1-k} \\ &\leq q^{C(n-s+2)}(a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)} \left(a^{s-2}(a-\xi) - 1\right)^k \left(a^{s-2}(a-\xi)\right)^{n-s+1-k}, \end{aligned}$$

where the second inequality is because  $\binom{s-1}{2} \leq C$ . Since  $\alpha \geq \left(\frac{a^{s-2}(a-\xi)-1}{a^{s-2}(a-\xi)}\right)^{1/(s-2)}$ , this is at most

$$(40) \quad q^{C(n-s+2)}(a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)+k(s-1)} \left(a^{s-2}(a-\xi)\right)^{n-s+1}.$$

Because  $C(n-s+2) \leq Cn - \binom{s-1}{2}$  and  $q^{-1} \leq \alpha$ , we have  $q^{C(n-s+2)} \leq q^{Cn} \alpha^{\binom{s-1}{2}}$ . Combining this with the fact that  $p_a(G) \leq p_a(R) + k(s-1) + \binom{s-1}{2}$  implies that (40) is at most

$$\begin{aligned} & q^{Cn}(a-\xi)^{\binom{n-s+1}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)} \alpha^{p_a(R)+k(s-1)+\binom{s-1}{2}} \left(a^{s-2}(a-\xi)\right)^{n-s+1} \\ &= q^{Cn}(a-\xi)^{\binom{n-s+1}{2}+(s-1)(n-s+1)} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n-s+1)+(s-2)(n-s+1)} \alpha^{p_a(R)+k(s-1)+\binom{s-1}{2}} \\ &\leq q^{Cn}(a-\xi)^{\binom{n}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n)} \alpha^{p_a(G)}. \end{aligned}$$

We now have that  $P(G) \leq q^{Cn}(a-\xi)^{\binom{n}{2}} \left(\frac{a}{a-\xi}\right)^{t_{s-1}(n)} \alpha^{p_a(G)}$ , as desired. Assume now  $G \notin A(n, s, q)$ . Then for all  $Y \in \binom{[n]}{s-1}$ ,  $S(Y) \leq a\binom{s-1}{2} + (1-\xi)b - 1$ . Thus  $G$  is an  $(n, s', q')$ -graph where  $s' = s-1$  and  $q' = a\binom{s-1}{2} + (1-\xi)b - 1$ . Suppose  $a = 1$ ,  $\xi = 0$ , and  $b = 0$ . Then  $q' = \binom{s'}{2} - 1$  and any  $(n, s', q')$ -graph must contain an edge of multiplicity 0. This implies  $P(G) = 0$  and (39) holds. We have the following three cases remaining, where  $b' = \max\{b-1, 0\}$ .

- (1)  $\xi = 0$ ,  $b = 0$ , and  $a \geq 2$ . In this case  $q' = a\binom{s'}{2} - 1$  and  $b' = 0$ .
- (2)  $\xi = 1$ ,  $b = 0$ , and  $a \geq 2$ . In this case  $q' = a\binom{s'}{2} - 1$  and  $b' = 0$ .
- (3)  $\xi = 0$ ,  $1 \leq b \leq s-2$ , and  $a \geq 1$ . In this case  $q' = a\binom{s'}{2} + b'$  and  $0 \leq b' \leq s' - 2$ .

It is clear that in all three of these cases,  $(s', q', b') \in \mathcal{I}$ , so by our induction hypothesis (on  $s$ ), there are  $\alpha' = \alpha(s', q', b') \leq \alpha$  and  $C' = C(s', q', b')$  such that

$$(41) \quad P(G) \leq \text{ex}_{\Pi}(n, s', q')(q')^{C'n}(\alpha')^{p_a(G)} \leq \text{ex}_{\Pi}(n, s', q')q^{C'n}\alpha^{p_a(G)},$$

where the inequality is because  $q' \leq q$  and  $\alpha' \leq \alpha$ . By Theorem 5.2.5(a) (Extremal) if cases 1 or 2 hold, and by Theorem 5.2.3 (Extremal) if case 3 holds, we have the following.

$$\text{ex}_{\Pi}(n, s', q') \leq (a - \xi)^{\binom{n}{2}} \left( \frac{a}{a - \xi} \right)^{t_{s'-1}(n)} \left( \frac{a+1}{a} \right)^{\lfloor \frac{b'}{b'+1}n \rfloor} \leq (a - \xi)^{\binom{n}{2}} \left( \frac{a}{a - \xi} \right)^{t_{s-1}(n)} \left( \frac{a+1}{a} \right)^{\frac{s-3}{s-2}n},$$

where the last inequality is because  $t_{s'-1}(n) \leq t_{s-1}(n)$  and  $\lfloor \frac{b'}{b'+1}n \rfloor \leq \frac{b'}{b'+1}n \leq \frac{s-3}{s-2}n$ . By choice of  $C$ ,  $(\frac{a+1}{a})^{\frac{s-3}{s-2}n} \leq q^{Cn/2}$ . Thus  $\text{ex}_{\Pi}(n, s', q') \leq (a - \xi)^{\binom{n}{2}} (\frac{a}{a - \xi})^{t_{s-1}(n)} q^{Cn/2}$ . Combining this with (41) implies

$$P(G) \leq (a - \xi)^{\binom{n}{2}} \left( \frac{a}{a - \xi} \right)^{t_{s-1}(n)} q^{Cn/2} q^{C'n} \alpha^{p_a(G)} \leq (a - \xi)^{\binom{n}{2}} \left( \frac{a}{a - \xi} \right)^{t_{s-1}(n)} q^{Cn} \alpha^{p_a(G)},$$

where the last inequality is because  $C' \leq C/2$ . Thus (39) holds.  $\square$

**Proof of Theorem 5.2.3 (Stability).** Let  $s \geq 2$ ,  $a \geq 1$ , and  $q = a \binom{s}{2} + b$  for some  $0 \leq b \leq s - 2$ . Fix  $\delta > 0$ . Given  $G \in F(n, s, q)$ , let  $p_G = p_a(G)$  and  $m_G = m_a(G)$ . Note that if  $G \in F(n, s, q)$ , then  $|\Delta(G, \mathbb{U}_a(n))| = m_G + p_G$ . Suppose first  $a = 1$ , so  $m_G = 0$ . Combining Lemma 5.4.2 with Lemma 5.4.1 implies there are  $\epsilon_1$  and  $M_1$  such that if  $n > M_1$  and  $G \in F(n, s, q)$  satisfies  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon_1}$ , then  $|\Delta(G, \mathbb{U}_a(n))| = p_G \leq \delta n^2$ . Assume now  $a > 1$ . Combining Lemma 5.4.2 with Lemma 5.4.1 implies there are  $\epsilon_1$  and  $M_1$  such that if  $n > M_1$  and  $G \in F(n, s, q)$  satisfies  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon_1}$ , then  $p_G \leq \delta' n^2$ , where

$$\delta' = \min \left\{ \frac{\delta}{2}, \frac{\delta \log(a/(a-1))}{4 \log q} \right\}.$$

Set  $\epsilon = \min\{\epsilon_1, \frac{\delta \log(a/(a-1))}{4 \log q}\}$ . Suppose  $n > M_1$  and  $G \in F(n, s, q)$  satisfies  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$ .

Our assumptions imply  $p_G \leq \delta' n^2 \leq \delta n^2/2$ . Observe that by definition of  $p_G$  and  $m_G$ ,

$$(42) \quad P(G) \leq a^{\binom{n}{2} - m_G} (a-1)^{m_G} q^{p_G} = a^{\binom{n}{2}} \left( \frac{a-1}{a} \right)^{m_G} q^{p_G}.$$

By Theorem 5.2.3(a)(Extremal),  $\text{ex}_{\Pi}(n, s, q) \geq a^{\binom{n}{2}}$ . Therefore  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon} \geq a^{\binom{n}{2}(1-\epsilon)}$ .

Combining this with (42) yields

$$a^{\binom{n}{2}(1-\epsilon)} \leq a^{\binom{n}{2}} \left( \frac{a-1}{a} \right)^{m_G} q^{p_G}.$$

Rearranging this, we obtain

$$\left(\frac{a}{a-1}\right)^{m_G} \leq a^{\epsilon \binom{n}{2}} q^{p_G} \leq q^{\epsilon \binom{n}{2} + p_G} \leq q^{\epsilon n^2 + p_G}.$$

Taking logs, dividing by  $\log(a/(a-1))$ , and applying our assumptions on  $p_G$  and  $\epsilon$  yields

$$m_G \leq \frac{\epsilon n^2 \log q}{\log(a/(a-1))} + \frac{p_G \log q}{\log(a/(a-1))} \leq \frac{\delta n^2}{4} + \frac{\delta n^2}{4} = \frac{\delta n^2}{2}.$$

Combining this with the fact that  $p_G \leq \frac{\delta n^2}{2}$  we have that  $|\Delta(G, \mathbb{U}_a(n))| \leq \delta n^2$ .  $\square$

**Proof of Theorem 5.2.5(a) (Stability).** Let  $s \geq 2$ ,  $a \geq 2$ , and  $q = a \binom{s}{2} - 1$ . Fix  $\delta > 0$ . Given  $G \in F(n, s, q)$ , let  $p_G = p_a(G)$ ,  $m_G = m_{a-1}(G)$ . Choose  $M_0$  and  $\mu$  such that  $\mu < \delta/2$  and so that Theorem 2.2.2 implies that any  $K_s$ -free graph with  $n \geq M_0$  vertices and at least  $(1 - \mu)t_{s-1}(n)$  edges can be made into  $T_{s-1}(n)$  by adding or removing at most  $\frac{\delta n^2}{3}$  edges. Set

$$A = \begin{cases} 2 & \text{if } a = 2 \\ \frac{a-1}{a-2} & \text{if } a > 2 \end{cases}$$

Combining Lemma 5.4.2 with Lemma 5.4.1 implies there are  $\epsilon_1, M_1$  so that if  $n > M_1$  and  $G \in F(n, s, q)$  satisfies  $P(G) \geq \text{ex}_\Pi(n, s, q)^{1-\epsilon_1}$ , then  $p_G \leq \delta' n^2$ , where

$$(43) \quad \delta' = \min \left\{ \frac{\delta}{3}, \frac{\mu \log(a/(a-1))}{2 \log q}, \frac{\delta \log A}{6 \log q} \right\}.$$

Let

$$\epsilon = \min \left\{ \epsilon_1, \frac{\delta \log A}{6 \log q}, \frac{\mu \log(a/(a-1))}{2 \log q} \right\} \quad \text{and} \quad M = \max\{M_0, M_1\}.$$

Suppose now that  $n > M$  and  $G \in F(n, s, q)$  satisfies  $P(G) \geq \text{ex}_\Pi(n, s, q)^{1-\epsilon}$ . By assumption,  $p_G \leq \delta' n^2 \leq \frac{\delta n^2}{3}$ . We now bound  $m_G$ . Note that if  $a = 2$  and  $P(G) \neq 0$ , then  $m_G = 0$ . If  $a > 2$ , observe that by definition of  $p_G$  and  $m_G$ ,

$$(44) \quad P(G) \leq q^{p_G} (a-2)^{m_G} a^{e_a(G)} (a-1)^{e_{a-1}(G)} \leq q^{p_G} \left(\frac{a-2}{a-1}\right)^{m_G} a^{e_a(G)} (a-1)^{\binom{n}{2} - e_a(G)},$$

where the last inequality is because  $e_{a-1}(G) + m_G \leq \binom{n}{2} - e_a(G)$ . Note that Turán's theorem and the fact that  $G$  is an  $(n, s, q)$ -graph implies that  $e_a(G) \leq t_{s-1}(n)$ , so

$$a^{e_a(G)} (a-1)^{\binom{n}{2} - e_a(G)} \leq a^{t_{s-1}(n)} (a-1)^{\binom{n}{2} - t_{s-1}(n)} = \text{ex}_\Pi(n, s, q),$$

where the last equality is from Theorem 5.2.5(a) (Extremal). Combining this with (44) yields

$$\text{ex}_\Pi(n, s, q)^{1-\epsilon} \leq P(G) \leq q^{p_G} \left( \frac{a-2}{a-1} \right)^{m_G} \text{ex}_\Pi(n, s, q).$$

Rearranging  $\text{ex}_\Pi(n, s, q)^{1-\epsilon} \leq q^{p_G} \left( \frac{a-2}{a-1} \right)^{m_G} \text{ex}_\Pi(n, s, q)$  and using that  $\text{ex}_\Pi(n, s, q) \leq q^{n^2}$ , we obtain

$$A^{m_G} = \left( \frac{a-1}{a-2} \right)^{m_G} \leq q^{p_G} \text{ex}_\Pi(n, s, q)^\epsilon \leq q^{p_G + \epsilon n^2}.$$

Taking logs, dividing by  $\log A$ , and applying our assumptions on  $p_G$  and  $\epsilon$  we obtain  $m_G < \delta n^2/3$ .

Using (44) and  $a^{t_{s-1}(n)}(a-1)^{\binom{n}{2}-t_{s-1}(n)} = \text{ex}_\Pi(n, s, q)$ , we have

$$\text{ex}_\Pi(n, s, q)^{1-\epsilon} \leq P(G) \leq q^{p_G} a^{e_a(G)} (a-1)^{\binom{n}{2}-e_a(G)} = q^{p_G} \text{ex}_\Pi(n, s, q) \left( \frac{a}{a-1} \right)^{e_a(G)-t_{s-1}(n)}.$$

Rearranging this we obtain

$$\left( \frac{a}{a-1} \right)^{t_{s-1}(n)-e_a(G)} \leq q^{p_G} \text{ex}_\Pi(n, s, q)^\epsilon \leq q^{p_G + \epsilon n^2}.$$

Taking logs, dividing by  $\log(a/(a-1))$ , and using the assumptions on  $p_G$  and  $\epsilon$  we obtain that

$$t_{s-1}(n) - e_a(G) \leq \frac{p_G \log q}{\log(a/(a-1))} + \frac{\epsilon n^2 \log q}{\log(a/(a-1))} \leq \frac{\mu n^2}{2} + \frac{\mu n^2}{2} = \mu n^2.$$

Let  $H$  be the graph with vertex set  $[n]$  and edge set  $E = E_a(G)$ . Then  $H$  is  $K_s$ -free, and has  $e_a(G)$  many edges. Since  $t_{s-1}(n) - e_a(G) \leq \mu n^2$ , Theorem 2.2.2 implies that  $H$  is  $\frac{\delta}{3}$ -close to some  $H' = T_{s-1}(n)$ . Define  $G' \in F(n, s, q)$  so that  $E_a(G') = E(H')$  and  $E_{a-1}(G') = \binom{n}{2} \setminus E_a(G')$ . Then  $G' \in \mathbb{T}_{s-1, a}(n)$  and

$$\Delta(G, G') \subseteq (E_a(G) \Delta E_a(G')) \cup \bigcup_{i \notin \{a, a-1\}} E_i(G) = \Delta(H, H') \cup \bigcup_{i \notin \{a, a-1\}} E_i(G).$$

This implies  $|\Delta(G, G')| \leq |\Delta(H, H')| + p_G + m_G \leq \frac{\delta}{3}n^2 + \frac{\delta}{3}n^2 + \frac{\delta}{3}n^2 = \delta n^2$ .  $\square$

**5.4.1. Proof of Theorem 5.2.5(b) (Stability).** In this subsection we prove Theorem 5.2.5(b) (Stability). We first prove two lemmas.

LEMMA 5.4.3. *Let  $s \geq 4$ ,  $a \geq 2$ , and  $q = a \binom{s}{2} - t$  for some  $2 \leq t \leq \frac{s}{2}$ . For all  $\lambda > 0$  there are  $M$  and  $\epsilon > 0$  such that the following holds. Suppose  $n > M$  and  $G \in F(n, s, q)$  satisfies  $P(G) > \text{ex}_\Pi(n, s, q)^{1-\epsilon}$ . Then  $k(G) < \lambda n$ , where  $k(G)$  is as defined in Lemma 5.3.10.*

PROOF. Fix  $\lambda > 0$ . Set  $\eta = a^{\frac{s-t-1}{s-t}}(a-1)^{\frac{1}{s-t}}$  and choose  $C$  and  $\alpha$  as in Lemma 5.3.10. Choose  $\epsilon > 0$  so that  $\alpha^{\lambda/2} = \eta^{-\epsilon}$ . By Theorem 5.2.5(b) (Extremal),  $\text{ex}_{\Pi}(n, s, q) = \eta^{\binom{n}{2} + o(n^2)}$ . Assume  $M$  sufficiently large so that for all  $n \geq M$ , (5.2.5) holds for all  $G \in F(n, s, q)$ ,  $\text{ex}_{\Pi}(n, s, q) < \eta^{n^2}$ ,  $C^{\lambda n} \leq \eta^{\epsilon n^2}$ , and  $C\alpha^n < 1$ . Fix  $n \geq M$  and suppose towards a contradiction that  $G \in F(n, s, q)$  satisfies  $P(G) > \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$  and  $k(G) \geq \lambda n$ . By Lemma 5.3.10 and the facts that  $C\alpha^n < 1$  and  $k(G) \geq 1$ , we obtain that

$$P(G) \leq C^{k(G)} \alpha^{nk(G)} \text{ex}_{\Pi}(n, s, q) = (C\alpha^n)^{k(G)} \text{ex}_{\Pi}(n, s, q) \leq (C\alpha^n)^{\lambda n} \text{ex}_{\Pi}(n, s, q).$$

By assumption on  $n$  and definition of  $\epsilon$ ,  $(C\alpha^n)^{\epsilon n} = C^{\lambda n} \alpha^{\lambda n^2} = C^{\lambda n} \eta^{-2\epsilon n^2} \leq \eta^{-\epsilon n^2}$ . Thus

$$P(G) \leq \eta^{-\epsilon n^2} \text{ex}_{\Pi}(n, s, q) < \text{ex}_{\Pi}(n, s, q)^{1-\epsilon},$$

where the last inequality is because by assumption,  $\text{ex}_{\Pi}(n, s, q) < \eta^{n^2}$ . But this contradicts our assumption that  $P(G) > \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$ .  $\square$

Given a multigraph  $G = (V, w)$ , let  $\mathcal{H}(G, s, q) = \{Y \in \binom{V}{s} : S(Y) > q\}$ . Observe that  $G$  is an  $(s, q)$ -graph if and only if  $\mathcal{H}(G, s, q) = \emptyset$ .

LEMMA 5.4.4. *Let  $s, q, m \geq 2$  be integers. For all  $0 < \delta < 1$ , there is  $0 < \lambda < 1$  and  $N$  such that  $n > N$  implies the following. If  $G = ([n], w)$  has  $\mu(G) \leq m$  and  $\mathcal{H}(G, s, q)$  contains strictly less than  $\lceil \lambda n \rceil$  pairwise disjoint elements, then  $G$  is  $\delta$ -close to an element in  $F(n, s, q)$ .*

PROOF. Fix  $0 < \delta < 1$ . Observe we can view any multigraph  $G$  with  $\mu(G) \leq m$  as an edge-colored graph with colors in  $\{0, \dots, m\}$ . One can use a multi-color version of Szemerédi's regularity lemma (see [7]) to prove a version of the triangle removal lemma for multigraphs: for all  $\delta > 0$  there is  $\epsilon$  and  $M$  such that if  $n > M$  and  $G = ([n], w)$  has  $\mu(G) \leq m$  and  $\mathcal{H}(G, s, q) \leq \epsilon \binom{n}{s}$ , then  $G$  is  $\delta$ -close to an element of  $F(n, s, q)$ . Since the proof of this statement is merely an adjustment of the proof of the triangle removal lemma for graphs, we omit it. Let  $\lambda := \epsilon/s$  and  $N = \max\{M, \frac{s}{1-\lambda s}\}$ . We claim this  $\lambda$  and  $N$  satisfy the desired conclusions. Suppose towards a contradiction that  $n > M$  and  $G = ([n], w)$  has  $\mu(G) \leq m$ ,  $\mathcal{H}(n, s, q)$  contains strictly less than  $\lceil \lambda n \rceil$  pairwise disjoint elements, but  $G$  is  $\delta$ -far from every element in  $F(n, s, q)$ . Then  $|\mathcal{H}(G, s, q)| > \epsilon \binom{n}{s}$  by choice of  $M$  and  $\lambda$ . By our choice of  $N$ ,  $\lceil \lambda n \rceil s \leq (\lambda n + 1)s \leq n$ . Then Proposition 11.6 in [56] and our assumptions imply  $|\mathcal{H}(G, s, q)| \leq (\lceil \lambda n \rceil - 1) \binom{n-1}{s-1}$ . But now

$$|\mathcal{H}(G, s, q)| \leq (\lceil \lambda n \rceil - 1) \binom{n-1}{s-1} < \lambda n \binom{n-1}{s-1} = \left(\frac{\epsilon n}{s}\right) \binom{s}{n} \binom{n}{s} = \epsilon \binom{n}{s},$$

a contradiction.  $\square$

**Proof of Theorem 5.2.5(b) (Stability).** Let  $s \geq 4$ ,  $a \geq 2$ , and  $q = a \binom{s}{2} - t$  for some  $2 \leq t \leq \frac{s}{2}$ . Fix  $\delta > 0$ . Let  $s' = s - t + 1$  and  $q' = a \binom{s'}{2} - 1$ . Note Theorem 5.2.5 (Extremal) implies that for sufficiently large  $n$ ,  $\mathcal{P}(n, s, q) = \mathbb{T}_{s'-1, a}(n)$ ,  $\text{ex}_{\Pi}(n, s', q') = \text{ex}_{\Pi}(n, s, q)$ , and  $\text{ex}_{\Pi}(s', q') = \text{ex}_{\Pi}(s, q) = \eta$ , where  $\eta = (a-1) \left(\frac{a}{a-1}\right)^{(s'-2)/(s'-1)}$ .

Apply Theorem 5.2.5 (a) (Stability) for  $(s', q')$  to  $\delta/2$  to obtain  $\epsilon_0$ . By replacing  $\epsilon_0$  if necessary, assume  $\epsilon_0 < 4\delta/\log \eta$ . Set  $\epsilon_1 = \epsilon_0 \log \eta / (8 \log q)$  and note  $\epsilon_1 < \delta/2$ . Apply Lemma 6.7.3 to  $\epsilon_1$  and  $m = q$  to obtain  $\lambda$  such that for large  $n$  the following holds. If  $G = ([n], w)$  has  $\mu(G) \leq q$  and  $\mathcal{H}(G, s', q')$  contains strictly less than  $\lceil \lambda n \rceil$  pairwise disjoint elements, then  $G$  is  $\epsilon_1$ -close to an element in  $F(n, s', q')$ . Finally, apply Lemma 5.4.3 for  $s, q, t$  to  $\lambda$  to obtain  $\epsilon_2 > 0$ .

Choose  $M$  sufficiently large for the desired applications of Theorems 5.2.5(a) (Stability) and 5.2.5(b) (Extremal) and Lemmas 5.4.3 and 6.7.3. Set  $\epsilon = \min\{\epsilon_2, \epsilon_0/2\}$ . Suppose  $n > M$  and  $G \in F(n, s, q)$  satisfies  $P(G) \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon}$ . Then Lemma 5.4.3 and our choice of  $\epsilon$  implies  $k(G) < \lambda n$ . Observe that by the definitions of  $s', q'$ ,

$$\left\{ Y \in \binom{[n]}{s-t+1} : S(Y) \geq a \binom{s-t+1}{2} \right\} = \left\{ Y \in \binom{[n]}{s'} : S(Y) \geq q' + 1 \right\} = \mathcal{H}(G, s', q').$$

Thus  $k(G) < \lambda n$  means  $\mathcal{H}(G, s', q')$  contains strictly less than  $\lceil \lambda n \rceil$  pairwise disjoint elements. Lemma 6.7.3 then implies  $G$  is  $\epsilon_1$ -close to some  $G' \in F(n, s', q')$ . Combining this with the definition of  $\epsilon_1$  yields

$$(45) \quad P(G') \geq P(G) q^{-|\Delta(G, G')|} \geq P(G) q^{-\epsilon_1 n^2} = P(G) \eta^{-\epsilon_0 n^2/8} \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon} \eta^{-(\epsilon_0/2) \binom{n}{2}}.$$

By Proposition 5.3.1,  $\text{ex}_{\Pi}(n, s, q) \geq \text{ex}_{\Pi}(s, q) \binom{n}{2} = \eta \binom{n}{2}$ . Combining this with (45) and the definition of  $\epsilon$  yields

$$(46) \quad P(G') \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon} \eta^{-(\epsilon_0/2) \binom{n}{2}} \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon-\epsilon_0/2} \geq \text{ex}_{\Pi}(n, s, q)^{1-\epsilon_0}.$$

Since  $\text{ex}_{\Pi}(n, s, q) = \text{ex}_{\Pi}(n, s', q')$ , (46) implies  $P(G') \geq \text{ex}_{\Pi}(n, s', q')^{1-\epsilon_0}$ , so Theorem 5.2.5(a) (Stability) implies  $G'$  is  $\delta/2$ -close to some  $G'' \in \mathbb{T}_{s'-1, a}(n) = \mathbb{T}_{s-t, a}(n)$ . Now we are done, since

$$|\Delta(G, G'')| \leq |\Delta(G, G')| + |\Delta(G', G'')| \leq \epsilon_1 n^2 + \delta n^2/2 \leq \delta n^2.$$

$\square$

### 5.5. Extremal Result for $(n, 4, 15)$ -graphs

In this section we prove Theorem 5.2.7 using two theorems, Theorems 5.5.2 and 5.5.3, which will be proved in Sections 5.7 and 5.8 respectively.

DEFINITION 5.5.1. Given  $G = (V, w)$  and  $i, j, k \in \mathbb{N}$ , an  $(i, j, k)$ -triangle in  $G$  is a set of three vertices  $x, y, z \in V$  such that  $\{w(xy), w(yz), w(xz)\} = \{i, j, k\}$ . Say that  $G$  omits  $(i, j, k)$ -triangles if there is no  $(i, j, k)$ -triangle in  $G$ . Given  $n \in \mathbb{N}$ , let  $A_{i,j,k}(n) = \{G \in F(n, 4, 15) : G \text{ omits } (i, j, k)\text{-triangles}\}$ .

Given  $n \in \mathbb{N}$ , let  $F_{\leq 3}(n, 4, 15) = \{G \in F(n, 4, 15) : \mu(G) \leq 3\}$ , and let

$$C(n) = F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8) \cap A_{3,1,1}(n) \cap A_{2,1,1}(n) \cap A_{3,2,1}(n).$$

Observe that for all  $n$ ,  $W(n) \subseteq C(n) \subseteq F(n, 4, 15)$ . The proof of Theorem 5.2.7 relies on understanding the structure of  $(4, 15)$ -graphs which are product-extremal subject to certain constraints.

Given a set  $\mathcal{F}$  of multigraphs, let

$$\mathcal{P}(\mathcal{F}) = \{G \in \mathcal{F} : P(G) \geq P(G') \text{ for all } G' \in \mathcal{F}\} \text{ and } \mathcal{S}(\mathcal{F}) = \{G \in \mathcal{F} : S(G) \geq S(G') \text{ for all } G' \in \mathcal{F}\}.$$

THEOREM 5.5.2. *There is an  $M$  such that for all  $n \geq M$ ,  $\mathcal{P}(C(n)) \subseteq W(n)$ .*

THEOREM 5.5.3. *There is an  $M$  such that for all  $n \geq M$ ,  $\mathcal{P}(C(n)) \subseteq \mathcal{P}(n, 4, 15)$ .*

Recall that if  $G = ([n], w) \in W(n)$ , then there is a partition  $L, R$  of  $[n]$  such that  $w(xy) = 1$  if and only if  $xy \in \binom{L}{2}$ . Given  $G \in W(n)$ , we let  $L(G)$  and  $R(G)$  denote this  $L$  and  $R$  respectively. Recall the definition of  $\gamma$  from Theorem 5.2.7.

LEMMA 5.5.4. *For all  $G \in \mathcal{P}(W(n))$ , we have  $P(G) = 2^{\gamma n^2 + O(n)}$ .*

PROOF. Let  $G = ([n], w) \in W(n)$ . Set  $h(y) = 2^{\binom{y}{2}} 3^{y(n-y)}$  and observe that if  $|L(G)| = n - y$  and  $|R(G)| = y$ , then  $P(G) = h(y)$ . Thus it suffices to show that  $\max_{y \in [n]} h(y) = 2^{\gamma n^2 + O(n)}$ . Basic calculus shows that  $h(y)$  has a global maximum at  $\tau = \beta n - (\log 2)/(2(2 \log 3 - \log 2))$ , where  $\beta = \frac{\log 3}{2 \log 3 - \log 2}$  is as in Theorem 5.2.7. This implies  $\max_{y \in \mathbb{N}} h(y) = \max\{h(\lfloor \tau \rfloor), h(\lceil \tau \rceil)\}$ . It is straightforward to check  $\max\{h(\lfloor \tau \rfloor), h(\lceil \tau \rceil)\} = \max\{h(\lfloor \beta n, \rfloor), h(\lceil \beta n \rceil)\}$ . By definition of  $\gamma$  and  $h$ , this implies  $\max_{y \in [n]} h(y) = 2^{\gamma n^2 + O(n)}$ .  $\square$

**Proof of Theorem 5.2.7.** Fix  $n$  sufficiently large and  $G \in \mathcal{P}(W(n))$ . Since  $W(n) \subseteq C(n)$ , Theorem 5.5.2 implies  $\mathcal{P}(C(n)) = \mathcal{P}(W(n))$ , so  $G \in \mathcal{P}(C(n))$ . Theorem 5.5.3 implies  $G \in \mathcal{P}(n, 4, 15)$ . Thus  $\mathcal{P}(W(n)) \subseteq \mathcal{P}(n, 4, 15)$  holds. Consequently  $\text{ex}_{\Pi}(n, 4, 15) = P(G) = 2^{\gamma n^2 + O(n)}$  by Lemma 5.5.4.  $\square$

### 5.6. A Key Lemma

In this section we prove a key lemma for Theorem 5.5.2, Lemma 5.6.2 below. Given multigraphs  $G = (V, w)$  and  $G' = (V', w')$ ,  $G'$  is a *submultigraph* of  $G$  if  $V' \subseteq V$  and  $G' = G[V']$ .

DEFINITION 5.6.1. Given  $t \geq 3$ , define  $C_t(3, 2)$  to be the multigraph  $([t], w)$  such that

$$w(12) = w(23) = \dots = w((t-1)t) = w(t1) = 3,$$

and  $w(ij) = 2$  for all other pairs  $i \neq j$ . For  $n \geq 3$ , set  $NC(n)$  (NC=“no cycles”) to be the set of  $G \in C(n)$  which contain no submultigraph isomorphic to  $C_t(3, 2)$  for any  $t \geq 3$ .

If  $G$  contains a submultigraph isomorphic to  $C_t(3, 2)$ , we will write  $C_t(3, 2) \subseteq G$ , if not, we will write  $C_t(3, 2) \not\subseteq G$ .

LEMMA 5.6.2. *There is  $M$  such that for  $n > M$ ,  $\mathcal{P}(NC(n)) \subseteq W(n)$ .*

We begin with some definitions. A *vertex-weighted graph* is a triple  $(V, E, f)$  where  $(V, E)$  is graph and  $f : V \rightarrow \mathbb{N}^{>0}$ . Given a multigraph  $G = (V, w)$ , let  $\sim_G$  be the binary relation on  $V$  defined by  $x \sim_G y \Leftrightarrow w^G(xy) = 1$ .

DEFINITION 5.6.3. A multigraph  $G$  is *neat* if  $\mu(G) \leq 3$  and  $G$  contains no  $(i, j, k)$ -triangle for  $(i, j, k) \in \{(1, 1, 2), (1, 1, 3), (1, 2, 3)\}$ .

Observe that all multigraphs in  $C(n)$  are neat. Neat multigraphs have the property that we can “mod out” by  $\sim_G$  in a coherent way.

PROPOSITION 5.6.4. *Suppose  $G = (V, w)$  is a neat multigraph. Then  $\sim_G$  forms an equivalence relation on  $V$ . Moreover, if  $\tilde{V} = \{V_1, \dots, V_t\}$  is the set of equivalence classes of  $V$  under  $\sim_G$ , then for each  $i \neq j$ , there is  $w_{ij} \in \{2, 3\}$  such that for all  $(x, y) \in V_i \times V_j$ ,  $w(xy) = w_{ij}$ .*

The proof is straightforward and left to the reader. Suppose  $G = (V, w)$  is a neat multigraph,  $\tilde{V} = \{V_1, \dots, V_t\}$  is the set of equivalence classes of  $V$  under  $\sim_G$ , and for each  $i \neq j$ ,  $w_{ij} \in \{2, 3\}$  is



from Proposition 5.6.4. Define the *vertex-weighted graph associated to  $G$  and  $\sim_G$*  to be  $\tilde{G} = (\tilde{V}, \tilde{E}, f)$  where  $\tilde{E} = \{V_i V_j \in \binom{\tilde{V}}{2} : w_{ij} = 3\}$  and  $f(V_i) = |V_i|$  for all  $i \in [t]$ . We will use the notation  $|\cdot|^G$  to denote this vertex-weight function  $f$ , and we will drop the superscript when  $G$  is clear from context. If  $H = (V, E)$  is a graph and  $X \subseteq V$ , then let  $H[X] = (X, E \cap \binom{X}{2})$ .

LEMMA 5.6.5. *Suppose  $G$  is a neat multigraph with vertex set  $[n]$ . Then  $G \in NC(n)$  if and only if  $\tilde{G}$  is a forest.*

PROOF. Suppose  $\tilde{G}$  is not a forest. Then there is  $X = \{V_{i_1}, \dots, V_{i_k}\} \subseteq \tilde{V}$  such that  $\tilde{G}[X]$  is a cycle of length  $k \geq 3$ . Choose some  $y_j \in V_{i_j}$  for each  $1 \leq i \leq k$  and let  $Y = \{y_1, \dots, y_k\}$ . Then by definition of  $\tilde{G}$ , we must have  $G[Y] \cong C_k(3, 2)$ . Thus  $G \notin NC(n)$ .

On the other hand, suppose  $G \notin NC(n)$ . Then because  $G$  is neat, we must have that either  $G \notin F(n, 4, 15)$  or  $C_t(3, 2) \subseteq G$  for some  $t \geq 3$ . Suppose  $G \notin F(n, 4, 15)$ . Then there is some  $Y \in \binom{[n]}{4}$  such that  $S^G(Y) > 15$ . Since  $\mu(G) \leq 3$ , this implies that either

- (i)  $\{w(xy) : xy \in \binom{Y}{2}\} = \{3, 3, 3, 3, 2, 2\}$  or
- (ii)  $\{w(xy) : xy \in \binom{Y}{2}\} = \{3, 3, 3, 3, 3, j\}$ , some  $j \in \{1, 2, 3\}$ .

Let  $X$  be the set of equivalence classes intersecting  $Y$ , that is  $X = \{V_i \in \tilde{V} : Y \cap V_i \neq \emptyset\}$ . In Case (i), because  $Y$  spans no edges of multiplicity 1 in  $G$ , the elements of  $Y$  must be in pairwise distinct equivalence classes under  $\sim_G$ . Thus in  $\tilde{G}$ ,  $|X| = 4$  and  $X$  spans exactly 4 edges. This implies  $\tilde{G}[X]$  is either a 4-cycle or contains a triangle. In Case (ii), if  $j = 1$ , then  $|X| = 3$  and  $\tilde{G}[X]$  is a triangle. If  $j \neq 1$ , then  $|X| = 4$  and spans at least 5 edges. This implies  $\tilde{G}[X]$  contains a triangle.  $\square$

DEFINITION 5.6.6. Given a vertex-weighted graph  $\tilde{G} = (\tilde{V}, E, |\cdot|)$ , set

$$f_\pi(\tilde{G}) = \prod_{UV \in E} 3^{|U||V|} \prod_{UV \in \binom{\tilde{V}}{2} \setminus E} 2^{|U||V|}.$$

Note that we have  $P(G) = f_\pi(\tilde{G})$  for all  $G \in C(n)$ .

Two vertex-weighted graphs  $(G_1, E_1, f_1)$  and  $(G_2, E_2, f_2)$ , are *isomorphic* if there is a graph isomorphism  $g : V(G_1) \rightarrow V(G_2)$  such that for all  $v \in V(G_1)$ ,  $f_1(v) = f_2(g(v))$ .

LEMMA 5.6.7. *Let  $H = (\tilde{V}, E, |\cdot|)$  be a vertex-weighted forest such that  $\sum_{V \in \tilde{V}} |V| = n$ . Then there is a multigraph  $G \in NC(n)$  such that  $\tilde{G}$  is isomorphic to  $H$ .*

PROOF. Let  $\tilde{V} = \{V_1, \dots, V_t\}$  and for each  $i$ , let  $x_i = |V_i|$ . Since  $\sum_{i=1}^t x_i = n$ , it is clear there exists a partition  $P_1, \dots, P_t$  of  $[n]$  such that for each  $i \in [t]$ ,  $|P_i| = x_i$ . Fix such a partition  $P_1, \dots, P_t$ . Define  $G = ([n], w)$  as follows. For each  $xy \in \binom{[n]}{2}$ ,

$$w(xy) = \begin{cases} 1 & \text{if } xy \in \binom{P_i}{2} \text{ for some } i \in [t] \\ 3 & \text{if } xy \in E(P_i, P_j) \text{ for some } i \neq j \text{ such that } V_i V_j \in E \\ 2 & \text{if } xy \in E(P_i, P_j) \text{ for some } i \neq j \text{ such that } V_i V_j \notin E. \end{cases}$$

By construction,  $G$  is a neat multigraph and  $\tilde{G}$  is isomorphic to  $H$ . Because  $H \cong \tilde{G}$  is a forest, Lemma 5.6.5 implies  $G \in NC(n)$ .  $\square$

Given a vertex-weighted graph,  $H = (\tilde{V}, E, |\cdot|)$  and  $V \in \tilde{V}$ , let  $d^H(V)$  to denote the degree of  $V$  in the graph  $(\tilde{V}, E)$ . Given a graph  $(\tilde{V}, E)$  and disjoint subsets  $\tilde{X}, \tilde{Y}$  of  $\tilde{V}$ , let  $E(\tilde{X}) = E \cap \binom{\tilde{X}}{2}$  and  $E(\tilde{X}, \tilde{Y}) = E \cap \{XY : X \in \tilde{X}, Y \in \tilde{Y}\}$ .

LEMMA 5.6.8. *Suppose  $H = (\tilde{V}, E, |\cdot|)$  is a vertex-weighted forest such that  $(\tilde{V}, E)$  is not a star. Then there is a vertex-weighted graph  $H' = (\tilde{V}, E', |\cdot|)$  such that  $(\tilde{V}, E')$  is a star, and*

$$f_\pi(H') \geq f_\pi(H).$$

Moreover, if  $f_\pi(H') = f_\pi(H)$ , then  $|V| = |W|$  where  $V$  is the center of the star  $(\tilde{V}, E')$  and  $W \in \tilde{V}$  is some vertex distinct from  $V$ .

PROOF. Let  $H = (\tilde{V}, E, |\cdot|)$  be a vertex-weighted forest. Fix  $V \in \tilde{V}$  with  $|V| = \max\{|X| : X \in \tilde{V}\}$ . We now define a sequence  $H_0, H_1, \dots, H_k$ , where for each  $i$ ,  $H_i = (\tilde{V}, E_i, |\cdot|)$ .

Step 0: Let  $\tilde{X}$  be the set of isolated points in  $H$ . If  $\tilde{X} = \emptyset$  set  $H_0 = H$  and go to the next step. If  $\tilde{X} \neq \emptyset$ , let  $E_0 = E \cup \{VX : X \in \tilde{X}\}$  and  $H_0 = (\tilde{V}, E_0, |\cdot|)$ . Clearly  $(\tilde{V}, E_0)$  is still a forest, since any cycle must contain a new edge, i.e. an edge of the form  $VX$ , some  $X \in \tilde{X}$ . But  $d^{H_0}(X) = 1$  for all  $X \in \tilde{X}$  implies no  $X \in \tilde{X}$  can be contained in a cycle in  $H_0$ . Further, note

$$f_\pi(H_0) = f_\pi(H) \left(\frac{3}{2}\right)^{\sum_{X \in \tilde{X}} |V||X|} > f_\pi(H).$$

If  $H_0$  is a star, end the construction and let  $k = 0$ , otherwise go to the next step.

Step  $i + 1$ : Suppose by induction we have defined  $H_0, \dots, H_i$  such that  $(\tilde{V}, E_i)$  is forest but not a star and contains no isolated points. Since  $(\tilde{V}, E_i)$  is not a star, it is in particular, not a star with center  $V$ . This implies the set  $\tilde{Y}_i := \tilde{V} \setminus (\{V\} \cup d^{H_i}(V)) \neq \emptyset$ . We show there is  $Y \in \tilde{Y}_i$  such that

$d^{H_i}(Y) = 1$ . Since there are no isolated points in  $(\tilde{V}, E_i)$ , every  $Y \in \tilde{Y}_i$  has  $d^{H_i}(Y) \geq 1$ . Suppose towards a contradiction that every  $Y \in \tilde{Y}_i$  had  $d^{H_i}(Y) \geq 2$ . Choose a maximal sequence of points  $\bar{Y} = (Y_1, \dots, Y_u)$  from  $\tilde{Y}_i$  with the property that  $Y_1 Y_2, \dots, Y_{u-1} Y_u \in E_i$ . Since  $Y_1$  and  $Y_u$  have degree at least two in  $(\tilde{V}, E_i)$  and because  $(\tilde{V}, E_i)$  is a forest, there are  $Z_1, Z_u \in \tilde{V} \setminus \bar{Y}$  such that  $Y_1 Z_1, Y_u Z_u \in E_i$ . Since  $Y_1, Y_u \in \tilde{Y}_i$ ,  $Z_1, Z_u \neq V$  and since  $\bar{Y}$  was maximal,  $Z_1, Z_u \notin \tilde{Y}_i$ . Thus  $Z_1, Z_u \in \tilde{V} \setminus (\tilde{Y}_i \cup \{V\})$  which implies  $V Z_1, V Z_u \in E_i$ . This yields that  $V, Z_1, Y_1, \dots, Y_u, Z_u, V$  is a cycle in  $(\tilde{V}, E_i)$ , a contradiction. Thus there exists  $Y \in \tilde{Y}_i$  such that  $d^{H_i}(Y) = 1$ . Fix such a  $Y \in \tilde{Y}_i$  and let  $W$  be the unique neighbor of  $Y$  in  $(\tilde{V}, E_i)$ . Define

$$E_{i+1} = (E_i \setminus \{YW\}) \cup \{VY\}.$$

and let  $H_{i+1} = (\tilde{V}, E_{i+1}, |\cdot|)$ . We first check  $(\tilde{V}, E_{i+1})$  is a forest. Since  $(\tilde{V}, E_i)$  is a forest, any cycle in  $(\tilde{V}, E_{i+1})$  will contain  $VY$ . However,  $d^{H_{i+1}}(Y) = 1$ , so  $Y$  cannot be contained in a cycle.

Note

$$f_\pi(H_{i+1}) = f_\pi(H_i) 3^{|V||Y|-|Y||W|} 2^{|Y||W|-|V||Y|} = f_\pi(H_i) \left(\frac{3}{2}\right)^{|Y|(|V|-|W|)} \geq f_\pi(H_i),$$

where the inequality holds because  $|V| \geq |W|$  by choice of  $V$ . Further, note that the inequality is strict unless  $|V| = |W|$ .

Clearly this process must end after some  $0 \leq k < |\tilde{V}|$  steps. If  $k = 0$ , then  $H_0 = H_k$  is a star and  $f_\pi(H_k) > f_\pi(H)$ . If  $k \geq 1$ , then the resulting  $H_k = (\tilde{V}, E_k, |\cdot|)$  will have the property that  $(\tilde{V}, E_k)$  is a star with center  $V$ . Since  $k \geq 1$ , one of the following holds.

- $f_\pi(H_1) > f_\pi(H_0)$ , so  $f_\pi(H_k) > f_\pi(H)$ , or
- $f_\pi(H_0) = f_\pi(H_1)$  and at step 1, we found a vertex  $W \neq V$  with  $|V| = |W|$ .

□

LEMMA 5.6.9. *Suppose  $G \in NC(n)$  and  $\tilde{G} = (\tilde{V}, E, |\cdot|)$  is the vertex-weighted graph associated to  $G$  and  $\sim_G$ . Suppose  $(\tilde{V}, E)$  is a star with center  $V$  and there is  $W \in \tilde{V} \setminus \{V\}$  such that  $|W| > 1$ . Then  $G \notin \mathcal{P}(NC(n))$ .*

PROOF. Let  $\tilde{V}' = (\tilde{V} \setminus \{W\}) \cup \{W_1, W_2\}$  and  $E' = (E \setminus \{VW\}) \cup \{VW_1, VW_2\}$ , where  $W_1, W_2$  are new vertices. Let  $H = (\tilde{V}', E', |\cdot|')$  where the vertex-weight function  $|\cdot|'$  is defined by  $|U|' = |U|$  for all  $U \in \tilde{V} \setminus \{W\}$ ,  $|W_1|' = |W| - 1$ , and  $|W_2|' = 1$ . By definition of  $H$ ,  $\sum_{U \in \tilde{V}'} |U|' = \sum_{U \in \tilde{V}} |U| = n$ . Since  $H$  is obtained from  $\tilde{G}$  by splitting the degree one vertex  $W$  into  $W_1$  and  $W_2$ , and  $\tilde{G}$  is a forest,  $H$  is also a forest. Thus  $H$  satisfies the hypotheses of Lemma 5.6.7, so there is an  $G' \in NC(n)$

such that  $\tilde{G}'$  is isomorphic to  $H$ . This and Definition 5.6.6 implies  $f_\pi(H) = f_\pi(\tilde{G}') = P(G')$ . Let  $\tilde{Z} = \tilde{V} \setminus \{V, W\}$ . Then

$$\begin{aligned} f_\pi(H) &= \left( \prod_{U \in \tilde{Z}} 3^{|U||V|} \prod_{UU' \in \binom{\tilde{Z}}{2}} 2^{|U||U'|} \right) \left( \prod_{U \in \tilde{Z}} 2^{|W_1||U|+|W_2||U|} \right) 3^{|V||W_1|'+|V||W_2|'} 2^{|W_1|'|W_2|'} \\ &= \left( \prod_{U \in \tilde{Z}} 3^{|U||V|} \prod_{UU' \in \binom{\tilde{Z}}{2}} 2^{|U||U'|} \right) \left( \prod_{U \in \tilde{Z}} 2^{|W||U|} \right) 3^{|V||W|} 2^{|W|-1} \\ &= f_\pi(\tilde{G}) 2^{|W|-1} \geq 2f_\pi(\tilde{G}). \end{aligned}$$

So  $G' \in NC(n)$  and  $P(G') = f_\pi(H) > f_\pi(\tilde{G}) = P(G)$  imply  $G \notin \mathcal{P}(NC(n))$ .  $\square$

**Proof of Lemma 5.6.2.** Let  $G = ([n], w) \in \mathcal{P}(NC(n))$ . Suppose first that  $G$  contains no edges of multiplicity 1. Then the graph  $H = ([n], E)$  where  $E = \{xy \in \binom{[n]}{2} : w(xy) = 3\}$  is  $C_4$ -free. By [28],  $|E| < 2n^{3/2}$ , so

$$P(G) \leq 3^{2n^{3/2}} 2^{\binom{n}{2} - 2n^{3/2}} \leq \left( 3^{2/\sqrt{n}} 2^{1/2 - 2/\sqrt{n}} \right)^{n^2}.$$

Choose  $M_1$  sufficiently large so that  $n > M_1$  implies  $3^{2/\sqrt{n}} 2^{1/2 - 2/\sqrt{n}} < 1.42$ . Since  $2^\gamma > 1.43$  we can choose  $M_2$  sufficiently large so that  $\max\{h(\lfloor \beta n \rfloor), h(\lceil \beta n \rceil)\} > 1.43n^2$  for  $n > M_2$ . Then if  $n > \max\{M_1, M_2\}$  and  $G_1 \in \mathcal{P}(W(n))$ ,

$$P(G) < 1.42n^2 < 1.43n^2 < \max\{h(\lfloor \beta n \rfloor), h(\lceil \beta n \rceil)\} = P(G_1).$$

This shows  $G \notin \mathcal{P}(NC(n))$ , a contradiction. So we may assume  $G$  contains some  $xy$  with  $w(xy) = 1$ . Consider now the vertex-weighted graph  $\tilde{G} = (\tilde{V}, E, |\cdot|)$  associated to  $G$  and  $\sim_G$ . Suppose  $(\tilde{V}, E)$  is a star with center  $V$ . If  $|W| = 1$  for all  $W \in \tilde{V} \setminus \{V\}$ , then  $G \in W(n)$  and we are done. If there is  $W \in \tilde{V} \setminus \{V\}$  such that  $|W| > 1$ , then Lemma 5.6.9 implies  $G \notin \mathcal{P}(NC(n))$ , a contradiction.

Suppose now  $(\tilde{V}, E)$  is not a star. Then Lemma 5.6.8 implies there is a vertex-weighted graph  $H = (\tilde{V}, E', |\cdot|)$  such that  $(\tilde{V}, E')$  is a star and  $f_\pi(H) \geq f_\pi(\tilde{G})$ . Since  $(\tilde{V}, E')$  is a star, it is a forest. Since  $(\tilde{V}, E, |\cdot|)$  is the vertex-weighted graph associated to  $G$  and  $\sim_G$ ,  $\sum_{U \in \tilde{V}} |U| = n$ . Thus  $H$  satisfies the hypotheses of Lemma 5.6.7, so there is  $G' \in NC(n)$  such that  $\tilde{G}' \cong H$ . Thus  $P(G') = f_\pi(H) \geq f_\pi(\tilde{G})$ , where the equality holds by Definition 5.6.6. Suppose  $f_\pi(H) > f_\pi(\tilde{G})$ . Then

$$P(G') = f_\pi(H) > f_\pi(\tilde{G}) = P(G),$$

contradicting that  $G \in \mathcal{P}(NC(n))$ . Thus we must have  $f_\pi(H) = f_\pi(\tilde{G})$ . By Lemma 5.6.8, this only happens if there is some  $W \neq V \in \tilde{V}$  such that  $|V| = |W|$ , where  $V$  is the center of the star  $(\tilde{V}, \tilde{E}')$ . Note that because  $G$  contains some  $xy$  with  $w(xy) = 1$ , there is some vertex  $U \in \tilde{V}$  such that  $|U| > 1$ . If  $U \neq V$ , then  $U \in \tilde{V} \setminus \{V\}$  and  $|U| > 1$ . If  $U = V$ , then  $W \in \tilde{V} \setminus \{V\}$  and  $|W| = |V| = |U| > 1$ . In either case Lemma 5.6.9 implies that  $G' \notin \mathcal{P}(NC(n))$ . Since  $P(G) = f_\pi(\tilde{G}) = f_\pi(H) = P(G')$ , this implies  $G \notin \mathcal{P}(NC(n))$ , a contradiction.  $\square$

### 5.7. Proof of Theorem 5.5.2.

In this section we prove Theorem 5.5.2. We first prove the following lemma.

LEMMA 5.7.1. *There is an  $M > 0$  such that for  $n \geq M$ ,  $\mathcal{P}(C(n)) \subseteq NC(n)$ .*

Note that if  $G \in C(n)$ , then  $C_3(3, 2) \not\subseteq G$  (since  $C(n) \subseteq F(n, 3, 8)$ ) and  $C_4(3, 2) \not\subseteq G$  (since  $S(C_4(3, 2)) = 16$ ). So to show some  $G \in C(n)$  is in  $NC(n)$ , we only need to show  $C_t(3, 2) \not\subseteq G$  for  $t \geq 5$ .

LEMMA 5.7.2. *Let  $5 \leq t \leq n$  and  $G = ([n], w) \in C(n)$ . Suppose  $C_t(3, 2) \subseteq G$ , and for all  $5 \leq t' < t$ ,  $C_{t'}(3, 2) \not\subseteq G$ . If  $X \in \binom{[n]}{t}$  is such that  $G[X] \cong C_t(3, 2)$ , then for all  $z \in [n] \setminus X$  either*

- (1)  $|\{x \in X : w(zx) = 3\}| \leq 1$  and  $P_z(X) \leq 3 \cdot 2^{t-1}$  or
- (2)  $|\{x \in X : w(zx) = 3\}| \geq 2$  and  $P_z(X) \leq 3^2 2^{t-3} < 3 \cdot 2^{t-1}$ .

PROOF. Let  $X = \{x_1, \dots, x_t\}$  where  $w(x_i x_{i+1}) = w(x_1 x_t) = 3$  for each  $i \in [t-1]$  and  $w(x_i x_j) = 2$  for all other pairs  $ij \in \binom{[t]}{2}$ . Since  $G \in C(n)$ ,  $C_3(3, 2), C_4(3, 2) \not\subseteq G$ . Combining this with our assumptions, we have that for all  $3 \leq t' < t$ ,  $C_{t'}(3, 2) \not\subseteq G$ . We will use throughout that  $\mu(G) \leq 3$  (since  $G \in C(n)$ ). Fix  $z \in [n] \setminus X$  and let  $Z = \{x \in X : w(zx) = 3\}$ . If  $|Z| \leq 1$ , then clearly 1 holds. So assume  $|Z| \geq 2$  and  $i_1 < \dots < i_\ell$  are such that  $Z = \{x_{i_1}, \dots, x_{i_\ell}\}$ . Without loss of generality, assume  $i_1 = 1$ . Set

$$I = \{(x_{i_j}, x_{i_{j+1}}) : 1 \leq j \leq \ell - 1\} \cup \{(x_{i_1}, x_{i_\ell})\}.$$

Given  $(x, y) \in I$ , let

$$d(x, y) = \begin{cases} i_{j+1} - i_j & \text{if } (x, y) = (x_{i_j}, x_{i_{j+1}}) \text{ some } 1 \leq j \leq \ell - 1 \\ t - i_\ell + 1 & \text{if } (x, y) = (x_{i_1}, x_{i_\ell}). \end{cases}$$

Note that because  $C_3(3, 2) \not\subseteq G$ ,  $2 \leq d(x, y) \leq t - 2$  for all  $(x, y) \in I$ . Suppose first that there is some  $(u, v) \in I$  such that  $d(u, v) = t - 2$ . Then since  $d(x, y) \geq 2$  for all  $(x, y) \in I$  we must have

that  $|I| = 1$  and either  $(u, v) = (x_{i_1}, x_{i_\ell}) = (x_1, x_{t-1})$  or  $(u, v) = (x_{i_1}, x_{i_\ell}) = (x_1, x_3)$ . Without loss of generality, assume  $(u, v) = (x_1, x_3)$ . Then we must have that  $w(zx_2) \leq 1$  since otherwise  $G[\{z, x_1, x_2, x_3\}] \cong C_4(3, 2)$ , a contradiction. This shows that  $P_z(X) \leq 3^2 \cdot 1 \cdot 2^{t-3} < 3 \cdot 2^{t-1}$ .

Suppose now that for all  $(x, y) \in I$ ,  $d(x, y) \leq t - 3$ . Given  $(x, y) \in I$ , say an element  $x_k$  is *between*  $x$  and  $y$  if either  $(x, y) = (x_{i_j}, x_{i_{j+1}})$  and  $i_j < k < i_{j+1}$  or  $(x, y) = (x_{i_1}, x_{i_\ell})$  and  $i_\ell < k$ . Then for each  $(x, y) \in I$ , there must be a  $x_k$  between  $x$  and  $y$  such that  $w(zx_k) \leq 1$ , since otherwise

$$\{z, x, y\} \cup \{u : u \text{ is between } x \text{ and } y\}$$

is a copy of  $C_{d(x,y)+2}(3, 2)$  in  $G$ , a contradiction since  $d(x, y) + 2 < t$ . This implies there are at least  $\ell$  elements  $u$  in  $X \setminus Z$  such that  $w(zu) \leq 1$ , so  $P_z(X) \leq 3^\ell 2^{t-2\ell} \leq 3^2 2^{t-4} < 3 \cdot 2^{t-1}$ .  $\square$

Given  $n, t \in \mathbb{N}$  set

$$f_\pi(n, t) = \min \left\{ 2^{\binom{\lceil \beta t \rceil}{2} + \lceil \beta t \rceil c} 3^{\lceil \beta t \rceil \lfloor (1-\beta)t \rfloor + c \lfloor (1-\beta)t \rfloor + \lceil \beta t \rceil (n-t-c)} : c \in \{\lfloor \beta(n-t) \rfloor, \lceil \beta(n-t) \rceil\} \right\}.$$

Given  $t \leq n$ ,  $G \in F(n, 4, 15)$ , and  $X \in \binom{[n]}{t}$ , define the multigraph  $\mathcal{G}_X = ([n], w)$  as follows. Choose  $B \in W(t)$  so that  $|R(B)| = \lceil \beta t \rceil$  and  $|L(B)| = \lfloor (1-\beta)t \rfloor$ , then define  $w$  on  $\binom{X}{2}$  to make  $\mathcal{G}_X[X] \cong B$ . Let  $Y = [n] \setminus X$ . Choose  $A \in \mathcal{P}(W(n-t))$  and define  $w$  on  $\binom{Y}{2}$  to make  $\mathcal{G}_X[Y] \cong A$ . Define  $w$  on the remaining pairs of vertices so that  $\mathcal{G}_X \in W(n)$  as follows. Let  $L_A, R_A$  and  $L_B, R_B$  be the partitions of  $Y$  and  $X$  respectively such that  $w(xy) = 1$  for all  $xy \in \binom{L_A}{2} \cup \binom{L_B}{2}$ . Set  $w(xy) = 3$  if  $xy \in E(L_B, R_A) \cup E(L_A, R_B)$ ,  $w(xy) = 2$  if  $xy \in E(R_A, R_B)$ , and  $w(xy) = 1$  if  $xy \in E(L_A, L_B)$ . We claim the following holds.

$$(47) \quad \text{For any } A' \in \mathcal{P}(W(n-t)), P(\mathcal{G}_X) \geq P(A') f_\pi(n, t).$$

By choice of  $B$ ,  $|L_B| = \lfloor (1-\beta)t \rfloor$  and  $|R_B| = \lceil \beta t \rceil$ . Let  $c = |R_A|$ . By definition,  $|L_A| = n - t - c$ , and since  $A \in \mathcal{P}(W(n-t))$ ,  $c \in \{\lfloor \beta(n-t) \rfloor, \lceil \beta(n-t) \rceil\}$  (by the proof of Lemma 5.5.4). Combining these observations with the definition of  $f_\pi(n, t)$  implies

$$2^{\binom{|R_B|}{2} + |R_A||R_B|} 3^{|L_B||R_B| + |R_B||L_A| + |L_B||R_A|} = 2^{\binom{\lceil \beta t \rceil}{2} + \lceil \beta t \rceil c} 3^{\lceil \beta t \rceil \lfloor (1-\beta)t \rfloor + c \lfloor (1-\beta)t \rfloor + \lceil \beta t \rceil (n-t-c)} \geq f_\pi(n, t).$$

Combining this with the definition of  $\mathcal{G}_X$ , we have

$$P(\mathcal{G}_X) = P(A) 2^{\binom{|R_B|}{2} + |R_A||R_B|} 3^{|L_B||R_B| + |R_B||L_A| + |L_B||R_A|} \geq P(A) f_\pi(n, t).$$

Since  $P(A) = P(A')$  for all  $A' \in \mathcal{P}(W(n-t))$ , this finishes the proof of (47).

DEFINITION 5.7.3. Given  $n, t \in \mathbb{N}$ , let  $h_\pi(n, t) = 3^n 2^{\binom{t}{2} + t(n-t) - n}$ .

LEMMA 5.7.4. Let  $5 \leq t \leq n$ ,  $G \in C(n)$ , and  $\nu > 0$ . Suppose  $X \in \binom{[n]}{t}$ ,  $G[X] \cong C_t(3, 2)$ , and there is some  $A \in \mathcal{P}(W(n-t))$  such that  $P(G[[n] \setminus X]) \leq \nu P(A)$ . Then  $P(G) \leq \nu((h_\pi(n, t)/f_\pi(n, t))P(\mathcal{G}_X))$ .

PROOF. Let  $Y = [n] \setminus X$ . Because  $G[X] \cong C_t(3, 2)$ ,  $P(G) = P(G[Y])3^t 2^{\binom{t}{2} - t} \prod_{z \in Y} P_z(X)$ . By Lemma 5.7.2, for each  $z \in Y$ ,  $P_z(X) \leq 3 \cdot 2^{t-1}$ . This implies

$$(48) \quad P(G) \leq P(G[Y])3^t 2^{\binom{t}{2} - t} (3 \cdot 2^{t-1})^{n-t} = P(G[Y])3^n 2^{\binom{t}{2} + t(n-t) - n} = P(G[Y])h_\pi(n, t).$$

By assumption,  $P(G[Y]) \leq \nu P(A)$ , so (48) implies  $P(G) \leq \nu P(A)h_\pi(n, t)$ . Combining this with (47) yields

$$P(G) \leq \nu P(A)h_\pi(n, t) = \nu P(A)f_\pi(n, t) \frac{h_\pi(n, t)}{f_\pi(n, t)} \leq \nu P(\mathcal{G}_X) \frac{h_\pi(n, t)}{f_\pi(n, t)}.$$

□

The following will be proved in the Appendix.

LEMMA 5.7.5. There are  $\gamma > 0$ ,  $K > 5$  and  $M_1$  such that the following holds.

- (1) For all  $K \leq t \leq n$ ,  $h_\pi(n, t) < f_\pi(n, t)$ .
- (2) For all  $5 \leq t \leq K$  and  $n \geq M_1$ ,  $h_\pi(n, t) < 2^{-\gamma n} f_\pi(n, t)$ .

LEMMA 5.7.6. Assume  $K$  is from Lemma 5.7.5 and  $K \leq t \leq n$ . Then the for all  $G \in C(n)$  the following holds. If  $C_t(3, 2) \subseteq G$  and  $C_{t'}(3, 2) \not\subseteq G$  for all  $t' < t$ , then for all  $G_1 \in \mathcal{P}(W(n))$ ,  $P(G) < P(G_1)$ .

PROOF. Let  $t \geq K$  and  $n = t + i$ . We proceed by induction on  $i$ . If  $i = 0$ , then  $G \cong C_t(3, 2)$  and so  $P(G) = 3^t 2^{\binom{t}{2} - t} = h_\pi(t, t)$ . Let  $H \in W(n)$  have  $|L(H)| = \lceil \beta n \rceil$  and  $|R(H)| = \lfloor (1 - \beta)n \rfloor$ . Then by definition of  $f_\pi(n, t)$ ,

$$P(H) = 2^{\binom{\lceil \beta n \rceil}{2}} 3^{\lceil \beta n \rceil \lfloor (1 - \beta)n \rfloor} = f_\pi(t, t) > h_\pi(t, t) = P(G),$$

where the inequality is by part (1) of Lemma 5.7.5. Since  $H \in W(n)$ , this implies  $P(G) < P(G_1)$  for all  $G_1 \in \mathcal{P}(W(n))$ .

For the induction step, suppose that the results holds for all  $0 \leq j < i$  and we wish to prove it for  $i$ . Let  $X \subseteq [n]$  be such that  $G[X] \cong C_t(3, 2)$ .

CLAIM 5.7.7. For any  $A \in \mathcal{P}(W(n-t))$ ,  $P(G[[n] \setminus X]) \leq P(A)$ .

*Proof.* Note that  $C_{t'}(3, 2) \not\subseteq G[[n] \setminus X]$  for all  $3 \leq t' < t$ . We have two cases.

(1) If  $C_{t'}(3, 2) \not\subseteq G[[n] \setminus X]$  for all  $t' \geq t$ , then  $G[[n] \setminus X]$  is isomorphic to an element  $D \in NC(n-t)$ .

By Lemma 5.6.2, for any  $A \in \mathcal{P}(W(n-t))$ ,  $P(G[[n] \setminus X]) = P(D) \leq P(A)$ .

(2) If  $C_{t'}(3, 2) \subseteq G[[n] \setminus X]$  for some  $t' \geq t$ , then fix  $t_0$  the smallest such  $t'$ . Our assumptions imply

$t_0 \geq t$  and  $t_0 \leq |[n] \setminus X| = n-t = i$ , so

$$n-t = t_0 + (n-t-t_0) = t_0 + i - t_0 = t_0 + j,$$

where  $0 \leq i - t_0 = j < i$ . Thus  $G[[n] \setminus X]$  is isomorphic to some  $D \in C(n-t)$  and  $D$  satisfies the hypotheses of the Lemma for  $n_0 = n-t$ ,  $t_0$ , and  $j$ . So by induction, this means that for any  $A \in \mathcal{P}(W(n_0)) = \mathcal{P}(W(n-t))$ ,  $P(G[[n] \setminus X]) = P(D) < P(A)$ . //

Claim 5.7.7 and Lemma 5.7.4 with  $\nu = 1$  imply  $P(G) \leq (h_\pi(n, t)/f_\pi(n, t))P(\mathcal{G}_X)$ . Since  $K \leq t \leq n$ , Lemma 5.7.5 part (1) implies  $(h_\pi(n, t)/f_\pi(n, t)) < 1$ , so this shows  $P(G) < P(\mathcal{G}_X)$ . Since  $\mathcal{G}_X \in W(n)$ , we have  $P(G) < P(\mathcal{G}_X) \leq P(G_1)$  for all  $G_1 \in \mathcal{P}(W(n))$ .  $\square$

LEMMA 5.7.8. *Let  $M_1$  and  $K$  be as in Lemma 5.7.5. There is  $M_2$  such that for all  $5 \leq t \leq K$ ,  $n \geq M_1 + K$ , and  $G \in C(n)$ , the following holds. If  $C_t(3, 2) \subseteq G$  and  $C_{t'}(3, 2) \not\subseteq G$  for all  $t' < t$ , then for all  $G_1 \in \mathcal{P}(W(n))$ ,  $P(G) \leq 2^{M_2}(h_\pi(n, t)/f_\pi(n, t))P(G_1)$ .*

PROOF. Choose  $M_2$  so that for all  $5 \leq t \leq K$  and  $t \leq n \leq M_1 + K$ ,  $\text{ex}_\Pi(n, 4, 15) \leq 2^{M_2}(h_\pi(n, t)/f_\pi(n, t))$ .

Set  $M = M_1 + K$  and fix  $5 \leq t \leq K$ . We show the lemma holds for all  $n \geq M$  by induction. If  $n = M$ , then by assumption,

$$P(G) \leq \text{ex}_\Pi(n, 4, 15) \leq 2^{M_2}(h_\pi(n, t)/f_\pi(n, t)) \leq 2^{M_2}(h_\pi(n, t)/f_\pi(n, t))P(G_1),$$

for all  $G_1 \in \mathcal{P}(W(n))$ . If  $n > M$ , let  $X \in \binom{[n]}{t}$  be such that  $G[X] \cong C_t(3, 2)$ .

CLAIM 5.7.9. *For any  $A \in \mathcal{P}(W(n-t))$ ,  $P(G[[n] \setminus X]) \leq 2^{M_2}P(A)$ .*

*Proof.* Note that  $C_{t'}(3, 2) \not\subseteq G[[n] \setminus X]$  for all  $t' < t$  and  $n-t \geq M_1$ . We have two cases.

(1) If  $C_{t'}(3, 2) \not\subseteq G[[n] \setminus X]$  for all  $t' \geq t$ , then  $G[[n] \setminus X]$  is isomorphic to an element  $D \in NC(n-t)$ .

By Lemma 5.6.2, for any  $A \in \mathcal{P}(W(n-t))$ ,  $P(G[[n] \setminus X]) = P(D) \leq P(A)$ .

(2) Suppose  $C_{t'}(3, 2) \subseteq G[[n] \setminus X]$  for some  $t' \geq t$ . If  $n-t < M$ , then  $t \leq t' \leq n-t < M = M_1 + K$

and our choice of  $M_2$  implies

$$P(G[[n] \setminus X]) \leq \text{ex}_\Pi(n-t, 4, 15) \leq 2^{M_2} \frac{h_\pi(n-t, t)}{f_\pi(n-t, t)} < 2^{M_2} P(A).$$



where the last inequality is because Lemma 5.7.5 part (2) implies  $\frac{h_\pi(n-t,t)}{f_\pi(n-t,t)} \leq 2^{-\gamma(n-t)} < 1$  (note we are using that  $n-t \geq M_1$ ). If  $n-t \geq M$ , then our induction hypothesis implies

$$P(G[[n] \setminus X]) \leq 2^{M_2} \frac{h_\pi(n-t,t)}{f_\pi(n-t,t)} P(A) < 2^{M_2} P(A),$$

where the last inequality holds because Lemma 5.7.5 part (2) implies  $\frac{h_\pi(n-t,t)}{f_\pi(n-t,t)} \leq 2^{-\gamma(n-t)} \leq 1$ .

//

Claim 5.7.9 and Lemma 5.7.4 with  $\nu = 2^{M_2}$  imply  $P(G) \leq 2^{M_2} (h_\pi(n,t)/f_\pi(n,t)) P(\mathcal{G}_X)$ . Since  $\mathcal{G}_X$  is in  $W(n)$ , we have  $P(G) \leq 2^{M_2} (h_\pi(n,t)/f_\pi(n,t)) P(\mathcal{G}_X) \leq 2^{M_2} (h_\pi(n,t)/f_\pi(n,t)) P(G_1)$  for all  $G_1 \in \mathcal{P}(W(n))$ .  $\square$

**Proof of Lemma 5.7.1.** Let  $\gamma$ ,  $K$ , and  $M_1$  be as in Lemma 5.7.5 and let  $M_2$  be as in Lemma 5.7.8. Choose  $M \geq M_1 + K$  sufficiently large so that  $2^{M_2 - \gamma n} < 1$  for all  $n \geq M$ . Suppose  $n > M$  and  $G \notin NC(n)$ . We show  $G \notin \mathcal{P}(C(n))$ . Clearly if  $G \notin C(n)$  we are done, so assume  $G \in C(n)$ . Since  $W(n) \subseteq C(n)$ , it suffices to show there is  $G_1 \in W(n)$  such that  $P(G_1) > P(G)$ . Since  $G \notin NC(n)$ , there is  $5 \leq t \leq n$  such that  $C_t(3,2) \subseteq G$  and for all  $t' < t$ ,  $C_{t'}(3,2) \not\subseteq G$ . If  $t \geq K$ , then Lemma 5.7.6 implies that for any  $G_1 \in \mathcal{P}(W(n))$ ,  $P(G) < P(G_1)$ . If  $5 \leq t < K$ , then Lemma 5.7.8 implies that for any  $G_1 \in \mathcal{P}(W(n))$ ,

$$P(G) \leq 2^{M_2} (h_\pi(n,t)/f_\pi(n,t)) P(G_1) \leq 2^{M_2 - \gamma n} P(G_1),$$

where the second inequality is because of Lemma 5.7.5 part (2). By our choice of  $M$ , this implies that for all  $G_1 \in W(n)$ ,  $P(G) < P(G_1)$ .  $\square$

**Proof of Theorem 5.5.2.** Suppose  $n$  is sufficiently large. By Lemma 5.7.1,  $\mathcal{P}(C(n)) \subseteq NC(n)$ . By definition,  $NC(n) \subseteq C(n)$ , so this implies  $\mathcal{P}(C(n)) = \mathcal{P}(NC(n))$ . Thus Lemma 5.6.2 implies  $\mathcal{P}(C(n)) = \mathcal{P}(NC(n)) \subseteq W(n)$  and Theorem 5.5.2 holds.  $\square$

### 5.8. Proof of Lemma 5.8.1

In this section we prove the following lemma, which will be used in the proof of Theorem 5.5.3.

LEMMA 5.8.1. *For all  $n$ , there is  $G \in C(n)$  such that  $G \in \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ .*

Suppose  $G = (V, w)$  and  $x \neq y \in V$ . Define  $G_{xy} = (V, w')$  to be the multigraph such that

- $G_{xy}[V \setminus \{x, y\}] = G[V \setminus \{x, y\}]$ ,

- $w'(xy) = 1$ , and
- for all  $u \in V \setminus \{x, y\}$ ,  $w'(xu) = w(yu)$ .

The idea is that  $G_{xy}$  is obtained from  $G$  by making the vertex  $x$  “look like” the vertex  $y$ . Given  $xy, vu \in \binom{V}{2}$ , define

$$G_{vu,xy} = (G_{uv})_{vu}.$$

Given  $G = (V, w)$  and  $y \in V$ , set  $p(y) = \prod_{x \in V \setminus \{y\}} w(xy)$ . We will use the following two equations for any  $xy \in \binom{V}{2}$  and  $\{u, v, z\} \in \binom{V}{3}$ .

$$(49) \quad P(G_{xy}) = \frac{p(y)}{p(x)w(xy)} P(G) \text{ and}$$

$$(50) \quad P(G_{vu,zu}) = \frac{p(u)^2 w(vz)}{p(v)p(z)w(uz)^2 w(uv)^2} P(G).$$

LEMMA 5.8.2. *Let  $G \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$  and  $uv, xy \in \binom{[n]}{2}$ . Then  $G_{uv}$  and  $G_{uv,xy}$  are both in  $F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ .*

PROOF. Let  $G = ([n], w)$  and  $G' := G_{uv} = ([n], w')$ , and given  $X \subseteq [n]$ , let  $S(X) = \sum_{xy \in \binom{X}{2}} w(xy)$  and  $S'(X) = \sum_{xy \in \binom{X}{2}} w'(xy)$ . We first show  $G' \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ . By definition of  $G_{uv}$  and because  $G \in F_{\leq 3}(n, 4, 15)$ ,  $\mu(G') \leq 3$ . We now check that  $G' \in F(n, 4, 15)$ . Suppose  $X \in \binom{[n]}{4}$ . If  $u \notin X$ , then  $S'(X) = S(X) \leq 15$ . If  $X \cap \{u, v\} = \{u\}$ , then  $S'(X) = S((X \setminus \{u\}) \cup \{v\}) \leq 15$ . So assume  $\{u, v\} \subseteq X$ , say  $X = \{u, v, z, z'\}$ . Because  $G \in F(n, 3, 8)$  and definition of  $G_{uv}$ , we must have that  $S'(\{v, z, z'\}) = S(\{v, z, z'\}) \leq 8$ . Combining this with the facts that  $w'(uv) = 1$  and  $\mu(G') \leq 3$ ,

$$S'(X) = S'(\{v, x, y\}) + w'(uv) + w'(ux) + w'(uy) \leq 8 + 1 + 3 + 3 = 15.$$

We now verify that  $G' \in F(n, 3, 8)$ . Suppose  $X \in \binom{[n]}{3}$ . If  $u \notin X$ , then  $S'(X) = S(X) \leq 8$ . If  $X \cap \{u, v\} = \{u\}$ , then  $S'(X) = S((X \setminus \{u\}) \cup \{v\}) \leq 8$ . So assume  $\{u, v\} \subseteq X$ , say  $X = \{u, v, z\}$ . Because  $\mu(G') \leq 3$ ,

$$S'(X) \leq w'(uv) + 3 + 3 = 1 + 3 + 3 = 7 \leq 8.$$

Consequently,  $G' \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ . Repeating the proof yields  $(G')_{xy} \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ .  $\square$

LEMMA 5.8.3. *Suppose  $G \in \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ . Then  $G$  contains no  $(3, 1, 1)$ -triangle or  $(2, 1, 1)$ -triangle.*

PROOF. Suppose towards a contradiction that  $G = ([n], w) \in \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$  but there is a set  $\{u, v, z\} \subseteq V(G)$  which is a  $(3, 1, 1)$ -triangle or a  $(2, 1, 1)$ -triangle. Assume  $w(uv) = w(uz) = 1$  and  $w(vz) \in \{2, 3\}$ . Without loss of generality assume  $p(v) \geq p(z)$ . Note that by Lemma 5.8.2,  $G_{uv}$  and  $G_{vu, zu}$  are in  $F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ . If  $p(v) > p(u)$ , then using (49) and  $w(uv) = 1$  we obtain

$$P(G_{uv}) = \frac{p(v)}{p(u)}P(G) > P(G),$$

which implies  $G \notin \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ . Therefore we may assume  $p(z) \leq p(v) \leq p(u)$ .

Using (50) and  $w(vz) \geq 2$ , we obtain

$$P(G_{vu, zu}) = \frac{w(vz)p(u)^2}{p(v)p(z)}P(G) \geq w(vz)P(G) \geq 2P(G) > P(G),$$

a contradiction. □

Given  $G \in F(n, 4, 15)$ , set  $\Gamma(G) = \{Y \in \binom{[n]}{3} : Y \text{ is a } (1, 2, 3)\text{-triangle in } G\}$ .

LEMMA 5.8.4. *Suppose  $G = ([n], w) \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$  and  $u, v, z \in [n]$  are such that  $w(uv) = 1$ ,  $w(uz) = 2$  and  $w(vz) = 3$ . Then either  $|\Gamma(G_{uv})| < |\Gamma(G)|$  or  $|\Gamma(G_{vu})| < |\Gamma(G)|$ .*

PROOF. Let  $X = \{u, v, z\}$ . Given  $y, y' \in X$ , set

$$\Gamma_y = \{x, x' \in [n] \setminus X : \{y, x, x'\} \in \Gamma(G)\} \quad \text{and} \quad \Gamma_{yy'} = \{x \in [n] \setminus X : \{y, y', x\} \in \Gamma(G)\}.$$

Observe that

$$\Gamma(G) = \Gamma(G[[n] \setminus X]) \cup \Gamma_u \cup \Gamma_v \cup \Gamma_z \cup \Gamma_{uv} \cup \Gamma_{vz} \cup \Gamma_{uz} \cup \{X\},$$

so  $|\Gamma(G)| = |\Gamma(G[[n] \setminus X])| + |\Gamma_u| + |\Gamma_v| + |\Gamma_z| + |\Gamma_{uv}| + |\Gamma_{vz}| + |\Gamma_{uz}| + 1$ . Let  $G_{uv} = ([n], w^{G_{uv}})$  and  $G_{vu} = ([n], w^{G_{vu}})$ . Note that for all  $x \in [n] \setminus \{u, v\}$ , we have  $w^{G_{uv}}(vx) = w^{G_{uv}}(ux)$  and  $w^{G_{vu}}(vx) = w^{G_{vu}}(ux)$ , so there are no  $(1, 2, 3)$ -triangles in  $G_{uv}$  or  $G_{vu}$  of the form  $\{u, v, x\}$ . If  $x \in [n] \setminus X$  is such that  $\{x, v, z\} \in \Gamma(G)$ , then  $\{x, v, z\}, \{x, u, z\} \in \Gamma(G_{uv})$ . Similarly, if  $x, y \in [n] \setminus X$  are such that  $\{x, y, v\} \in \Gamma(G)$ , then  $\{x, y, v\}, \{x, y, u\} \in \Gamma(G_{uv})$ . Combining these observations, we have that  $|\Gamma(G_{uv})| = |\Gamma(G[[n] \setminus X])| + |\Gamma_z| + 2|\Gamma_v| + 2|\Gamma_{vz}|$ . The same argument with the roles of  $u$  and  $v$  switched implies  $|\Gamma(G_{vu})| = |\Gamma(G[[n] \setminus X])| + |\Gamma_z| + 2|\Gamma_u| + 2|\Gamma_{uz}|$ . Suppose first that  $|\Gamma_v| + |\Gamma_{vz}| \leq |\Gamma_u| + |\Gamma_{uz}|$ . Then

$$|\Gamma(G_{uv})| \leq |\Gamma(G[[n] \setminus X])| + |\Gamma_z| + |\Gamma_u| + |\Gamma_v| + |\Gamma_{uz}| + |\Gamma_{vz}| \leq |\Gamma(G)| - 1.$$

If on the other hand,  $|\Gamma_v| + |\Gamma_{vz}| \geq |\Gamma_u| + |\Gamma_{uz}|$ , then the same argument with the roles of  $u$  and  $v$  switched implies  $|\Gamma(G_{vu})| \leq |\Gamma(G)| - 1$ .  $\square$

LEMMA 5.8.5. *For any  $G \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$  there is  $H \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8) \cap A_{1,2,3}(n)$  such  $P(H) \geq P(G)$ .*

PROOF. Suppose  $G \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$  satisfies  $\Gamma(G) \neq \emptyset$ . We give a procedure for defining  $H(G) \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$  such that either  $P(H(G)) > P(G)$  or  $P(G) = P(H(G))$  and  $|\Gamma(H(G))| < |\Gamma(G)|$ . Choose some  $\{u, v, z\} \in \Gamma(G)$ , say  $w(uv) = 1$ ,  $w(uz) = 2$ , and  $w(vz) = 3$ . Suppose  $p(v) < p(u)$ . Then Lemma 5.8.2 implies  $G_{uv} \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ , and (49) and  $w(uv) = 1$  imply  $P(G_{uv}) = (p(v)/p(u))P(G) > P(G)$ , so set  $H(G) = G_{uv}$ . If  $p(u) < p(v)$ , the same argument with the roles of  $u$  and  $v$  switched implies  $G_{vu} \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$  and  $P(G_{vu}) > P(G)$ , so set  $H(G) = G_{vu}$ . If  $p(u) = p(v)$ , use Lemma 5.8.4 to choose  $H(G) = G_{uv}$  or  $H(G) = G_{vu}$  such that  $|\Gamma(H(G))| < |\Gamma(G)|$  and set  $P(G) = P(H(G))$ .

Now fix  $G \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)$ . Define a sequence  $G_1, \dots, G_k$  as follows. Set  $G_1 = H(G)$ . Suppose  $i > 1$  and  $G_1, \dots, G_i$  have been defined. If  $\Gamma(G_i) = \emptyset$ , set  $k = i$ . If  $\Gamma(G_i) \neq \emptyset$ , set  $G_{i+1} = H(G_i)$ . Clearly this algorithm will end after at some finite number of steps. The resulting  $G_k$  will contain no  $(1, 2, 3)$ -triangles and will satisfy  $P(G_k) \geq P(G)$ .  $\square$

**Proof of Lemma 5.8.1.** Suppose  $G \in \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ . Lemma 5.8.5 implies there is  $H \in F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8) \cap A_{1,2,3}(n)$  such that  $P(H) \geq P(G)$ . Since  $G \in \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ , this implies  $H \in \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ . Lemma 5.8.3 implies  $H \in A_{2,1,1}(n) \cap A_{3,1,1}(n)$ . Therefore  $H \in C(n)$ .  $\square$

### 5.9. Proof of Theorem 5.5.3

In this section we prove Theorem 5.5.3. We will need the following computational lemma, which is proved in the appendix. Given  $n, t$ , let  $k_\pi(n, t) = 15^t 2^{\binom{t}{2} + t(n-t) - t}$ .

LEMMA 5.9.1. *There is  $M$  such that for all  $n \geq M$  and  $2 \leq t \leq n$ ,  $k_\pi(n, t) < f_\pi(n, t)$ .*

**Proof of Theorem 5.5.3.** We first prove the following.

$$(51) \quad \text{There is } M \text{ such that for all } n \geq M, \mathcal{P}(F(n, 4, 15)) \subseteq F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8).$$

Let  $M$  be from Lemma 5.9.1. Let  $n > M$  and  $G \in \mathcal{P}(F(n, 4, 15)) \setminus (F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8))$ . If  $G \notin F(n, 3, 8)$ , let  $D_1, \dots, D_k$  be a maximal collection of pairwise disjoint elements of  $\binom{[n]}{3}$  such that

$S(D_i) \geq 9$  for each  $i$ , and set  $D = \bigcup_{i=1}^k D_i$ . If  $G \in F(n, 3, 8)$ , set  $D = \emptyset$ . If  $\mu(G[[n] \setminus D]) > 3$ , choose  $e_1, \dots, e_m$  a maximal collection of pairwise disjoint elements of  $\binom{[n] \setminus D}{2}$  such that  $S(e_i) \geq 4$  for each  $i$  and set  $C = \bigcup_{i=1}^m e_i$ . If  $\mu(G[[n] \setminus D]) \leq 3$ , set  $C = \emptyset$ . Let  $X = D \cup C$  and  $\ell = |X| = 3k + 2m$ . Note that by assumption  $X$  is nonempty, so we must have  $\ell \geq 2$ . We now make a few observations. If  $D \neq \emptyset$ , then for each  $D_i$  and  $z \in [n] \setminus D_i$ ,

$$S_z(D_i) \leq S(D_i \cup \{z\}) - S(D_i) \leq 15 - 9 = 6 = 2 \cdot 3,$$

which implies  $P_z(D_i) \leq 2^3$ . By maximality of the collection  $D_1, \dots, D_k$ ,  $G[[n] \setminus D]$  is a  $(3, 8)$ -graph. Thus if  $C \neq \emptyset$ , then for each  $i$  and  $z \in [n] \setminus (D \cup e_i)$ ,

$$S_z(e_i) \leq S(e_i \cup \{z\}) - 4 \leq 8 - 4 = 4 = 2 \cdot 2,$$

which implies  $P_z(e_i) \leq 2^2$ . Since  $\mu(G) \leq 15$ , for each  $D_i$  and  $e_j$ ,  $P(D_i) \leq 15^3$  and  $P(e_j) \leq 15$ . Let  $Y = [n] \setminus X$ . Our observations imply that  $P(G)$  is at most the following.

$$(52) \quad P(Y) \left( \prod_{i=1}^k P(D_i) \right) \left( \prod_{i=1}^m P(e_i) \right) 2^{\binom{\ell}{2} + \ell(n-\ell) - |X|} \leq P(Y) 15^{3k+m} 2^{\binom{\ell}{2} + \ell(n-\ell) - \ell} \leq P(Y) k_\pi(n, \ell).$$

Note that  $G[Y]$  is isomorphic to an element of  $F_{\leq 3}(n - \ell, 4, 15) \cap F(n - \ell, 3, 8)$ . By Lemma 5.8.1, Lemma 5.7.1, and Lemma 5.5.4,  $\mathcal{P}(W(n - \ell)) \subseteq \mathcal{P}(F_{\leq 3}(n, 4, 15 - \ell) \cap F(n, 3, 8 - \ell))$ , which implies that for any  $A \in \mathcal{P}(W(n - \ell))$ ,  $P(G[Y]) \leq P(A)$ . Combining this with (47) yields  $P(\mathcal{G}_X) \geq P(G[Y]) f_\pi(n, \ell)$ . This, along with the bound on  $P(G)$  in (52), implies

$$\frac{P(G)}{P(\mathcal{G}_X)} \leq \frac{P(G[Y]) k_\pi(n, \ell)}{P(G[Y]) f_\pi(n, \ell)} = \frac{k_\pi(n, \ell)}{f_\pi(n, \ell)} < 1$$

where the last inequality is by choice of  $M$  and Lemma 5.9.1. So  $P(G) < P(\mathcal{G}_X)$ , a contradiction. Thus (7.2.5) holds. Combining this with Lemma 5.8.1 we have that for sufficiently large  $n$ ,

$$\mathcal{P}(n, 4, 15) \subseteq \mathcal{P}(F_{\leq 3}(n, 4, 15) \cap F(n, 3, 8)) \subseteq \mathcal{P}(C(n)).$$

□

### 5.10. Concluding Remarks

The arguments used to prove Theorem 5.2.7 can be adapted to prove a version for sums. Straight-forward calculus shows that for  $G \in W(n)$ , the sum  $S(G)$  is maximized when  $|L(G)| \approx (2/3)n$ .

THEOREM 5.10.1. *There is  $M$  such that for all  $n \geq M$ , we have  $\mathcal{P}(W(n)) \subseteq \mathcal{S}(n, 4, 15)$ . Consequently*

$$\text{ex}_\Sigma(n, 4, 15) = \max \left\{ 2 \binom{\lfloor \frac{2n}{3} \rfloor}{2} + 3 \binom{\lfloor \frac{2n}{3} \rfloor}{1} \binom{\lceil \frac{n}{3} \rceil}{1}, 2 \binom{\lceil \frac{2n}{3} \rceil}{2} + 3 \binom{\lceil \frac{2n}{3} \rceil}{1} \binom{\lfloor \frac{n}{3} \rfloor}{1} \right\} = \frac{8}{3} \binom{n}{2} + O(n).$$

We would like to point out that the asymptotic value for  $\text{ex}_\Sigma(n, 4, 15)$  was already known as a consequence of [53]. Our contribution is in determining the exact value for large  $n$ . We have not been able to prove a stability result here, but one can prove the following result that shows nearly product-extremal  $(n, 4, 15)$ -graphs are far from nearly sum-extremal ones.

COROLLARY 5.10.2. *There is  $\delta > 0$  such that for all sufficiently large  $n$ , the following holds. Suppose  $G \in \mathcal{P}(n, 4, 15)$  and  $G' \in \mathcal{S}(n, 4, 15)$ . Then  $G$  and  $G'$  are  $\delta$ -far from one another.*

PROOF. Assume  $M$  is sufficiently large and  $\delta$  is sufficiently small. Let  $n \geq M$  and suppose towards a contradiction that  $G \in \mathcal{P}(n, 4, 15)$  and  $G' \in \mathcal{S}(n, 4, 15)$  are  $\delta$ -close. Since  $\mu(G), \mu(G') \leq 15$ , this implies

$$S(G) \geq S(G') - 15|\Delta(G, G')| \geq S(G') - 15\delta n^2.$$

Using the asymptotic value of  $\text{ex}_\Sigma(n, 4, 15)$ , this implies  $S(G) \geq \frac{8}{3} \binom{n}{2} - 15\delta n^2$ . On the other hand, fix  $H \in \mathcal{P}(W(n))$  and let  $L = L(H)$  and  $R = R(H)$ . Theorem 5.2.7 implies  $P(G) = P(H)$ . Note (7.2.5) implies that  $\mu(G) \leq 3$ . Thus  $P(G) = P(H) = 2^{\binom{|R|}{2}} 3^{|L||R|} = 2^{|E_2(G)|} 3^{|E_3(G)|}$ . Since 2 and 3 are relatively prime, this implies  $|E_2(G)| = \binom{|R|}{2}$ ,  $|E_3(G)| = |L||R|$ , and  $|E_1(G)| = \binom{|L|}{2}$ . So

$$S(G) = \binom{|L|}{2} + 2 \binom{|R|}{2} + 3|L||R| = \binom{n}{2} + \binom{|R|}{2} + 2|L||R|.$$

Because  $H \in \mathcal{P}(W(n))$ ,  $|R(H)| \leq \beta n + 1$  and  $|L(H)| \leq (1 - \beta)n + 1$ . Therefore

$$S(G) \leq \binom{n}{2} + \binom{\beta n + 1}{2} + 2(\beta n + 1)((1 - \beta)n + 1) = n^2 \left( \frac{1}{2} + 2\beta - \frac{3}{2}\beta^2 \right) - n \left( \frac{4 + \beta}{2} \right) + 2.$$

But a straightforward computation shows  $\frac{1}{2} + 2\beta - \frac{3}{2}\beta^2 < 8/6$ , so since  $n$  is large and  $\delta$  is small,

$$S(G) < n^2 \left( \frac{1}{2} + 2\beta - \frac{3}{2}\beta^2 \right) < \frac{8}{3} \binom{n}{2} - 15\delta n^2,$$

a contradiction. □

### 5.11. Appendix

For ease of notation, we will write  $x = \beta$  for the rest of this section. For any  $r \in \mathbb{R}$ ,  $\binom{r}{2} = \frac{r^2-r}{2}$ .

Recall that given  $n, t \in \mathbb{N}$

$$f_\pi(n, t) = \min \left\{ 2^{\binom{\lceil \beta t \rceil}{2} + \lceil \beta t \rceil c} 3^{\lceil \beta t \rceil \lfloor (1-\beta)t \rfloor + c \lfloor (1-\beta)t \rfloor + \lceil \beta t \rceil (n-t-c)} : c \in \{ \lfloor \beta(n-t) \rfloor, \lceil \beta(n-t) \rceil \} \right\}.$$

Given  $2 \leq t \leq n$ , let

$$f_*(n, t) = 2^{\binom{xt}{2} + x^2 t(n-t)} 3^{2xt(1-x)(n-t) + x(1-x)t^2}.$$

PROPOSITION 5.11.1. *For all  $2 \leq t \leq n$ ,  $f_\pi(n, t) \geq f_*(n, t) 2^{-xt-3/2} 3^{-t-1}$ .*

PROOF. By definition of  $x$ ,  $x(2 \log 3 - \log 2) = \log 3$ . Dividing both sides of this by  $\log 2$  and rearranging yields

$$(53) \quad -x - \log_2 3 + 2x \log_2 3 = 0.$$

Fix  $2 \leq t \leq n$  and let  $a = \lceil xt \rceil - xt$ . Define  $\eta(u, v, z, w) = 2^{\binom{u}{2} + uz} 3^{uw + vz + uv}$  and observe that

$$(54) \quad \begin{aligned} f_\pi(n, t) &= \min \{ \eta(\lceil xt \rceil, \lfloor (1-x)t \rfloor, y, n-t-y) : y \in \lceil x(n-t) \rceil, \lfloor x(n-t) \rfloor \} \\ &= \min \{ \eta(xt+a, (1-x)t-a, y, n-t-y) : y \in \lceil x(n-t) \rceil, \lfloor x(n-t) \rfloor \}. \end{aligned}$$

Note that for all  $y \in \{ \lceil x(n-t) \rceil, \lfloor x(n-t) \rfloor \}$ ,  $y \geq x(n-t) - 1$  and  $n-t-y \geq (1-x)(n-t) - 1$ .

Combining this with (54) and the definition of  $\eta(u, v, z, w)$ , we have

$$(55) \quad f_\pi(n, t) \geq \eta(xt+a, (1-x)t-a, x(n-t)-1, (1-x)(n-t)-1).$$

We leave it to the reader to verify that the righthand side of (55) is equal to  $f_*(n, t) 2^{g_1(n, t)} 3^{g_2(n, t)}$ ,

where  $g_1(n, t) = \frac{a^2}{2} - \frac{3a}{2} - xt + axn$  and  $g_2(n, t) = -2axn + an - t - a^2$ . Observe

$$\begin{aligned} g_1(n, t) + g_2(n, t) \log_2 3 &= an \left( x + \log_2 3 - 2x \log_2 3 \right) + \frac{a^2}{2} - \frac{3a}{2} - xt - (t+a^2) \log_2 3 \\ &= \frac{a^2}{2} - \frac{3a}{2} - xt - (t+a^2) \log_2 3, \end{aligned}$$

where the second equality is by (53). Since  $0 \leq a \leq 1$ ,  $\frac{a^2}{2} - \frac{3a}{2} = \frac{a}{2}(a-3) \geq \frac{a}{2}(-3) \geq -3/2$  and  $-a^2 \geq -1$ . So

$$g_1(n, t) + g_2(n, t) \log_2 3 \geq -\frac{3}{2} - xt - (t+1) \log_2 3.$$

Thus  $f_\pi(n, t) \geq f_*(n, t) 2^{g_1(n, t)} 3^{g_2(n, t)} \geq f_*(n, t) 2^{-\frac{3}{2}-xt} 3^{-t-1}$ , as desired.  $\square$

Recall that given  $n, t \in \mathbb{N}$ , let  $h_\pi(n, t) = 3^n 2^{\binom{t}{2} + t(n-t) - n}$ .

PROPOSITION 5.11.2. *Let  $2 \leq t \leq n$ . Then  $h_\pi(n, t)/f_\pi(n, t) \leq 2^{C_1(n, t)} 3^{C_2(n, t)}$ , where*

$$C_1(n, t) = \frac{t^2}{2}(x^2 - 1) + \frac{t}{2}(3x - 1) + tn(1 - x^2) - n + \frac{3}{2} \quad \text{and} \quad C_2(n, t) = n - 2x(1 - x)tn + x(1 - x)t^2 + t + 1.$$

PROOF. Fix  $2 \leq t \leq n$ . Proposition 5.11.1 and the definition of  $h_\pi(n, t)$  implies

$$(56) \quad \frac{h_\pi(n, t)}{f_\pi(n, t)} \leq \frac{3^n 2^{\binom{t}{2} + t(n-t) - n}}{f_*(n, t) 2^{-3/2 - xt} 3^{-t-1}}.$$

Plugging in  $f_*(n, t)$  to the right hand side of (56) yields that  $h_\pi(n, t)/f_\pi(n, t) \leq 2^{C_1(n, t)} 3^{C_2(n, t)}$  where

$$C_1(n, t) = \binom{t}{2} + t(n - t) - n - \left( \binom{xt}{2} + x^2 t(n - t) - 3/2 - xt \right) \text{ and}$$

$$C_2(n, t) = n - \left( x(1 - x)t^2 + 2x(1 - x)t(n - t) - t - 1 \right).$$

Simplifying these expressions finishes the proof. □

We now prove the following three inequalities.

$$(I) \quad 2^{1-x^2} < 3^{1.5x(1-x)}.$$

$$(II) \quad 3^{(2/3)x(1-x)} < 2^{(1-x^2)/2}.$$

$$(III) \quad 5(1 - x^2 - 2x(1 - x) \log_2 3) + \log_2 3 - 1 < 0.$$

We will use the following bounds for  $\log 2$  and  $\log 3$  which come from the On-Line Encyclopedia of Integer Sequences, published electronically at <http://oeis.org> (Sequences A002162 and A002391 respectively).

$$(57) \quad .693 < \log 2 < .694 \quad \text{and} \quad 1.098 < \log 3 < 1.099.$$

For (I), note that  $2^{1-x^2} = 2^{(1-x)(1+x)} < 3^{1.5x(1-x)} \Leftrightarrow 2^{1+x} < 3^{1.5x} \Leftrightarrow (1+x) \log 2 < 1.5x \log 3$ .

Solving for  $x$  yields that this is equivalent to

$$(58) \quad \frac{\log 2}{1.5 \log 3 - \log 2} = \frac{2 \log 2}{3 \log 3 - 2 \log 2} < x = \frac{\log 3}{2 \log 3 - \log 2}.$$

Clearing out the denominators, (58) holds if and only if

$$(59) \quad 4 \log 3 \log 2 - 2(\log 2)^2 < 3(\log 3)^2 - 2 \log 2 \log 3 \Leftrightarrow 6 \log 2 \log 3 - 3(\log 3)^2 - 2(\log 2)^2 < 0.$$



By (57),  $6 \log 2 \log 3 - 3(\log 3)^2 - 2(\log 2)^2 < 6(.694)(1.099) - 3(1.098)^2 - 2(.693)^2 < 0$ . Thus the righthand inequality in (59) holds, which finishes the proof of (I). For (II), note that

$$3^{(2/3)x(1-x)} < 2^{(1-x^2)/2} = 2^{(1-x)(1+x)/2} \Leftrightarrow 3^{2x/3} < 2^{(1+x)/2} \Leftrightarrow \frac{2x}{3} \log 3 < \frac{(1+x) \log 2}{2}.$$

Rearranging and plugging in for  $x$ , this becomes

$$\frac{\log 3}{2 \log 3 - \log 2} = x < \frac{3 \log 2}{4 \log 3 - 3 \log 2}.$$

By clearing denominators, we have that this inequality holds if and only if

$$(60) \quad 4(\log 3)^2 - 3 \log 3 \log 2 < 6 \log 3 \log 2 - 3(\log 2)^2 \Leftrightarrow 4(\log 3)^2 - 9 \log 2 \log 3 + 3(\log 2)^2 < 0.$$

By (57),  $4(\log 3)^2 - 9 \log 2 \log 3 + 3(\log 2)^2 < 4(1.099)^2 - 9(.693)(1.098) + 3(.694)^2 < 0$ . Thus the righthand inequality in (60) holds, which finishes the proof of (II). We now prove (III). By rearranging the left hand side, (III) is equivalent to

$$5x^2(2 \log_2 3 - 1) - 10x \log_2 3 + \log_2 3 + 4 < 0.$$

Multiplying by  $\log 2$ , this becomes  $5x^2(2 \log 3 - \log 2) - 10x \log 3 + \log 3 + 4 \log 2 < 0$ . Plugging in for  $x$  and simplifying, this is equivalent to

$$(61) \quad \frac{-5(\log 3)^2}{2 \log 3 - \log 2} + \log 3 + 4 \log 2 < 0 \Leftrightarrow -3(\log 3)^2 + 7 \log 2 \log 3 - 4(\log 2)^2 < 0,$$

where the “ $\Leftrightarrow$ ” is from clearing the denominators of, then rearranging the lefthand inequality. By (57),  $-3(\log 3)^2 + 7 \log 2 \log 3 - 4(\log 2)^2 < -3(1.098)^2 + 7(.694)(1.099) - 4(.693)^2 < 0$ , thus the righthand inequality in (61) holds, which finishes the proof of (III).

**Proof of Lemma 5.7.5.** Given  $n, t \in \mathbb{N}$ , let  $p(n, t) = (-\frac{x}{6}(1-x)t + 2)n + 2$ . Choose  $K$  sufficiently large so that  $n \geq t \geq K$  implies  $p(n, t) \leq p(n, K) < 0$ . We prove part 1 for this  $K$ . By Proposition 5.11.2,  $h_\pi(n, t)/f_\pi(n, t) \leq 2^{C_1(n, t)} 3^{C_2(n, t)}$ . Note that

$$C_1(n, t) = (1-x^2)tn + D_1(n, t) \quad \text{and} \quad C_2(n, t) = -1.5x(1-x)tn + D_2(n, t)$$

where  $D_1(n, t) = \frac{t^2}{2}(x^2 - 1) + \frac{t}{2}(3x - 1) - n + 3/2$  and  $D_2(n, t) = -.5x(1-x)tn + x(1-x)t^2 + n + t + 1$ .

Therefore

$$2^{C_1(n, t)} 3^{C_2(n, t)} = \left( \frac{2^{1-x^2}}{3^{1.5x(1-x)}} \right)^{tn} 2^{D_1(n, t)} 3^{D_2(n, t)} \leq 2^{D_1(n, t)} 3^{D_2(n, t)},$$

where the inequality is because by (I),  $\frac{2^{1-x^2}}{3^{1.5x(1-x)}} \leq 1$ . Now note that

$$D_1(n, t) = \frac{t^2}{2}(x^2 - 1) + E_1(n, t) \quad \text{and} \quad D_2(n, t) = -(x/3)(1-x)tn + x(1-x)t^2 + E_2(n, t),$$

where  $E_1(n, t) = \frac{t}{2}(3x - 1) - n + 3/2$  and  $E_2(n, t) = -(x/6)(1-x)tn + n + t + 1$ . Since  $n \geq t$ , we have

$$-(x/3)(1-x)tn + x(1-x)t^2 \leq -(x/3)(1-x)t^2 + x(1-x)t^2 = (2x/3)(1-x)t^2,$$

so  $D_2(n, t) \leq (2x/3)(1-x)t^2 + E_2(n, t)$ . Thus

$$2^{D_1(n, t)} 3^{D_2(n, t)} \leq \left( \frac{3^{(2/3)x(1-x)}}{2^{(1-x^2)/2}} \right)^{t^2} 2^{E_1(n, t)} 3^{E_2(n, t)} \leq 2^{E_1(n, t)} 3^{E_2(n, t)},$$

where the last inequality is because by (II),  $\frac{3^{(2/3)x(1-x)}}{2^{(1-x^2)/2}} \leq 1$ . Note that since  $3x - 1 < 2$ ,  $n \geq t$  and  $3/2 \leq \log_2 3$ ,

$$E_1(n, t) = \frac{t}{2}(3x - 1) - n + 3/2 \leq t - t + \log_2 3 = \log_2 3.$$

Since  $5 \leq t \leq n$ ,  $E_2(n, t) = -(x/6)(1-x)tn + n + t + 1 \leq -(x/6)(1-x)tn + 2n + 1$ . Therefore,

$$2^{E_1(n, t)} 3^{E_2(n, t)} \leq 2^{\log_2 3} 3^{-(x/6)(1-x)tn + 2n + 1} = 3^{-(x/6)(1-x)tn + 2n + 2} = 3^{p(n, t)} < 1.$$

where the inequality is by assumption on  $K \leq t \leq n$ . This finishes the proof of part 1. For part 2, set

$$\gamma = -\frac{1}{2} \left( 5(1-x^2 - 2x(1-x)\log_2 3) + \log_2 3 - 1 \right).$$

Observe that (III) implies  $\gamma > 0$ . Fix  $5 \leq t \leq K$  and  $n \geq t$ . By Proposition 5.11.2, for any  $5 \leq t \leq n$ ,  $h_\pi(n, t)/f_\pi(n, t) \leq 2^{C_1(n, t)} 3^{C_2(n, t)}$ . Clearly there are polynomials  $q_1(t)$  and  $q_2(t)$  such that

$$C_1(n, t) = tn(1-x^2) - n + q_1(t) \quad \text{and} \quad C_2(n, t) = n - 2x(1-x)tn + q_2(t).$$

Let  $q(t) = q_1(t) + q_2(t)\log_2 3$  and let  $T$  be sufficiently large so that for all  $5 \leq t \leq K$ ,  $|q(t)| \leq T$ .

Then for all  $5 \leq t \leq K$  and  $n \geq t$ ,

$$2^{C_1(n, t)} 3^{C_2(n, t)} = 2^{n(t(1-x^2 - 2x(1-x)\log_2 3) + \log_2 3 - 1) + q(t)} \leq 2^{n(t(1-x^2 - 2x(1-x)\log_2 3) + \log_2 3 - 1) + T}.$$

By (III),  $(1-x^2 - 2x(1-x)\log_2 3) < 1 - \log_2 3 < 0$  so for all  $t \geq 5$ ,

$$t(1-x^2 - 2x(1-x)\log_2 3) + \log_2 3 - 1 \leq 5(1-x^2 - 2x(1-x)\log_2 3) + \log_2 3 - 1 = -2\gamma.$$

Combining all this, we have that for all  $5 \leq t \leq K$  and  $n \geq t$ ,  $h_\pi(n, t)/f_\pi(n, t) \leq 2^{-2\gamma n + T}$ . Choose  $M_1$  sufficiently large so that for all  $5 \leq t \leq K$  and  $n > M_1$ ,  $-2\gamma n + T \leq -\gamma n$ . Then for all  $n \geq M_1$  and  $5 \leq t \leq K$ ,  $h_\pi(n, t)/f_\pi(n, t) \leq 2^{-\gamma n}$ . This finishes the proof of part 2.  $\square$

**Proof of Lemma 5.9.1.** Recall we want to show there is  $M$  such that for all  $n \geq M$  and  $2 \leq t \leq n$ ,  $k_\pi(n, t) < f_\pi(n, t)$ , where  $k_\pi(n, t) = 15^t 2^{\binom{t}{2} + t(n-t) - t}$ . Let  $K$  be from Lemma 5.7.5 and recall the proof of Lemma 5.7.5 showed that for all  $K \leq t \leq n$ ,  $h_\pi(n, t)/f_\pi(n, t) \leq 3^{p(n, t)}$ , where

$$p(n, t) = -(x/6)(1-x)tn + 2n + 2.$$

Choose  $K' \geq K$  such that  $K' \leq t \leq n$  implies  $p(n, t) < -100n + 2 < -98n$ . Suppose now that  $K' \leq t \leq n$ . Then by definition of  $k_\pi(n, t)$  and since  $h_\pi(n, t)/f_\pi(n, t) \leq 3^{p(n, t)} < 3^{-98n}$ ,

$$\frac{k_\pi(n, t)}{f_\pi(n, t)} = \frac{(15/2)^t (2/3)^n h_\pi(n, t)}{f_\pi(n, t)} \leq (15/2)^t (2/3)^n 3^{p(n, t)} \leq 3^{p(n, t) + 4n} < 3^{-94n} < 1.$$

Thus the Lemma holds for all  $K' \leq t \leq n$ . Suppose now that  $2 \leq t \leq K'$  and  $n \geq t$ . By Proposition 5.11.1 and definition of  $k_\pi(n, t)$ ,

$$\frac{k_\pi(n, t)}{f_\pi(n, t)} \leq \frac{15^t 2^{\binom{t}{2} + t(n-t) - t}}{f_*(n, t) 2^{-xt - 3/2} 3^{-t-1}} = 2^{G_1(n, t)} 3^{G_2(n, t)},$$

where  $G_1(n, t)$  and  $G_2(n, t)$  are the appropriate polynomials in  $n$  and  $t$ . Using the definition of  $f_*(n, t)$ , we see that for some polynomials  $r_1(t)$  and  $r_2(t)$  in  $t$ ,

$$G_1(n, t) = tn - x^2 tn + r_1(t) \quad \text{and} \quad G_2(n, t) = -2x(1-x)tn + r_2(t).$$

Let  $r(t) = r_1(t) + r_2(t) \log_2 3$  and let  $T'$  be such that for all  $2 \leq t' \leq K'$ ,  $|r(t)| \leq T'$ . Then for all  $2 \leq t \leq K'$ ,

$$G_1(n, t) + G_2(n, t) \log_2 3 \leq tn(1 - x^2 - 2x(1-x) \log_2 3) + T'.$$

By (III),  $1 - x^2 - 2x(1-x) \log_2 3 < 0$ , so we can choose  $M$  sufficiently large so that if  $n > M$ , then  $n(1 - x^2 - 2x(1-x) \log_2 3) + T' < 0$ . Then for all  $2 \leq t \leq K'$  and  $n \geq M, t$ ,

$$\frac{k_\pi(n, t)}{f_\pi(n, t)} \leq 2^{nt(1-x^2-2x(1-x) \log_2 3) + T'} < 2^{n(1-x^2-2x(1-x) \log_2 3) + T'} < 1.$$

Thus the lemma holds for all  $n \geq \max\{M, K'\}$  and  $2 \leq t \leq n$ .  $\square$

## Hereditary Properties of First-Order Structures

### 6.1. Introduction

Given a first-order language  $\mathcal{L}$ , we say a class  $\mathcal{H}$  of  $\mathcal{L}$ -structures has the *hereditary property* if for all  $A \in \mathcal{H}$ ,  $B \subseteq_{\mathcal{L}} A$  implies  $B \in \mathcal{H}$ . Suppose  $\mathcal{L}$  is a finite relational language. A *hereditary  $\mathcal{L}$ -property* is a nonempty class of  $\mathcal{L}$ -structures which has the hereditary property and which is closed under isomorphisms. This is the natural generalization of existing notions of hereditary properties of various combinatorial structures. Indeed, for appropriately chosen  $\mathcal{L}$ , all of the examples mentioned Chapter 2 are hereditary properties of  $\mathcal{L}$ -structures. These include hereditary properties of graphs (studied in [2, 3, 11, 12, 13, 26, 27, 93]), hereditary properties of  $k$ -uniform hypergraphs (studied in [45]), hereditary properties of colored  $k$ -uniform hypergraphs (studied in [61]), hereditary properties of directed graphs and of posets (studied in [8, 9]), as well as the specific families of directed graphs from [73], metric spaces from [22, 82], and  $H$ -free hypergraphs from [19, 20, 83, 89]. In this chapter, we unify under a general framework, certain definitions, theorems, and proof techniques which have arisen from the study of these examples. We now give a brief description of our results.

Given a finite relational language  $\mathcal{L}$  and a hereditary  $\mathcal{L}$ -property  $\mathcal{H}$ , we define an invariant associated to  $\mathcal{H}$ , called the *asymptotic density of  $\mathcal{H}$* . Our main theorem, Theorem 6.4.4, gives an asymptotic enumeration of  $\mathcal{H}_n$  in terms of the asymptotic density of  $\mathcal{H}$ , where  $\mathcal{H}_n$  denotes the set of elements from  $\mathcal{H}$  with domain  $[n]$ . We will show in Chapter 7 that this generalizes enumeration theorems for hereditary properties of combinatorial structures appearing in [20, 45, 61, 66, 73, 82]. The tools we use include a new adaptation of the hypergraph containers theorem to the setting of arbitrary  $\mathcal{L}$ -structures (Theorem 6.4.9) and a version of the graph removal lemma for  $\mathcal{L}$ -structures which was proved by Aroskar and Cummings in [6] (Theorem 6.7.5). Our proof strategy is based on a series of enumeration results for combinatorial structures which employ the hypergraph containers theorem, namely results in [18, 92] for  $H$ -free graphs, in [73] for directed graphs, in [22] for metric spaces, and in [81] for graphs. We will also define generalizations of extremal graphs and graph stability theorems, and prove the existence of a stability theorem implies the existence of an approximate

structure theorem. This result, Theorem 6.4.7, generalizes arguments appearing in many papers, for just a few examples see [19, 20, 73, 85] and Chapter 4 of this thesis.

We end this introduction by pointing out that while our main theorem, Theorem 6.4.4, gives an enumeration theorem for a hereditary  $\mathcal{L}$ -property in terms of its asymptotic density, determining its asymptotic density is often a hard combinatorial problem. Similarly, while Theorem 6.4.7 shows that a stability theorem and an understanding of extremal structure implies an approximate structure theorem, proving a specific family  $\mathcal{H}$  has a stability theorem and understanding its extremal structures are difficult problems which must usually be solved for each  $\mathcal{H}$  individually.

## 6.2. Preliminaries

In this section we give preliminaries required to state our main theorems.

**6.2.1. Notation and Setup.** In this section we fix some notational conventions and definitions. We will use the word “collection” to denote either a set or a class. Suppose  $\ell \geq 1$  is an integer and  $X$  is a set. Let  $Perm(\ell)$  be the set of permutations of  $[\ell]$ . We let  $\mathcal{P}(X)$  or  $2^X$  denote the power set of  $X$ . Given a finite tuple  $\bar{x} = (x_1, \dots, x_\ell)$  and  $\mu \in Perm(\ell)$ , let  $\cup \bar{x} = \{x_1, \dots, x_\ell\}$ ,  $|\bar{x}| = \ell$ , and  $\mu(\bar{x}) = (x_{\mu(1)}, \dots, x_{\mu(\ell)})$ . An *enumeration of  $X$*  is a tuple  $\bar{x} = (x_1, \dots, x_{|X|})$  such that  $\cup \bar{x} = X$ . Given  $x \neq y \in X$ , we will write  $xy$  as shorthand for the set  $\{x, y\}$ . Set

$$X^\ell = \{(x_1, \dots, x_\ell) \in X^\ell : x_i \neq x_j \text{ for each } i \neq j\} \quad \text{and} \quad \binom{X}{\ell} = \{Y \subseteq X : |Y| = \ell\}.$$

Suppose  $\mathcal{L}$  is a finite relational first-order language. Let  $r_{\mathcal{L}}$  denote the maximum arity of any relation symbol in  $\mathcal{L}$ . Given a formula  $\phi$  and a tuple of variables  $\bar{x}$ , we write  $\phi(\bar{x})$  to denote that the free variables in  $\phi$  are all in the set  $\cup \bar{x}$ . Similarly, if  $p$  is a set of formulas, we will write  $p(\bar{x})$  if every formula in  $p$  has free variables in the set  $\cup \bar{x}$ . We will sometimes abuse notation and write  $\bar{x}$  instead of  $\cup \bar{x}$  when it is clear from context what is meant.

Suppose  $M$  is an  $\mathcal{L}$ -structure. Then  $dom(M)$  denotes the underlying set of  $M$ , and the *size* of  $M$  is  $|dom(M)|$ . If  $\mathcal{L}' \subseteq \mathcal{L}$ ,  $M \upharpoonright_{\mathcal{L}'}$  is the  $\mathcal{L}'$ -structure with underlying set  $dom(M)$  such that for all  $\ell \geq 1$ , if  $\bar{a} \in dom(M)^\ell$  and  $R$  is an  $\ell$ -ary relation symbol from  $\mathcal{L}'$ , then  $M \upharpoonright_{\mathcal{L}'} \models R(\bar{a})$  if and only if  $M \models R(\bar{a})$ . We call  $M \upharpoonright_{\mathcal{L}'}$  the *reduct of  $M$  to  $\mathcal{L}'$* . Given  $X \subseteq dom(M)$ ,  $M[X]$  is the  $\mathcal{L}$ -structure with domain  $X$  such that for all  $\ell \geq 1$ , if  $\bar{a} \in X^\ell$  and  $R$  is an  $\ell$ -ary relation symbol from  $\mathcal{L}$ , then  $M[X] \models R(\bar{a})$  if and only if  $M \models R(\bar{a})$ . We call  $M[X]$  the  $\mathcal{L}$ -structure induced by  $M$  on  $X$ . Given

a tuple  $\bar{a} \in \text{dom}(M)^\ell$ , the *quantifier-free type* of  $\bar{a}$  is

$$qftp_{\mathcal{L}}^M(\bar{a}) = \{\phi(x_1, \dots, x_\ell) : \phi(x_1, \dots, x_\ell) \text{ is a quantifier-free } \mathcal{L}\text{-formula and } M \models \phi(\bar{a})\}.$$

If  $\bar{x} = (x_1, \dots, x_\ell)$  and  $p(\bar{x})$  is a set of quantifier-free  $\mathcal{L}$ -formulas, then  $p$  is called a *quantifier-free  $\ell$ -type* if there is some  $\mathcal{L}$ -structure  $N$  and a tuple  $\bar{a} \in \text{dom}(N)^\ell$  such that  $N \models \phi(\bar{a})$  for all  $\phi(\bar{x}) \in p$ . In this case we say  $\bar{a}$  *realizes  $p$  in  $N$* . If there is some  $\bar{a} \in \text{dom}(N)^\ell$  realizing  $p$  in  $N$ , we say  $p$  *is realized in  $N$* . A quantifier-free  $\ell$ -type  $p(\bar{x})$  is *complete* if for every quantifier-free formula  $\phi(\bar{x})$ , either  $\phi(\bar{x})$  or  $\neg\phi(\bar{x})$  is in  $p(\bar{x})$ . Note that any type of the form  $qftp_{\mathcal{L}}^M(\bar{a})$  is complete. All types and formulas we consider will be quantifier-free, so for the rest of the chapter, any use of the words type and formula means quantifier-free type and quantifier-free formula.

If  $X$  and  $Y$  are both  $\mathcal{L}$ -structures, let  $X \subseteq_{\mathcal{L}} Y$  denote that  $X$  is a  $\mathcal{L}$ -substructure of  $Y$ . Given an  $\mathcal{L}$ -structure  $H$ , we say that  $M$  is  *$H$ -free* if there is no  $A \subseteq_{\mathcal{L}} M$  such that  $A \cong_{\mathcal{L}} H$ . Suppose  $\mathcal{H}$  is a collection of  $\mathcal{L}$ -structures. We say  $M$  is  *$\mathcal{H}$ -free* if  $M$  is  $H$ -free for all  $H \in \mathcal{H}$ . For each positive integer  $n$ , let  $\mathcal{H}(n)$  denote the collection of all elements in  $\mathcal{H}$  of size  $n$ , and let  $\mathcal{H}_n$  denote the set of elements in  $\mathcal{H}$  with domain  $[n]$ .  $\mathcal{H}$  is *trivial* if there is  $N$  such that  $\mathcal{H}(n) = \emptyset$  for all  $n \geq N$ . Otherwise  $\mathcal{H}$  is *non-trivial*.

We now define a modified version of the traditional type space, which is appropriate for working with families of finite structures instead of with complete first-order theories. Given  $\bar{x} = (x_1, \dots, x_\ell)$ , an  $\ell$ -type  $p(\bar{x})$  is *proper* if it contains the formulas  $x_i \neq x_j$  for each  $i \neq j$ .

**DEFINITION 6.2.1.** Suppose  $\mathcal{F}$  is a collection of  $\mathcal{L}$ -structures and  $\ell \geq 1$  is an integer. Define  $S_\ell(\mathcal{F})$  to be the set of all complete, proper, quantifier-free  $\ell$ -types which are realized in some element of  $\mathcal{F}$ . Let  $S_\ell(\mathcal{L})$  denote the set of all complete, proper, quantifier-free  $\ell$ -types.

We would like to emphasize some important differences between this and the usual type space. First, the elements of these type spaces are proper and contain only quantifier-free formulas. Second, these type spaces are defined relative to families of finite structures instead of complete first-order theories.

It will at times be convenient to expand our languages to contain constant symbols naming elements of the structures under consideration. If  $V$  is a set, let  $C_V$  denote the set of constant symbols  $\{c_v : v \in V\}$ . Given  $\bar{v} = (v_1, \dots, v_\ell) \in V^\ell$ , let  $c_{\bar{v}} = (c_{v_1}, \dots, c_{v_\ell})$ . Suppose  $M$  is an  $\mathcal{L}$ -structure. The *diagram of  $M$* , denoted  $\text{Diag}(M)$ , is the following set of sentences in the language  $\mathcal{L} \cup C_{\text{dom}(M)}$ .

$$\text{Diag}(M) = \{\phi(c_{\bar{a}}) : \phi(\bar{x}) \text{ is a quantifier-free } \mathcal{L}\text{-formula, } \cup \bar{a} \subseteq \text{dom}(M), \text{ and } M \models \phi(\bar{a})\}.$$

If  $A \subseteq \text{dom}(M)$ , the *diagram of  $A$  in  $M$*  is the following set of sentences in the language  $\mathcal{L} \cup C_A$ .

$$\text{Diag}^M(A) = \{\phi(c_{\bar{a}}) : \phi(\bar{x}) \text{ is a quantifier-free } \mathcal{L}\text{-formula, } \cup \bar{a} \subseteq A, \text{ and } M \models \phi(\bar{a})\}.$$

Observe that if  $A = \{a_1, \dots, a_r\} \subseteq \text{dom}(M)$  and  $p(\bar{x}) \in S_r(\mathcal{L})$  is such that  $p(\bar{x}) = \text{qftp}_{\mathcal{L}}^M(a_1, \dots, a_r)$ , then  $\text{Diag}^M(A) = p(c_{a_1}, \dots, c_{a_r})$ . Given a set of constants  $C$ , a collection of  $\mathcal{L}$ -structures  $\mathcal{F}$ , and  $\ell \geq 1$ , set

$$S_\ell(C) = \{p(\bar{c}) : p(\bar{x}) \in S_\ell(\mathcal{L}) \text{ and } \bar{c} \in C^\ell\} \quad \text{and} \quad S_\ell(C, \mathcal{F}) = \{p(\bar{c}) : p(\bar{x}) \in S_\ell(\mathcal{F}) \text{ and } \bar{c} \in C^\ell\}.$$

We would like to emphasize that if  $p(\bar{c}) \in S_\ell(C)$ , then  $\bar{c} \in C^\ell$  is a tuple of  $\ell$  *distinct* constants. Note that by this definition, if  $A \in \binom{\text{dom}(M)}{\ell}$ , then  $\text{Diag}^M(A) \in S_\ell(C_{\text{dom}(M)})$ .

**6.2.2. Facts about hereditary properties.** Suppose  $\mathcal{L}$  is a finite relational language. In this subsection we state some well known facts about hereditary  $\mathcal{L}$ -properties. First we recall that hereditary  $\mathcal{L}$ -properties are the same as families of structures with forbidden configurations. This fact will be used throughout the chapter.

DEFINITION 6.2.2. If  $\mathcal{F}$  is a collection of finite  $\mathcal{L}$ -structures, let  $\text{Forb}(\mathcal{F})$  be the class of all finite  $\mathcal{L}$ -structures which are  $\mathcal{F}$ -free.

It is easy to check that for any collection  $\mathcal{F}$  of finite  $\mathcal{L}$ -structures,  $\text{Forb}(\mathcal{F})$  is a hereditary  $\mathcal{L}$ -property. The converse to this statement is also true in the sense of Observation 6.2.3 below. This fact is standard, but we include a proof for completeness.

OBSERVATION 6.2.3. If  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property, then there is a class of finite  $\mathcal{L}$ -structures  $\mathcal{F}$  which is closed under isomorphism and such that  $\mathcal{H} = \text{Forb}(\mathcal{F})$ .

PROOF. Let  $\mathcal{F}$  be the class of all finite  $\mathcal{L}$ -structures  $F$  such that  $\text{prob}(F, M) = 0$  for all  $M \in \mathcal{H}$ . Clearly  $\mathcal{F}$  is closed under  $\mathcal{L}$ -isomorphism. We show  $\mathcal{H} = \text{Forb}(\mathcal{F})$ . Suppose  $M \in \mathcal{H}$  but  $M \notin \text{Forb}(\mathcal{F})$ . Then there is some  $F' \subseteq_{\mathcal{L}} M$  and  $F \in \mathcal{F}$  such that  $F \cong_{\mathcal{L}} F'$ . Since  $\mathcal{F}$  is closed under  $\mathcal{L}$ -isomorphism,  $F' \in \mathcal{F}$ . But then  $M \in \mathcal{H}$  and  $F' \subseteq_{\mathcal{L}} M$  implies  $F' \notin \mathcal{F}$  by definition of  $\mathcal{F}$ , a contradiction. Conversely, suppose  $M \in \text{Forb}(\mathcal{F})$  but  $M \notin \mathcal{H}$ . Because  $M$  is  $\mathcal{F}$ -free, we must have  $M \notin \mathcal{F}$ . By definition of  $\mathcal{F}$ , this implies there is some  $M' \in \mathcal{H}$  such that  $M \subseteq_{\mathcal{L}} M'$ . Because  $\mathcal{H}$  has the hereditary property, this implies  $M \in \mathcal{H}$ , a contradiction.  $\square$

A sentence  $\phi$  is universal if it is of the form  $\forall \bar{x} \psi(\bar{x})$  where  $\psi(\bar{x})$  is quantifier-free. The following well known fact is another reason hereditary  $\mathcal{L}$ -properties are natural objects of study. Given a finite

$\mathcal{L}$ -structure  $F$ , let  $\bar{x} \cong_{\mathcal{L}} F$  be short hand for the  $\mathcal{L}$ -formula  $\phi(x_1, \dots, x_{|\text{dom}(F)|})$  which says  $\bar{x}$  is a copy of  $F$ .

OBSERVATION 6.2.4.  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property if and only if there is a set of universal sentences  $\Phi$  such that  $\mathcal{H}$  is the class of finite models of  $\Phi$ .

PROOF. Suppose first  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Let  $\mathcal{F}$  be as in Observation 6.2.3 such that  $\mathcal{H} = \text{Forb}(\mathcal{F})$ . Let  $\{F_i : i \in \omega\}$  contain one representative of each isomorphism class in  $\mathcal{F}$ , and let  $\theta_i$  be the sentence saying  $\forall \bar{x} \neg(\bar{x} \cong_{\mathcal{L}} F_i)$ . Then clearly  $\mathcal{H} = \text{Forb}(\mathcal{F})$  is the class of all finite models of  $\Phi := \{\theta_i : i \in \omega\}$ . Conversely, suppose  $\Phi$  is a set of universal sentences and  $\mathcal{H}$  is the class of all finite models of  $\Phi$ . Clearly  $\mathcal{H}$  is closed under isomorphism. Suppose now  $B \in \mathcal{H}$  and  $A \subseteq_{\mathcal{L}} B$ . Let  $\phi \in \Phi$ . Then  $B \models \phi$  by assumption. Since  $\phi$  is universal, it is preserved under substructures, so  $A \models \phi$ . Thus  $A \models \phi$  for all  $\phi \in \Phi$ , so  $A \in \mathcal{H}$ . This shows  $\mathcal{H}$  has the hereditary property, and thus is a hereditary  $\mathcal{L}$ -property.  $\square$

**6.2.3. Distance between first-order structures.** In this section we define a notion of distance between finite first-order structures. The following is based on definitions in [6].

DEFINITION 6.2.5. Suppose  $\mathcal{L}$  is a first-order language,  $B$  is a finite  $\mathcal{L}$ -structure of size  $\ell$ , and  $M$  is a finite  $\mathcal{L}$ -structure of size  $L$ .

- The *set of copies of  $B$  in  $M$*  is  $\text{cop}(B, M) = \{A : A \subseteq_{\mathcal{L}} M \text{ and } A \cong_{\mathcal{L}} B\}$ .
- The *induced structure density of  $B$  in  $M$*  is  $\text{prob}(B, M) = |\text{cop}(B, M)| / \binom{L}{\ell}$
- If  $\mathcal{B}$  is a set of finite  $\mathcal{L}$ -structures, let

$$\text{cop}(\mathcal{B}, M) = \bigcup_{B \in \mathcal{B}} \text{cop}(B, M) \quad \text{and} \quad \text{prob}(\mathcal{B}, M) = \max\{p(B, M) : B \in \mathcal{B}\}.$$

If  $\mathcal{B}$  is a *class* of finite  $\mathcal{L}$ -structures, define  $\text{cop}(\mathcal{B}, M) = \text{cop}(\mathcal{B}', M)$  and  $\text{prob}(\mathcal{B}, M) = \text{prob}(\mathcal{B}', M)$ , where  $\mathcal{B}'$  is any set containing one representative of each isomorphism type in  $\mathcal{B}$ .

We now state our definition for the distance between two finite first-order structures. It is a simplified version of the distance notion appearing in [6]. We will discuss the relationship between the two notions in Section 6.7.



DEFINITION 6.2.6. Let  $\mathcal{L}$  be a finite relational first-order language with  $r_{\ell} = r \geq 2$ . Suppose  $M$  and  $N$  are two finite  $\mathcal{L}$ -structures with the same underlying set  $V$  of size  $n$ . Let

$$\text{diff}(M, N) = \left\{ A \in \binom{V}{r} : \text{for some (equivalently, any) enumeration } \bar{a} \text{ of } A, \text{qftp}^M(\bar{a}) \neq \text{qftp}^N(\bar{a}) \right\} \text{ and}$$

$$\text{dist}(M, N) = \frac{|\text{diff}(M, N)|}{\binom{n}{r}}$$

We say that  $M$  and  $N$  are  $\delta$ -close if  $\text{dist}(M, N) \leq \delta$ .

Observe that in the notation of Definition 6.2.6,  $\text{diff}(M, N) = \{A \in \binom{V}{r} : \text{Diag}^M(A) \neq \text{Diag}^N(A)\}$ .

### 6.3. $\mathcal{L}_{\mathcal{H}}$ -structures

**From now on,  $\mathcal{L}$  is a fixed finite relational language and  $r := r_{\mathcal{L}} \geq 2$ .** For this section,  $\mathcal{H}$  is a nonempty collection of finite  $\mathcal{L}$ -structures. In this section we introduce a language  $\mathcal{L}_{\mathcal{H}}$  associated to  $\mathcal{L}$  and  $\mathcal{H}$ . Structures in this new language play key roles in our main theorems.

DEFINITION 6.3.1. Define  $\mathcal{L}_{\mathcal{H}} = \{R_p(\bar{x}) : p(\bar{x}) \in S_r(\mathcal{H})\}$  to be the relational language with one  $r$ -ary relation for each  $p(\bar{x})$  in  $S_r(\mathcal{H})$ .

The goal of this section is to formalize how an  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  with the right properties can serve as a “template” for building  $\mathcal{L}$ -structures with the same underlying set as  $M$ . We now give an example or a hereditary property and its corresponding auxiliary language as in Definition 6.3.1.

EXAMPLE 6.3.2. To avoid confusion, we will use  $\mathcal{P}$  to refer to specific hereditary properties in example settings. Suppose  $\mathcal{L} = \{R_1(x, y), R_2(x, y), R_3(x, y)\}$ . Let  $\mathcal{P}$  be the class of all finite metric spaces with distances in  $\{1, 2, 3\}$ , considered as  $\mathcal{L}$ -structures in the natural way (i.e.  $R_i(x, y)$  if and only if  $d(x, y) = i$ ). It is easy to see that  $\mathcal{P}$  is a hereditary  $\mathcal{L}$ -property. Since  $r_{\mathcal{L}} = 2$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(x, y) : p(x, y) \in S_2(\mathcal{P})\}$ . For each  $i \in [3]$ , set

$$q_i(x, y) := \{x \neq y\} \cup \{R_i(x, y), R_i(y, x)\} \cup \{\neg R_j(x, y), \neg R_j(y, x) : j \neq i\},$$

and let  $p_i(x, y)$  be the unique quantifier-free 2-type containing  $q_i(x, y)$ . Informally, the type  $p_i(x, y)$  says “the distance between  $x$  and  $y$  is equal to  $i$ .” We leave it as an exercise to check  $S_2(\mathcal{P}) = \{p_i(x, y) : i \in [3]\}$  (recall  $S_2(\mathcal{P})$  contains only *proper* types). Thus  $\mathcal{L}_{\mathcal{P}} = \{R_{p_i}(x, y) : i \in [3]\}$ .

Observe that in an arbitrary  $\mathcal{L}_{\mathcal{H}}$ -structure may, the relation symbols in  $\mathcal{L}_{\mathcal{H}}$  have nothing to do with the properties of the type space  $S_r(\mathcal{H})$ . For instance, in the notation of Example 6.3.2, we

can easily build an  $\mathcal{L}_{\mathcal{P}}$ -structure  $M$  so that for some  $a, b \in \text{dom}(M)$ ,  $M \models R_{p_1}(a, b) \wedge \neg R_{p_1}(b, a)$ , even though  $p_1(x, y) = p_1(y, x)$  in  $S_2(\mathcal{P})$ . This kind of behavior will be an undesirable for various technical reasons. We now define the class of  $\mathcal{L}_{\mathcal{H}}$ -structures which are most nicely behaved for our purposes, and where in particular, this bad behavior does not happen.

**DEFINITION 6.3.3.** An  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  with domain  $V$  is *complete* if for all  $A \in \binom{V}{r}$  there is an enumeration  $\bar{a}$  of  $A$  and  $R_p \in \mathcal{L}_{\mathcal{H}}$  such that  $M \models R_p(\bar{a})$ .

**DEFINITION 6.3.4.** An  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  with domain  $V$  is an  $\mathcal{L}_{\mathcal{H}}$ -*template* if it is complete and the following hold.

- (1) If  $p(\bar{x}) \in S_r(\mathcal{H})$  and  $\bar{a} \in V^r \setminus V^x$ , then  $M \models \neg R_p(\bar{a})$ .
- (2) If  $p(\bar{x}), p'(\bar{x}) \in S_r(\mathcal{H})$  and  $\mu \in \text{Perm}(r)$  are such that  $p(\bar{x}) = p'(\mu(\bar{x}))$ , then for every  $\bar{a} \in V^x$ ,  $M \models R_p(\bar{a})$  if and only if  $M \models R_{p'}(\mu(\bar{a}))$ .

The idea behind this definition is that  $\mathcal{L}_{\mathcal{H}}$ -templates are  $\mathcal{L}_{\mathcal{H}}$ -structures which most accurately reflect the properties of  $S_r(\mathcal{H})$ .

**EXAMPLE 6.3.5.** Let  $\mathcal{L}$  and  $\mathcal{P}$  be as in Example 6.3.2. We now define an  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  with domain  $V = \{u, v, w\}$ . Define  $G \models \bigwedge_{i=1}^3 (R_{p_i}(u, v) \wedge R_{p_i}(v, u))$ ,

$$G \models \neg R_{p_3}(w, v) \wedge \neg R_{p_3}(v, w) \wedge \bigwedge_{i=1}^2 (R_{p_i}(w, v) \wedge R_{p_i}(v, w)),$$

$$G \models R_{p_1}(w, u) \wedge R_{p_1}(u, w) \wedge \bigwedge_{i=2}^3 (\neg R_{p_i}(w, u) \wedge \neg R_{p_i}(u, w)),$$

and for  $x \in V$ , define  $G \models \bigwedge_{i=1}^3 \neg R_{p_i}(x, x)$ . We leave it to the reader to verify  $G$  is a  $\mathcal{L}_{\mathcal{P}}$ -template.

While  $\mathcal{L}_{\mathcal{H}}$ -templates are important for the main results of this chapter, many of the definitions and facts in the rest of this section will be presented for  $\mathcal{L}_{\mathcal{H}}$ -structures with weaker assumptions.

**6.3.1. Choice functions and subpatterns.** In this subsection, we give crucial definitions for how we can use  $\mathcal{L}_{\mathcal{H}}$ -structures to build  $\mathcal{L}$ -structures.

**DEFINITION 6.3.6.** Suppose  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -structure with domain  $V$ .

- (1) Given  $A \in \binom{V}{r}$ , the *set of choices for  $A$  in  $M$*  is

$$Ch_M(A) = \{p(c_{a_1}, \dots, c_{a_r}) \in S_r(C_V, \mathcal{H}) : \{a_1, \dots, a_r\} = A \text{ and } M \models R_p(a_1, \dots, a_r)\}.$$

- (2) A *choice function for  $M$*  is a function  $\chi : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  such that for each  $A \in \binom{V}{r}$ ,  $\chi(A) \in Ch_M(A)$ . Let  $Ch(M)$  denote the set of all choice functions for  $M$ .

In the notation of Definition 6.3.6, clearly there exists a choice function for  $M$  if and only if  $Ch_M(A) \neq \emptyset$  for all  $A \in \binom{V}{r}$ . Observe that  $Ch_M(A) \neq \emptyset$  for all  $A \in \binom{V}{r}$  if and only if  $M$  is complete. Therefore  $Ch(M) \neq \emptyset$  if and only if  $M$  is complete. This is why  $\mathcal{L}_{\mathcal{H}}$ -templates are defined to be complete.

EXAMPLE 6.3.7. Recall that if  $x$  and  $y$  are distinct elements of a set, then  $xy$  is shorthand for the set  $\{x, y\}$ . Let  $\mathcal{L}$ ,  $\mathcal{P}$ ,  $V$ , and  $G$  be as in Example 6.3.5. Note that  $C_V = \{c_u, c_v, c_w\}$  and

$$S_2(C_V, \mathcal{P}) = \{p_i(c_u, c_v) : i \in [3]\} \cup \{p_i(c_v, c_w) : i \in [3]\} \cup \{p_i(c_u, c_w) : i \in [3]\}.$$

By definition of  $G$ ,  $Ch_G(uv) = \{p_1(c_u, c_v), p_2(c_u, c_v), p_3(c_u, c_v)\}$ ,  $Ch_G(vw) = \{p_1(c_v, c_w), p_2(c_v, c_w)\}$ , and  $Ch_G(uw) = \{p_1(c_u, c_w)\}$ . Therefore choice functions for  $G$  are all the functions

$$\chi : \{uv, vw, uw\} \rightarrow \{p_i(c_u, c_v) : i \in [3]\} \cup \{p_i(c_v, c_w) : i \in [3]\} \cup \{p_i(c_u, c_w) : i \in [3]\}$$

with the properties that  $\chi(uv) \in \{p_1(c_u, c_v), p_2(c_u, c_v), p_3(c_u, c_v)\}$ ,  $\chi(vw) \in \{p_1(c_v, c_w), p_2(c_v, c_w)\}$  and  $\chi(uw) = p_1(c_u, c_w)$ . Clearly there are six choice functions for  $G$ .

The following observation is immediate from the definition of  $\mathcal{L}_{\mathcal{H}}$ -template.

OBSERVATION 6.3.8. *If  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template with domain  $V$ , then for all  $\bar{a} \in V^r$  and  $R_p \in \mathcal{L}_{\mathcal{H}}$ ,  $M \models R_p(\bar{a})$  if and only if  $|\cup \bar{a}| = r$  and  $p(c_{\bar{a}}) \in Ch_M(\cup \bar{a})$ .*

The following fact is one reason why  $\mathcal{L}_{\mathcal{H}}$ -templates are convenient.

PROPOSITION 6.3.9. *Suppose  $M_1$  and  $M_2$  are  $\mathcal{L}_{\mathcal{H}}$ -templates with domain  $V$  such that for all  $A \in \binom{V}{r}$ ,  $Ch_{M_1}(A) = Ch_{M_2}(A)$ . Then  $M_1$  and  $M_2$  are the same  $\mathcal{L}_{\mathcal{H}}$ -structure.*

PROOF. We show that for all  $\bar{a} \in V^r$  and  $R_p \in \mathcal{L}_{\mathcal{H}}$ ,  $M_1 \models R_p(\bar{a})$  if and only if  $M_2 \models R_p(\bar{a})$ . Fix  $\bar{a} \in V^r$  and  $R_p \in \mathcal{L}_{\mathcal{H}}$ . Suppose first that  $|\cup \bar{a}| < r$ . By part (1) of Definition 6.3.4,  $M_1 \models \neg R_p(\bar{a})$  and  $M_2 \models \neg R_p(\bar{a})$ . So assume  $|\cup \bar{a}| = r$ . By Observation 6.3.8,  $M_1 \models R_p(\bar{a})$  if and only if  $p(c_{\bar{a}}) \in Ch_{M_1}(\cup \bar{a})$  and  $M_2 \models R_p(\bar{a})$  if and only if  $p(c_{\bar{a}}) \in Ch_{M_2}(\cup \bar{a})$ . Since  $Ch_{M_1}(\cup \bar{a}) = Ch_{M_2}(\cup \bar{a})$ , this implies  $M_1 \models R_p(\bar{a})$  if and only if  $M_2 \models R_p(\bar{a})$ .  $\square$

The next example shows Proposition 6.3.9 can fail when we are not dealing with  $\mathcal{L}_{\mathcal{H}}$ -templates.

EXAMPLE 6.3.10. Let  $\mathcal{L}$ ,  $\mathcal{P}$ ,  $V$ , and  $G$  be as in Example 6.3.13. Let  $G'$  be the  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$  which agrees with  $G$  on  $V^2 \setminus \{(v, u), (w, v), (w, u)\}$  and where

$$G' \models \bigwedge_{i=1}^3 (\neg R_{p_i}(v, u) \wedge \neg R_{p_i}(w, v) \wedge \neg R_{p_i}(w, u)).$$

We leave it to the reader to check that for all  $xy \in \binom{V}{2}$ ,  $Ch_{G'}(xy) = Ch_G(xy)$ . However,  $G$  and  $G'$  are distinct  $\mathcal{L}_{\mathcal{P}}$ -structures because, for instance,  $G \models R_{p_1}(v, u)$  while  $G' \models \neg R_{p_1}(v, u)$ . Observe that  $G'$  is not an  $\mathcal{L}_{\mathcal{P}}$ -template because  $G' \models R_{p_1}(u, v) \wedge \neg R_{p_1}(v, u)$  while  $p_1(x, y) = p_1(y, x)$ .

The definition shows how choice functions give rise to  $\mathcal{L}$ -structures.

DEFINITION 6.3.11. Suppose  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure with domain  $V$ ,  $N$  is an  $\mathcal{L}$ -structure such that  $dom(N) \subseteq V$ , and  $\chi \in Ch(M)$  is a choice function for  $M$ .

- (1)  $N$  is a  $\chi$ -subpattern of  $M$ , denoted  $N \leq_{\chi} M$ , if for every  $A \in \binom{dom(N)}{r}$ ,  $\chi(A) = Diag^N(A)$ .
- (2)  $N$  is a full  $\chi$ -subpattern of  $M$ , denoted  $N \sqsubseteq_{\chi} M$ , if  $N \leq_{\chi} M$  and  $dom(N) = V$ .

When  $N \sqsubseteq_{\chi} M$ , we say  $\chi$  chooses  $N$ . We say  $N$  is a subpattern of  $M$ , denoted  $N \leq_p M$ , if  $N \leq_{\chi} M$  for some choice function  $\chi$  for  $M$ . We say  $N$  is a full subpattern of  $M$ , denoted  $N \leq_p M$ , if  $N \sqsubseteq_{\chi} M$  for some choice function  $\chi$  for  $M$ . The subscript in  $\leq_p$  and  $\sqsubseteq_p$  is for ‘‘pattern.’’

OBSERVATION 6.3.12. Suppose  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure,  $\chi \in Ch(M)$ , and  $G$  is an  $\mathcal{L}$ -structure such that  $G \leq_{\chi} M$ . If  $G'$  is another  $\mathcal{L}$ -structure such that  $G' \sqsubseteq_{\chi} M$ , then  $G$  and  $G'$  are the same  $\mathcal{L}$ -structure. If  $\chi' \in Ch(M)$  satisfies  $G \sqsubseteq_{\chi'} M$ , then  $\chi = \chi'$ .

PROOF. By definition,  $G \leq_{\chi} M$  and  $G' \sqsubseteq_{\chi} M$  imply that  $Diag^G(A) = \chi(A) = Diag^{G'}(A)$  for all  $A \in \binom{V}{r}$ . This implies

$$Diag(G) = \bigcup_{A \in \binom{V}{r}} Diag^G(A) = \bigcup_{A \in \binom{V}{r}} Diag^{G'}(A) = Diag(G'),$$

which clearly implies  $G$  and  $G'$  are the same  $\mathcal{L}$ -structure. Similarly,  $G \sqsubseteq_{\chi} M$  and  $G \sqsubseteq_{\chi'} M$  imply that for all  $A \in \binom{V}{r}$ ,  $\chi(A) = Diag^G(A) = \chi'(A)$ . Thus  $\chi = \chi'$ .  $\square$

EXAMPLE 6.3.13. Let  $\mathcal{L}$ ,  $\mathcal{P}$ ,  $V$  and  $G$  be as in Example 6.3.5. We give two examples of subpatterns of  $G$ . Let  $\chi$  be the function from  $\binom{V}{2} \rightarrow S_2(C_V, \mathcal{P})$  defined by  $\chi(uv) = p_1(c_u, c_v)$ ,  $\chi(vw) = p_2(c_v, c_w)$ , and  $\chi(uw) = p_1(c_u, c_w)$ . Clearly  $\chi$  is a choice function for  $G'$ . Let  $H$  be the  $\mathcal{L}$ -structure with domain  $V$  such that  $H \models p_1(u, v) \cup p_2(v, w) \cup p_1(u, w)$ . Then by definition of  $H$ ,  $Diag^H(uv) = p_1(c_u, c_v)$ ,

$Diag^H(vw) = p_2(v, w)$  and  $Diag^H(uw) = p_1(c_u, c_w)$ . In other words,  $H \leq_{\chi} G$ . Since  $dom(H) = dom(G) = V$ ,  $H \leq_{\chi} G$ . Note that  $H$  is a metric space, that is,  $H \in \mathcal{P}$ .

Let  $\chi'$  be the function from  $\binom{V}{2} \rightarrow S_2(C_V, \mathcal{P})$  defined by  $\chi'(uv) = p_3(c_u, c_v)$ ,  $\chi'(vw) = p_1(c_v, c_w)$ , and  $\chi'(uw) = p_1(c_u, c_w)$ . Clearly  $\chi'$  is a choice function for  $G$ . Let  $H'$  be the  $\mathcal{L}$ -structure with domain  $V$  such that  $H' \models p_3(u, v) \cup p_1(v, w) \cup p_1(u, w)$ . Then as above, it is easy to see that  $H' \leq_{\chi'} G$ , and since  $dom(H') = V$ ,  $H' \leq_{\chi'} G$ . However,  $H'$  is *not* a metric space, that is,  $H' \notin \mathcal{P}$ .

This example demonstrates although  $\mathcal{L}_{\mathcal{H}}$ -templates are well behaved in certain ways, an  $\mathcal{L}_{\mathcal{H}}$ -template can have full subpatterns that are not in  $\mathcal{H}$ . We will give further definitions to address this in Section 6.3.3.

**6.3.2. Errors and counting subpatterns.** In this subsection we characterize when an  $\mathcal{L}_{\mathcal{H}}$ -structure has the property that every choice function gives rise to a subpattern. This is important for counting subpatterns of  $\mathcal{L}_{\mathcal{H}}$ -structures.

**DEFINITION 6.3.14.** Given  $r < \ell < 2r$ , an *error of size  $\ell$*  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  of size  $\ell$  with the following properties. There are  $\bar{a}_1, \bar{a}_2 \in dom(M)^c$  such that  $dom(M) = \cup \bar{a}_1 \cup \cup \bar{a}_2$  and for some  $p_1(\bar{x}), p_2(\bar{x}) \in S_r(\mathcal{H})$ ,  $M \models R_{p_1}(\bar{a}_1) \wedge R_{p_2}(\bar{a}_2)$  but  $p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$  is unsatisfiable.

**EXAMPLE 6.3.15.** Let  $\mathcal{L} = \{E(x, y, z), R_1(x, y), R_2(x, y), R_3(x, y)\}$  consist of one ternary relation  $E$  and three binary relations  $R_1, R_2, R_3$ . Suppose  $\mathcal{P}$  is the class of all finite  $\mathcal{L}$ -structures  $M$  such that the restriction of  $M$  to  $\{R_1, R_2, R_3\}$  is a metric space with distances in  $\{1, 2, 3\}$  (we put no restrictions on how  $E$  must behave). Let  $p_i(x, y)$  be the quantifier-free  $\{R_1, R_2, R_3\}$ -types from Examples 6.3.2, 6.3.5, and 6.3.13. Let  $\bar{x} = (x_1, x_2, x_3)$  and set  $q_0(\bar{x}) = \{E(x_i, x_j, x_k) : \{x_i, x_j, x_k\} \subseteq \{x_1, x_2, x_3\}\}$ . Then set  $q_1(\bar{x})$  and  $q_2(\bar{x})$  to be the complete quantifier-free types satisfying the following.

$$q_0(\bar{x}) \cup p_1(x_1, x_2) \cup p_1(x_1, x_3) \cup p_1(x_2, x_3) \subseteq q_1(\bar{x}) \text{ and}$$

$$q_0(\bar{x}) \cup p_2(x_1, x_2) \cup p_1(x_1, x_3) \cup p_1(x_2, x_3) \subseteq q_2(\bar{x}).$$

It is easy to check that  $q_i(\bar{x}) \in S_3(\mathcal{P})$  for  $i = 1, 2$ . Note  $q_1$  and  $q_2$  agree about how  $E$  behaves, but disagree on how the binary relations in  $\mathcal{L}$  behave. Let  $V = \{t, u, v, w\}$  be a set of size 4. Choose  $G$  to be the  $\mathcal{L}_{\mathcal{P}}$ -structure which satisfies  $G \models R_{q_1}(x, y, z)$  if and only if  $x, y$  and  $z$  are distinct,  $G \models R_{q_2}(x, y, z)$  if and only if  $(x, y, z) = (t, u, v)$ , and  $G \models \neg R_q(x, y, z)$  for all  $q \in S_3(\mathcal{P}) \setminus \{p_1, p_2\}$ .

By construction,  $G$  is a complete  $\mathcal{L}_{\mathcal{P}}$ -structure. Let  $\bar{a}_1 = (u, v, w)$  and  $\bar{a}_2 = (t, u, v)$ . Then  $dom(G) = \cup \bar{a}_1 \cup \cup \bar{a}_2$  and  $G \models R_{p_1}(\bar{a}_1) \wedge R_{p_2}(\bar{a}_2)$ . However,  $p_1(c_u, c_v) \subseteq q_1(c_u, c_v, c_w) = q_1(c_{\bar{a}_1})$  implies  $q_1(c_{\bar{a}_1})$

contains the formula  $R_1(c_u, c_v)$  while  $p_2(c_u, c_v) \subseteq q_2(c_u, c_v, c_w) = q_2(c_{\bar{a}_2})$  implies  $q_2(c_{\bar{a}_2})$  contains the formula  $\neg R_1(c_u, c_v)$ . Therefore  $q_1(c_{\bar{a}_1}) \cup q_2(c_{\bar{a}_2})$  is unsatisfiable, and  $G$  is an error of size 4.

DEFINITION 6.3.16. Let  $\mathcal{E}$  be the class of  $\mathcal{L}_{\mathcal{H}}$ -structures which are errors of size  $\ell$  for some  $r < \ell < 2r$ .

An  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  is *error-free* if it is  $\mathcal{E}$ -free. Error-free  $\mathcal{L}_{\mathcal{H}}$ -structures will be important for the following reason.

PROPOSITION 6.3.17. *Suppose  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure with domain  $V$ . Then  $M$  is error-free if and only if for every choice function  $\chi$  for  $M$ , there is an  $\mathcal{L}$ -structure  $N$  such that  $N \trianglelefteq_{\chi} M$ .*

PROOF. Suppose first that there exists a choice function  $\chi : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  such that there are no  $\chi$ -subpatterns of  $M$ . This means  $\Gamma := \bigcup_{A \in \binom{V}{r}} \chi(A)$  is not satisfiable. So there is an atomic formula  $\psi(\bar{x})$  and a tuple  $\bar{c} \subseteq C_V^r$  such that  $\psi(\bar{c}) \in \Gamma$  and  $\neg\psi(\bar{c}) \in \Gamma$ . For each  $A \in \binom{V}{r}$ , because  $\chi(A) \in S_r(C_A, \mathcal{H})$ , exactly one of  $\psi(\bar{c})$  or  $\neg\psi(\bar{c})$  is in  $\chi(A)$ . This implies there must be distinct  $A_1, A_2 \in \binom{V}{r}$  such that  $\bar{c} \subseteq A_1 \cap A_2$  and  $\psi(\bar{c}) \in \chi(A_1)$  and  $\neg\psi(\bar{c}) \in \chi(A_2)$ . Note  $A_1 \neq A_2$  and  $A_1 \cap A_2 \neq \emptyset$  imply that if  $\ell := |A_1 \cup A_2|$ , then  $r < \ell < 2r$ . Let  $N$  be the  $\mathcal{L}_{\mathcal{H}}$ -structure  $M[A_1 \cup A_2]$ . We show  $N$  is an error of size  $\ell$ . By definition of  $\chi$  being a choice function, there are  $p_1, p_2 \in S_r(C_V, \mathcal{H})$ ,  $\bar{a}_1$  and  $\bar{a}_2$  such that  $\bar{c} \subseteq A_1$ ,  $\bar{c} \subseteq A_2$ ,  $p_1(c_{\bar{a}_1}) = \chi(A_1)$ ,  $p_2(c_{\bar{a}_2}) = \chi(A_2)$ ,  $M \models R_{p_1}(\bar{a}_1)$ , and  $M \models R_{p_2}(\bar{a}_2)$ . By definition,  $N \subseteq_{\mathcal{L}_{\mathcal{H}}} M$ , thus  $N \models R_{p_1}(\bar{a}_1) \wedge R_{p_2}(\bar{a}_2)$ . Note

$$\{\psi(\bar{c}), \neg\psi(\bar{c})\} \subseteq \chi(A_1) \cup \chi(A_2) = p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$$

implies  $p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$  is unsatisfiable. Thus  $N \in \mathcal{E}$  and  $N \subseteq_{\mathcal{L}_{\mathcal{H}}} M$  implies  $M$  is not error-free.

Suppose on the other hand that  $M$  is not error-free. Say  $r < \ell < 2r$  and  $N$  is an error of size  $\ell$  in  $M$ . Then  $N \subseteq_{\mathcal{L}_{\mathcal{H}}} M$  and there are  $\bar{a}_1, \bar{a}_2 \in \text{dom}(N)^r$  and types  $p_1(\bar{x}), p_2(\bar{x}) \in S_r(\mathcal{H})$  such that  $\text{dom}(N) = \bar{a}_1 \cup \bar{a}_2$ ,  $N \models R_{p_1}(\bar{a}_1) \wedge R_{p_2}(\bar{a}_2)$ , and  $p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$  is unsatisfiable. We define a function  $\chi : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  as follows. Set  $\chi(\bar{a}_1) = p_1(c_{\bar{a}_1})$  and  $\chi(\bar{a}_2) = p_2(c_{\bar{a}_2})$ . For every  $A' \in \binom{V}{r} \setminus \{\bar{a}_1, \bar{a}_2\}$ , choose  $\chi(A')$  to be any element of  $Ch_M(A')$  (note  $Ch_M(A')$  is nonempty since  $M$  is complete). By construction,  $\chi$  is a choice function for  $M$ . Suppose there is  $G \trianglelefteq_{\chi} M$ . Then  $p_1(c_{\bar{a}_1}) = \text{Diag}^G(\bar{a}_1)$  and  $p_2(c_{\bar{a}_2}) = \text{Diag}^G(\bar{a}_2)$  implies  $G \models p_1(\bar{a}_1) \cup p_2(\bar{a}_2)$ , contradicting that  $p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$  is unsatisfiable. Thus  $\chi$  is a choice function for  $M$  such that there are no  $\chi$ -subpatterns of  $M$ . This finishes the proof.  $\square$

DEFINITION 6.3.18. Given a finite  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$ , let  $\text{sub}(M) = |\{G : G \trianglelefteq_p M\}|$  be the number of full subpatterns of  $M$ .

This definition and the following observation will be crucial to our enumeration theorem.

OBSERVATION 6.3.19. *If  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure with finite domain  $V$ , then*

$$\text{sub}(M) \leq \prod_{A \in \binom{V}{r}} |\text{Ch}_M(A)|,$$

*and equality holds if and only if  $M$  is error-free.*

PROOF. By definition of a choice function,  $|\text{Ch}(M)| = \prod_{A \in \binom{V}{r}} |\text{Ch}_M(A)|$ . By definition of subpattern, for each  $G \trianglelefteq_p M$ , there is  $\chi_G \in \text{Ch}(M)$  which chooses  $G$ . Observation 6.3.12 implies the map  $f : G \mapsto \chi_G$  is a well-defined injection from  $\{G : G \trianglelefteq_p M\}$  to  $\text{Ch}(M)$ . Thus

$$\text{sub}(M) = |\{G : G \trianglelefteq_p M\}| \leq |\text{Ch}(M)| = \prod_{A \in \binom{V}{r}} |\text{Ch}_M(A)|.$$

We now show equality holds if and only if  $M$  is error-free. Suppose first  $M$  is error-free. We claim  $f$  is surjective. Fix  $\chi \in \text{Ch}(M)$ . Since  $M$  is error-free, Lemma 6.3.17 implies that there is an  $\mathcal{L}$ -structure  $G_\chi$  such that  $G_\chi \trianglelefteq_\chi M$ . So  $G_\chi \in \{G : G \trianglelefteq_p M\}$  implies  $f(G_\chi)$  exists. By Observation 6.3.12, we must have  $f(G_\chi) = \chi$ . Thus  $f$  is surjective, and consequently  $\text{sub}(M) = |\text{Ch}(M)| = \prod_{A \in \binom{V}{r}} |\text{Ch}_M(A)|$ . Conversely, suppose equality holds. Then  $f$  is an injective map from a finite set to another finite set of the same size, thus it must be surjective. This implies that for all  $\chi \in \text{Ch}(M)$ , there is an  $\mathcal{L}$ -structure  $G$  such that  $G \trianglelefteq_\chi M$ . By Lemma 6.3.17, this implies  $M$  is error-free.  $\square$

REMARK 6.3.20. *Suppose  $\mathcal{L}$  contains no relations of arity less than  $r$ . If  $\mathcal{M}$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure with finite domain  $V$ , then  $\text{sub}(M) = \prod_{A \in \binom{V}{r}} |\text{Ch}_M(A)|$ .*

PROOF. Our assumption on  $\mathcal{L}$  implies  $\mathcal{E} = \emptyset$  by definition. Thus, if  $\mathcal{M}$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure with finite domain  $V$ , it is error-free, so Observation 6.3.19 implies  $\text{sub}(M) = \prod_{A \in \binom{V}{r}} |\text{Ch}_M(A)|$ .  $\square$

Remark 6.3.20 applies to most examples we are interested in, including graphs, (colored)  $k$ -uniform hypergraphs for any  $k \geq 2$ , directed graphs, and discrete metric spaces.

**6.3.3.  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -structures and  $\mathcal{L}_{\mathcal{H}}$ -templates.** In this subsection we consider  $\mathcal{L}_{\mathcal{H}}$ -structures with the property that all choice functions give rise to subpatterns in  $\mathcal{H}$ .

DEFINITION 6.3.21. An  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  is  *$\mathcal{H}$ -random* if it is complete and for every  $\chi \in \text{Ch}(M)$ , there is an  $\mathcal{L}$ -structure  $N \in \mathcal{H}$  such that  $N \trianglelefteq_\chi M$ .

Observe that by Proposition 6.3.17, any  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -structure is error-free. The difference between being error-free and being  $\mathcal{H}$ -random is as follows. If an  $\mathcal{L}_{\mathcal{H}}$ -structure is error-free, then it must have at least one full subpattern, however some or all its subpatterns may not be in  $\mathcal{H}$ . On the other hand, if an  $\mathcal{L}_{\mathcal{H}}$ -structure is  $\mathcal{H}$ -random, then it must have at least one full subpattern, and further, all its full subpatterns must also be in  $\mathcal{H}$ .

EXAMPLE 6.3.22. Let  $\mathcal{L}$ ,  $\mathcal{P}$ ,  $V$ ,  $H'$  and  $G$  be as in Example 6.3.13. Observe that  $G$  is *not*  $\mathcal{P}$ -random, since  $H' \trianglelefteq_{\mathcal{P}} G$ , but  $H' \notin \mathcal{P}$ . We now define an  $\mathcal{L}_{\mathcal{P}}$ -structure  $G''$  which is  $\mathcal{P}$ -random. Let  $G''$  be any  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$  such that for all  $(x, y) \in V^2$ ,

$$G'' \models R_{p_1}(x, y) \wedge R_{p_2}(x, y) \wedge \neg R_{p_3}(x, y) \wedge \bigwedge_{i=1}^3 \neg R_{p_i}(x, x).$$

It is easy to check that  $G''$  is a  $\mathcal{L}_{\mathcal{P}}$ -template and for all  $xy \in \binom{V}{2}$ ,  $Ch_{G''}(xy) = \{p_1(c_x, c_y), p_2(c_x, c_y)\}$ . Suppose  $\chi'' \in Ch(G'')$ . Then for all  $xy \in \binom{V}{2}$ ,  $\chi''(xy) \in \{p_1(c_x, c_y), p_2(c_x, c_y)\}$ . Let  $M$  be the  $\mathcal{L}$ -structure such that  $M \trianglelefteq_{\chi''} G''$ , that is,  $dom(M) = V$  and for each  $xy \in \binom{V}{2}$ ,  $M \models p_i(x, y)$  if and only if  $\chi''(xy) = p_i(c_x, c_y)$ . Then for all  $xy \in \binom{V}{2}$ ,  $M \models p_1(x, y)$  or  $M \models p_2(x, y)$ . Since there is no way to violate the triangle inequality using distances in  $\{1, 2\}$ ,  $M$  is a metric space. Thus we have shown that for every  $\chi'' \in Ch(G'')$ , there is an  $\mathcal{L}$ -structure  $M \in \mathcal{P}$  such that  $M \trianglelefteq_{\chi''} G''$ . Thus  $G''$  is  $\mathcal{P}$ -random.

The most important  $\mathcal{L}_{\mathcal{H}}$ -structures for the rest of the chapter are  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -templates. We now fix notation for these special  $\mathcal{L}_{\mathcal{H}}$ -structures.

DEFINITION 6.3.23. Suppose  $V$  is a set, and  $n$  is an integer. Then

- $\mathcal{R}(V, \mathcal{H})$  is the set of all  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -templates with domain  $V$  and
- $\mathcal{R}(n, \mathcal{H})$  is the class of all  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -templates of size  $n$ .

In the above notation,  $\mathcal{R}$  is for “random.” Note that if  $\mathcal{H}(n) = \emptyset$  for some  $n$ , then  $\mathcal{R}(n, \mathcal{H}) = \emptyset$ .

## 6.4. Main Results

In this section we state the main results of this chapter. Recall that  $\mathcal{L}$  is a fixed finite relational language of maximum arity  $r \geq 2$ . We now define our generalization of extremal graphs. By convention, set  $\max \emptyset = 0$ .



DEFINITION 6.4.1. Suppose  $\mathcal{H}$  is a collection of finite  $\mathcal{L}$ -structures. Given  $n$ , set

$$ex(n, \mathcal{H}) = \max\{sub(M) : M \in \mathcal{R}(n, \mathcal{H})\}.$$

We say  $M \in \mathcal{R}(n, \mathcal{H})$  is *extremal* if  $sub(M) = ex(n, \mathcal{H})$ . If  $V$  is a set and  $n \in \mathbb{N}$ , then

- $\mathcal{R}_{ex}(V, \mathcal{H})$  is the set of extremal elements of  $\mathcal{R}(V, \mathcal{H})$  and
- $\mathcal{R}_{ex}(n, \mathcal{H})$  is the class of extremal elements of  $\mathcal{R}(n, \mathcal{H})$ .

The main idea is that when  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property,  $ex(n, \mathcal{H})$  is the correct generalization of the extremal number of a graph, and elements of  $\mathcal{R}_{ex}(n, \mathcal{H})$  are the correct generalizations extremal graphs of size  $n$ .

DEFINITION 6.4.2. Suppose  $\mathcal{H}$  is a nonempty collection of finite  $\mathcal{L}$ -structures. When it exists, set

$$\pi(\mathcal{H}) = \lim_{n \rightarrow \infty} ex(n, \mathcal{H})^{1/\binom{n}{r}}$$

Using techniques similar to those in [26] we will show the following.

THEOREM 6.4.3. *If  $\mathcal{H}$  is hereditary  $\mathcal{L}$ -property, then  $\pi(\mathcal{H})$  exists.*

We now state our approximate enumeration theorem in terms of the asymptotic density.

THEOREM 6.4.4 (Enumeration). *Suppose  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Then the following hold.*

- (1) *If  $\pi(\mathcal{H}) > 1$ , then  $|\mathcal{H}_n| = \pi(\mathcal{H})^{\binom{n}{r} + o(n^r)}$ .*
- (2) *If  $\pi(\mathcal{H}) \leq 1$ , then  $|\mathcal{H}_n| = 2^{o(n^r)}$ .*

The notion  $\pi(\mathcal{H})$  is related to many existing notions of asymptotic density for various combinatorial structures, and Theorem 6.4.4 can be seen as generalizing many existing enumeration theorems. Some of these connections will be discussed in Chapter 7. We say a hereditary  $\mathcal{L}$ -property  $\mathcal{H}$  is *fast-growing* if  $\pi(\mathcal{H}) > 1$ . In this case, we informally say  $M \in \mathcal{R}(n, \mathcal{H})$  is *almost extremal* if  $sub(M) \geq ex(n, \mathcal{H})^{1-\epsilon}$  for some small  $\epsilon$ . Our next theorem shows that almost all elements in a fast-growing hereditary  $\mathcal{L}$ -property  $\mathcal{H}$  are close to subpatterns of almost extremal elements of  $\mathcal{R}(n, \mathcal{H})$ . Given  $\epsilon > 0$ ,  $n$ , and a collection  $\mathcal{H}$  of  $\mathcal{L}$ -structures, let

$$E(n, \mathcal{H}) = \{G \in \mathcal{H}_n : G \leq_p M \text{ for some } M \in \mathcal{R}_{ex}(n, \mathcal{H})\} \text{ and}$$

$$E(\epsilon, n, \mathcal{H}) = \{G \in \mathcal{H}_n : G \leq_p M \text{ for some } M \in \mathcal{R}(n, \mathcal{H}) \text{ with } sub(M) \geq ex(n, \mathcal{H})^{1-\epsilon}\}.$$

Given  $\delta > 0$ , let  $E^\delta(n, \mathcal{H})$  and  $E^\delta(\epsilon, n, \mathcal{H})$  denote the set of  $G \in \mathcal{H}_n$  which are  $\delta$ -close to any element of  $E(n, \mathcal{H})$  and  $E(\epsilon, n, \mathcal{H})$ , respectively.

**THEOREM 6.4.5.** *Suppose  $\mathcal{H}$  is a fast-growing hereditary  $\mathcal{L}$ -property. For all  $\epsilon, \delta > 0$  there is  $\beta > 0$  such that for sufficiently large  $n$ ,*

$$\frac{|\mathcal{H}_n \setminus E^\delta(\epsilon, n, \mathcal{H})|}{|\mathcal{H}_n|} \leq 2^{-\beta \binom{n}{r}}.$$

We now define our generalization of a graph stability theorem.

**DEFINITION 6.4.6.** Suppose  $\mathcal{H}$  is a nontrivial collection of  $\mathcal{L}$ -structures. We say  $\mathcal{H}$  has a *stability theorem* if for all  $\delta > 0$  there is  $\epsilon > 0$  and  $N$  such that  $n > N$  implies the following. If  $M \in \mathcal{R}(n, \mathcal{H})$  satisfies  $\text{sub}(M) \geq \text{ex}(n, \mathcal{H})^{1-\epsilon}$ , then  $M$  is  $\delta$ -close to some  $M' \in \mathcal{R}_{\text{ex}}(n, \mathcal{H})$ .

Our next result, Theorem 6.4.7 below, shows that if a fast-growing hereditary  $\mathcal{L}$ -property  $\mathcal{H}$  has a stability theorem, we can strengthen Theorem 6.4.5 to say that almost all elements in  $\mathcal{H}_n$  are approximately subpatterns of elements of  $\mathcal{R}_{\text{ex}}(n, \mathcal{H})$ .

**THEOREM 6.4.7.** *Suppose  $\mathcal{H}$  is a fast growing hereditary  $\mathcal{L}$ -property with a stability theorem. Then for all  $\delta > 0$ , there is a  $\beta > 0$  such that for sufficiently large  $n$ ,*

$$\frac{|\mathcal{H}_n \setminus E^\delta(n, \mathcal{H})|}{|\mathcal{H}_n|} \leq 2^{-\beta \binom{n}{r}}.$$

When one has a good understanding of the structure of elements in  $\mathcal{R}_{\text{ex}}(n, \mathcal{H})$ , Theorem 6.4.7 gives us a good description of the approximate structure asymptotic structure of  $\mathcal{H}_n$ . The main new tool we will use to prove our main theorems is Theorem 6.4.9 below, which is an adaptation of the hypergraph containers theorem to the setting of  $\mathcal{L}$ -structures.

**DEFINITION 6.4.8.** If  $F$  is an  $\mathcal{L}$ -structure, let  $\tilde{F}$  be the set of  $\mathcal{L}_{\mathcal{H}}$ -structures  $M$  such that  $F \triangleleft_p M$ . If  $\mathcal{F}$  is a collection of  $\mathcal{L}$ -structures, let  $\tilde{\mathcal{F}} = \bigcup_{F \in \mathcal{F}} \tilde{F}$ .

**THEOREM 6.4.9.** *Suppose  $0 < \epsilon < 1$  and  $k \geq r$  is an integer. Then there exist positive constants  $c = c(k, r, \mathcal{L}, \epsilon)$  and  $m = m(k, r) > 1$  such that for all sufficiently large  $n$  the following holds. Assume  $\mathcal{F}$  is a collection of finite  $\mathcal{L}$ -structures each of size at most  $k$  and  $\mathcal{B} := \text{Forb}(\mathcal{F}) \neq \emptyset$ . For any  $n$ -element set  $W$ , there is a collection  $\mathcal{C}$  of  $\mathcal{L}_{\mathcal{B}}$ -templates with domain  $W$  such that*

- (1) *For all  $\mathcal{F}$ -free  $\mathcal{L}$ -structures  $M$  with domain  $W$ , there is  $C \in \mathcal{C}$  such that  $M \triangleleft_p C$ ,*
- (2) *For all  $C \in \mathcal{C}$ ,  $\text{prob}(\tilde{\mathcal{F}}, C) \leq \epsilon$  and  $\text{prob}(\mathcal{E}, C) \leq \epsilon$ .*

$$(3) \log |\mathcal{C}| \leq cn^{r-\frac{1}{m}} \log n.$$

We will combine Theorem 6.4.9 with a general version of the graph removal lemma proved by Aroskar and Cummings in [6] to prove a supersaturation theorem for hereditary  $\mathcal{L}$ -properties (Theorem 6.4.10 below), and a version of the hypergraph containers theorem for hereditary  $\mathcal{L}$ -properties (Theorem 6.4.11 below).

**THEOREM 6.4.10 (Supersaturation).** *Suppose  $\mathcal{H}$  is a non-trivial hereditary  $\mathcal{L}$ -property and  $\mathcal{F}$  is as in Observation 6.2.3 so that  $\mathcal{H} = \text{Forb}(\mathcal{F})$ . Then for all  $\delta > 0$  there are  $\epsilon > 0$  and  $K$  such that for all sufficiently large  $n$ , if  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template of size  $n$  such that  $\text{prob}(\tilde{\mathcal{F}}(K) \cup \mathcal{E}(K), M) < \epsilon$ , then*

- (1) *If  $\pi(\mathcal{H}) > 1$ , then  $\text{sub}(M) \leq \text{ex}(n, \mathcal{H})^{1+\delta}$ .*
- (2) *If  $\pi(\mathcal{H}) \leq 1$ , then  $\text{sub}(M) \leq 2^{\delta \binom{n}{r}}$ .*

**THEOREM 6.4.11.** *Suppose  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Then there is  $m = m(\mathcal{H}, r_{\mathcal{L}}) > 1$  such that the following holds. For every  $\delta > 0$  there is a constant  $c = c(\mathcal{H}, \mathcal{L}, \delta)$  such that for all sufficiently large  $n$  there is a set of  $\mathcal{L}_{\mathcal{H}}$ -templates  $\mathcal{C}$  with domain  $[n]$  satisfying the following properties.*

- (1) *For every  $H \in \mathcal{H}_n$ , there is  $C \in \mathcal{C}$  such that  $H \leq_p C$ .*
- (2) *For every  $C \in \mathcal{C}$ , there is  $C' \in \mathcal{R}([n], \mathcal{H})$  such that  $\text{dist}(C, C') \leq \delta$ .*
- (3)  *$\log |\mathcal{C}| \leq cn^{r-\frac{1}{m}} \log n$ .*

## 6.5. Proofs of Main Theorems

In this section we prove our main results using Theorems 6.4.9, 6.4.10, and 6.4.11. For the rest of the section,  $\mathcal{H}$  is a fixed hereditary  $\mathcal{L}$ -property.

**LEMMA 6.5.1.** *Suppose  $N$  is an  $\mathcal{L}$ -structure and  $\tilde{N}$  is the  $\mathcal{L}_{\mathcal{H}}$ -structure such that  $\text{dom}(\tilde{N}) = \text{dom}(N)$  and for each  $\bar{a} \in \text{dom}(\tilde{N})^r$  and  $p(\bar{x}) \in S_r(\mathcal{H})$ ,  $\tilde{N} \models R_p(\bar{a})$  if and only if  $N \models p(\bar{a})$ . Then  $\tilde{N}$  is an  $\mathcal{L}_{\mathcal{H}}$ -template and  $N$  is the unique full subpattern of  $\tilde{N}$ .*

**PROOF.** Let  $V = \text{dom}(N) = \text{dom}(\tilde{N})$ . We first verify  $\tilde{N}$  is an  $\mathcal{L}_{\mathcal{H}}$ -template. By the definition of  $\tilde{N}$ , for all  $A \in \binom{V}{r}$ ,  $\text{Ch}_{\tilde{N}}(A) = \{\text{Diag}^N(A)\}$ . Therefore  $\tilde{N}$  is complete. If  $\bar{a} \in V^r \setminus V^{\perp}$  and  $p \in S_r(\mathcal{H})$ , then because  $p$  is a proper type,  $N \not\models p(\bar{a})$ . Thus by definition of  $\tilde{N}$ ,  $\tilde{N} \models \neg R_p(\bar{a})$  and  $\tilde{N}$  satisfies part (1) of Definition 6.3.4. Suppose  $p(\bar{x}), p'(\bar{x}) \in S_r(\mathcal{H})$  and  $\mu \in \text{Perm}(r)$  are such that  $p(\bar{x}) = p'(\mu(\bar{x}))$ . Then for all  $\bar{a} \in V^r$ ,  $\tilde{N} \models R_p(\bar{a})$  if and only if  $N \models p(\bar{a})$  if and only if  $N \models p'(\mu(\bar{a}))$  if and only if  $\tilde{N} \models R_{p'}(\mu(\bar{a}))$ . Thus  $\tilde{N}$  satisfies part (2) of Definition 6.3.4, so  $\tilde{N}$  is an  $\mathcal{L}_{\mathcal{H}}$ -template. Define

$\chi : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  by setting  $\chi(A) = \text{Diag}^N(A)$  for each  $A \in \binom{V}{r}$ . It is clear that  $\chi$  is a choice function for  $\tilde{N}$ , and  $N \preceq_{\chi} \tilde{N}$ . By definition of  $\tilde{N}$ ,  $\chi$  is the *only* choice function for  $\tilde{N}$ , so any full subpattern of  $\tilde{N}$  must be chosen by  $\chi$ . By Observation 6.3.12,  $\chi$  chooses at most one  $\mathcal{L}$ -structure, so  $N$  is the unique full subpattern of  $\tilde{N}$ .  $\square$

We now prove Theorem 6.4.4. The proof is based on the method of proof in [26].

**Proof of Theorem 6.4.3.** Let  $b_n = \text{ex}(n, \mathcal{H})^{1/\binom{n}{r}}$ . If  $\mathcal{H}$  is trivial, then for sufficiently large  $n$ ,  $\mathcal{R}(n, \mathcal{H}) = \emptyset$  so by convention,  $\text{ex}(n, \mathcal{H}) = 0$ . Thus, for sufficiently large  $n$ ,  $b_n = 0$  and  $\pi(\mathcal{H})$  exists and is equal to zero.

Assume now  $\mathcal{H}$  is nontrivial. We show that the sequence  $b_n$  is bounded below and non-increasing. Since  $\mathcal{H}$  is non-trivial and has the hereditary property,  $\mathcal{H}_n \neq \emptyset$  for all  $n$ . Fix  $n \geq 1$  and choose any  $N \in \mathcal{H}_n$ . Let  $\tilde{N}$  be the  $\mathcal{L}_{\mathcal{H}}$ -structure defined as in Lemma 6.5.1 for  $N$ . Then  $\tilde{N}$  is an  $\mathcal{L}_{\mathcal{H}}$ -template, and its only full subpattern is  $N$ . Since  $N \in \mathcal{H}$ , this implies  $\tilde{N} \in \mathcal{R}(n, \mathcal{H})$  and  $\text{sub}(\tilde{N}) = 1$ . So we have shown  $b_n \geq 1$  for all  $n \geq 1$ .

We now show the  $b_n$  are non-increasing. Fix  $n \geq 2$  and let  $M \in \mathcal{R}(n, \mathcal{H})$  be such that  $\text{sub}(M) \geq 1$ . Let  $V = \text{dom}(M)$ . Fix  $a \in V$  and set  $V_a = V \setminus \{a\}$  and  $M_a = M[V_a]$ . We claim  $M_a \in \mathcal{R}(n-1, \mathcal{H})$ . Because  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template, the definition of  $M_a$  implies  $M_a$  is also an  $\mathcal{L}_{\mathcal{H}}$ -template. Suppose  $\chi \in \text{Ch}(M_a)$ . We want to show there exists  $N_a \in \mathcal{H}$  with  $N_a \preceq_{\chi} M_a$ . Define a function  $\chi' : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  as follows. For  $A \in \binom{V_a}{r}$ , set  $\chi'(A) = \chi(A)$ , and for  $A \in \binom{V}{r} \setminus \binom{V_a}{r}$ , choose  $\chi'(A)$  to be any element of  $\text{Ch}_{M_a}(A) = \text{Ch}_M(A)$  (this is possible since  $M$  is complete). Note that for each  $A \in \binom{V_a}{r}$ ,  $\chi(A) \in \text{Ch}_M(A)$ , so  $\chi' \in \text{Ch}(M)$ . Because  $M$  is  $\mathcal{H}$ -random, there is  $N \in \mathcal{H}$  such that  $N \preceq_{\chi'} M$ . Let  $N_a = N[V_a]$ . Because  $\mathcal{H}$  has the hereditary property and  $N_a \subseteq_{\mathcal{L}} N$ ,  $N_a \in \mathcal{H}$ . For each  $A \in \binom{V_a}{r}$ ,  $\text{Diag}^{N_a}(A) = \text{Diag}^N(A) = \chi'(A) = \chi(A)$ , so  $N_a \preceq_{\chi} M_a$ . Thus we have verified that  $M_a \in \mathcal{R}(n-1, \mathcal{H})$ . By definition of  $b_{n-1}$ , this implies  $\text{sub}(M_a)^{1/\binom{n-1}{r}} \leq b_{n-1}$ . Because  $M_a$  is  $\mathcal{H}$ -random, Lemma 6.3.17 implies it is error-free, so Observation 6.3.19 implies  $\text{sub}(M_a) = \prod_{A \in \binom{V_a}{r}} |\text{Ch}_{M_a}(A)|$ . Then observe that

$$\text{sub}(M) = \left( \prod_{a \in V} \prod_{A \in \binom{V_a}{r}} |\text{Ch}_{M_a}(A)| \right)^{1/(n-r)} = \left( \prod_{a \in V} \text{sub}(M_a) \right)^{1/(n-r)}.$$

Since  $\text{sub}(M_a) \leq b_{n-1}^{\binom{n-1}{r}}$ , this implies

$$\text{sub}(M) \leq \left( \prod_{a \in V} b_{n-1}^{\binom{n-1}{r}} \right)^{1/(n-r)} = b_{n-1}^{n \binom{n-1}{r} / (n-r)} = b_{n-1}^{\binom{n}{r}}.$$

Thus for all  $M \in \mathcal{R}(n, \mathcal{H})$ ,  $sub(M)^{1/\binom{n}{r}} \leq b_{n-1}$ . So by definition,  $b_n \leq b_{n-1}$ .  $\square$

The following observations follow from the proof of Theorem 6.4.3.

OBSERVATION 6.5.2. *Assume  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property.*

- (a) *For all  $n$ ,  $ex(n, \mathcal{H})^{1/\binom{n}{r}} \geq \pi(\mathcal{H})$  (since  $(b_n)_{n \in \mathbb{N}}$  is non-increasing and converges to  $\pi(\mathcal{H})$ ).*  
 (b) *Either  $\mathcal{H}$  is trivial and  $\pi(\mathcal{H}) = 0$  or  $\mathcal{H}$  is non-trivial and  $\pi(\mathcal{H}) \geq 1$ .*

**Proof of Theorem 6.4.4.** Assume  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Recall we want to show the following.

- (1) If  $\pi(\mathcal{H}) > 1$ , then  $|\mathcal{H}_n| = \pi(\mathcal{H})^{\binom{n}{r} + o(n^r)}$ .  
 (2) If  $\pi(\mathcal{H}) \leq 1$ , then  $|\mathcal{H}_n| = 2^{o(n^r)}$ .

Assume first that  $\mathcal{H}$  is trivial. Then by Observation 6.5.2(b),  $\pi(\mathcal{H}) = 0$ . Since  $|\mathcal{H}_n| = 0$  for all sufficiently large  $n$ ,  $|\mathcal{H}_n| = 2^{o(n^2)}$  holds, as desired. Assume now  $\mathcal{H}$  is non-trivial, so  $\pi(\mathcal{H}) \geq 1$  by Observation 6.5.2(b). We show that for all  $0 < \eta < 1$ , either  $\pi(\mathcal{H}) = 1$  and  $|\mathcal{H}_n| \leq 2^{\eta n^r}$  or  $\pi(\mathcal{H}) > 1$  and  $\pi(\mathcal{H})^{\binom{n}{2}} \leq |\mathcal{H}_n| \leq \pi(\mathcal{H})^{\binom{n}{r} + \eta n^r}$ . Fix  $0 < \eta < 1$ . Let  $\mathcal{F}$  be as in Observation 6.2.3 for  $\mathcal{H}$  so that  $\mathcal{H} = \text{Forb}(\mathcal{F})$ . Choose  $\epsilon > 0$  and  $K$  as in Theorem 6.4.10 for  $\delta = \eta/4$ . Replacing  $K$  if necessary, assume  $K \geq r$ . Apply Theorem 6.4.9 to  $\epsilon$  and  $\mathcal{F}(K)$  to obtain  $m = m(K, r) > 1$  and  $c = c(r, K, \epsilon)$ . Assume  $n$  is sufficiently large. Theorem 6.4.9 with  $W = [n]$  and  $\mathcal{B} := \text{Forb}(\mathcal{F}(K))$  implies there is a collection  $\mathcal{C}$  of  $\mathcal{L}_{\mathcal{B}}$ -templates with domain  $[n]$  such that the following hold.

- (i) For all  $\mathcal{F}(K)$ -free  $\mathcal{L}$ -structures  $M$  with domain  $[n]$ , there is  $C \in \mathcal{C}$  such that  $M \trianglelefteq_p C$ ,  
 (ii) For all  $C \in \mathcal{C}$ ,  $\text{prob}(\widetilde{\mathcal{F}(K)}, C) \leq \epsilon$  and  $\text{prob}(\mathcal{E}, C) \leq \epsilon$ .  
 (iii)  $\log |\mathcal{C}| \leq cn^{r - \frac{1}{m}} \log n$ .

Note that because  $K \geq r$ ,  $\mathcal{H} = \text{Forb}(\mathcal{F})$  and  $\mathcal{B} = \text{Forb}(\mathcal{F}(K))$  imply we must have  $S_r(\mathcal{H}) = S_r(\mathcal{B})$ . Consequently all  $\mathcal{L}_{\mathcal{B}}$ -templates are also  $\mathcal{L}_{\mathcal{H}}$ -templates. In particular the elements in  $\mathcal{C}$  are all  $\mathcal{L}_{\mathcal{H}}$ -templates. Therefore, (ii) and Theorem 6.4.10 imply that for all  $C \in \mathcal{C}$ , either  $sub(C) \leq ex(n, \mathcal{H})^{1+\eta/4}$  (case  $\pi(\mathcal{H}) > 1$ ) or  $sub(C) \leq 2^{\eta \binom{n}{r}/4}$  (case  $\pi(\mathcal{H}) = 1$ ). Note every element in  $\mathcal{H}_n$  is  $\mathcal{F}$ -free, so is also  $\mathcal{F}(K)$ -free. This implies by (i) that every element of  $\mathcal{H}_n$  is a full subpattern of some  $C \in \mathcal{C}$ . Therefore we can construct every element in  $\mathcal{H}_n$  as follows.

- Choose a  $C \in \mathcal{C}$ . There are at most  $|\mathcal{C}| \leq 2^{cn^{r - \frac{1}{m}} \log n}$  choices.

- Choose a full subpattern of  $C$ . There are at most  $\text{sub}(C) \leq \text{ex}(n, \mathcal{H})^{1+\eta/4}$  choices if  $\pi(\mathcal{H}) > 1$  and at most  $\text{sub}(C) \leq 2^{\eta \binom{n}{r}/4}$  choices if  $\pi(\mathcal{H}) = 1$ .

This implies

$$(62) \quad |\mathcal{H}_n| \leq \begin{cases} 2^{cn^{r-\frac{1}{m}} \log n \text{ex}(n, \mathcal{H})^{1+\eta/4}} & \text{if } \pi(\mathcal{H}) > 1 \\ 2^{cn^{r-\frac{1}{m}} \log n 2^{\eta \binom{n}{r}/4}} & \text{if } \pi(\mathcal{H}) = 1. \end{cases}$$

If  $\pi(\mathcal{H}) > 1$ , then we may assume  $n$  is sufficiently large so that  $\text{ex}(n, \mathcal{H}) \leq \pi(\mathcal{H})^{(1+\eta/4)\binom{n}{r}}$  (see Observation 6.5.2(a)). Combining this with (62), we have that when  $\pi(\mathcal{H}) > 1$ ,

$$|\mathcal{H}_n| \leq 2^{cn^{r-\frac{1}{m}} \log n \pi(\mathcal{H})^{(1+\eta/4)^2 \binom{n}{r}}} \leq \pi(\mathcal{H})^{\binom{n}{r} + \eta \binom{n}{r}},$$

where the last inequality is because  $\pi(\mathcal{H}) > 1$ ,  $(1 + \eta/4)^2 < 1 + \eta$ , and  $n$  is sufficiently large. If  $\pi(\mathcal{H}) = 1$ , then (62) implies

$$|\mathcal{H}_n| \leq 2^{cn^{r-\frac{1}{m}} \log n 2^{\eta \binom{n}{r}/2}} \leq 2^{\eta \binom{n}{r}},$$

where the last inequality is because  $n$  is sufficiently large. Thus, we have shown  $|\mathcal{H}_n| \leq 2^{\eta n^r}$  when  $\pi(\mathcal{H}) = 1$  and  $|\mathcal{H}_n| \leq \pi(\mathcal{H})^{\binom{n}{r} + \eta n^r}$  when  $\pi(\mathcal{H}) > 1$ . We just have left to show that when  $\pi(\mathcal{H}) > 1$ , then  $|\mathcal{H}_n| \geq \pi(\mathcal{H})^{\binom{n}{r}}$ . This holds because for any  $M \in \mathcal{R}_{\text{ex}}([n], \mathcal{H})$ , all  $\text{ex}(n, \mathcal{H})$  many subpatterns of  $M$  are in  $\mathcal{H}_n$ . Thus  $|\mathcal{H}_n| \geq \text{ex}(n, \mathcal{H}) \geq \pi(\mathcal{H})^{\binom{n}{r}}$ , where the second inequality is by Observation 6.5.2(a). This finishes the proof.  $\square$

We now prove lemmas needed for Theorems 6.4.5 and 6.4.7.

LEMMA 6.5.3. *Suppose  $C$  and  $C'$  are  $\mathcal{L}_{\mathcal{H}}$ -templates with the same domain  $V$ . Then for all  $A \in \binom{V}{r}$ ,  $A \in \text{diff}(C, C')$  if and only if  $\text{Ch}_C(A) \neq \text{Ch}_{C'}(A)$ .*

PROOF. Fix  $A \in \binom{V}{r}$ . Suppose first  $A \in \text{diff}(C, C')$ . Then there is  $p \in S_r(\mathcal{H})$  and an enumeration  $\bar{a}$  of  $A$  such that  $C \models R_p(\bar{a})$  and  $C' \models \neg R_p(\bar{a})$ . This implies  $p(c_{\bar{a}}) \in \text{Ch}_C(A)$ . Suppose by contradiction  $p(c_{\bar{a}}) \in \text{Ch}_{C'}(A)$ . Then there is  $p'(\bar{x}) \in S_r(\mathcal{H})$  and  $\mu \in \text{Perm}(r)$  such that  $p'(\mu(\bar{x})) = p(\bar{x})$  and  $C' \models R_{p'}(\mu(\bar{a}))$ . Because  $C'$  is an  $\mathcal{L}_{\mathcal{H}}$ -template, this implies  $C' \models R_p(\bar{a})$ , a contradiction.

Suppose now  $\text{Ch}_C(A) \neq \text{Ch}_{C'}(A)$ . Then there is  $p(\bar{x}) \in S_r(\mathcal{H})$  and an enumeration  $\bar{a}$  of  $A$  such that  $p(c_{\bar{a}}) \in \text{Ch}_C(A)$  and  $p(c_{\bar{a}}) \notin \text{Ch}_{C'}(A)$ . Since  $p(c_{\bar{a}}) \in \text{Ch}_C(A)$ , by definition there is  $p'(\bar{x})$  and  $\mu \in \text{Perm}(r)$  such that  $p'(\mu(\bar{x})) = p(\bar{x})$  and  $C \models R_{p'}(\mu(\bar{a}))$ . Since  $p(c_{\bar{a}}) \notin \text{Ch}_{C'}(A)$  and  $C'$

is an  $\mathcal{L}_{\mathcal{H}}$ -template,  $C' \models \neg R_{p'}(\mu(\bar{a}))$ . This shows  $qftp_{\mathcal{L}_{\mathcal{H}}}^C(\bar{a}) \neq qftp_{\mathcal{L}_{\mathcal{H}}}^{C'}(\bar{a})$ , so  $A \in \text{diff}(C, C')$ , as desired.  $\square$

LEMMA 6.5.4. *Suppose  $\mathcal{H}$  is a non-trivial hereditary  $\mathcal{L}$ -property. Then there is  $\gamma = \gamma(\mathcal{H}) > 0$  such that for all  $\delta > 0$  and  $n \geq r$ , if  $C$  and  $C'$  are  $\mathcal{L}_{\mathcal{H}}$ -templates with domain  $[n]$  such that  $C'$  is error-free and  $\text{dist}(C, C') \leq \delta$ , then the following holds.*

- (1) *If  $\pi(\mathcal{H}) > 1$ , then  $\text{sub}(C) \leq \text{sub}(C') \text{ex}(n, \mathcal{H})^{\gamma\delta}$ .*
- (2) *If  $\pi(\mathcal{H}) = 1$ , then  $\text{sub}(C) \leq \text{sub}(C') 2^{\gamma\delta \binom{n}{r}}$ .*

PROOF. Fix  $n \geq r$  and assume  $C$  and  $C'$  are  $\mathcal{L}_{\mathcal{H}}$ -templates with domain  $[n]$  such that  $C'$  is error-free and  $\text{dist}(C, C') \leq \delta$ . Then by definition of  $\text{dist}(C, C')$ ,  $|\text{diff}(C, C')| \leq \delta \binom{n}{r}$ . By Lemma 6.5.3,

$$(63) \quad \text{diff}(C, C') = \left\{ A \in \binom{V}{r} : Ch_C(A) \neq Ch_{C'}(A) \right\}.$$

Note that for every  $A \in \binom{V}{r}$ ,  $|Ch_C(A)| \leq |S_r(\mathcal{H})|$  (by definition of  $Ch_C(A)$ ) and  $1 \leq |Ch_{C'}(A)|$  (since  $C'$  is complete). Thus  $\frac{|Ch_C(A)|}{|Ch_{C'}(A)|} \leq |S_r(\mathcal{H})|$ . By Observation 6.3.19 and (63),

$$\begin{aligned} \text{sub}(C) &\leq \prod_{A \in \binom{V}{r}} |Ch_C(A)| = \left( \prod_{A \notin \text{diff}(C, C')} |Ch_{C'}(A)| \right) \left( \prod_{A \in \text{diff}(C, C')} |Ch_C(A)| \right) \\ &= \left( \prod_{A \in \binom{V}{r}} |Ch_{C'}(A)| \right) \left( \prod_{A \in \text{diff}(C, C')} \frac{|Ch_C(A)|}{|Ch_{C'}(A)|} \right). \end{aligned}$$

Combining this with  $\frac{|Ch_C(A)|}{|Ch_{C'}(A)|} \leq |S_r(\mathcal{H})|$  and  $|\text{diff}(C, C')| \leq \delta \binom{n}{r}$  yields

$$(64) \quad \text{sub}(C) \leq \left( \prod_{A \in \binom{V}{r}} |Ch_{C'}(A)| \right) |S_r(\mathcal{H})|^{\delta \binom{n}{r}} = \text{sub}(C') |S_r(\mathcal{H})|^{\delta \binom{n}{r}},$$

where the equality is by Observation 6.3.19 and because  $C'$  is error-free. If  $\pi(\mathcal{H}) > 1$ , choose  $\gamma > 0$  such that  $|S_r(\mathcal{H})| = \pi(\mathcal{H})^\gamma$  (this is possible since  $\pi(\mathcal{H}) > 1$  implies  $|S_r(\mathcal{H})| > 1$ ). Recall from Observation 6.5.2(a) that for all  $n$ ,  $\text{ex}(n, \mathcal{H}) \geq \pi(\mathcal{H})^{\binom{n}{r}}$ . Combining this with our choice of  $\gamma$  and (64), we have

$$\text{sub}(C) \leq \text{sub}(C') |S_r(\mathcal{H})|^{\delta \binom{n}{r}} = \text{sub}(C') \pi(\mathcal{H})^{\gamma\delta \binom{n}{r}} \leq \text{sub}(C') \text{ex}(n, \mathcal{H})^{\gamma\delta}.$$

If  $\pi(\mathcal{H}) = 1$ , choose  $\gamma > 0$  such that  $|S_r(\mathcal{H})| \leq 2^\gamma$  (this is possible since  $\mathcal{H}$  nontrivial implies  $|S_r(\mathcal{H})| \geq 1$ ). Combining our choice of  $\gamma$  with (64) implies

$$\text{sub}(C) \leq \text{sub}(C') |S_r(\mathcal{H})|^{\delta \binom{n}{r}} = \text{sub}(C') 2^{\gamma\delta \binom{n}{r}}.$$

□

LEMMA 6.5.5. *Suppose  $C$  is an  $\mathcal{L}_{\mathcal{H}}$ -template with domain  $W$  of size  $n \geq r$  and  $G \trianglelefteq_p C$ . If  $D \in \mathcal{R}(W, \mathcal{H})$  is such that  $\text{dist}(C, D) \leq \delta$ , then there is  $G' \in \mathcal{H}$  such that  $G' \trianglelefteq_p D$  and  $\text{dist}(G, G') \leq \delta$ .*

PROOF. Fix  $C$  and  $D$  satisfying the hypotheses. Because  $\text{dist}(C, D) \leq \delta$ , we have  $|\text{diff}(C, D)| \leq \delta \binom{n}{r}$ . By Lemma 6.5.3,

$$(65) \quad \text{diff}(C, D) = \left\{ A \in \binom{W}{r} : \text{Ch}_C(A) \neq \text{Ch}_D(A) \right\}.$$

Define a function  $\chi : \binom{W}{r} \rightarrow S_r(C_W)$  as follows. For each  $A \in \binom{W}{r} \setminus \text{diff}(C, D)$ , set  $\chi(A) = \text{Diag}^G(A)$ . For each  $A \in \text{diff}(C, D)$ , choose  $\chi(A)$  to be any element of  $\text{Ch}_D(A)$  (which is nonempty because  $D$  is an  $\mathcal{L}_{\mathcal{H}}$ -template). Since  $G \trianglelefteq_p C$ , for all  $A \in \binom{W}{r}$ ,  $\text{Diag}^G(A) \in \text{Ch}_C(A)$ . Thus, by definition of  $\chi$  and (65), for all  $A \in \binom{W}{r} \setminus \text{diff}(C, D)$ ,  $\chi(A) = \text{Diag}^G(A) \in \text{Ch}_C(A) = \text{Ch}_D(A)$ . For  $A \in \text{diff}(C, D)$ ,  $\chi(A) \in \text{Ch}_D(A)$  by assumption. Thus  $\chi \in \text{Ch}(D)$ . Because  $D$  is  $\mathcal{H}$ -random, there is  $G' \in \mathcal{H}$  such that  $G' \trianglelefteq_{\chi} D$ . We show  $\text{dist}(G, G') \leq \delta$ . By definition of  $\chi$  and since  $G' \trianglelefteq_{\chi} D$ , we have that for all  $A \in \binom{W}{r}$ , if  $A \notin \text{diff}(C, D)$ , then  $\text{Diag}^{G'}(A) = \chi(A) = \text{Diag}^G(A)$ , which implies  $A \notin \text{diff}(G, G')$ . Thus  $\text{diff}(G, G') \subseteq \text{diff}(C, D)$  so  $|\text{diff}(G, G')| \leq \delta \binom{n}{r}$  and  $\text{dist}(G, G') \leq \delta$  by definition. □

**Proof of Theorem 6.4.5.** Let  $\mathcal{H}$  be a fast-growing hereditary  $\mathcal{L}$ -property. Fix  $\epsilon$  and  $\delta > 0$ . Given  $n$ , let  $A(n, \epsilon, \delta) = \mathcal{H}_n \setminus E^{\delta}(\epsilon, n, \mathcal{H})$ . Recall, we want to show there is  $\beta > 0$  such that for sufficiently large  $n$ ,

$$(66) \quad \frac{|A(n, \epsilon, \delta)|}{|\mathcal{H}_n|} \leq 2^{-\beta \binom{n}{r}}.$$

Let  $\gamma > 0$  be as in Lemma 6.5.4 for  $\mathcal{H}$ . Choose  $K > 2r$  sufficiently large so that  $1 - \epsilon + \gamma\delta/K < 1 - \epsilon/2$ . Apply Theorem 6.4.11 to  $\frac{\delta}{K}$  to obtain constants  $c$  and  $m > 1$ . Assume  $n$  is sufficiently large. Then Theorem 6.4.11 implies there is a collection  $\mathcal{C}$  of  $\mathcal{L}_{\mathcal{H}}$ -templates with domain  $[n]$  such that the following hold.

- (i) For every  $H \in \mathcal{H}_n$ , there is  $C \in \mathcal{C}$  such that  $H \trianglelefteq_p C$ .
- (ii) For every  $C \in \mathcal{C}$ , there is  $C' \in \mathcal{R}([n], \mathcal{H})$  such that  $\text{dist}(C, C') \leq \delta$ .
- (iii)  $\log |\mathcal{C}| \leq cn^{r - \frac{1}{m}} \log n$ .

Suppose  $G \in A(n, \epsilon, \delta)$ . By (i), there is  $C \in \mathcal{C}_n$  such that  $G \trianglelefteq_p C$ . By (ii), there is  $M_C \in \mathcal{R}([n], \mathcal{H})$  such that  $\text{dist}(C, M_C) \leq \frac{\delta}{K}$ . By Lemma 6.5.5, there is  $G' \trianglelefteq_p M_C$  such that  $\text{dist}(G, G') \leq \frac{\delta}{K} \leq \delta$ . Since  $\text{dist}(G, G') \leq \delta$ ,  $G' \trianglelefteq_p M_C$ , and  $G \in A(n, \epsilon, \delta) = \mathcal{H}_n \setminus E^{\delta}(\epsilon, n, \mathcal{H})$ , we must have by definition of



$E^\delta(\epsilon, n, \mathcal{H})$  that  $\text{sub}(M_C) < \text{ex}(n, \mathcal{H})^{1-\epsilon}$ . Note  $M_C \in \mathcal{R}([n], \mathcal{H})$  implies  $M_C$  is error-free, so Lemma 6.5.4 and the fact that  $\text{dist}(C, M_C) \leq \delta/K$  imply  $\text{sub}(C) \leq \text{sub}(M_C)\text{ex}(n, \mathcal{H})^{\gamma\delta/K}$ . Combining this with the fact that  $\text{sub}(M_C) < \text{ex}(n, \mathcal{H})^{1-\epsilon}$  we have that

$$\text{sub}(C) < \text{ex}(n, \mathcal{H})^{1-\epsilon}\text{ex}(n, \mathcal{H})^{\gamma\delta/K} = \text{ex}(n, \mathcal{H})^{1-\epsilon+\gamma\delta/K} \leq \text{ex}(n, \mathcal{H})^{1-\epsilon/2},$$

where the second inequality is by assumption on  $K$ . Therefore every  $G \in A(n, \epsilon, \delta)$  can be constructed as follows.

- Choose  $C \in \mathcal{C}_n$  with  $\text{sub}(C) < \text{ex}(n, \mathcal{H})^{1-\epsilon/2}$ . There are at most  $|\mathcal{C}_n| \leq 2^{cn^{r-\frac{1}{m}} \log n}$  ways to do this, where the bound is by (iii). Since  $n$  is large and  $\pi(\mathcal{H}) > 1$ , we may assume  $2^{cn^{r-\frac{1}{m}} \log n} \leq \pi(\mathcal{H})^{\epsilon \binom{n}{r}/4}$ .
- Choose a full subpattern of  $C$ . There are at most  $\text{sub}(C) < \text{ex}(n, \mathcal{H})^{1-\epsilon/2}$  ways to do this.

Combining these bounds we have that  $|A(n, \epsilon, \delta)| \leq \pi(\mathcal{H})^{\epsilon \binom{n}{r}/4} \text{ex}(n, \mathcal{H})^{1-\epsilon/2}$ . Recall that  $|\mathcal{H}_n| \geq \text{ex}(n, \mathcal{H})$  holds since for any  $M \in \mathcal{R}_{\text{ex}}([n], \mathcal{H})$ , all  $\text{ex}(n, \mathcal{H})$ -many full subpatterns of  $M$  are all in  $\mathcal{H}_n$ . Therefore

$$(67) \quad \frac{|A(n, \epsilon, \delta)|}{|\mathcal{H}_n|} \leq \frac{\pi(\mathcal{H})^{\epsilon \binom{n}{r}/4} \text{ex}(n, \mathcal{H})^{1-\epsilon/2}}{\text{ex}(n, \mathcal{H})} = \pi(\mathcal{H})^{\epsilon \binom{n}{r}/4} \text{ex}(n, \mathcal{H})^{-\epsilon/2} \leq \pi(\mathcal{H})^{-\epsilon \binom{n}{r}/4},$$

where the last inequality is because  $\pi(\mathcal{H})^{\binom{n}{r}} \leq \text{ex}(n, \mathcal{H})$ . Therefore we have  $|A(n, \epsilon, \delta)|/|\mathcal{H}_n| \leq 2^{-\beta \binom{n}{r}}$ , where  $\beta = \frac{\epsilon \log \pi(\mathcal{H})}{4 \log 2}$ . Note  $\beta > 0$  since  $\pi(\mathcal{H}) > 1$ .  $\square$

**Proof of Theorem 6.4.7.** Suppose  $\mathcal{H}$  is a fast growing hereditary  $\mathcal{L}$ -property with a stability theorem. Fix  $\delta > 0$ . Given  $n$ , let  $B(n, \delta) = \mathcal{H}_n \setminus E^\delta(n, \mathcal{H})$ . Recall we want to show there is  $\beta > 0$  such that for sufficiently large  $n$ ,

$$\frac{|B(n, \delta)|}{|\mathcal{H}_n|} \leq 2^{-\beta \binom{n}{r}}$$

By Theorem 6.4.5, it suffices to show that there are  $\epsilon_1, \delta_1 > 0$  such that for all sufficiently large  $n$ ,  $B(n, \delta) \subseteq A(n, \epsilon_1, \delta_1) = \mathcal{H}_n \setminus E^{\delta_1}(\epsilon_1, n, \mathcal{H})$ .

Because  $\mathcal{H}$  has a stability theorem, there is  $N$  and  $\epsilon$  such that if  $n \geq N$  and  $H \in \mathcal{R}([n], \mathcal{H})$  satisfies  $\text{sub}(H) \geq \text{ex}(n, \mathcal{H})^{1-\epsilon}$ , then there is  $H' \in \mathcal{R}_{\text{ex}}([n], \mathcal{H})$  with  $\text{dist}(H, H') \leq \frac{\delta}{4}$ . Fix  $n > N$ . We claim  $B(n, \delta) \subseteq A(n, \epsilon, \delta/2)$ . Suppose  $G \in \mathcal{H}_n \setminus A(n, \epsilon, \delta/2)$ . Then  $G$  is  $\delta/2$ -close to some  $G'$  such that  $G' \preceq_p H'$ , for some  $H \in \mathcal{R}([n], \mathcal{H})$  satisfying  $\text{sub}(H) \geq \text{ex}(n, \mathcal{H})^{1-\epsilon}$ . By choice of  $\epsilon$  and  $N$ , this implies there  $H' \in \mathcal{R}_{\text{ex}}([n], \mathcal{H})$  such that  $\text{dist}(H, H') \leq \frac{\delta}{4}$ . Lemma 6.5.5 implies there is some

$G'' \trianglelefteq_p H'$  such that  $\text{dist}(G', G'') \leq 2\frac{\delta}{4} = \frac{\delta}{2}$ . Then

$$\text{dist}(G, G'') \leq \text{dist}(G, G') + \text{dist}(G', G'') \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

This implies that  $G \notin B(n, \delta)$ . So  $\mathcal{H}_n \setminus A(n, \epsilon, \delta/2) \subseteq \mathcal{H}_n \setminus B(n, \delta)$  implies  $B(n, \delta) \subseteq A(n, \epsilon, \delta/2)$ , as desired.  $\square$

### 6.6. Characterization of $\mathcal{H}$ -random $\mathcal{L}_{\mathcal{H}}$ -templates

In this section we give an equivalent characterization for when an  $\mathcal{L}_{\mathcal{H}}$ -structure is an  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -template, where  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. The results in this section will be used in the proofs of our remaining results, Theorems 6.4.9, 6.4.10, and 6.4.11. For the rest of this section,  $\mathcal{H}$  is a fixed nonempty collection of finite  $\mathcal{L}$ -structures.

**DEFINITION 6.6.1.** Define FLAW to be the class of all  $\mathcal{L}_{\mathcal{H}}$ -structures of size  $r$  which are not  $\mathcal{L}_{\mathcal{H}}$ -templates. Elements of FLAW are called flaws.

**LEMMA 6.6.2.** *An  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template if and only if its FLAW-free.*

**PROOF.** Let  $\text{dom}(M) = V$ . It is straightforward from Definition 6.3.4 to check that  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template if and only if for all  $A \in \binom{V}{r}$ ,  $M[A]$  is an  $\mathcal{L}_{\mathcal{H}}$ -template. By definition of FLAW,  $M$  is FLAW-free if and only if for all  $A \in \binom{V}{r}$ ,  $M[A]$  is an  $\mathcal{L}_{\mathcal{H}}$ -template. This finishes the proof.  $\square$

We are now ready to prove the main result of this section.

**PROPOSITION 6.6.3.** *Suppose  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property, and  $\mathcal{F}$  is the class of finite  $\mathcal{L}$ -structures from Observation 6.2.3 such that  $\text{Forb}(\mathcal{F}) = \mathcal{H}$ . Then a complete  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  is  $\mathcal{H}$ -random if and only if  $M$  is  $\tilde{\mathcal{F}}$ -free and error-free.*

**PROOF.** By Observation 6.2.3,  $\mathcal{F}$  is closed under isomorphism. Fix a complete  $\mathcal{L}_{\mathcal{H}}$ -structure  $M$  and let  $V = \text{dom}(M)$ . Suppose first that  $M$  is  $\mathcal{H}$ -random. Then  $M$  is complete and for every choice function  $\chi$  for  $M$ , there is  $N \in \mathcal{H}$  such that  $N \trianglelefteq_{\chi} M$ . This implies by Proposition 6.3.17 that  $M$  is error-free. Suppose by contradiction  $M$  is not  $\tilde{\mathcal{F}}$ -free. Combining the assumption that  $\mathcal{F}$  is closed under isomorphism and the definition of  $\tilde{\mathcal{F}}$ , this implies there is  $B \subseteq V$  and  $F \in \mathcal{F}$  such that  $M[B] \in \tilde{\mathcal{F}}$ . By definition of  $\tilde{\mathcal{F}}$ , there is  $\chi_B \in \text{Ch}(M[B])$  such that  $F \trianglelefteq_{\chi_B} M[B]$ . Define a function  $\chi : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  as follows. For each  $A \in \binom{B}{r}$ , set  $\chi(A) = \chi_B(A)$ . Clearly,  $\chi_B \in \text{Ch}(M[B])$  implies that for all  $A \in \binom{B}{r}$ ,  $\chi_B(A) \in \text{Ch}_M(A)$ . For each  $A \in \binom{V}{r} \setminus \binom{B}{r}$ , define  $\chi(A)$  to be

any element of  $Ch_M(A)$  (this is possible since  $M$  is complete by assumption). By construction,  $\chi \in Ch(M)$ . Because  $M$  is  $\mathcal{H}$ -random, there is  $D \in \mathcal{H}$  such that  $D \sqsubseteq_\chi M$ . By choice of  $\mathcal{F}$ ,  $D \in \mathcal{H}$  implies  $D$  is  $\mathcal{F}$ -free, which implies  $D$  is  $F$ -free since  $F \in \mathcal{F}$ . We claim  $D[B] \cong_{\mathcal{L}} F$ , a contradiction. For each  $A \in \binom{B}{r}$ ,  $D \sqsubseteq_\chi M$ ,  $F \sqsubseteq_{\chi_B} M[B]$ , and the definition of  $\chi$  imply

$$Diag^D(A) = \chi(A) = \chi_B(A) = Diag^F(A).$$

Thus  $Diag(D[B]) = \bigcup_{A \in \binom{B}{r}} Diag^D(A) = \bigcup_{A \in \binom{B}{r}} Diag^F(A) = Diag(F)$  implies  $D[B] \cong_{\mathcal{L}} F$ .

For the converse, suppose  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure which is  $\tilde{\mathcal{F}}$ -free and error-free. Suppose by contradiction  $M$  is not  $\mathcal{H}$ -random. Then there is a choice function  $\chi$  for  $M$  such that there is no  $N \in \mathcal{H}$  with  $N \sqsubseteq_\chi M$ . Since  $M$  is error-free, Proposition 6.3.17 implies there is some  $\mathcal{L}$ -structure  $N$  such that  $N \sqsubseteq_\chi M$ . Thus we must have  $N \notin \mathcal{H}$ . By choice of  $\mathcal{F}$  from Observation 6.2.3,  $N$  is not  $\mathcal{F}$ -free. This along with the fact that  $\mathcal{F}$  is closed under isomorphism implies there is  $B \subseteq V$  such that  $N[B] \in \mathcal{F}$ . But  $N \sqsubseteq_p M$  implies  $N[B] \sqsubseteq_p M[B]$  (this is straightforward to check). Since  $N[B] \in \mathcal{F}$ , this implies  $M[B] \in \tilde{\mathcal{F}}$  by definition of  $\tilde{\mathcal{F}}$ , contradicting that  $M$  is  $\tilde{\mathcal{F}}$ -free.  $\square$

**COROLLARY 6.6.4.** *Suppose  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property, and  $\mathcal{F}$  is the class of finite  $\mathcal{L}$ -structures from Observation 6.2.3 such that  $Forb(\mathcal{F}) = \mathcal{H}$ . Let  $M$  be an  $\mathcal{L}_{\mathcal{H}}$ -structure. Then  $M \in \mathcal{R}(dom(M), \mathcal{H})$  if and only if  $M$  is  $\tilde{\mathcal{F}}$ -free, error-free, and FLAW-free.*

**PROOF.** By definition,  $M \in \mathcal{R}(dom(M), \mathcal{H})$  if and only if  $M$  is an  $\mathcal{H}$ -random  $\mathcal{L}_{\mathcal{H}}$ -template. By Lemma 6.6.2 and Proposition 6.6.3, this holds if and only if  $M$  is  $\tilde{\mathcal{F}}$ -free, error-free, and FLAW-free.  $\square$

### 6.7. Graph Removal and Proofs of Theorems 6.4.10 and 6.4.11.

In this section we will use a version of the graph removal lemma from [6] to prove Theorem 6.4.10 and to prove Theorem 6.4.11 from Theorem 6.4.9. We now state definitions required to quote the graph removal lemma from [6]. Throughout the rest of this section,  $\mathcal{L}_0$  is a fixed finite relational language with  $r_{\mathcal{L}_0} = r$ . Note  $\mathcal{L}_0$  is not necessarily the same as  $\mathcal{L}$ , although we are assuming  $r_{\mathcal{L}_0} = r_{\mathcal{L}} = r$ . Given a partition  $p$  of a finite set  $X$ , let  $\|p\|$  denote the number of parts in  $p$ .

**DEFINITION 6.7.1.** Let  $Index = \{(R, p) : R \in \mathcal{L}_0 \text{ and } p \text{ is a partition of } [\ell] \text{ where } \ell \text{ is the arity of } R\}$ . Suppose  $(R, p) \in Index$  and  $R$  has arity  $\ell$ .

- (1)  $C_p(x_1, \dots, x_\ell)$  is the subtuple of  $(x_1, \dots, x_\ell)$  obtained by replacing each  $x_i$  with  $x_{p(i)}$  where  $p(i) = \min\{j : x_j \text{ is in the same part of } p \text{ as } i\}$ , then deleting all but the first occurrence of each variable in the tuple  $(x_{p(1)}, \dots, x_{p(\ell)})$ .
- (2)  $R_p(C(\bar{x}))$  is the  $\|p\|$ -ary relation obtained from  $R(x_1, \dots, x_\ell)$  by replacing each  $x_i$  with  $x_{p(i)}$  where  $p(i) = \min\{j : x_j \text{ is in the same part of } p \text{ as } i\}$ .
- (3) If  $N$  is an  $\mathcal{L}_0$ -structure, define  $DH_p^R(N) = \{\bar{a} \in \text{dom}(N)^{\|p\|} : N \models R_p(\bar{a})\}$ .

Now we can define the notion of distance between two  $\mathcal{L}_0$ -structures from [6].

DEFINITION 6.7.2. Given  $(R, p) \in \text{Index}$  and  $M, N$  two finite  $\mathcal{L}_0$ -structures with the same universe  $W$ , set

$$d_p^R(M, N) = \frac{|DH_p^R(M) \Delta DH_p^R(N)|}{|W|^{\|p\|}} \quad \text{and set} \quad d(M, N) = \sum_{(R,p) \in \text{Index}} d_p^R(M, N).$$

We will see below in Lemma 6.7.4 that this notion of distance,  $d(M, N)$ , is related to our notion of distance,  $\text{dist}(M, N)$ . We first state the graph removal lemma from [6], as it appears there.

THEOREM 6.7.3 (Theorem 2 from [6]). *Suppose  $\mathcal{A}$  is a collection of finite  $\mathcal{L}_0$ -structures. For every  $\delta > 0$  there exists  $\epsilon > 0$  and  $K$  such that the following holds. For all sufficiently large finite  $\mathcal{L}_0$ -structures  $M$ , if  $\text{prob}(\mathcal{A}(K), M) < \epsilon$ , then there is an  $\mathcal{L}_0$ -structure  $M'$  with  $\text{dom}(M') = \text{dom}(M)$  such that  $d(M', M) < \delta$  and  $\text{prob}(\mathcal{A}, M') = 0$ .*

The following relationship between  $d(M, N)$  and  $\text{dist}(M, N)$  will allow us to restate this graph removal lemma. Given a tuple  $\bar{x} = (x_1, \dots, x_\ell)$ , a *subtuple* of  $\bar{x}$  is any tuple  $\bar{x}' = (x_{i_1}, \dots, x_{i_{\ell'}}$  where  $1 \leq i_1 < \dots < i_{\ell'} \leq \ell$ . If  $\ell' < \ell$ , we say  $\bar{x}'$  is a *proper subtuple* of  $\bar{x}$ , and denote this by  $\bar{x}' \subsetneq \bar{x}$ .

LEMMA 6.7.4. *If  $M$  and  $N$  are  $\mathcal{L}_0$ -structures with the same domain finite  $W$  of size at least  $2r$ , then*

$$\text{dist}(M, N) \leq (r!)^3 2^r d(M, N).$$

PROOF. Let  $n = |W|$ . Note that  $n \geq 2r$  implies for all  $1 \leq \ell \leq r$ ,

$$(68) \quad \frac{n!}{(n-\ell)!} = n \cdot (n-1) \cdots (n-\ell+1) \geq (n-\ell+1)^\ell \geq (n/2)^\ell = n^\ell / 2^\ell.$$

Given  $1 \leq \ell \leq r$ , define

$$\text{diff}^\ell(M, N) = \{\bar{a} \in W^\ell : \text{qftp}_{\mathcal{L}_0}^M(\bar{a}) \neq \text{qftp}_{\mathcal{L}_0}^N(\bar{a}) \text{ and for all } \bar{a}' \subsetneq \bar{a}, \text{qftp}_{\mathcal{L}_0}^M(\bar{a}') = \text{qftp}_{\mathcal{L}_0}^N(\bar{a}')\}.$$

Observe that elements in  $\text{diff}(M, N)$  are *sets* of elements from  $W$ , while elements in  $\text{diff}^\ell(M, N)$  are *tuples* of elements of  $W$ . Clearly if  $A \in \text{diff}(M, N)$ , there is some  $\ell \in [r]$  and a tuple  $\bar{a} \in A^\ell$  such that  $\bar{a} \in \text{diff}^\ell(M, N)$ . Define  $\Psi : \text{diff}(M, N) \rightarrow \bigcup_{\ell \in [r]} \text{diff}^\ell(M, N)$  to be any map which sends each  $A \in \text{diff}(M, N)$  to some such tuple. Given  $\ell \in [r]$  and  $\bar{a} = (a_1, \dots, a_\ell) \in \text{diff}^\ell(M, N)$ , note that

$$\Psi^{-1}(\bar{a}) \subseteq \{A \in \binom{W}{r} : \cup \bar{a} \subseteq A\}.$$

Since the right hand side has size  $\binom{n-\ell}{r-\ell}$ , we have that for all  $\bar{a} \in \text{diff}^\ell(M, N)$ ,  $|\Psi^{-1}(\bar{a})| \leq \binom{n-\ell}{r-\ell}$ .

For each  $\ell \in [r]$ , we now define a map  $f_\ell : \text{diff}^\ell(M, N) \rightarrow \bigcup_{(R,p) \in \text{Index}, ||p||=\ell} DH_p^R(M) \Delta DH_p^R(N)$ . Let  $\bar{a} \in \text{diff}^\ell(M, N)$ . Since  $\bar{a} \in \text{diff}^\ell(M, N)$ , there is a relation  $R(x_1, \dots, x_t) \in \mathcal{L}_0$  and a map  $h : [\ell] \rightarrow [t]$  such that  $M \models R(a_{h(1)}, \dots, a_{h(t)})$  and  $N \models \neg R(a_{h(1)}, \dots, a_{h(t)})$  or vice versa. If  $h$  is not surjective, then some permutation of  $C_p(a_{h(1)}, \dots, a_{h(t)})$  is a proper subtuple  $\bar{a}'$  of  $\bar{a}$  such that  $qftp_{\mathcal{L}_0}^M(\bar{a}') \neq qftp_{\mathcal{L}_0}^N(\bar{a}')$ . But this contradicts that  $\bar{a} \in \text{diff}^\ell(M, N)$ . Thus  $h$  is surjective. Let  $p$  be the partition of  $[t]$  with parts  $h^{-1}(\{1\}), \dots, h^{-1}(\{\ell\})$ . Since  $h$  is surjective, the parts are all nonempty, so  $||p|| = \ell$ . Then by definition,  $C_p(a_{h(1)}, \dots, a_{h(t)}) \in DH_p^R(M) \Delta DH_p^R(N)$ . Define  $f_\ell(\bar{a}) = C_p(a_{h(1)}, \dots, a_{h(t)})$ . Observe that  $\cup C_p(a_{h(1)}, \dots, a_{h(t)}) = \cup \bar{a}$  implies

$$f_\ell^{-1}(f_\ell(\bar{a})) \subseteq \{\bar{b} \in W^\ell : \cup \bar{b} = \cup \bar{a}\},$$

so  $|f_\ell^{-1}(f_\ell(\bar{a}))| \leq \ell!$ . Thus  $f_\ell : \text{diff}^\ell(M, N) \rightarrow \bigcup_{(R,p) \in \text{Index}, ||p||=\ell} DH_p^R(M) \Delta DH_p^R(N)$  and

$$(69) \quad \text{for all } \bar{c} \in \bigcup_{(R,p) \in \text{Index}, ||p||=\ell} DH_p^R(M) \Delta DH_p^R(N), \quad |f_\ell^{-1}(\bar{c})| \leq \ell!.$$

Now define a map  $\beta : \text{diff}(M, N) \rightarrow \bigcup_{(R,p) \in \text{Index}} DH_p^R(M) \Delta DH_p^R(N)$  as follows. Given  $A \in \text{diff}(M, N)$ , apply  $\Psi$  to obtain  $\Psi(A) \in \text{diff}^\ell(M, N)$  for some  $\ell \in [r]$ . Then define

$$\beta(\bar{a}) := f_\ell(\Psi(\bar{a})) \in \bigcup_{(R,p) \in \text{Index}, ||p||=\ell} DH_p^R(M) \Delta DH_p^R(N).$$

Suppose  $\bar{c} \in \bigcup_{(R,p) \in \text{Index}} DH_p^R(M) \Delta DH_p^R(N)$  and  $\ell := |\bar{c}|$ . Then  $\bar{c} \in DH_p^R(M) \Delta DH_p^R(N)$  for some  $(R, p) \in \text{Index}$  with  $||p|| = \ell$ . By definition of  $\beta$ ,  $\beta^{-1}(\bar{c}) = \Psi^{-1}(f_\ell^{-1}(\bar{c}))$ . Combining (69) and the fact that  $|\Psi^{-1}(\bar{a})| \leq \binom{n-\ell}{r-\ell}$  for all  $\bar{a} \in \text{diff}^\ell(M, N)$ , we have that

$$|\beta^{-1}(\bar{c})| = |\Psi^{-1}(f_\ell^{-1}(\bar{c}))| \leq \binom{n-\ell}{r-\ell} \ell!.$$

This shows that  $|\text{diff}(M, N)| \leq \sum_{\ell \in [r]} \sum_{(R,p) \in \text{Index}, ||p||=\ell} \binom{n-\ell}{r-\ell} \ell! |DH_p^R(M) \Delta DH_p^R(N)|$ . Dividing both sides of this by  $\binom{n}{r}$ , we obtain the following.

$$(70) \quad \text{dist}(M, N) \leq \sum_{\ell \in [r]} \sum_{(R,p) \in \text{Index}, ||p||=\ell} \frac{\binom{n-\ell}{r-\ell} \ell!}{\binom{n}{r}} |DH_p^R(M) \Delta DH_p^R(N)|.$$

Note that for all  $1 \leq \ell < r$ ,

$$\frac{\binom{n-\ell}{r-\ell} \ell!}{\binom{n}{r}} = \frac{n!}{(n-\ell)!} \frac{\ell! r!}{(r-\ell)!} \leq \frac{2^\ell}{n^\ell} \frac{\ell! r!}{(r-\ell)!} < \frac{(r!)^3 2^r}{n^\ell},$$

where the first inequality is by (68) and the last is because  $\ell < r$ . If  $\ell = r$ , then

$$\frac{\binom{n-\ell}{r-\ell} \ell!}{\binom{n}{r}} = \frac{r!}{\binom{n}{r}} = \frac{(r!)^3 (n-r)!}{n!} \leq \frac{(r!)^3 2^r}{n^r},$$

where the inequality is by (68). Thus for all  $\ell \in [r]$ ,  $\frac{\binom{n-\ell}{r-\ell} \ell!}{\binom{n}{r}} \leq \frac{(r!)^3 2^r}{n^r}$ . Combining this with (70) yields

$$\text{dist}(M, N) \leq (r!)^3 2^r \sum_{\ell \in [r]} \sum_{(R,p) \in \text{Index}, ||p||=\ell} \frac{|DH_p^R(M) \Delta DH_p^R(N)|}{n^\ell} = (r!)^3 2^r d(M, N).$$

□

We will use the following version of Theorem 6.7.3 adapted to our notation.

**THEOREM 6.7.5.** *Suppose  $\mathcal{A}$  is a collection of finite  $\mathcal{L}_0$ -structures. For every  $\delta > 0$  there exists  $\epsilon > 0$  and  $K$  such that the following holds. For all sufficiently large finite  $\mathcal{L}_0$ -structures  $M$ , if  $\text{prob}(\mathcal{A}(K), M) < \epsilon$ , then there is an  $\mathcal{L}_0$ -structure  $M'$  with  $\text{dom}(M') = \text{dom}(M)$  such that  $\text{dist}(M', M) < \delta$  and  $\text{prob}(\mathcal{A}, M') = 0$ .*

**PROOF.** Fix  $\delta > 0$ . Let  $\delta' = \frac{\delta}{(r!)^3 2^r}$  and choose  $K = K(\delta')$  and  $\epsilon = \epsilon(\delta')$  by applying Theorem 6.7.3 to  $\delta'$  and  $\mathcal{A}$ . Suppose  $n$  is sufficiently large so that Theorem 6.7.3 applies to structures of size  $n$ . Suppose  $M$  is an  $\mathcal{L}_0$ -structure of size  $n$  such that  $\text{prob}(\mathcal{A}(K), M) < \epsilon$ . Then Theorem 6.7.3 implies there is an  $\mathcal{L}_0$ -structure  $M'$  with  $\text{dom}(M') = \text{dom}(M)$  such that  $d(M', M) < \delta'$  and  $\text{prob}(\mathcal{A}, M') = 0$ . Combining this with Lemma 6.7.4, we have  $\text{dist}(M', M) \leq (r!)^3 2^r d(M', M) < (r!)^3 2^r \delta' = \delta$ . □

**Proof of Theorem 6.4.10.** Let  $\mathcal{H}$  be a nontrivial hereditary  $\mathcal{L}$ -property and let  $\mathcal{F}$  be as in Observation 6.2.3 so that  $\mathcal{H} = \text{Forb}(\mathcal{F})$ . Recall we want to show that for all  $\delta > 0$ , there are  $\epsilon > 0$  and  $K$  such that for sufficiently large  $n$  the following holds. For any  $\mathcal{L}_{\mathcal{H}}$ -template  $M$  of size  $n$ , if  $\text{prob}(\tilde{\mathcal{F}}(K) \cup \mathcal{E}(K), M) \leq \epsilon$  then

- (1) If  $\pi(\mathcal{H}) > 1$ , then  $\text{sub}(M) \leq \text{ex}(n, \mathcal{H})^{1+\delta}$ .
- (2) If  $\pi(\mathcal{H}) \leq 1$ , then  $\text{sub}(M) \leq 2^{\delta \binom{n}{r}}$ .

Fix  $\delta > 0$ . Let  $\mathcal{A} = \tilde{\mathcal{F}} \cup \mathcal{E} \cup \text{FLAW}$ . Apply Lemma 6.5.4 to  $\mathcal{H}$  to obtain  $\gamma > 0$ . Apply Theorem 6.7.5 to obtain  $K$  and  $\epsilon$  for  $\delta/2\gamma$  and  $\mathcal{A}$ . Suppose  $n$  is sufficiently large and  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template of size  $n$  satisfying  $\text{prob}(\tilde{\mathcal{F}}(K) \cup \mathcal{E}(K), M) < \epsilon$ . Because  $M$  is an  $\mathcal{L}_{\mathcal{H}}$ -template, Lemma 6.6.2 implies for all  $B \in \text{FLAW}$ ,  $\text{prob}(B, M) = 0$ . These facts imply  $\text{prob}(\mathcal{A}(K), M) < \epsilon$ , so by Theorem 6.7.5, there is an  $\mathcal{L}_{\mathcal{H}}$ -structure  $M'$  with  $\text{dom}(M) = \text{dom}(M')$  such that  $\text{prob}(\mathcal{A}, M') = 0$  and  $\text{dist}(M, M') \leq \delta/2\gamma$ . Since  $\text{prob}(\mathcal{A}, M') = 0$ , Corollary 6.6.4 implies  $M' \in \mathcal{R}(n, \mathcal{H})$ . Thus  $\text{sub}(M') \leq \text{ex}(n, \mathcal{H})$  holds by definition of  $\text{ex}(n, \mathcal{H})$ . Combining this with Lemma 6.5.4 (note  $M' \in \mathcal{R}(n, \mathcal{H})$  implies  $M'$  is error-free), we have the following.

- (1) If  $\pi(\mathcal{H}) > 1$ , then  $\text{sub}(M) \leq \text{sub}(M') \text{ex}(n, \mathcal{H})^{\gamma(\delta/2\gamma)} = \text{sub}(M') \text{ex}(n, \mathcal{H})^{\delta/2} \leq \text{ex}(n, \mathcal{H})^{1+\delta/2}$ .
- (2) If  $\pi(\mathcal{H}) = 1$ , then  $\text{sub}(M) \leq \text{sub}(M') 2^{\gamma(\delta/2\gamma) \binom{n}{r}} = \text{sub}(M') 2^{\delta \binom{n}{r}/2} \leq \text{ex}(n, \mathcal{H}) 2^{\delta/2 \binom{n}{r}}$ .

We are done in the case where  $\pi(\mathcal{H}) > 1$ . If  $\pi(\mathcal{H}) = 1$ , assume  $n$  is sufficiently large so that  $\text{ex}(n, \mathcal{H}) \leq 2^{\delta/2 \binom{n}{r}}$ . Then (2) implies  $\text{sub}(M) \leq 2^{\delta \binom{n}{r}}$  as desired.  $\square$

**Proof of Theorem 6.4.11 from Theorem 6.4.9.** Suppose  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Let  $\mathcal{F}$  be the class of finite  $\mathcal{L}$ -structures from Observation 6.2.3 so that  $\mathcal{H} = \text{Forb}(\mathcal{F})$ . Then for each  $n$ ,  $\mathcal{H}_n$  is the set of all  $\mathcal{F}$ -free  $\mathcal{L}$ -structures with domain  $[n]$ . Let  $\mathcal{A} = \tilde{\mathcal{F}} \cup \mathcal{E} \cup \text{FLAW}$ . Fix  $\delta > 0$  and choose  $K$  and  $\epsilon$  as in Theorem 6.7.5 for  $\delta$  and the family  $\mathcal{A}$ . By replacing  $K$  if necessary, assume  $K \geq r$ . Apply Theorem 6.4.9 to  $\mathcal{B} := \mathcal{F}(K)$  to obtain  $c = c(K, r, \mathcal{L}, \epsilon)$ ,  $m = m(K, r)$ . Observe the choice of  $K$  depended on  $\mathcal{H}$  and  $r = r_{\mathcal{L}}$ , so  $m = m(\mathcal{H}, r_{\mathcal{L}})$ . Let  $n$  be sufficiently large. Then Theorem 6.4.9 applied to  $W = [n]$  implies there is a collection  $\mathcal{C}$  of  $\mathcal{L}_{\mathcal{B}}$ -templates with domain  $[n]$  such that the following hold.

- (i) For all  $\mathcal{F}(K)$ -free  $\mathcal{L}$ -structures  $M$  with domain  $[n]$ , there is  $C \in \mathcal{C}$  such that  $M \preceq_p C$ .
- (ii) For all  $C \in \mathcal{C}$ ,  $\text{prob}(\widetilde{\mathcal{F}(K)}, C) \leq \epsilon$  and  $\text{prob}(\mathcal{E}(K), C) \leq \epsilon$ .
- (iii)  $\log |\mathcal{C}| \leq cn^{r - \frac{1}{m}} \log n$ .

We show this  $\mathcal{C}$  satisfies the conclusions of Theorem 6.4.11 with  $c$ ,  $m$  and  $\delta$ . Note that because  $K \geq r$ ,  $S_r(\mathcal{H}) = S_r(\mathcal{B})$ , so all  $\mathcal{L}_{\mathcal{B}}$ -templates are also  $\mathcal{L}_{\mathcal{H}}$ -templates. In particular the elements in  $\mathcal{C}$  are all  $\mathcal{L}_{\mathcal{H}}$ -templates. Clearly (iii) implies part (3) of Theorem 6.4.11 holds. For part (1), since any  $H \in \mathcal{H}_n$  is  $\mathcal{F}$ -free, it is also  $\mathcal{F}(K)$ -free, so (i) implies there is  $C \in \mathcal{C}$  such that  $H \preceq_p C$ . This shows part (1) of Theorem 6.4.11 holds. For part (2), fix  $C \in \mathcal{C}$ . Since  $C$  is an  $\mathcal{L}_{\mathcal{H}}$ -template, Lemma 6.6.2 implies

$\text{prob}(G, C) = 0$  for all  $G \in \text{FLAW}$ . Then (ii) implies that for all  $G \in \widetilde{\mathcal{F}(K)} \cup \mathcal{E}$ ,  $\text{prob}(G, C) \leq \epsilon$ . Since  $\widetilde{\mathcal{F}(K)} = \tilde{\mathcal{F}}(K)$ , these facts imply that for all  $G \in \mathcal{A}(K)$ ,  $\text{prob}(G, C) \leq \epsilon$ . Thus Theorem 6.7.5 implies there is an  $\mathcal{L}_{\mathcal{H}}$ -structure  $C'$  with  $\text{dom}(C) = \text{dom}(C') = [n]$  such that  $\text{dist}(C, C') \leq \delta$  and  $\text{prob}(\mathcal{A}, C') = 0$ . Since  $\text{prob}(\mathcal{A}, C') = 0$ ,  $C'$  is a FLAW-free,  $\tilde{\mathcal{F}}$ -free, and error-free  $\mathcal{L}_{\mathcal{H}}$ -structure with domain  $[n]$ , so by Corollary 6.6.4,  $C' \in \mathcal{R}([n], \mathcal{H})$ . This finishes the proof.  $\square$

## 6.8. A Reduction

We have now proved all the results in this chapter except Theorem 6.4.9. In this section we prove Theorem 6.4.9 by reducing it to another result, Theorem 6.8.14 (which is proved in Section 6.9).

**6.8.1. Preliminaries.** In this subsection we give preliminaries necessary for the statement of Theorem 6.8.14. Many of these notions are similar to definitions from Section 6.3. However, we will see that our proofs necessitate this more syntactic treatment.

DEFINITION 6.8.1. Suppose  $C$  is a set of constants and  $\sigma \subseteq S_r(C)$ .

- $V(\sigma) = \{c \in C : c \text{ appears in some } p(\bar{c}) \in \sigma\}$ .
- Given  $A \in \binom{V(\sigma)}{r}$ , let  $Ch_\sigma(A) = \{p(\bar{c}) \in \sigma : \cup \bar{c} = A\}$ . Elements of  $Ch_\sigma(A)$  are *choices for A*.
- We say  $\sigma$  is *complete* if  $Ch_\sigma(A) \neq \emptyset$ , for all  $A \in \binom{V(\sigma)}{r}$ .

EXAMPLE 6.8.2. Let  $\mathcal{L}$  and  $\mathcal{P}$  be as in Example 6.3.2 (i.e. metric spaces with distances in  $[3]$ ). Let  $W = \{u, v, w\}$  and  $\sigma = \{p_1(c_u, c_v), p_2(c_u, c_v), p_2(c_u, c_w)\} \subseteq S_2(C_W, \mathcal{P})$ . Then  $V(\sigma) = \{c_u, c_v, c_w\}$  and it is easy to check  $Ch_\sigma(c_u c_v) = \{p_1(c_u, c_v), p_2(c_u, c_v)\}$ ,  $Ch_\sigma(c_u c_w) = \{p_2(c_u, c_w)\}$ , and  $Ch_\sigma(c_v c_w) = \emptyset$ . Observe, this  $\sigma$  is not complete.

DEFINITION 6.8.3. Suppose  $C$  is a set of  $n$  constants and  $\sigma \subseteq S_r(C)$ . Given  $m \leq n$ ,  $\sigma$  is a *syntactic  $m$ -diagram* if  $|V(\sigma)| = m$  and for all  $A \in \binom{V(\sigma)}{r}$ ,  $|Ch_\sigma(A)| = 1$ .

EXAMPLE 6.8.4. Let  $\mathcal{L}$  and  $\mathcal{P}$  be as in Example 6.8.2, and let  $W = \{t, u, v, w\}$  be a set of size 4. Set  $\sigma' = \{p_1(c_u, c_v), p_2(c_u, c_w), p_3(c_v, c_w)\} \subseteq S_2(C_W, \mathcal{P})$ . Then  $V(\sigma') = \{c_u, c_v, c_w\}$  and  $Ch_{\sigma'}(c_u c_v) = \{p_1(c_u, c_v)\}$ ,  $Ch_{\sigma'}(c_u c_w) = \{p_2(c_u, c_w)\}$ , and  $Ch_{\sigma'}(c_v c_w) = \{p_3(c_v, c_w)\}$ . This shows  $\sigma'$  is a syntactic 3-diagram.

Observe that if  $\sigma$  is a syntactic  $m$ -diagram, then by definition,  $|V(\sigma)| = m$  and  $|\sigma| = \binom{m}{r}$ . Given a tuple of constants  $\bar{c} = (c_1, \dots, c_k)$ , a first-order language  $\mathcal{L}_0$  containing  $\{c_1, \dots, c_k\}$ , and an  $\mathcal{L}_0$ -structure  $M$ , let  $\bar{c}^M$  denote the tuple  $(c_1^M, \dots, c_k^M) \in \text{dom}(M)^k$ .



DEFINITION 6.8.5. Suppose  $C$  is a set of constants and  $\sigma \subseteq S_r(C)$ .

- (1) If  $M$  is an  $\mathcal{L} \cup V(\sigma)$ -structure, write  $M \models \sigma^M$  if  $M \models p(\bar{c}^M)$  for all  $p(\bar{c}) \in \sigma$ . Call  $\sigma$  *satisfiable* if there exists an  $\mathcal{L} \cup V(\sigma)$ -structure  $M$  such that  $M \models \sigma^M$ .
- (2) If  $M$  is an  $\mathcal{L} \cup C$ -structure, the *type-diagram* of  $M$  is the set

$$Diag^{tp}(M, C) = \{p(\bar{c}) \in S_r(C) : M \models p(\bar{c}^M)\}.$$

Suppose that  $M$  is an  $\mathcal{L}$ -structure with  $dom(M) = W$ . The *canonical type-diagram* of  $M$  is the set

$$Diag^{tp}(M) = \{p(c_{\bar{a}}) \in S_r(C_W) : M \models p(\bar{a})\}.$$

In other words,  $Diag^{tp}(M) = Diag^{tp}(M, C_W)$  where  $M$  is considered with its natural  $\mathcal{L} \cup C_W$ -structure. Observe that  $Diag^{tp}(M)$  is always a syntactic  $|dom(M)|$ -diagram. The difference between  $Diag^{tp}(M)$  and  $Diag(M)$  is that elements of  $Diag^{tp}(M)$  are types (with constants plugged in for the variables) while the elements of  $Diag(M)$  are formulas (with constants plugged in for the variables). Clearly  $Diag(M)$  and  $Diag^{tp}(M)$  are logically equivalent.

EXAMPLE 6.8.6. Let  $\mathcal{L}$  and  $\mathcal{P}$  be as in Example 6.8.4. Let  $M$  be the  $\mathcal{L}$ -structure with domain  $W = \{u, v, w\}$  such that  $M \models p_1(u, v) \cup p_2(u, w) \cup p_3(v, w)$ . Then  $Diag^{tp}(M) = \{p_1(c_u, c_v), p_2(c_u, c_w), p_3(c_v, c_w)\}$ , while  $Diag(M)$  is the set of all  $\mathcal{L} \cup C_W$ -sentences implied by  $p_1(c_u, c_v) \cup p_2(c_u, c_w) \cup p_3(c_v, c_w)$ .

We now make a few observations which will be used in the remainder of the chapter.

OBSERVATION 6.8.7. *Suppose  $M$  is an  $\mathcal{L}$ -structure with domain  $W$  of size  $n$ . Then the following hold.*

- (1) *Suppose  $m \leq n$ ,  $\sigma \subseteq S_r(C_W)$  is a syntactic  $m$ -diagram, and  $N$  is an  $\mathcal{L} \cup V(\sigma)$ -structure of size  $m$ . Then  $N \models \sigma^N$  if and only if  $\sigma = Diag^{tp}(N, V(\sigma))$ .*
- (2) *Suppose  $N$  is an  $\mathcal{L} \cup C_W$ -structure of size  $n$  and  $N \models Diag^{tp}(M)$ . Then  $M \cong_{\mathcal{L}} N$ .*
- (3) *If  $\sigma \subseteq S_r(C_W)$  and  $Diag^{tp}(M) \subseteq \sigma$ , then  $\sigma$  is complete.*

PROOF. (1): Suppose first  $\sigma = Diag^{tp}(N, V(\sigma))$ . Then by Definition 6.8.5,  $N \models \sigma^N$ . Conversely, suppose  $N \models \sigma^N$ . By Definition 6.8.5, this implies  $\sigma \subseteq Diag^{tp}(N, V(\sigma))$ . To show the reverse inclusion, suppose  $p(c_{\bar{a}}) \in Diag^{tp}(N, V(\sigma))$ . By Definition 6.8.5,  $N \models p(\bar{c}^N)$ . Let  $A = \cup \bar{c}^N \in \binom{dom(N)}{r}$  (since  $p \in S_r(\mathcal{L})$  is proper and  $N \models p(\bar{c}^N)$ , the coordinates of  $\bar{c}^N$  must all be distinct). Since  $\sigma$  is a syntactic  $m$ -diagram,  $|Ch_{\sigma}(A)| = 1$ , so there is  $p'(\bar{x}) \in S_r(\mathcal{L})$  and  $\mu \in Perm(r)$  such that

$p'(\mu(\bar{c})) \in \sigma$ . Since  $N \models \sigma^N$ , this implies  $N \models p'(\mu(\bar{c}^N))$ . Clearly  $N \models p(\bar{c}^N)$  and  $N \models p'(\mu(\bar{c}^N))$  implies  $p(\bar{c}^N) = p'(\mu(\bar{c}^N))$ . So we have  $p(\bar{c}) \in \sigma$  as desired.

(2): Clearly the map  $f : W \rightarrow \text{dom}(N)$  sending  $a \mapsto c_a^N$  is an elementary embedding of  $M$  into  $N$ . Since by assumption,  $M$  and  $N$  both have size  $n$ , it must be a bijection, and thus an  $\mathcal{L}$ -isomorphism.

(3): Observe that for each  $A \in \binom{C^W}{r}$ ,  $\text{Diag}^M(A) \in \text{Ch}_\sigma(A)$  implies  $\text{Ch}_\sigma(A) \neq \emptyset$ .  $\square$

DEFINITION 6.8.8. Suppose  $\mathcal{A}$  is a collection of finite  $\mathcal{L}$ -structures and  $C$  is a set of constant symbols.

- (1) We say  $\sigma \subseteq S_r(C)$  is  $\mathcal{A}$ -satisfiable if there is an  $\mathcal{L} \cup V(\sigma)$ -structure  $M$  such that  $M \upharpoonright_{\mathcal{L}} \in \mathcal{A}$  and  $M \models \sigma^M$ .
- (2) Define  $\text{Diag}^{tp}(\mathcal{A}, C) = \{\sigma \subseteq S_r(C) : \sigma \text{ is a syntactic type diagram which is } \mathcal{A}\text{-satisfiable}\}$ .
- (3) Given  $\sigma \subseteq S_r(C)$ , set  $\text{Span}(\sigma) = \{\sigma' \subseteq \sigma : \sigma' \text{ is a syntactic type diagram}\}$ .

EXAMPLE 6.8.9. Let  $\mathcal{L}$  and  $\mathcal{P}$  be as in Example 6.8.4. Let  $C = \{c_1, c_2, c_3\}$  be a set of three constant symbols. Then  $\sigma \subseteq S_2(C, \mathcal{P})$  is a syntactic 3-diagram if and only if  $\sigma = \{p_i(c_1, c_2), p_j(c_1, c_3), p_k(c_2, c_3)\}$  for some  $i, j, k \in [3]$ . Clearly such a  $\sigma$  is  $\mathcal{P}$ -satisfiable if and only if  $|i - j| \leq k \leq i + j$ , that is, if and only if the numbers  $i, j, k$  do not violate the triangle inequality. Thus  $\text{Diag}^{tp}(\mathcal{P}, C)$  consists of sets of the form  $\sigma = \{p_i(c_1, c_2), p_j(c_1, c_3), p_k(c_2, c_3)\}$  where  $i, j, k \in [3]$  satisfy  $|i - j| \leq k \leq i + j$ .

Suppose now that  $\sigma = \{p_1(c_1, c_2), p_2(c_1, c_2), p_3(c_1, c_2), p_1(c_2, c_3), p_1(c_1, c_3)\}$ . Then  $\text{Span}(\sigma)$  consists of the following syntactic 3-diagrams.

- (1)  $\{p_1(c_1, c_2), p_1(c_2, c_3), p_1(c_1, c_3)\}$ .
- (2)  $\{p_2(c_1, c_2), p_1(c_2, c_3), p_1(c_1, c_3)\}$ .
- (3)  $\{p_3(c_1, c_2), p_1(c_2, c_3), p_1(c_1, c_3)\}$ .

Observe that (1) and (2) are  $\mathcal{P}$ -satisfiable, while (3) is not.

For the rest of this subsection,  $\mathcal{H}$  is a fixed collection of finite  $\mathcal{L}$ -structures.

LEMMA 6.8.10. Suppose  $X \subseteq W$  are finite sets,  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure with domain  $X$ , and  $\chi \in \text{Ch}(M)$ . Set  $\sigma := \{\chi(A) : A \in \binom{V}{r}\} \subseteq S_r(C_W, \mathcal{H})$ . Then

- (1)  $\sigma$  is a syntactic  $|X|$ -diagram.
- (2) If  $F \trianglelefteq_X M$  then  $\sigma = \text{Diag}^{tp}(F)$ .

PROOF. Clearly  $V(\sigma) = C_X$ . Let  $m = |C_X|$ . Note  $\binom{C^X}{r} = \{C_A : A \in \binom{X}{r}\}$ . By definition of  $\sigma$ , for each  $A \in \binom{X}{r}$ ,  $\{\chi(A)\} = \text{Ch}_\sigma(C_A)$ . Thus  $|\text{Ch}_\sigma(C_A)| = 1$  for all  $A \in \binom{X}{r}$  and  $\sigma$  is a syntactic

$m$ -diagram. This shows 1 holds. For 2, suppose  $F \trianglelefteq_\chi M$ . This means  $\text{dom}(F) = X$  and for all  $A \in \binom{X}{r}$ ,  $\text{Diag}^F(A) = \chi(A)$ . Clearly this implies  $F \models \sigma^F$ , where  $F$  is considered with its natural  $C_X$ -structure. Part 1 of Observation 6.8.7 then implies  $\sigma = \text{Diag}^{tp}(F)$ .  $\square$

DEFINITION 6.8.11. Given an integer  $\ell$  and a set of constants  $C$ , set

$$\text{Err}_\ell(C) = \{\sigma \subseteq S_r(C) : |V(\sigma)| = \ell \text{ and } \sigma \text{ is complete and unsatisfiable}\}.$$

We call the elements of  $\text{Err}_\ell(C)$  *syntactic  $C$ -errors of size  $\ell$* .

EXAMPLE 6.8.12. Let  $\mathcal{L} = \{R, E\}$  and  $\mathcal{P}$  be as in Example 6.3.15. Let  $C = \{c_1, c_2, c_3, c_4\}$  be a set of constants. Recall from Example 6.8.12, that  $p_1(c_2, c_3, c_4) \cup p_2(c_1, c_2, c_3)$  is unsatisfiable. Therefore an example of a syntactic  $C$ -error of size 4 is the set  $\{p_1(c_2, c_3, c_4), p_2(c_1, c_2, c_3), p_1(c_1, c_3, c_4), p_1(c_1, c_2, c_4)\}$ .

LEMMA 6.8.13. *Suppose  $W$  is finite a set,  $r + 1 \leq \ell < 2r$ , and  $M$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure which is an error of size  $\ell$  and with domain  $X \subseteq W$ . Then there is a choice function  $\chi \in \text{Ch}(M)$  such that  $\{\chi(A) : A \in \binom{X}{r}\}$  is a syntactic  $C_W$ -error of size  $\ell$ .*

PROOF. Since  $M$  is an error of size  $\ell$  then there are  $\bar{a}_1, \bar{a}_2 \in V^r$  such that  $\cup \bar{a}_1 \cup \cup \bar{a}_2 = V$  and  $p_1(\bar{x}), p_2(\bar{x}) \in S_r(\mathcal{H})$  such that  $M \models R_{p_1}(\bar{a}_1) \wedge R_{p_2}(\bar{a}_2)$  but  $p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$  is unsatisfiable. Define a function  $\chi : \binom{V}{r} \rightarrow S_r(C_V, \mathcal{H})$  as follows. Set  $\chi(\cup \bar{a}_1) = p(c_{\bar{a}_1})$  and  $\chi(\cup \bar{a}_2) = p(c_{\bar{a}_2})$ . For all other  $A \in \binom{V}{r}$  choose any  $\chi(A) \in \text{Ch}_M(A)$  (this is possible because  $M$  is a complete). By construction,  $\chi \in \text{Ch}(M)$ . By part 1 of Lemma 6.8.10,  $\sigma := \{\chi(A) : A \in \binom{V}{r}\}$  is a syntactic  $\ell$ -diagram. Because  $\sigma$  contains  $p_1(c_{\bar{a}_1}) \cup p_2(c_{\bar{a}_2})$ , it is unsatisfiable. Thus by definition,  $\sigma$  is a syntactic  $C_W$ -error of size  $\ell$ .  $\square$

**6.8.2. Proof of Theorem 6.4.9.** In this section we state Theorem 6.8.14 and use it to prove Theorem 6.4.9.

THEOREM 6.8.14. *Let  $0 < \epsilon < 1$ . For all  $k \geq r$ , there is a positive constant  $c = c(k, r, \mathcal{L}, \epsilon)$  and  $m = m(k, r) > 1$  such that for all sufficiently large  $n$  the following holds. Suppose  $\mathcal{F}$  is a collection of finite  $\mathcal{L}$ -structures, each of size at most  $k$ , and  $\mathcal{H} = \text{Forb}(\mathcal{F}) \neq \emptyset$ . For any set  $W$  of size  $n$ , there is a set  $\Sigma \subseteq \mathcal{P}(S_r(C_W, \mathcal{H}))$  such that the following hold.*

- (1) *For all  $\mathcal{F}$ -free  $\mathcal{L}$ -structures  $M$  with domain  $W$ , there is  $\sigma \in \Sigma$  such that  $\text{Diag}^{tp}(M) \subseteq \sigma$ .*
- (2) *For all  $\sigma \in \Sigma$  the following hold. For each  $1 \leq \ell \leq k$ ,  $|\text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell}$ , and for each  $r + 1 \leq \ell \leq 2r$ ,  $|\text{Err}_\ell(C_W) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell}$ .*

$$(3) \log |\Sigma| \leq cn^{r-\frac{1}{m}} \log n.$$

Given a collection  $\mathcal{H}$  of finite  $\mathcal{L}$ -structures, we now define a way of building an  $\mathcal{L}_{\mathcal{H}}$ -template from a complete subset of  $S_r(C_W, \mathcal{H})$ .

DEFINITION 6.8.15. Suppose  $\mathcal{H}$  is a nonempty collection of  $\mathcal{L}$ -structures,  $W$  is a set, and  $\sigma \subseteq S_r(C_W, \mathcal{H})$  is such that  $V(\sigma) = C_W$ . Define an  $\mathcal{L}_{\mathcal{H}}$ -structure  $D_\sigma$  as follows. Set  $\text{dom}(D_\sigma) = W$  and for each  $\bar{a} \in W^r$ , define  $D_\sigma \models R_p(\bar{a})$  if and only if  $p(c_{\bar{a}}) \in Ch_\sigma(\cup \bar{a})$ .

In the notation of Definition 6.8.15, note that for all  $A \in \binom{W}{r}$ ,  $Ch_{D_\sigma}(A) = Ch_\sigma(A)$  (here  $Ch_{D_\sigma}(A)$  is in the sense of Definition 6.3.6 and  $Ch_\sigma(A)$  is in the sense of Definition 6.8.1). We now prove two lemmas.

LEMMA 6.8.16. *Suppose  $\mathcal{F}$  is collection of finite  $\mathcal{L}$ -structures and  $\mathcal{H} = \text{Forb}(\mathcal{F}) \neq \emptyset$ . For any set  $W$  and complete  $\sigma \subseteq S_r(C_W, \mathcal{H})$ ,  $D_\sigma$  is an  $\mathcal{L}_{\mathcal{H}}$ -template.*

PROOF. First, observe that  $D_\sigma$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure since for each  $A \in \binom{W}{r}$ ,  $Ch_{D_\sigma}(A) = Ch_\sigma(C_A)$ , and  $Ch_\sigma(C_A) \neq \emptyset$  because  $\sigma$  is complete by assumption (in the sense of Definition 6.8.1). Suppose now  $\bar{a} \in W^r \setminus W^\perp$ . Then because  $S_r(\mathcal{H})$  contains only proper types, there is no  $p(\bar{x}) \in S_r(\mathcal{H})$  such that  $p(c_{\bar{a}}) \in S_r(C_W, \mathcal{H})$ . Thus  $D_\sigma \models \neg R_p(\bar{a})$  for all  $p(\bar{x}) \in S_r(\mathcal{H})$ , so  $D_\sigma$  satisfies part (1) of Definition 6.3.4. Suppose  $p(\bar{x}), p'(\bar{x}) \in S_r(\mathcal{H})$  and  $\mu \in \text{Perm}(r)$  are such that  $p(\bar{x}) = p'(\mu(\bar{x}))$ . Suppose  $a \in W^\perp$ . Then by definition of  $D_\sigma$ ,  $D_\sigma \models R_p(\bar{a})$  if and only if  $p(c_{\bar{a}}) \in \sigma$ . Since  $p(c_{\bar{a}}) = p'(c_{\mu(\bar{a})})$ ,  $p(c_{\bar{a}}) \in \sigma$  if and only if  $p'(c_{\mu(\bar{a})}) \in \sigma$ . By definition of  $D_\sigma$ ,  $p'(c_{\mu(\bar{a})}) \in \sigma$  if and only if  $D_\sigma \models R_{p'}(\mu(\bar{a}))$ . Thus we've shown  $D_\sigma \models R_p(\bar{a})$  if and only if  $D_\sigma \models R_{p'}(\mu(\bar{a}))$ , so  $D_\sigma$  satisfies part (2) of Definition 6.3.4. This finishes the verification that  $D_\sigma$  is an  $\mathcal{L}_{\mathcal{H}}$ -template.  $\square$

LEMMA 6.8.17. *Suppose  $k \geq r$ ,  $W$  is a finite set,  $\mathcal{H}$  is a nonempty collection of finite  $\mathcal{L}$ -structures, and  $\sigma \subseteq S_r(C_W, \mathcal{H})$  is complete. Suppose  $\mathcal{F}$  is a collection of finite  $\mathcal{L}$ -structures, each of size at most  $k$ . Then for each  $1 \leq \ell \leq k$ , there is an injection*

$$\Phi : \text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma) \rightarrow \text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cap \text{Span}(\sigma).$$

and for each  $r+1 \leq \ell \leq 2r$ , there is an injection

$$\Theta : \text{cop}(\mathcal{E}(\ell), D_\sigma) \rightarrow \text{Err}_\ell(C_W) \cap \text{Span}(\sigma).$$

PROOF. Without loss of generality, assume  $\mathcal{F}$  is closed under isomorphism (we can do this because it does not change either the sets  $\text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma)$  or  $\text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cap \text{Span}(\sigma)$ ). Suppose  $1 \leq \ell \leq k$

and  $G \in \text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma)$ . Then  $G \subseteq_{\mathcal{L}_{\mathcal{H}}} D_\sigma$  and  $G \cong_{\mathcal{L}_{\mathcal{H}}} B$ , for some  $B \in \tilde{\mathcal{F}}(\ell)$ . It is straightforward to check that since  $\mathcal{F}$  is closed under isomorphism, this implies  $G \in \tilde{\mathcal{F}}(\ell)$ . So without loss of generality we may assume that  $B = G$ . This implies there is some  $F \in \mathcal{F}(\ell)$  such that  $F \trianglelefteq_p G$ . Choose any such  $F$  and let  $\chi \in Ch(G)$  be such that  $F \trianglelefteq_\chi G$ . Define  $\Phi(G) = \{\chi(A) : A \in \binom{\text{dom}(G)}{r}\}$ . By part 2 of Lemma 6.8.10,  $\Phi(G) = \text{Diag}^{tp}(F)$ . Thus by definition,  $\Phi(G) \in \text{Diag}^{tp}(\mathcal{F}(\ell), C_W)$ . By definition of  $D_\sigma$  and because  $\chi \in Ch(G)$ ,  $G \subseteq_{\mathcal{L}_{\mathcal{H}}} D_\sigma$  implies  $\Phi(G) \subseteq \sigma$ , so  $\Phi(G) \in \text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cap \text{Span}(\sigma)$ , as desired. To see that  $\Phi$  is injective, note that for all  $G \in \text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma)$ ,  $V(\Phi(G)) = \text{dom}(G)$ . Therefore if  $G_1 \neq G_2 \in \text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma)$ ,  $\text{dom}(G_1) \neq \text{dom}(G_2)$  implies  $V(\Phi(G_1)) \neq V(\Phi(G_2))$ , so  $\Phi(G_1) \neq \Phi(G_2)$ .

Suppose now  $r + 1 \leq \ell \leq 2r$  and  $G \in \text{cop}(\mathcal{E}(\ell), D_\sigma)$ . Then  $G$  is a complete  $\mathcal{L}_{\mathcal{H}}$ -structure which is an error of size  $\ell$ . Lemma 6.8.13 implies there is  $\chi \in Ch(G)$  such that  $\{\chi(A) : A \in \binom{\text{dom}(G)}{r}\}$  is a syntactic  $C_W$ -error of size  $\ell$ . Set  $\Theta(G) = \{\chi(A) : A \in \binom{\text{dom}(G)}{r}\}$ . Then this shows  $\Theta(G) \in \text{Err}_\ell(C_W)$ . By definition of  $D_\sigma$  and because  $\chi \in Ch(G)$ ,  $G \subseteq_{\mathcal{L}_{\mathcal{H}}} D_\sigma$  implies  $\Theta(G) \subseteq \sigma$ , so  $\Theta(G) \in \text{Err}_\ell(C_W) \cap \text{Span}(\sigma)$ , as desired. To see that  $\Theta$  is injective, note that for all  $G \in \text{cop}(\mathcal{E}(\ell), D_\sigma)$ ,  $V(\Theta(G)) = \text{dom}(G)$ . Therefore if  $G_1 \neq G_2 \in \text{cop}(\mathcal{E}(\ell), D_\sigma)$ ,  $\text{dom}(G_1) \neq \text{dom}(G_2)$  implies  $V(\Theta(G_1)) \neq V(\Theta(G_2))$ , so  $\Theta(G_1) \neq \Theta(G_2)$ .  $\square$

**Proof of Theorem 6.4.9 from Theorem 6.8.14.** Let  $0 < \epsilon < 1$  and let  $k \geq r$  be an integer. Choose the constants  $c = c(k, r, \mathcal{L}, \epsilon)$  and  $m = m(k, r)$  to be the ones given by Theorem 6.8.14. Suppose  $\mathcal{F}$  is a collection of finite  $\mathcal{L}$ -structures, each of size at most  $k$ , and  $\mathcal{B} := \text{Forb}(\mathcal{F}) \neq \emptyset$ . Suppose  $n$  is sufficiently large and  $W$  is a set of size  $n$ . Theorem 6.8.14 applied to  $\mathcal{B}$  implies there exists a set  $\Sigma \subseteq \mathcal{P}(S_r(C_W, \mathcal{B}))$  such that the following hold.

- (i) For all  $\mathcal{F}$ -free  $\mathcal{L}$ -structures  $M$  with domain  $W$ , there is  $\sigma \in \Sigma$  such that  $\text{Diag}^{tp}(M) \subseteq \sigma$ .
- (ii) For all  $\sigma \in \Sigma$  the following hold. For each  $1 \leq \ell \leq k$ ,  $|\text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell}$ ,  
and for each  $r + 1 \leq \ell \leq 2r$ ,  $|\text{Err}_\ell(C_W) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell}$ .
- (iii)  $\log |\Sigma| \leq cn^{r - \frac{1}{m}} \log n$ .

Set  $\mathcal{D} = \{D_\sigma : \sigma \in \Sigma\}$ , where for each  $\sigma \in \Sigma$ ,  $D_\sigma$  is the  $\mathcal{L}_{\mathcal{B}}$ -structure from Definition 6.8.15. We claim this  $\mathcal{D}$  satisfies conclusions of Theorem 6.4.9. First note (i) and part 3 of Observation 6.8.7 imply that every  $\sigma \in \Sigma$  is complete in the sense of Definition 6.8.1. Therefore Lemma 6.8.16 implies each  $D_\sigma \in \mathcal{D}$  is an  $\mathcal{L}_{\mathcal{B}}$ -template. We now verify parts (1)-(3) of Theorem 6.4.9 hold for this  $\mathcal{D}$ .

Clearly  $|\mathcal{D}| \leq |\Sigma|$ , so (iii) implies part (3) of Theorem 6.4.9 is satisfied. Suppose now  $M$  is an  $\mathcal{F}$ -free  $\mathcal{L}$ -structure with  $\text{dom}(M) = W$ . By (i), there is  $\sigma \in \Sigma$  such that  $\text{Diag}^{tp}(M) \subseteq \sigma$ . We

claim that  $M \leq_p D_\sigma$ . Let  $A \in \binom{W}{r}$  and suppose  $p(\bar{x}) \in S_r(\mathcal{H})$  is such that  $M \models p(\bar{a})$  for some enumeration  $\bar{a}$  of  $A$ . Then  $\text{Diag}^M(A) = p(c_{\bar{a}}) \in \text{Diag}^{tp}(M) \subseteq \sigma$  implies by definition of  $D_\sigma$ ,  $D_\sigma \models R_p(\bar{a})$ , so  $p(c_{\bar{a}}) \in \text{Ch}_{D_\sigma}(A)$ . This shows  $M \leq_p D_\sigma$ .  $M \leq_p D_\sigma$  because by assumption  $\text{dom}(M) = \text{dom}(D_\sigma) = W$ . Thus part (1) of Theorem 6.4.9 is satisfied.

We now verify part (2) of Theorem 6.4.9. Let  $D_\sigma \in \mathcal{D}$ . We need to show  $\text{prob}(\tilde{\mathcal{F}}, D_\sigma) \leq \epsilon$  and  $\text{prob}(\mathcal{E}, D_\sigma) \leq \epsilon$ . For each  $1 \leq \ell \leq k$ , we have

$$|\text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma)| \leq |\text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell},$$

where the first inequality is because of Lemma 6.8.17 and the second inequality is by (ii). This implies that for all  $1 \leq \ell \leq k$ ,  $|\text{cop}(\tilde{\mathcal{F}}(\ell), D_\sigma)| \leq \epsilon \binom{n}{\ell}$ , so  $\text{prob}(\tilde{\mathcal{F}}(\ell), D_\sigma) \leq \epsilon$ . Since every element in  $\tilde{\mathcal{F}}$  has size at most  $k$ , we have  $\text{prob}(\tilde{\mathcal{F}}, D_\sigma) \leq \epsilon$ . Similarly, for each  $r+1 \leq \ell \leq 2r$ ,

$$|\text{cop}(\mathcal{E}(\ell), D_\sigma)| \leq |\text{Err}_\ell(C_W) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell},$$

where the first inequality is by Lemma 6.8.17 and the second inequality is by (ii). This implies for all  $r+1 \leq \ell \leq 2r$ ,  $|\text{cop}(\mathcal{E}(\ell), D_\sigma)| \leq \epsilon \binom{n}{\ell}$ , so  $\text{prob}(\mathcal{E}(\ell), D_\sigma) \leq \epsilon$ . Since every element in  $\mathcal{E}$  has size at least  $r+1$  and at most  $2r$ , we have  $\text{prob}(\mathcal{E}, D_\sigma) \leq \epsilon$ . This finishes the proof.  $\square$

### 6.9. Applying Hypergraph Containers to Prove Theorem 6.8.14

In this section we prove Theorem 6.8.14. We will use the hypergraph containers theorem. We begin with a definition.

DEFINITION 6.9.1. Suppose  $K$  is a positive integer and  $\mathcal{A}$  is a collection of finite  $\mathcal{L}$ -structures each of size at most  $K$ . Set

$$\text{cl}_K(\mathcal{A}) = \{M : M \text{ is an } \mathcal{L}\text{-structure of size } K \text{ such that } \text{prob}(\mathcal{A}, M) > 0\}.$$

Observe that in the above notation, an  $\mathcal{L}$ -structure of size at least  $K$  is  $\mathcal{A}$ -free if and only if it is  $\text{cl}_K(\mathcal{A})$ -free. We now state a key lemma.

LEMMA 6.9.2. *Assume  $k \geq r$  and  $\mathcal{F}$  is a nonempty collection of  $\mathcal{L}$ -structures, each of size at most  $k$ . Suppose  $\mathcal{H} := \text{Forb}(\mathcal{F}) \neq \emptyset$  and  $W$  is a finite set of size  $n$  for some  $n \geq k$ . Fix  $0 < \epsilon < 1/2$ . Suppose  $\sigma \subseteq S_r(C_W, \mathcal{H})$  is complete and satisfies  $V(\sigma) = C_W$ . Then*

$$|(\text{Diag}^{tp}(\text{cl}_k(\mathcal{F}), C_W) \cup \text{Err}_k(C_W)) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{k}$$

implies that for all  $\ell \leq k$ ,  $|(Diag^{tp}(\mathcal{F}(\ell), C_W) \cup Err_\ell(C_W)) \cap Span(\sigma)| \leq \epsilon \binom{n}{\ell}$ .

PROOF. For  $\ell < k$ , set  $\Gamma(\ell) = (Diag^{tp}(\mathcal{F}(\ell), C_W) \cup Err_\ell(C_W)) \cap Span(\sigma)$  and let

$$\Gamma(k) = (Diag^{tp}(cl_k(\mathcal{F}), C_W) \cup Err_k(C_W)) \cap Span(\sigma).$$

We want to show that  $|\Gamma(k)| \leq \epsilon \binom{n}{k}$  implies that for all  $\ell \in [k]$ ,  $|\Gamma(\ell)| \leq \epsilon \binom{n}{\ell}$ . Fix  $\ell \in [k]$ . We claim the following holds.

$$(71) \quad \text{For all } S_0 \in \Gamma(\ell), |\{S_1 \in \Gamma(k) : S_0 \subseteq S_1\}| \geq \binom{n-\ell}{r-\ell}.$$

Suppose  $S_0 \in \Gamma(\ell)$ . Consider the following procedure for constructing a set  $S_1 \subseteq S_r(C_W, \mathcal{H})$ .

- Choose  $X \in \binom{C_W}{k}$  such that  $V(S_0) \subseteq X$ . There are  $\binom{n-\ell}{k-\ell}$  choices.
- For each  $A \in \binom{X}{r} \setminus \binom{V(S_0)}{r}$ , choose some  $p_A \in Ch_\sigma(A)$  (this is possible since  $\sigma$  is complete).
- Set  $S_1 = S_0 \cup \{p_A : A \in \binom{X}{r} \setminus \binom{V(S_0)}{r}\}$ .

Suppose  $S_1$  is constructed from  $S_0$  in this way. We claim  $S_1 \in \Gamma(k)$ . By construction,  $S_1$  is a syntactic  $k$ -diagram and  $S_1 \subseteq \sigma$ , so  $S_1 \in Span(\sigma)$ . If  $S_1$  is unsatisfiable, then by definition  $S_1 \in Err_k(C_W) \cap Span(\sigma) \subseteq \Gamma(k)$ , so we are done. Suppose now  $S_1$  is satisfiable and  $M$  is an  $\mathcal{L} \cup V(S_1)$ -structure such that  $M \models S_1^M$ . Let  $N = M[V(S_0)^M]$ . Then considered as an  $\mathcal{L} \cup V(S_0)$ -structure,  $N \models S_0^N$ , so part 1 of Observation 6.8.7 implies  $Diag^{tp}(N) = S_0$ . On the other hand,  $S_0 \in Diag^{tp}(\mathcal{F}(\ell), C_W)$  implies there is  $F \in \mathcal{F}(\ell)$  which can be made into an  $\mathcal{L} \cup V(S_0)$ -structure such that  $F \models S_0^F$ . Part (2) of Observation 6.8.7 then implies  $N \cong_{\mathcal{L}} F$ . Since  $M$  has size  $k$  and contains  $N$  as a substructure, there is  $F' \in cl_k(\mathcal{F})$  such that  $M \cong_{\mathcal{L}} F'$ . Clearly this implies that there is an expansion of  $F'$  into an  $\mathcal{L} \cup V(S_1)$ -structure such that  $F' \models S_1^{F'}$ . Thus by definition,  $S_1 \in Diag^{tp}(cl_k(\mathcal{F}), C_W)$ , and we have shown that  $S_1$  is in  $\Gamma(k)$ . Observe that every distinct choice of  $X$  produces a distinct  $S_1$ , this construction produces at least  $\binom{n-\ell}{k-\ell}$  distinct  $S_1 \in \Gamma(k)$  such that  $S_0 \subseteq S_1$ , so we have proved (71) holds for all  $\ell \in [k]$ . Then (71) implies the following procedure constructs every element in  $S_0 \in \Gamma(\ell)$  at least  $\binom{n-\ell}{k-\ell}$  times:

- Choose  $S_1 \in \Gamma(k)$ . There are  $|\Gamma(k)|$  choices.
- Choose  $S_0 \subseteq S_1$  such that  $S_0 \in \Gamma(\ell)$  (if one exists). There are at most  $\binom{V(S_1)}{\ell} = \binom{k}{\ell}$  choices.

This gives an upper bound of  $|\Gamma(\ell)| \leq |\Gamma(k)| \binom{k}{\ell}$ , and since each element of  $\Gamma(\ell)$  gets counted at least  $\binom{n-\ell}{k-\ell}$  times, we have

$$|\Gamma(\ell)| \leq |\Gamma(k)| \binom{k}{\ell} / \binom{n-\ell}{k-\ell} \leq \epsilon \binom{n}{k} \binom{k}{\ell} / \binom{n-\ell}{k-\ell} = \epsilon \binom{n}{\ell},$$

where the second inequality is because  $|\Gamma(k)| \leq \epsilon \binom{n}{k}$  by assumption.  $\square$

We now present a computational lemma which will be used in the proof of Theorem 6.8.14.

LEMMA 6.9.3. *For all integers  $2 \leq x < y$ ,  $m(y, x) := \max \left\{ \frac{\binom{\ell}{x} - 1}{\ell - x} : x < \ell \leq y \right\} > 1$ .*

PROOF. We show that for all  $2 \leq x < y$ ,  $\binom{y}{x} > y - x + 1$ . Since by definition,  $m(y, x) \geq \frac{\binom{y}{x} - 1}{y - x}$ , this will imply  $m(y, x) > 1$ , as desired. Fix  $x \geq 2$ . We induct on  $t$  where  $y = x + t$ . Suppose first  $y = x + 1$ . Then  $\binom{y}{x} = \frac{(x+1)!}{x!} = x + 1$ . By assumption on  $x$ ,  $x + 1 \geq 3 > 2 = y - x + 1$ . Assume now that  $y > x + 1$  and suppose by induction the claim holds for  $y - 1$ . Then  $\binom{y}{x} = \frac{(y-1)!y}{x!(y-x-1)!(y-x)} = \binom{y-1}{x} \frac{y}{y-x}$ . By our induction hypothesis,

$$\binom{y-1}{x} \frac{y}{y-x} \geq ((y-1-x) + 1) \binom{y-1}{y-x} = (y-x) \frac{y}{y-x} = y > y - x + 1,$$

where the last inequality is because  $x \geq 2$ . Thus  $\binom{y}{x} > y - x + 1$ , as desired.  $\square$

### Defining the Hypergraph.

We now give a procedure for defining a special hypergraph given a collection of  $\mathcal{L}$ -structures and a sufficiently large finite set satisfying certain properties. Assume we are given the following.

- A nonempty collection,  $\mathcal{F}$ , of finite  $\mathcal{L}$ -structures, each of size at most  $k$ , where  $k \geq r$  is an integer.
- A set  $W$  of size  $n$ , where  $n \geq k$  is an integer.

Let  $\mathcal{H}$  be the class of all finite  $cl_k(\mathcal{F})$ -free  $\mathcal{L}$ -structures. Define the hypergraph  $H = H(\mathcal{F}, W)$  as follows.

$$V(H) = S_r(C_W, \mathcal{H}) \text{ and}$$

$$E(H) = \text{Diag}^{tp}(cl_k(\mathcal{F}), C_W) \cup \text{Err}_k(C_W).$$

We now make a few observations about  $H$ . First, note that the edges of  $H$  are syntactic  $k$ -diagrams, so  $H$  is a  $\binom{k}{r}$ -uniform hypergraph. By definition  $|V(H)| = \binom{n}{r} |S_r(\mathcal{H})|$ . If  $X$  and  $X'$  are both in  $\binom{C_W}{k}$ , then since relabeling constants does not change the satisfiability properties of a collection of



$\mathcal{L} \cup C_W$ -sentences, we must have  $|Diag^{tp}(Cl_k(\mathcal{F}), X) \cup Err_k(X)| = |Diag^{tp}(Cl_k(\mathcal{F}), X') \cup Err_k(X')|$ .

Therefore, the following is well defined.

DEFINITION 6.9.4. Let  $\alpha = \alpha(\mathcal{F})$  be such that for all  $X \in \binom{C_W}{k}$ ,  $|Diag^{tp}(Cl_k(\mathcal{F}), X) \cup Err_k(X)| = \alpha$ .

We claim that  $|E(H)| = \alpha \binom{n}{k}$ . Indeed, any  $\sigma \in E(H)$  can be constructed as follows.

- Choose  $X \in \binom{C_W}{k}$ . There are  $\binom{n}{k}$  choices.
- Choose an element  $\sigma \in Diag^{tp}(Cl_k(\mathcal{F}), C_W) \cup Err_k(C_W)$  such that  $V(\sigma) = X$ , i.e. choose an element  $\sigma \in Diag^{tp}(Cl_k(\mathcal{F}), X) \cup Err_k(X)$ . There are  $\alpha$  choices.

Each of these choices lead to distinct subsets  $\sigma \in E(H)$ , so this shows  $|E(H)| = \alpha \binom{n}{k}$ . Note that because  $\mathcal{F} \neq \emptyset$ ,  $\alpha \geq 1$ . On the other hand, there are at most  $|S_r(\mathcal{H})| \binom{k}{r}$  syntactic  $k$ -diagrams  $\sigma$  with  $V(\sigma) = X$ , so  $\alpha \leq |S_r(\mathcal{H})| \binom{k}{r} \leq |S_r(\mathcal{L})| \binom{k}{r}$ . We now make a key observation about this hypergraph.

PROPOSITION 6.9.5. *For any  $\mathcal{F}$ -free  $\mathcal{L}$ -structure  $M$  with domain  $W$ ,  $Diag^{tp}(M)$  is an independent subset of  $V(H)$ .*

PROOF. Suppose towards a contradiction that  $Diag^{tp}(M)$  contains an edge  $\sigma \in E(H)$ . Then  $\sigma$  is a syntactic  $k$ -diagram which is either in  $Err_k(C_W)$  or  $Diag^{tp}(cl_k(\mathcal{F}), C_W)$ . Clearly  $\sigma \notin Err_k(C_W)$ , since  $M \models \sigma^M$  implies  $\sigma$  is satisfiable. Thus  $\sigma \in Diag^{tp}(cl_k(\mathcal{F}), C_W)$ . So there is an  $\mathcal{L} \cup V(\sigma)$ -structure  $B$  such that  $B \upharpoonright_{\mathcal{L}} \in cl_k(\mathcal{F})$  and  $Diag^{tp}(B, V(\sigma)) = \sigma$ . Let  $A = \{a : c_a \in V(\sigma)\} \subseteq W$  and let  $N = M[A]$ . Suppose  $p(c_{\bar{a}}) \in \sigma$ . Since  $\sigma \subseteq Diag^{tp}(M)$ ,  $M \models p(\bar{a})$ . Since  $N \subseteq_{\mathcal{L}} M$  and  $\cup \bar{a} \subseteq A = dom(N)$ , we have  $N \models p(\bar{a})$ . This shows that with its canonical  $\mathcal{L} \cup V(\sigma)$ -structure,  $N \models \sigma^N$ . Since  $\sigma$  is a syntactic  $k$ -diagram and  $N$  has size  $k$ , part 1 of Observation 6.8.7 implies  $\sigma = Diag^{tp}(N)$ . Now  $Diag^{tp}(N) = \sigma = Diag^{tp}(B, V(\sigma))$  implies by part 2 of Observation 6.8.7, that  $N \cong_{\mathcal{L}} B$ . But now  $N$  is an  $\mathcal{L}$ -substructure of  $M$  isomorphic to an element of  $cl_k(\mathcal{F})$ , contradicting our assumption that  $M$  is  $\mathcal{F}$ -free (since  $|dom(M)| = n \geq k$  implies  $M$  is  $cl_k(\mathcal{F})$ -free if and only if  $M$  is  $\mathcal{F}$ -free).  $\square$

Observe that by definition of  $H$ , if  $S \subseteq V(H)$ , then

$$(72) \quad E(H[S]) = \left( Diag^{tp}(cl_k(\mathcal{F}), C_W) \cup Err_k(C_W) \right) \cap Span(S).$$

We are now ready to prove Theorem 6.8.14. At this point the reader may wish to review Theorem 2.2.7 and its corresponding notation in Chapter 2, as this is the key tool used in this proof.

**Proof of Theorem 6.8.14.** Clearly it suffices to show the theorem holds for all  $0 < \epsilon < 1/2$ . We claim that further, it suffices to show the theorem holds for any  $k \geq 2r$ . Indeed, suppose  $k < 2r$  and Theorem 6.8.14 holds for all  $k' \geq 2r$ . Suppose  $\mathcal{F}$  is a nonempty collection of  $\mathcal{L}$ -structures, each of size at most  $k$  and  $\mathcal{H} := \text{Forb}(\mathcal{F}) \neq \emptyset$ . Then  $\mathcal{F}$  is also a collection of  $\mathcal{L}$ -structures, each of size at most  $k' = 2r$ . Apply Theorem 6.8.14 to  $k' = 2r$  implies to obtain constants  $c = c(2r, r, \mathcal{L}, \epsilon)$  and  $m = m(2r, r)$ . Since  $k < 2r$ , it is clear the conclusions of Theorem 6.8.14 for  $\mathcal{H}$  and  $2r$  imply the conclusions of Theorem 6.8.14 for  $\mathcal{H}$  and  $k$ . Thus we may take  $c(k, r, \epsilon) = c(2r, r, \epsilon)$  and  $m(k, r) = m(2r, r)$ . We now prove the theorem holds for all  $0 < \epsilon < 1/2$  and  $k \geq 2r$ .

Fix  $0 < \epsilon < 1/2$  and  $k \geq 2r$ . Apply Theorem 2.2.7 to  $\ell = \binom{k}{r}$  to obtain the constant  $c_0 = c_0(\binom{k}{r})$  and set

$$m = m(k, r) = \max \left\{ \frac{\binom{\ell}{r} - 1}{\ell - r} : r < \ell \leq k \right\}.$$

By Lemma 6.9.3, since  $2 \leq r < k$ ,  $m > 1$ . Set  $\epsilon' = \epsilon / |S_r(\mathcal{L})|^{\binom{k}{r}}$  and choose  $0 < \gamma < 1$  sufficiently small so that

$$(73) \quad 2^{\binom{k}{2}+1} |S_r(\mathcal{L})| r! (k-r)^{k-r} \gamma \leq \frac{\epsilon'}{12 \binom{k}{2}!}.$$

Now set  $c = c(k, r, \mathcal{L}, \epsilon) = (c_0 \log(\frac{1}{\epsilon})) / (|S_r(\mathcal{L})| \gamma m)$ . Observe that  $c$  depends on  $\mathcal{L}$ , not just  $r_{\mathcal{L}}$ . Let  $M \geq k$  be such that  $n \geq M$  implies  $(n-r)^{k-r} \geq n^{k-r}/2$ , and  $n^{-\frac{1}{m}} \gamma^{-1} < 1/2$ . We show Lemma 6.8.14 holds for this  $c$  and  $m$  for all  $n \geq M$ .

Fix  $n \geq M$ . Suppose  $\mathcal{F}$  a nonempty collection of finite  $\mathcal{L}$ -structures, each of size at most  $k$ , such that  $\mathcal{H} := \text{Forb}(\mathcal{F}) \neq \emptyset$ . Let  $W$  be a set of size  $n$  and let  $H = H(\mathcal{F}, W)$  be the  $\binom{k}{r}$ -uniform hypergraph described above. Set  $\tau = n^{-\frac{1}{m}} \gamma^{-1}$ . By our assumptions we have that  $0 < \epsilon', \tau < \frac{1}{2}$ . We show that  $\delta(H, \tau) \leq \frac{\epsilon'}{12 \binom{k}{r}!}$  so that we may apply Theorem 2.2.7 to  $H$ . Let  $\alpha = \alpha(\mathcal{F})$  be as in Definition 6.9.4 so that  $E(H) = \alpha \binom{n}{k}$  and let  $N = |V(H)|$ . Given  $2 \leq j \leq \binom{k}{r}$ , set

$$(74) \quad f(j) = \min \left\{ \ell : \binom{\ell}{r} \geq j \right\}.$$

Observe that for each  $2 \leq j \leq \binom{k}{r}$ ,  $r < f(j) \leq k$ . Indeed,  $f(j) > r$  holds since  $\binom{f(j)}{r} \geq j \geq 2$ , and  $f(j) \leq k$  holds since  $k \in \{ \ell : \binom{\ell}{r} \geq j \}$ . Thus by definition of  $m$ , for each  $2 \leq j \leq \binom{k}{r}$ ,

$$(75) \quad m \geq \frac{\binom{f(j)}{r} - 1}{f(j) - r} \geq \frac{j - 1}{f(j) - r},$$

where the inequality is because by (74),  $\binom{f(j)}{r} \geq j$ . We now show that for all  $\sigma \subseteq V(H)$  with  $2 \leq |\sigma| \leq \binom{k}{r}$ ,  $d(\sigma) \leq \alpha n^{k-f(|\sigma|)}$ . Fix  $\sigma \subseteq V(H)$  so that  $2 \leq |\sigma| \leq \binom{k}{r}$ .

Observe that if  $|V(\sigma)| > k$ , then  $\{e \in E(H) : \sigma \subseteq e\} = \emptyset$  since every  $e \in E(H)$  is a syntactic  $k$ -diagram and therefore satisfies  $|V(e)| = k$ . So in this case  $d(\sigma) = 0 \leq \alpha n^{k-f(|\sigma|)}$ . Similarly, if there is  $A \in \binom{V(\sigma)}{r}$  such that  $|Ch_\sigma(A)| \geq 2$ , then  $\{e \in E(H) : \sigma \subseteq e\} = \emptyset$ , since every  $e \in E(H)$  is a syntactic  $k$ -diagram and therefore satisfies  $|Ch_e(A)| = 1$ . So in this case,  $d(\sigma) = 0 \leq \alpha n^{k-f(|\sigma|)}$ . Suppose now  $|V(\sigma)| \leq k$  and  $|Ch_\sigma(A)| \leq 1$  for all  $A \in \binom{V(\sigma)}{r}$ . This implies  $|\sigma| \leq \binom{|V(\sigma)|}{r}$ , so by (74),  $f(|\sigma|) \leq |V(\sigma)|$ . Every edge in  $H$  containing  $\sigma$  can be constructed as follows.

- Choose a set  $X \in \binom{C_k^w}{k}$  such that  $V(\sigma) \subseteq X$  (this makes sense since  $|V(\sigma)| \leq k$ ). There are  $\binom{n-|V(\sigma)|}{k-|V(\sigma)|}$  ways to do this.
- Choose an element of  $Diag^{tp}(cl_k(\mathcal{F}), X) \cup Err_k(X)$  containing  $\sigma$ . By definition of  $\alpha$ , there are at most  $\alpha$  choices for this.

Therefore,  $d(\sigma) = |\{e \in E(H) : \sigma \subseteq e\}| \leq \alpha \binom{n-|V(\sigma)|}{k-|V(\sigma)|} \leq \alpha n^{k-|V(\sigma)|} \leq \alpha n^{k-f(|\sigma|)}$ , where the last inequality is because  $f(|\sigma|) \leq |V(\sigma)|$ . Thus we have shown that for any  $2 \leq j \leq \binom{k}{r}$  and  $\sigma \subseteq V(H)$  such that  $|\sigma| = j$ ,  $d(\sigma) \leq \alpha n^{k-f(j)}$ . Thus given  $2 \leq j \leq \binom{k}{r}$  and a vertex  $v \in V(H)$ ,

$$d^{(j)}(v) = \max\{d(\sigma) : v \in \sigma, |\sigma| = j\} \leq \alpha n^{k-f(j)}.$$

Note the average degree of  $H$  is

$$d = \binom{k}{r} |E(H)| / |V(H)| = \frac{\binom{k}{r} \alpha \binom{n}{k}}{|S_r(\mathcal{H})| \binom{n}{k}} = \frac{\alpha}{|S_r(\mathcal{H})|} \binom{n-r}{k-r} \geq \frac{\alpha}{|S_r(\mathcal{H})|} \binom{n-r}{k-r}^{k-r}.$$

Combining this with our assumption  $n$ , we obtain the following inequality.

$$(76) \quad d \geq \frac{\alpha}{|S_r(\mathcal{H})|} \binom{n-r}{k-r}^{k-r} = \frac{\alpha}{|S_r(\mathcal{H})| (k-r)^{k-r}} (n-r)^{k-r} \geq \frac{\alpha}{2|S_r(\mathcal{H})| (k-r)^{k-r}} n^{k-r}.$$

Combining all of these computations we have the following.

$$\delta_j = \frac{\sum_{v \in V(H)} d^{(j)}(v)}{Nd\tau^{j-1}} \leq \frac{Nn^{k-f(j)}}{Nd\tau^{j-1}} = \frac{n^{k-f(j)+(j-1)\frac{1}{m}} \gamma^{j-1}}{d}.$$

Using our lower bound for  $d$  from (76), this implies

$$\delta_j \leq 2|S_r(\mathcal{H})| (k-r)^{k-r} \gamma^{j-1} \alpha^{-1} n^{k-f(j)+\frac{j-1}{m}-k+r} = 2|S_r(\mathcal{H})| (k-r)^{k-r} \gamma^{j-1} \alpha^{-1} n^{r-f(j)+\frac{j-1}{m}}.$$

By (75),  $\frac{j-1}{m} \leq f(j) - r$ , so this implies  $\delta_j$  is at most

$$2|S_r(\mathcal{H})| (k-r)^{k-r} \gamma^{j-1} \alpha^{-1} n^{r-f(j)+f(j)-r} = 2|S_r(\mathcal{H})| (k-r)^{k-r} \gamma^{j-1} \alpha^{-1} \leq 2|S_r(\mathcal{H})| (k-r)^{k-r} \gamma \alpha^{-1},$$

where the last inequality is because  $j \geq 2$  and  $\gamma < 1$ . Therefore

$$(77) \quad \delta(H, \tau) = 2^{\binom{k}{2}-1} \sum_{j=2}^{\binom{k}{r}} 2^{-(j-1)} \delta_j \leq 2^{\binom{k}{2}-1} 2 |S_r(\mathcal{H})| (k-r)^{k-r} \gamma \alpha^{-1} \sum_{j=2}^{\binom{k}{r}} 2^{-(j-1)}.$$

Note that because  $\mathcal{F} \neq \emptyset$ ,  $\alpha^{-1} \leq 1$ . If  $t = \binom{k}{2}$ , then  $\sum_{j=2}^{\binom{k}{r}} 2^{-(j-1)} < \sum_{j=0}^t 2^{-t}$ . Using the formula for summing finite geometric series,  $\sum_{j=0}^t 2^{-t} = \frac{1-2^{-t-1}}{1-2^{-1}} = 2(1-2^{-t-1}) < 2$ . Plugging these two bounds into (77) yields

$$\delta(H, \tau) \leq 2^{\binom{k}{2}-1} 2 |S_r(\mathcal{H})| (k-r)^{k-r} \gamma 2 = 2^{\binom{k}{2}+1} |S_r(\mathcal{H})| (k-r)^{k-r} \gamma \leq 2^{\binom{k}{2}+1} |S_r(\mathcal{L})| (k-r)^{k-r} \gamma.$$

By (73), the right hand side above is at most  $\frac{\epsilon'}{12 \binom{k}{r}!}$ , so we have shown that  $\delta(H, \tau) \leq \frac{\epsilon'}{12 \binom{k}{r}!}$ . Thus Theorem 2.2.7 implies there is  $\Sigma_0 \subseteq \mathcal{P}(V(H))$  with the following properties.

- (i) For every independent set  $I \subseteq V(H)$ , there is  $\sigma \in \Sigma_0$  such that  $I \subseteq \sigma$ .
- (ii) For every  $\sigma \in \Sigma_0$ ,  $e(H[\sigma]) \leq \epsilon' e(H)$ .
- (iii)  $\log |\Sigma_0| \leq c_0 \log(1/\epsilon') N \tau \log(1/\tau)$ .

Define  $\Sigma = \{\sigma \in \Sigma_0 : \exists \text{ an } \mathcal{F}\text{-free } \mathcal{L}\text{-structure } M \text{ with domain } W \text{ such that } \text{Diag}^{tp}(M) \subseteq \sigma\}$ . Observe that every  $\sigma \in \Sigma$  is complete by part 3 of Observation 6.8.7. We show  $\Sigma$  satisfies the conclusions (1)-(3) of Theorem 6.8.14. Suppose  $M$  is an  $\mathcal{F}$ -free  $\mathcal{L}$ -structure with domain  $W$ . Proposition 6.9.5 implies  $\text{Diag}^{tp}(M)$  is an independent subset of  $V(H)$ , so by (i) and definition of  $\Sigma$ , there is  $\sigma \in \Sigma$  such that  $\text{Diag}^{tp}(M) \subseteq \sigma$ . Thus part (1) of Theorem 6.8.14 holds.

For all  $\sigma \in \Sigma$ , by (72),  $(\text{Diag}^{tp}(cl_k(\mathcal{F}), C_W) \cup \text{Err}_k(C_W)) \cap \text{Span}(\sigma) = E(H[\sigma])$ . So (ii) implies

$$|(\text{Diag}^{tp}(cl_k(\mathcal{F}), C_W) \cup \text{Err}_k(C_W)) \cap \text{Span}(\sigma)| \leq \epsilon' e(H).$$

By definition of  $\epsilon'$  and because  $\alpha \leq |S_r(\mathcal{L})|^{\binom{k}{r}}$ ,

$$\epsilon' e(H) = \epsilon' \alpha \binom{n}{k} = \frac{\epsilon}{|S_r(\mathcal{L})|^{\binom{k}{r}}} \alpha \binom{n}{k} \leq \epsilon \binom{n}{k}.$$

Thus  $|(\text{Diag}^{tp}(cl_k(\mathcal{F}), C_W) \cup \text{Err}_k(C_W)) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{k}$ . By Lemma 6.9.2, for all  $1 \leq \ell \leq k$ ,

$$|(\text{Diag}^{tp}(\mathcal{F}(\ell), C_W) \cup \text{Err}_\ell(C_W)) \cap \text{Span}(\sigma)| \leq \epsilon \binom{n}{\ell}.$$

Since  $k \leq 2r$ , this immediately implies part (2) of Theorem 6.8.14 holds. By (iii), definition of  $c$ , and because  $\Sigma \subseteq \Sigma_0$  we have that

$$\begin{aligned} |\Sigma| \leq |\Sigma_0| &\leq c_0 \log(1/\epsilon') N \tau \log(1/\tau) = c_0 \log(1/\epsilon') |S_r(\mathcal{H})| \binom{n}{r} \tau \log(1/\tau) \\ &\leq c_0 \log(1/\epsilon') |S_r(\mathcal{L})| \binom{n}{r} \tau \log(1/\tau) = c\gamma m \binom{n}{r} \tau \log\left(\frac{1}{\tau}\right). \end{aligned}$$

This shows  $|\Sigma| \leq c\gamma mn^r \tau \log\left(\frac{1}{\tau}\right)$ . By definition of  $\tau$ ,

$$c\gamma mn^r \tau \log\left(\frac{1}{\tau}\right) = cmn^{r-\frac{1}{m}} \left(\frac{\log n}{m} + \log \gamma\right) = cn^{r-\frac{1}{m}} (\log n + m \log \gamma) \leq cn^{r-\frac{1}{m}} \log n,$$

where the last inequality is because  $\gamma \leq 1 \leq m$  implies  $m \log \gamma \leq 0$ . This shows part (3) of Theorem 6.8.14 holds, so we are done.  $\square$

## 6.10. Conjectures

We end with some questions and conjectures. Returning to the metric spaces of Chapter 4, it was shown there that the following is true.

**THEOREM 6.10.1** (Mubayi-Terry Chapter 4). *If  $r \geq 2$  is even, then  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  has a 0-1 law in the language  $\mathcal{L}_r$ .*

It was then conjectured in Chapter 4 that this theorem is false in the case when  $r$  is odd. In Section 7.2 of the next chapter, we will show that in the case when  $r$  is even, the hereditary property associated to the family  $M_r$  has a stability theorem in the sense of Definition 6.4.6, while when  $r$  is odd, this is false. These phenomena lead us to make the following conjecture.

**CONJECTURE 6.10.2.** *Suppose  $\mathcal{L}$  is a finite relational language and  $\mathcal{H}$  is a fast-growing hereditary  $\mathcal{L}$ -property with  $r_{\mathcal{L}} \geq 2$ , such that  $\bigcup_{n \in \mathbb{N}} \mathcal{H}_n$  has a 0-1 law. Then  $\mathcal{H}$  has a stability theorem.*

The idea behind this conjecture is that if  $\mathcal{H}$  has nice asymptotic structure, it should reflect the structure of elements of  $\mathcal{R}_{ex}(n, \mathcal{H})$ . Another phenomenon which can be observed from known examples is that the structures in  $\mathcal{R}_{ex}(n, \mathcal{H})$  are not very complicated. The following questions are various ways of asking if this is always the case.

**QUESTION 6.10.3.** *Suppose  $\mathcal{L}$  is a finite relational language and  $\mathcal{H}$  is a fast-growing hereditary  $\mathcal{L}$ -property with  $r_{\mathcal{L}} \geq 2$ . For each  $n$ , let  $\mathcal{P}_n = \mathcal{R}_{ex}([n], \mathcal{H})$ . Does  $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$  always have a 0-1 law?*

QUESTION 6.10.4. *Suppose  $\mathcal{L}$  is a finite relational language with  $r_{\mathcal{L}} \geq 2$ , and  $\mathcal{H}$  is a fast-growing hereditary  $\mathcal{L}$ -property. Is there a finite  $k = k(\mathcal{H})$  such that for all  $n$  and  $M \in \mathcal{R}_{ex}(n, \mathcal{H})$ , every atomic formula  $\phi(x; y)$  in  $\mathcal{L}_{\mathcal{H}}$  does not have the  $k$ -order property?*

A weaker version of this question is the following.

QUESTION 6.10.5. *Suppose  $\mathcal{L}$  is a finite relational language with  $r_{\mathcal{L}} \geq 2$ , and  $\mathcal{H}$  is a fast-growing hereditary  $\mathcal{L}$ -property. Is there a finite  $k = k(\mathcal{H})$  such that for all  $n$  and  $M \in \mathcal{R}_{ex}(n, \mathcal{H})$ , every atomic formula  $\phi(x; y)$  in  $\mathcal{L}_{\mathcal{H}}$  has VC-dimension bounded by  $k$  in  $M$ ?*

## Examples

In this chapter we give examples of how our general framework from Chapter 6 applies to specific examples. All the structure and enumeration results in this chapter have been previously proved using purely combinatorial techniques, so this chapter is largely expository. Our goal is to demonstrate that the machinery developed in Chapter 6 is sufficiently general to cover a large number of results. In particular, we will consider examples in the setting of graphs, metric spaces, multigraphs, digraphs,  $k$ -uniform hypergraphs, and colored  $k$ -uniform hypergraphs. We now give a restatement of Theorem 6.4.4 which will be convenient for some of our examples.

**THEOREM 7.0.6** (Restatement of Theorem 6.4.4). *Suppose  $\mathcal{L}$  is a finite relational language with  $r = r_{\mathcal{L}} \geq 2$  and  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Then  $|\mathcal{H}_n| = ex(n, \mathcal{H})2^{o(n^r)}$ .*

**PROOF.** By Theorem 6.4.4, it suffices to show

$$ex(n, \mathcal{H})2^{o(n^r)} = \begin{cases} \pi(\mathcal{H})^{1+o(n^r)} & \text{if } \pi(\mathcal{H}) > 1. \\ 2^{o(n^r)} & \text{if } \pi(\mathcal{H}) = 1. \end{cases}$$

This is obvious by definition of  $\pi(\mathcal{H}) = \lim_{n \rightarrow \infty} ex(n, \mathcal{H})^{1/\binom{n}{r}}$ . □

### 7.1. $K_s$ -free graphs

In this section we consider structure and enumeration results by Kolaitis, Prömel, and Rothschild from [66] for  $K_s$ -free graphs, where  $s \geq 3$ . We begin with this example because it is well-known and the least complicated of the examples we consider in this chapter.

**7.1.1. Statements of Results from [66].** Let  $Forb(n, K_s)$  be the set of  $K_s$ -free graphs with vertex set  $[n]$  and let  $Col_s(n)$  be the set of  $s$ -colorable (i.e.  $s$ -partite) graphs with vertex set  $[n]$ . Given two graphs  $G = (V, E)$  and  $G' = (V, E')$  let  $\Delta(G, G') = E \Delta E'$ . We say that  $G$  and  $G'$  are  $\delta$ -close in the classical sense if  $|\Delta(G, G')| \leq \delta n^2$ . For this section, if  $G = (V, E)$  and  $G' = (V', E')$ , we will say  $G$  is a *subgraph* of  $G'$  if  $V = V'$  and  $E \subseteq E'$ . Given  $\delta > 0$ , let  $Col_s^\delta(n)$  be the set of graphs with vertex set  $[n]$  which are  $\delta$ -close in the classical sense to an element in  $Col_s(n)$ . The following

are approximate versions of the precise enumeration and structural result appearing in [66]. They are immediately implied by the results in [66].

**THEOREM 7.1.1** (Kolaitis-Prömel-Rothschild [66]). *Let  $s \geq 3$ . Then  $|Forb(n, K_s)| = 2^{t_{s-1}(n)+o(n^2)}$ .*

**THEOREM 7.1.2** (Kolaitis-Prömel-Rothschild [66]). *Let  $s \geq 3$ . For all  $\delta > 0$  there is  $\beta > 0$  such that for sufficiently large  $n$ ,*

$$(78) \quad \frac{|Forb(n, K_s) \setminus Col_{s-1}^\delta(n)|}{|Forb(n, K_s)|} \leq 2^{-\beta n^2}.$$

We will reprove these theorems using our machinery along with the stability theorem of Erdős and Simonovits (Theorem 2.2.2) as well as Turán’s Theorem (Theorem 2.2.1).

**7.1.2. Preliminaries.** In this subsection we interpret the basic definitions from Chapter 6 in the setting of graphs. Let  $\mathcal{L} = \{R(x, y)\}$  be the language of graphs. Fix  $s \geq 3$  and let  $\mathcal{P}$  be the class of all finite  $K_s$ -free graphs. Since  $r_{\mathcal{L}} = 2$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(x, y) : p \in S_2(\mathcal{P})\}$ . Let

$$q_1(x, y) = \{x \neq y, R(x, y), R(y, x)\} \text{ and}$$

$$q_2(x, y) = \{x \neq y, \neg R(x, y), \neg R(y, x)\}.$$

Then let  $p_1(x, y)$  and  $p_2(x, y)$  be the unique quantifier-free 2-types containing  $q_1(x, y)$  and  $q_2(x, y)$ , respectively. Clearly,  $S_2(\mathcal{P}) = \{p_1(x, y), p_2(x, y)\}$ , so  $\mathcal{L}_{\mathcal{P}} = \{R_{p_1}(x, y), R_{p_2}(x, y)\}$ .

**DEFINITION 7.1.3.** Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$ . The *graph associated to  $G$*  is  $\Psi(G) := (V, E)$  where for all  $xy \in \binom{V}{2}$ ,  $xy \in E$  if and only if  $G \models R_{p_1}(x, y) \vee R_{p_1}(y, x)$  (in other words,  $xy \in E$  if and only if  $p_1(c_x, c_y) \in Ch_G(xy)$ ).

**DEFINITION 7.1.4.** A complete  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  is *downward closed* if  $G \models \forall x \forall y R_{p_1}(x, y) \rightarrow R_{p_2}(x, y)$ .

The idea is that if  $G$  is downward closed and the type which says “put an edge between  $x$  and  $y$ ” is a choice according to  $G$ , then so is the type which says “do not put an edge between  $x$  and  $y$ .”

Given a graph  $(V, E)$ , define  $\Psi^{-1}(V, E)$  to be the  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  with domain  $V$  satisfying

$$G \models \forall x \forall y (x \neq y \rightarrow R_{p_2}(x, y)),$$

and for each  $(x, y) \in V^2$ ,  $G \models R_{p_1}(x, y)$  if and only if  $x \neq y$  and  $xy \in E$ . We leave it to the reader to verify that for any graph  $(V, E)$ ,  $\Psi(\Psi^{-1}(V, E)) = (V, E)$ , and  $\Psi^{-1}(V, E)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template.



LEMMA 7.1.5. *Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$ . Then  $\Psi(G) \trianglelefteq_p G$  and if a graph  $G'$  satisfies  $G' \trianglelefteq_p G$ , then  $G'$  is a subgraph of  $\Psi(G)$  with vertex set  $V$ . If  $G$  is also downward closed, then any subgraph  $G'$  of  $\Psi(G)$  satisfies  $G' \trianglelefteq_p G$ .*

PROOF. Fix  $\Psi(G) = (V, E)$ . We first verify that  $\Psi(G) \trianglelefteq_p G$ . Let  $xy \in \binom{V}{2}$ . We want to show  $Diag^{\Psi(G)}(xy) \in Ch_G(xy)$ . If  $xy \in E$ , then  $Diag^{\Psi(G)}(xy) = p_1(c_x, c_y)$  and by definition of  $\Psi(G)$ , we must have  $G \models R_{p_1}(x, y) \vee R_{p_1}(y, x)$ . By definition, this implies  $p_1(c_x, c_y) \in Ch_G(xy)$ , so we are done. If  $xy \notin E$ , then  $Diag^{\Psi(G)}(xy) = p_2(c_x, c_y)$  and by definition of  $\Psi(G)$ , we must have  $G \models \neg R_{p_1}(x, y) \wedge \neg R_{p_1}(y, x)$ . Since  $G$  is complete, this implies  $G \models R_{p_2}(x, y) \vee R_{p_2}(y, x)$ . By definition, this implies  $p_2(c_x, c_y) \in Ch_G(xy)$ , so we are done.

Suppose  $G' = (V, E') \trianglelefteq_p G$ . We want to show  $G'$  is a subgraph of  $\Psi(G)$ , that is,  $E' \subseteq E$ . Fix  $uv \in E'$ . Then  $Diag^{G'}(u, v) = p_1(c_u, c_v)$ . Since  $G' \trianglelefteq_p G$ , this implies  $p_1(c_u, c_v) \in Ch_G(uv)$ . By definition of  $\Psi(G)$ , this implies  $uv \in E$ , as desired.

Suppose now  $G$  is also downward closed and  $G'$  is a subgraph of  $\Psi(G)$ . Then  $G' = (V, E')$  where  $E' \subseteq E$ . We want to show  $G' \trianglelefteq_p G$ . It suffices to check that for all  $uv \in \binom{V}{2}$ , if  $p(x, y) = qftp_{\mathcal{L}}^{G'}(u, v)$ , then  $p(c_u, c_v) \in Ch_G(uv)$ . Fix  $uv \in \binom{V}{2}$  and set  $p(x, y) = qftp_{\mathcal{L}}^{G'}(u, v)$ . Since  $G$  is a complete downward closed  $\mathcal{L}_{\mathcal{P}}$ -structure we have two possible cases:

- (1)  $Ch_G(uv) = \{p_2(c_u, c_v)\}$ .
- (2)  $Ch_G(uv) = \{p_1(c_u, c_v), p_2(c_u, c_v)\}$ .

If Case (1) happens, then we are done since in this case  $Ch_G(uv)$  must contain  $p(c_u, c_v)$ . Suppose now Case (2) happens. Then  $uv \notin E$  by definition of  $\Psi(G)$ . Since  $G'$  is a subgraph of  $\Psi(G)$ , this implies  $uv \notin E'$ . Thus  $p(c_u, c_v) = p_2(c_u, c_v) \in Ch_G(uv)$ , as desired.  $\square$

COROLLARY 7.1.6. *Suppose  $G$  is a complete finite  $\mathcal{L}_{\mathcal{P}}$ -template. Then  $sub(G) \leq 2^{e(\Psi(G))}$ , and if  $G$  is also downward closed, then equality holds.*

PROOF. By Lemma 7.1.5,  $sub(G)$  is at most the number of subgraphs of  $\Psi(G)$ , so  $sub(G) \leq 2^{e(\Psi(G))}$ . If  $G$  is also downward closed, the Lemma 7.1.5 implies  $sub(G)$  is exactly the number of subgraphs of  $\Psi(G)$ , so  $sub(G) = 2^{e(\Psi(G))}$ .  $\square$

COROLLARY 7.1.7. *If  $(V, E)$  is a  $K_s$ -free graph, then  $\Psi^{-1}(V, E)$  is a downward closed,  $\mathcal{P}$ -random  $\mathcal{L}_{\mathcal{P}}$ -template.*

PROOF. We leave it to the reader to check that by definition of  $\Psi^{-1}$ ,  $\Psi^{-1}(V, E)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. To show  $\Psi^{-1}(V, E)$  is  $\mathcal{P}$ -random, suppose  $M \trianglelefteq_p \Psi^{-1}(V, E)$ . We want to show  $M \in \mathcal{P}$ . By Lemma 7.1.5,  $M$  is a subgraph of  $\Psi(\Psi^{-1}(V, E)) = (V, E)$ . Since  $(V, E)$  is  $K_s$ -free, this implies  $M$  is  $K_s$ -free, so  $M \in \mathcal{P}$ .  $\square$

**7.1.3. Counting.** In this subsection we prove Theorem 7.1.1. Let  $\text{ex}_{\text{edge}}(n, K_s)$  be the maximal number of edges in a  $K_s$ -free graph with  $n$  vertices. This number is usually denoted “ $\text{ex}(n, K_s)$ ”, but we use  $\text{ex}_{\text{edge}}(n, K_s)$  to avoid confusion with our notation  $\text{ex}(n, \mathcal{P})$  from Definition 6.4.1. Recall that a  $K_s$ -free graph  $G$  of size  $n$  is called extremal if  $e(G) = \text{ex}_{\text{edge}}(n, K_s)$ . To avoid confusion with the notion of extremal  $\mathcal{L}_{\mathcal{P}}$ -templates from Definition 6.4.1, we will call a  $K_s$ -free graph with  $n$  vertices *edge-extremal* if  $e(G) = \text{ex}_{\text{edge}}(n, K_s)$ . We can now prove the following.

PROPOSITION 7.1.8. *Suppose  $n \geq 2$  is an integer. If  $V$  is a set of size  $n$  and  $G \in \mathcal{R}_{\text{ex}}(V, \mathcal{P})$ , then  $\Psi(G)$  is an edge-extremal  $K_s$ -free graph of size  $n$ . Consequently,  $\text{ex}(n, \mathcal{P}) = 2^{\text{ex}_{\text{edge}}(n, K_s)}$ .*

PROOF. Suppose  $G \in \mathcal{R}_{\text{ex}}(V, \mathcal{P})$ . By Lemma 7.1.5,  $\Psi(G) \trianglelefteq_p G$ , so since  $G$  is  $\mathcal{P}$ -random,  $\Psi(G) \in \mathcal{P}$ . Thus  $\Psi(G)$  is  $K_s$ -free. Suppose  $\Psi(G) = (V, E)$  were not edge-extremal. Then there is a  $K_s$ -free graph  $(V, E')$  such that  $|E'| > |E|$ . Let  $G' = \Psi^{-1}(V, E')$ . By Corollary 7.1.7,  $G'$  is a downward element of  $\mathcal{R}(V, \mathcal{P})$ . Then Corollary 7.1.6 implies

$$\text{sub}(G) \leq 2^{|E|} < 2^{|E'|} = \text{sub}(G'),$$

contradicting that  $G$  is extremal. We now show  $\text{ex}(n, \mathcal{P}) = 2^{\text{ex}_{\text{edge}}(n, K_s)}$ . Suppose  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ , so  $\text{sub}(G) = \text{ex}(n, \mathcal{P})$ . By what we've just shown,  $\Psi(G)$  is an extremal  $K_s$ -free graph, that is,  $e(\Psi(G)) = \text{ex}_{\text{edge}}(n, K_s)$ . We claim that because  $G \in \mathcal{R}_{\text{ex}}(V, \mathcal{P})$ ,  $G$  is downward closed. Suppose not. Then there is  $uv \in \binom{V}{2}$  such that  $\text{Ch}_G(uv) = \{p_2(c_u, c_v)\}$ . Let  $G^*$  be the  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$  which agrees with  $G$  everywhere, except on  $uv$ , where  $\text{Ch}_{G^*}(uv) = \{p_1(c_u, c_v), p_2(c_u, c_v)\}$ . Then  $G^*$  is also a  $\mathcal{P}$ -random  $\mathcal{L}_{\mathcal{P}}$ -template, since the only change we have made is to allow the possibility of a subpattern taking an additional *nonedge*, which cannot create a  $K_s$ . But then  $G^* \in \mathcal{R}([n], \mathcal{P})$  and

$$\text{sub}(G^*) = \frac{|\text{Ch}_{G^*}(uv)|}{|\text{Ch}_G(uv)|} \text{sub}(G) = 2 \text{sub}(G) > \text{sub}(G),$$

contradicting that  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . Therefore,  $G$  is downward closed so Corollary 7.1.6 implies  $\text{sub}(G) = 2^{e(\Psi(G))}$ . Combining what we've shown yields

$$\text{ex}(n, \mathcal{P}) = \text{sub}(G) = 2^{e(\Psi(G))} = 2^{\text{ex}_{\text{edge}}(n, K_s)}.$$

□

Recall Turán's Theorem says that  $\text{ex}_{\text{edge}}(n, K_s) = (1 - \frac{1}{s-1} + o(1))\binom{n}{2}$  and the only edge-extremal  $K_s$ -free graphs on  $n$  vertices are the  $(s-1)$ -partite Turán graphs of size  $n$ . Recall  $T_{s-1}(n)$  is the set of  $(s-1)$ -partite Turán graphs on  $[n]$ . We can now compute  $\pi(\mathcal{P})$ .

PROPOSITION 7.1.9.  $\pi(\mathcal{P}) = 2^{1 - \frac{1}{s-1}}$ .

PROOF. Recall  $\pi(\mathcal{P}) = \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{P})^{1/\binom{n}{2}}$ . By Proposition 7.1.8 and Turán's theorem,

$$\text{ex}(n, \mathcal{P}) = 2^{\text{ex}_{\text{edge}}(n, K_s)} = 2^{(1 - \frac{1}{s-1} + o(1))\binom{n}{2}}.$$

Therefore,  $\pi(\mathcal{P}) = \lim_{n \rightarrow \infty} 2^{(1 - \frac{1}{s-1} + o(1))\binom{n}{2}/\binom{n}{2}} = 2^{1 - \frac{1}{s-1}}$ , as desired. □

**Proof of Theorem 7.1.1.** Proposition 7.1.9 along with Theorem 6.4.4 implies

$$|\mathcal{P}_n| = \pi(\mathcal{P})^{(1+o(1))\binom{n}{2}} = 2^{(1 - \frac{1}{s-1} + o(1))\binom{n}{2}}.$$

□

Thus we have shown Theorem 7.1.1 is a special case of Theorem 6.4.4.

**7.1.4. Stability and Approximate Structure.** In this subsection we show  $\mathcal{P}$  has a stability theorem and prove Theorem 7.1.2. Suppose  $G$  and  $G'$  are two graphs with the same finite vertex set  $V$ . Observe that considering them as  $\mathcal{L}$ -structures,

$$\text{diff}^2(G, G') = \{(x, y) \in V^2 : xy \in \Delta(G, G')\},$$

so  $|\text{diff}^2(G, G')| = 2|\Delta(G, G')|$ . So  $\text{dist}(G, G') = 2|\Delta(G, G')|/n^2$  and we see that  $G$  and  $G'$  are  $\delta$ -close in the classical sense if and only if they are  $2\delta$ -close in the sense of Definition 6.2.6. When we just say  $\delta$ -close, we always mean in the sense of Definition 6.2.6.

LEMMA 7.1.10. *Suppose  $V$  is a finite set and  $G, G' \in \mathcal{R}(V, \mathcal{P})$  are downward closed. Then for all  $\delta > 0$ ,  $G$  and  $G'$  are  $\delta$ -close if and only if  $\Psi(G)$  and  $\Psi(G')$  are  $\delta$ -close.*

PROOF. Because  $G$  and  $G'$  are both  $\mathcal{L}_{\mathcal{P}}$ -templates, Lemma 6.5.3 implies that for all  $(x, y) \in V^2$ ,  $qftp^G(x, y) = qftp^{G'}(x, y)$  if and only if  $Ch_G(x, y) = Ch_{G'}(x, y)$ . By definition of  $\Psi(G)$  and  $\Psi(G')$  and because  $G$  and  $G'$  are downward closed, this implies  $\text{diff}(G, G') = \text{diff}(\Psi(G), \Psi(G'))$ . Clearly this implies the desired conclusion. □

PROPOSITION 7.1.11.  $\mathcal{P}$  has a stability theorem.

PROOF. Fix  $\eta > 0$  and set  $\delta = \eta/16$ . Choose  $M$  and  $\epsilon < \eta/16$  so that Theorem 2.2.2 implies any  $K_s$ -free graph with  $n > M$  vertices and at least  $(1 - \epsilon)\text{ex}_{\text{edge}}(n, K_s)$ -many edges is  $\delta$ -close (in the classical sense) to an  $(s - 1)$ -partite Turán graph. By Proposition 7.1.9, we may assume that  $M$  is sufficiently large so that  $n > M$  implies

$$(79) \quad 2^{(1 - \frac{1}{s-1})\binom{n}{2} + (2 - \frac{s-2}{s-1})\epsilon n^2 + \epsilon \frac{s-2}{s-1}n} \geq \text{ex}(n, \mathcal{P}) \geq 2^{(1 - \frac{1}{s-1} - \epsilon/2)\binom{n}{2}}.$$

Suppose now that  $n \geq M$  and  $G \in \mathcal{R}([n], \mathcal{P})$  is such that  $\text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}$ . We want to show there is  $G' \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$  such that  $\text{dist}(G, G') \leq \eta$ . We first claim there is a downward closed  $G^* \in \mathcal{R}([n], \mathcal{P})$  such that

- (1) For all  $xy \in \binom{V}{2}$ ,  $\text{Ch}_G(xy) \subseteq \text{Ch}_{G^*}(xy)$ .
- (2)  $|\text{diff}(G, G^*)| \leq 2\epsilon n^2$ .

Indeed, let  $G^*$  be the  $\mathcal{L}_{\mathcal{P}}$  structure with domain  $[n]$  such that  $G^* \models \forall x \forall y (x \neq y \rightarrow R_{p_2}(x, y))$  and for all  $(x, y) \in V^2$ ,  $G^* \models R_{p_1}(x, y)$  if and only if  $G \models R_{p_1}(x, y)$ . We leave it to the reader to verify that because  $G \in \mathcal{R}([n], \mathcal{P})$ ,  $G^* \in \mathcal{R}([n], \mathcal{P})$  and that by definition of  $G^*$ , (1) holds and  $G^*$  is downward closed. Suppose towards a contradiction that (2) is false. Then there is  $uv \in \binom{V}{2}$  is such that  $\text{Ch}_G(uv) \subsetneq \text{Ch}_{G^*}(uv)$ . Recall from the proof of Lemma 7.1.5, the only options for  $\text{Ch}_G(uv)$  and  $\text{Ch}_{G^*}(uv)$  are  $\{p_1(c_u, c_v)\}$ ,  $\{p_2(c_u, c_v)\}$ , and  $\{p_1(c_u, c_v), p_2(c_u, c_v)\}$ . Therefore  $\text{Ch}_G(uv) \subsetneq \text{Ch}_{G^*}(uv)$  implies we must have  $|\text{Ch}_G(uv)| = 1$  and  $|\text{Ch}_{G^*}(uv)| = 2$ . Therefore,

$$\text{sub}(G^*) = \text{sub}(G) \left( \prod_{uv \in \text{diff}(G, G^*)} \frac{|\text{Ch}_{G^*}(uv)|}{|\text{Ch}_G(uv)|} \right) = \text{sub}(G) 2^{\frac{1}{2}|\text{diff}(G, G^*)|} \geq \text{ex}(n, \mathcal{P})^{1-\epsilon} 2^{2\epsilon n^2},$$

where the last inequality is by assumption on  $G$ . Combining this with the lower bound in (79) implies

$$\text{sub}(G^*) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon} 2^{2\epsilon n^2} \geq 2^{(1 - \frac{1}{s-1} - \epsilon/2)\binom{n}{2} (1-\epsilon) + 2\epsilon n^2} = 2^{(1 - \frac{1}{s-1})\binom{n}{2} + (2 - \frac{s-2}{s-1})\epsilon n^2 + \epsilon \frac{s-2}{s-1}n}.$$

But this contradicts the upper bound in (79) since  $G^* \in \mathcal{R}([n], \mathcal{P})$  implies  $\text{sub}(G^*) \leq \text{ex}(n, \mathcal{P})$ . Thus (2) holds. Then we have by (1), our assumptions, and Proposition 7.1.8

$$(80) \quad \text{sub}(G^*) \geq \text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon} = 2^{\text{ex}_{\text{edge}}(n, K_s)(1-\epsilon)}.$$

Proposition 7.1.5 implies  $\Psi(G^*) \trianglelefteq_p G^*$  (since  $\Psi(G^*)$  is a subgraph of itself), so because  $G^*$  is  $\mathcal{P}$ -random,  $\Psi(G^*) \in \mathcal{P}$ . Proposition 7.1.6 and (80) imply

$$e(\Psi(G^*)) = \log_2(\text{sub}(G^*)) \geq \text{ex}_{\text{edge}}(n, K_s)(1 - \epsilon)$$

Therefore, Theorem 2.2.2 implies  $\Psi(G^*)$  is  $\delta$ -close (in the classical sense) to an edge-extremal  $K_s$ -free graph  $H$  with vertex set  $[n]$ , which implies  $\Psi(G^*)$  and  $H$  are  $2\delta$ -close. By Proposition 7.1.7,  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$ , so Corollary 7.1.6 along with the fact that  $\Psi(\Psi^{-1}(H)) = H$  implies  $\text{sub}(\Psi^{-1}(H)) = 2^{e(H)} = 2^{\text{ex}_{\text{edge}}(n, K_s)}$ , where the second equality is because  $H$  is edge-extremal. By Proposition 7.1.8, this shows  $\text{sub}(\Psi^{-1}(H)) = \text{ex}(n, \mathcal{P})$ , so  $\Psi^{-1}(H) \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . By Lemma 7.1.10, because  $\Psi(G^*)$  and  $H = \Psi(\Psi^{-1}(H))$  are  $2\delta$ -close,  $G^*$  and  $\Psi^{-1}(H)$  are  $2\delta$ -close. Combining what we have shown with our assumptions yields

$$|\text{diff}(G, \Psi^{-1}(H))| \leq |\text{diff}(G, G^*)| + |\text{diff}(G^*, \Psi^{-1}(H))| \leq 2\epsilon n^2 + 2\delta n^2 \leq \eta/8n^2 + \eta/8n^2 = \eta/4n^2 \leq \eta \binom{n}{2},$$

where the last inequality is since we may assume  $n \geq 2$ . This implies  $\text{dist}(G, \Psi^{-1}(H)) \leq \eta$ .  $\square$

REMARK 7.1.12. *In the notation of Theorem 6.4.7, for all  $\delta > 0$ ,  $E(n, \mathcal{P}) \subseteq \text{Col}_{s-1}(n)$ .*

PROOF. Recall  $E(n, \mathcal{P})$  is the set of  $\mathcal{L}$ -structures  $H$  with domain  $[n]$  such that  $H \trianglelefteq_p G$  for some  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . Suppose  $H \in E(n, \mathcal{P})$  and let  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$  be such that  $H \trianglelefteq_p G$ . Lemma 7.1.5 implies  $H$  is a subgraph of  $\Psi(G)$ . Lemma 7.1.10 implies  $\Psi(G)$  is an edge-extremal  $K_s$ -free graph. By Turán's Theorem,  $\Psi(G) \in T_{s-1}(n) \subseteq \text{Col}_{s-1}(n)$ . Since subgraphs of  $(s-1)$ -partite graphs are also  $(s-1)$ -partite, this implies  $H \in \text{Col}_{s-1}(n)$ .  $\square$

**Proof of Theorem 7.1.2.** Fix  $\delta > 0$ . Choose  $\beta > 0$  for  $\delta$  using Theorem 6.4.7. Then Theorem 6.4.7 implies that for sufficiently large  $n$ ,  $|\mathcal{P}_n \setminus E^\delta(n, \mathcal{P})|/|\mathcal{P}_n| \leq 2^{-\beta \binom{n}{2}}$ . By Remark 7.1.12,  $E(n, \mathcal{P}) \subseteq \text{Col}_{s-1}(n)$ . This implies  $E^\delta(n, \mathcal{P}) \subseteq \text{Col}_{s-1}^\delta(n)$ , so (78) follows.  $\square$

## 7.2. Metric Spaces

In this section we consider the metric spaces examined in Chapter 4. We recall a few definitions from Chapter 4. Let  $r \geq 2$  an integer. An  $r$ -graph is a pair  $(V, c)$  such that  $c : \binom{V}{r} \rightarrow 2^{[r]}$ . An  $r$ -graph  $(V, c)$  *simple* if for all  $xy \in \binom{V}{r}$ ,  $|c(xy)| \leq 1$  and is *complete* if for all  $xy \in \binom{V}{r}$ ,  $|c(xy)| \geq 1$ . We will sometimes abuse notation when  $(V, c)$  is a simple complete  $r$ -graph by thinking of  $c$  instead as a function from  $\binom{V}{r} \rightarrow [r]$  with  $c(xy) = i$  instead of  $c(xy) = \{i\}$ . An  $r$ -graph  $(V, c)$  is a *metric  $r$ -graph*

if it contains no violating triangles, and given an integer  $n$ ,  $M_r(n)$  is the set of simple complete metric  $r$ -graphs with vertex set  $[n]$ .

**7.2.1. Preliminaries.** Given an integer  $r \geq 2$ , let  $\mathcal{L}_r = \{R_1(x, y), \dots, R_r(x, y)\}$  be the language of  $r$ -graphs from Chapter 4. We consider elements of  $M_r(n)$  as  $\mathcal{L}_r$ -structures by interpreting  $R_i(x, y)$  if and only if  $d(x, y) = i$ . Let  $\mathcal{M}_r$  denote the class of  $\mathcal{L}_r$ -structures obtained by closing  $M_r = \bigcup_{n \in \mathbb{N}} M_r(n)$  under isomorphism. Clearly  $\mathcal{M}_r$  is a hereditary  $\mathcal{L}_r$ -property, and  $(\mathcal{M}_r)_n = M_r(n)$ . For the rest of the section,  $r \geq 2$  is a fixed integer,  $\mathcal{P} = \mathcal{M}_r$ , and  $\mathcal{L} = \mathcal{L}_r$ . Observe that since  $r_{\mathcal{L}} = 2$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(x, y) : p(x, y) \in S_2(\mathcal{P})\}$ . For each  $i \in [r]$ , set

$$q_i(x, y) := \{x \neq y\} \cup \{R_i(x, y)\} \cup \{\neg R_j(x, y) : j \neq i\},$$

and let  $p_i(x, y)$  be the unique quantifier-free 2-type containing  $q_i(x, y)$ . It is straightforward to check that  $S_2(\mathcal{P}) = \{p_i(x, y) : i \in [r]\}$ , so  $\mathcal{L}_{\mathcal{P}} = \{R_{p_i}(x, y) : i \in [r]\}$ .

**DEFINITION 7.2.1.** Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -structure with underlying set  $V$ . The  $r$ -graph associated to  $G$  is  $\Psi(G) := (V, c)$ , where for each  $xy \in \binom{V}{2}$ ,  $c(xy) = \{i : G \models R_{p_i}(x, y) \vee R_{p_i}(y, x)\}$  (in other words,  $i \in c(x, y)$  if and only if  $p_i(c_x, c_y) \in Ch_G(xy)$ ).

Given an  $r$ -graph  $(V, c)$ , define  $\Psi^{-1}(V, c)$  to be the  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  which has domain  $V$  and such that for each  $(x, y) \in V^2$  and  $i \in [r]$ ,  $G \models R_{p_i}(x, y)$  if and only if  $x \neq y$  and  $i \in c(xy)$ . We leave it to the reader to verify that  $\Psi^{-1}(V, c)$  is an  $\mathcal{L}_{\mathcal{P}}$ -template and  $\Psi(\Psi^{-1}(V, c)) = (V, c)$ . Further, if  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template, then  $\Psi^{-1}(\Psi(G)) = G$ .

**PROPOSITION 7.2.2.** Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$  and  $\Psi(G) = (V, c)$ . Then  $G' \preceq_p G$  if and only if  $G'$  is a simple complete  $r$ -graph  $(V, d)$  with the property that for all  $xy \in \binom{V}{2}$ ,  $d(xy) \in c(xy)$ .

**PROOF.** Suppose  $M \preceq_p G$ . Let  $\chi$  be a choice function for  $G$  which chooses  $M$ . Define an  $r$ -graph  $(V, d)$  as follows. For each  $uv \in \binom{V}{2}$ , set  $d(uv) = i$  where  $i$  is the unique element of  $[r]$  such that  $\chi(uv) = p_i(c_u, c_v)$ . Clearly  $(V, d)$  is a simple, complete  $r$ -graph. Fix  $uv \in \binom{V}{2}$ . By definition of a choice function for  $G$  and by definition of  $(V, d)$ , if  $i = d(uv)$ , then

$$p_i(c_u, c_v) = \chi(uv) \in Ch_G(uv) = \{p_j(c_u, c_v) : G \models R_{p_j}(u, v) \vee R_{p_j}(v, u)\}.$$

This implies  $d(uv) = i \in \{j : G \models R_{p_j}(u, v) \vee R_{p_j}(v, u)\} = c(uv)$ , where the last equality is by definition of  $\Psi(G) = (V, c)$ . Thus  $d(uv) \in c(uv)$  for all  $uv \in \binom{V}{2}$ , as desired.

Suppose on the other hand that  $(V, d)$  is a simple complete  $r$ -graph such that for all  $xy \in \binom{V}{2}$ ,  $d(xy) \in c(xy)$ . We want to show that considered as an  $\mathcal{L}$ -structure,  $M \trianglelefteq_p G$ . Define a function  $\chi : \binom{V}{2} \rightarrow S_2(C_V, \mathcal{P})$  as follows. For each  $uv \in \binom{V}{2}$ , if  $d(u, v) = i$ , set  $\chi(uv) = p_i(c_u, c_v)$ . By assumption  $d(u, v) \in c(u, v) = \{j : G \models R_{p_j}(x, y)\}$ . Thus  $p_i \in \{p_j : G \models R_{p_j}(u, v)\} = Ch_G(uv)$ . This verifies that  $\chi$  is a choice function for  $G$ . By definition of  $\chi$ ,  $M \trianglelefteq_\chi G$ .  $\square$

**COROLLARY 7.2.3.** *If  $(V, c)$  is a complete metric  $r$ -graph, then  $\Psi^{-1}(V, C) \in \mathcal{R}(V, \mathcal{P})$ . If  $G \in \mathcal{R}(V, \mathcal{P})$ , then  $\Psi(G)$  is a complete metric  $r$ -graph.*

**PROOF.** Suppose  $(V, c)$  is a complete metric  $r$ -graph. We leave it to the reader to check  $\Psi^{-1}(V, c)$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$ . Since  $(V, c)$  is a metric  $r$ -graph, all simple complete  $r$ -graphs  $(V, d)$  with the property that for all  $xy \in \binom{V}{2}$ ,  $d(xy) \in c(xy)$  are metric spaces. By Proposition 7.2.2, this implies all full subpatterns of  $\Psi^{-1}(V, c)$  are metric spaces, which implies  $\Psi^{-1}(V, c)$  is  $\mathcal{P}$ -random by definition. Suppose now  $G \in \mathcal{R}(V, \mathcal{P})$ . Let  $\Psi(G) = (V, c)$ . We leave it to the reader to verify that  $G$  a complete  $\mathcal{L}_{\mathcal{P}}$ -structure implies  $(V, c)$  is a complete  $r$ -graph. To show  $(V, c)$  is a metric  $r$ -graph, let  $xyz \in \binom{V}{3}$  and let  $d(xy), d(yz), d(xz)$  be in  $c(xy), c(yz), c(xz)$  respectively. Extend  $d$  to a function  $d : \binom{V}{2} \rightarrow [r]$  such that for all  $uv \in \binom{V}{2}$ ,  $d(uv) \in c(uv)$  (this is possible since  $(V, c)$  is complete). By Proposition 7.2.2,  $(V, d) \trianglelefteq_p G$ , so since  $G$  is  $\mathcal{P}$ -random,  $(V, d)$  is a metric space. Thus  $|d(yz) - d(xz)| \leq d(xy) \leq d(yz) + d(xz)$ , so  $\{x, y, z\}$  cannot be a violating triangle in  $(V, c)$ . This shows  $(V, c)$  is a metric  $r$ -graph.  $\square$

Recall from Chapter 4 that if  $G = (V, c)$  is an  $r$ -graph, then  $W(R) = \prod_{xy \in \binom{V}{2}} |c(xy)|$ .

**PROPOSITION 7.2.4.** *Suppose  $G$  is a finite  $\mathcal{L}_{\mathcal{P}}$ -template. Then  $sub(G) = W(\Psi(G))$ .*

**PROOF.** Let  $\Psi(G) = (V, c)$ . By Proposition 7.2.2, the full subpatterns of  $G$  are exactly the simple complete  $r$ -graphs  $(V, d)$  with the property that for all  $xy \in \binom{V}{2}$ ,  $d(xy) \in c(xy)$ . Clearly the number of such  $r$ -graphs is  $\prod_{uv \in \binom{V}{2}} |c(uv)|$ . This shows  $sub(G) = \prod_{uv \in \binom{V}{2}} |c(uv)| = W(\Psi(G))$ .  $\square$

**7.2.2. Counting.** In this section we prove the approximate enumeration result from Chapter 4, Corollary 4.1.6, using the tools of Chapter 6. Recall the definitions of  $C_r(n)$ ,  $\tilde{C}_r(n)$ ,  $m(r)$ ,  $\tilde{M}_r(n)$ , and (when  $r$  is odd)  $L_r$  and  $U_r$  from Chapter 4. If  $G \in \tilde{M}_r(n)$ , say that  $G$  is *product-extremal* if

$$W(G) = \max\{W(G') : G' \in \tilde{M}_r(n)\}.$$

**PROPOSITION 7.2.5.** *Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $[n]$ . Then  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$  if and only if  $\Psi(G)$  is a product-extremal element of  $\tilde{M}_r(n)$ .*

PROOF. Suppose first  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . By Corollary 7.2.3, and definition of  $\tilde{M}_r(n)$ ,  $\Psi(G) \in \tilde{M}_r(n)$ . Suppose towards a contradiction that  $\Psi(G)$  is not product-extremal. Then there is  $H \in \tilde{M}_r(n)$  such that  $W(H) > W(\Psi(G))$ . Corollary 7.2.3 implies  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$  and Proposition 7.2.4 implies  $sub(\Psi^{-1}(H)) = W(\Psi(\Psi^{-1}(H))) = W(H) > W(\Psi(G)) = sub(G)$ , contradicting that  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . Conversely, suppose  $\Psi(G) \in \tilde{M}_r(n)$  is product-extremal. By Corollary 7.2.3,  $\Psi^{-1}(\Psi(G)) = G \in \mathcal{R}([n], \mathcal{P})$ . Suppose towards a contradiction  $G \notin \mathcal{R}_{ex}([n], \mathcal{P})$ . Then there is  $G' \in \mathcal{R}([n], \mathcal{P})$  such that  $sub(G') > sub(G)$ . Corollary 7.2.3 implies  $\Psi(G') \in \tilde{M}_r(n)$  and Proposition 7.2.4 implies  $W(\Psi(G')) = sub(G') > sub(G) = W(\Psi(G))$ , contradicting that  $\Psi(G)$  is product-extremal.  $\square$

Recall from Chapter 5 that given  $s \geq 2$  and  $q \geq 0$ , an  $(s, q)$ -graph is a multigraph  $(V, w)$  such that for all  $X \in \binom{V}{s}$ ,  $\sum_{xy \in \binom{X}{2}} w(xy) \leq q$ . Given a multigraph  $G = (V, w)$ ,  $P(G) = \prod_{xy \in \binom{V}{2}} w(xy)$ .

LEMMA 7.2.6. *If  $r \geq 2$  is even, then for all  $n \geq 3$ , the unique  $r$ -graph in  $\tilde{C}_r(n)$  is the only product-extremal element of  $\tilde{M}_r(n)$ .*

PROOF. Suppose  $G = (V, c) \in \tilde{M}_r(n)$  is product-extremal. Let  $H = ([n], w)$  be the multigraph defined by  $w(xy) = |c(xy)|$  for all  $xy \in \binom{[n]}{2}$ . Observe  $P(H) = W(G)$ . We claim that  $H$  is a  $(3, 3m(r))$ -graph. Suppose towards a contradiction there were three distinct points  $x, y, z \in [n]$  such that  $w(x, y) + w(y, z) + w(x, z) > 3m(r)$ . Then

$$w(x, y) + w(y, z) + w(x, z) = |c(x, y)| + |c(y, z)| + |c(x, z)| > 3m(r),$$

and  $\min\{|c(x, y)|, |c(y, z)|, |c(x, z)|\} \geq 1$  imply by Lemma 4.4.9 from Chapter 4 that  $G$  contains a violating triangle, contradicting that  $G \in \tilde{M}_r(n)$ . Thus  $H$  is a  $(3, 3m(r))$ -graph, so by Theorem 5.2.3 in Chapter 5,  $W(G) = P(H) \leq m(r) \binom{n}{2}$  and equality holds if and only if  $w(x, y) = m(r)$  for all  $xy \in \binom{[n]}{2}$ . Since  $G$  is extremal, this implies  $P(H) = W(G) = m(r) \binom{n}{2}$ , and for all  $xy \in \binom{[n]}{2}$ ,  $w(xy) = m(r) = |c(xy)|$ . Part (1) of Corollary 4.4.15 in Chapter 4 implies that the only metric  $r$ -graph  $([n], c')$  satisfying  $|c'(x, y)| = m(r)$  for all  $xy \in \binom{[n]}{2}$  is the unique element of  $\tilde{C}_r(n)$ . Thus  $G$  is the unique element of  $\tilde{C}_r(n)$ .  $\square$

If  $r \geq 3$  is odd, define  $\tilde{E}_r(n)$  to be the set of  $r$ -graphs  $([n], c)$  such that there is a set  $\{e_1, \dots, e_{\lfloor n/2 \rfloor}\}$  of pairwise disjoint elements in  $\binom{[n]}{2}$  such that  $c(e_i) = U_r \cup L_r$  and if  $xy \in \binom{[n]}{2} \setminus \{e_1, \dots, e_{\lfloor n/2 \rfloor}\}$  then  $c(xy) = U_r$ . Observe that  $\tilde{E}_r(n) \subseteq \tilde{C}_r(n)$ .



LEMMA 7.2.7. *If  $r \geq 3$  is odd, then for all  $n \geq 3$ , all product-extremal elements of  $\tilde{M}_r(n)$  are in  $\tilde{E}_r(n)$ .*

PROOF. Let  $G = ([n], c) \in \tilde{M}_r(n)$  and let  $H$  be the multigraph  $([n], w)$  where  $w(xy) = |c(xy)|$  for all  $xy \in \binom{[n]}{2}$ . Observe  $P(H) = W(G)$ . We claim  $H$  is a  $(3, 3m(r) + 1)$ -graph. Suppose by contradiction there are three distinct points  $x, y, z \in [n]$  such that  $w(x, y) + w(y, z) + w(x, z) > 3m(r) + 1$ . Then

$$w(x, y) + w(y, z) + w(x, z) = |c(x, y)| + |c(y, z)| + |c(x, z)| > 3m(r) + 1,$$

and  $\min\{|c(x, y)|, |c(y, z)|, |c(x, z)|\} \geq 1$  imply by Lemma 4.4.9 from Chapter 4 that  $G$  contains a violating triangle, contradicting that  $G \in \tilde{M}_r(n)$ . Thus  $H$  is a  $(3, 3m(r) + 1)$ -graph, so by Theorem 5.2.3 in Chapter 5,  $W(G) = P(H) \leq m(r) \binom{n}{r} \left(\frac{m(r)+1}{m(r)}\right)^{\lfloor n/2 \rfloor}$  and equality holds if and only if  $H \in U_{1, m(r)}(n)$ , where recall,  $H \in U_{1, m(r)}(n)$  if and only if there is a set  $\{e_1, \dots, e_{\lfloor n/2 \rfloor}\}$  of pairwise disjoint elements in  $\binom{[n]}{2}$  such that  $w(e_i) = m(r) + 1$  for each  $e_i$  and for all  $xy \in \binom{[n]}{2} \setminus \{e_1, \dots, e_{\lfloor n/2 \rfloor}\}$ ,  $w(x, y) = m(r)$ . Since  $G$  is product-extremal, we must have  $\text{sub}(G) = m(r) \binom{n}{r} \left(\frac{m(r)+1}{m(r)}\right)^{\lfloor n/2 \rfloor} = P(H)$  and  $H \in U_{1, m(r)}(n)$ . So there is a set  $\{e_1, \dots, e_{\lfloor n/2 \rfloor}\}$  of pairwise disjoint elements in  $\binom{[n]}{2}$  such that  $|c(e_i)| = m(r) + 1$  for each  $e_i$  and for all  $xy \in \binom{[n]}{2} \setminus \{e_1, \dots, e_{\lfloor n/2 \rfloor}\}$ ,  $|c(x, y)| = m(r)$ . Combining this with part (2) of Lemma 4.4.14, we must have that for each  $e_i$ ,  $c(e_i) = U_r \cup L_r$  and for all other edges  $c(x, y) = U_r$ , that is,  $G$  is in  $\tilde{E}_r(n)$ .  $\square$

COROLLARY 7.2.8. *Let  $n \geq 2$ . If  $r \geq 2$  is even, then  $\text{ex}(n, \mathcal{P}) = m(r) \binom{n}{2}$ . If  $r \geq 3$  is odd, then  $\text{ex}(n, \mathcal{P}) = m(r) \binom{n}{r} \left(\frac{m(r)+1}{m(r)}\right)^{\lfloor n/2 \rfloor}$ . Consequently,  $\pi(\mathcal{P}) = m(r)$ .*

PROOF. That  $\text{ex}(n, \mathcal{P}) = \max\{W(G) : G \in \tilde{M}_r(n)\}$  follows from Propositions 7.2.4 and 7.2.5. When  $r$  is even, this implies by Lemma 7.2.6 that  $\text{ex}(n, \mathcal{P}) = W(G)$  where  $G \in \tilde{C}_r(n)$ . By definition of  $\tilde{C}_r(n)$  when  $r$  is even, this implies  $\text{ex}(n, \mathcal{P}) = m(r) \binom{n}{2}$ . If  $r$  is odd, then Lemma 7.2.7 implies  $\text{ex}(n, \mathcal{P}) = W(G)$  where  $G \in \tilde{E}_r(n)$ . By definition of  $\tilde{E}_r(n)$ , this implies  $\text{ex}(n, \mathcal{P}) = m(r) \binom{n}{r} \left(\frac{m(r)+1}{m(r)}\right)^{\lfloor n/2 \rfloor}$ . That  $\pi(\mathcal{P}) = m(r)$  holds in both cases now follows from the definition of  $\pi(\mathcal{P}) = \lim_{n \rightarrow \infty} \text{ex}(n, \mathcal{P})^{1/\binom{n}{2}}$ .  $\square$

We can now recover our approximate enumeration theorem from Chapter 4, Corollary 4.1.6.

**Proof of Corollary 4.1.6 from Chapter 4.** Theorem 6.4.4 implies  $|\mathcal{P}_n| = \pi(\mathcal{P})^{(1+o(1))\binom{n}{2}}$ . By definition of  $\mathcal{P}$ ,  $M_r(n) = \mathcal{P}_n$ , and Corollary 7.2.8 implies  $\pi(\mathcal{P}) = m(r)$ . Thus

$$|\mathcal{P}_n| = |M_r(n)| = m(r)^{(1+o(1))\binom{n}{2}}.$$

□

**7.2.3. Stability and Approximate Structure.** In this section we consider the stability theorems and approximate structural results from Chapter 4. Recall the following definition of  $\delta$ -closeness for  $r$ -graphs from Chapter 4.

**DEFINITION 7.2.9.** Suppose  $G = (V, c)$  and  $G' = (V, c')$  are complete  $r$ -graphs with the same vertex set  $V$  of size  $n$ . Set  $\Delta(G, G') = \{e \in \binom{V}{2} : c(e) \neq c'(e)\}$ . We say  $G$  and  $G'$  are  $\delta$ -close in the sense of Chapter 4 if  $|\Delta(G, G')| \leq \delta n^2$ .

Just “ $\delta$ -close” will always mean in the sense of Definition 6.2.6. Observe that for complete  $r$ -graphs  $G$  and  $G'$  with the same finite vertex set,  $\text{dist}(G, G') = 2|\Delta(G, G')|$ . So  $G$  and  $G'$  are  $\delta$ -close if and only if they are  $\frac{\delta}{2}$ -close in the sense of Chapter 4. We can now recover the approximate structure theorems from Chapter 4, Theorem 4.1.5.

**Proof of Theorem 4.1.5 from Chapter 4.** Fix  $\delta > 0$ . Choose  $\epsilon > 0$  and  $M_1$  from Theorem 4.4.13 in Chapter 4 such that if  $n > M_1$  and  $H \in \tilde{M}_r(n)$  satisfies  $W(H) > m(r)^{(1-\epsilon)\binom{n}{2}}$  then  $H$  is  $\delta/2$ -close in the sense of Chapter 4 to an element of  $\tilde{C}_r(n)$ . Now let  $\beta > 0$  and  $M_2$  be as in Theorem 6.4.5 from Chapter 6 applied to  $\delta/2$  and  $\epsilon$ . Let  $N = \max\{M_1, M_2\}$ . We show for all  $n > N$ ,

$$\frac{|M_r(n) \setminus C_r^\delta(n)|}{|M_r(n)|} \leq 2^{-\beta n^2}.$$

By Theorem 6.4.5 in Chapter 6, it suffices to show that for all  $n \geq N$ ,  $E^{\delta/2}(\epsilon, n, \mathcal{P}) \subseteq C_r^\delta(n)$ . Suppose  $H = (V, d) \in E^{\delta/2}(\epsilon, n, \mathcal{P})$  and let  $H' = (V, d') \in E(\epsilon, n, \mathcal{P})$  such that  $\text{dist}(H, H') \leq \delta/2$ . By definition of  $E(\epsilon, n, \delta)$ , there is  $G' \in \mathcal{R}([n], \mathcal{P})$  such that  $H' \trianglelefteq_p G'$  and  $\text{sub}(G') > \epsilon x(n, \mathcal{P})^{1-\epsilon}$ . Recall that we showed in the proof of Theorem 6.4.3 that for all  $n$ ,  $\text{ex}(n, \mathcal{P}) \geq \pi(\mathcal{P})^{\binom{n}{2}}$ , so  $\text{sub}(G') > \pi(\mathcal{P})^{(1-\epsilon)\binom{n}{2}}$ . Corollary 7.2.3 implies  $\Psi(G') := (V, c') \in \tilde{M}_r(n)$  and Proposition 7.2.4 implies  $W(\Psi(G')) = \text{sub}(G') \geq \pi(\mathcal{P})^{(1-\epsilon)\binom{n}{2}}$ . By Theorem 4.4.13 in Chapter 4, this implies  $\Psi(G')$  is  $\delta/2$ -close in the sense of Chapter 4 to some  $M = ([n], c) \in \tilde{C}_r(n)$ . Define  $H'' = ([n], d'')$  as follows. If  $xy \notin \Delta(\Psi(G'), M) \cup \Delta(H, H')$ , let  $d''(xy) = d'(xy) = d(xy)$ , and if  $xy \in \Delta(\Psi(G'), M) \cup \Delta(H, H')$ , let  $d''(xy)$  be any element of  $c(xy)$ . We claim that for all  $xy$ ,  $d''(xy) \in c(xy)$ . If  $xy \in \Delta(\Psi(G'), M) \cup \Delta(H, H')$ , this is by definition of  $d''(xy)$ . If  $xy \notin \Delta(\Psi(G'), M) \cup \Delta(H, H')$ , then  $d''(xy) = d'(xy)$  and  $c(xy) = c'(xy)$ . Since,  $H' = (V, d') \trianglelefteq_p G' = (V, c')$ , Proposition 7.2.2 implies  $d'(xy) \in c'(xy)$ , thus  $d''(xy) \in c(xy)$ . Therefore,  $G'' \in \tilde{C}_r(n)$  implies  $H'' \in C_r(n)$ . We claim  $\text{dist}(H, H'') \leq \delta$ . By definition of  $H''$ ,

$$\text{diff}(H, H'') \subseteq \text{diff}(H, H') \cup \text{diff}(\Psi(G'), M),$$

so  $\text{dist}(H, H'') \leq \text{dist}(H, H') + \text{dist}(\Psi(G'), M) = 2|\Delta(H, H')| + 2|\Delta(\Psi(G'), M)| \leq \delta n^2$  and we have that  $H \in C_r^\delta(n)$ .  $\square$

We leave the following lemma to the reader.

LEMMA 7.2.10. *If  $M$  and  $N$  are complete  $r$ -graphs with the same vertex set  $V$ , then  $\text{dist}(M, N) = \text{dist}(\Psi^{-1}(M), \Psi^{-1}(N))$ .*

We now show that when  $r$  is even,  $\mathcal{P}$  has a stability theorem in the sense of Chapter 6, but when  $r$  is odd, this is not the case.

THEOREM 7.2.11. *If  $r \geq 2$  is even, then  $\mathcal{P}$  has a stability theorem.*

PROOF. Fix  $\delta > 0$ . By Theorem 4.4.13 in Chapter 4, there is  $\epsilon > 0$  and  $M$  such that for all  $n > M$  if  $H \in \tilde{M}_r(n)$  satisfies  $P(H) > m(r)^{(1-\epsilon)\binom{n}{2}}$ , then  $H$  is  $\delta/2$ -close in the sense of Definition 7.2.9 to the unique  $r$ -graph  $H' \in \tilde{C}_r(n)$ . Suppose now that  $n > M$  and  $G \in \mathcal{R}([n], \mathcal{P})$  satisfying  $\text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}$ . We want to show there is  $G' \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$  such that  $\text{dist}(G, G') \leq \delta$ . Recall that we showed in the proof of Theorem 6.4.3 that for all  $n$ ,  $\pi(\mathcal{P})^{\binom{n}{2}} \leq \text{ex}(n, \mathcal{P})$ . Thus our assumptions imply  $\text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon} \geq \pi(\mathcal{P})^{(1-\epsilon)\binom{n}{2}}$ . Proposition 7.2.8 implies  $\pi(\mathcal{P}) = m(r)$  and Corollary 7.2.3 implies  $\Psi(G) \in \tilde{M}_r(n)$ , so Proposition 7.2.4 implies

$$W(\Psi(G)) = \text{sub}(G) \geq \pi(\mathcal{P})^{(1-\epsilon)\binom{n}{2}} = m(r)^{(1-\epsilon)\binom{n}{2}}.$$

Thus Theorem 4.4.13 implies  $\Psi(G)$  is  $\delta/2$ -close in the sense of Chapter 4 to the unique  $r$ -graph  $H' \in \tilde{C}_r(n)$ . By Lemma 7.2.6,  $H'$  is a product extremal element of  $\tilde{M}_r(n)$ . By Corollary 7.2.3,  $\Psi^{-1}(H') \in \mathcal{R}([n], \mathcal{P})$ , and since  $\Psi(\Psi^{-1}(H')) = H'$ , Proposition 7.2.5 implies  $\Psi^{-1}(H') \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . By Lemma 7.2.10,  $\text{dist}(\Psi^{-1}(H'), G) = \text{dist}(H', \Psi(G)) = \frac{2}{n^2}|\Delta(H', \Psi(G))| \leq \delta n^2$ , where the inequality is because  $H'$  and  $\Psi(G)$  are  $\delta$ -close in the sense of Chapter 4.  $\square$

COROLLARY 7.2.12. *When  $r \geq 3$  is odd,  $\mathcal{P}$  does not have a stability theorem.*

PROOF. Let  $A = ([n], c)$  be such that for all  $xy \in \binom{[n]}{2}$ ,  $c(xy) = L_r$ . Then by definition,  $A \in \tilde{M}_r(n)$  and  $W(A) = m(r)^{\binom{n}{2}}$ . By Corollary 7.2.3,  $\Psi^{-1}(A) \in \mathcal{R}([n], \mathcal{P})$  and by Proposition 7.2.4,  $\text{sub}(\Psi^{-1}(A)) = W(A) = m(r)^{\binom{n}{2}}$ . Let  $B \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . By Proposition 7.2.7,  $\Psi(B) \in \tilde{E}_r(n)$ . By definition of  $A$  and  $\tilde{E}_r(n)$ ,  $\Delta(A, \Psi(B)) = \binom{[n]}{2}$ , so  $\text{dist}(A, \Psi(B)) = 1 > \delta$ . However, for all  $\epsilon > 0$ ,  $\pi(\mathcal{P}) = m(r)$  implies that if  $n$  is sufficiently large,

$$\text{sub}(\Psi^{-1}(A)) = W(A) = m(r)^{\binom{n}{2}} \geq \text{ex}(n, \mathcal{P})^{1-\epsilon},$$

so  $\Psi^{-1}(A) \in E(\epsilon, n, \mathcal{P})$ . This shows that  $\mathcal{P}$  does not have a stability theorem in the sense of Definition 6.4.6 of Chapter 6.  $\square$

### 7.3. Multigraphs

In this section we use the results from Chapter 5 to prove approximate structure and enumeration theorems for multigraphs. These results were originally proved in joint work of the author and D. Mubayi in the specific setting of multigraphs (i.e. not using the machinery of Chapter 6). This work will appear in forthcoming paper by the author and D. Mubayi. Our notation follows that found in Chapter 5. Recall that given  $n, s, q$ ,  $F(n, s, q)$  is the set of  $(n, s, q)$ -graphs with vertex set  $[n]$ . If  $G = (V, w)$  is a multigraph, the multiplicity of  $G$  is  $\mu(G) = \max\{w(xy) : xy \in \binom{V}{2}\}$  and  $P(G) = \prod_{xy \in \binom{V}{2}} w(xy)$ . If  $G = (V, w)$  and  $G' = (V', w')$ ,  $G$  is a *full subgraph* of  $G'$  if  $V = V'$  and for all  $xy \in \binom{V}{2}$ ,  $w(xy) \leq w'(xy)$ .

**7.3.1. Preliminaries.** Given an integer  $q \geq 1$ , let  $\mathcal{L}_{mg}^q = \{R_i(x, y) : i = 0, \dots, q\}$  consist of  $q + 1$  binary relations. We consider a multigraph  $G = (V, w)$  of multiplicity at most  $q$  as an  $\mathcal{L}_{mg}^q$ -structure by interpreting, for each  $(x, y) \in V^2$ ,  $G \models R_i(x, y)$  if and only if  $w(xy) = i$ . Observe that by definition of a multigraph, this means that for all  $x \in V$  and  $0 \leq i \leq q$ ,  $G \models \neg R_i(x, x)$ .

Fix integers  $s \geq 2$  and  $q \geq 1$ . Let  $\mathcal{L} = \mathcal{L}_{mg}^q$  and let  $\mathcal{P}$  be the class of finite  $(s, q)$ -graphs, considered as  $\mathcal{L}$ -structures. Clearly  $\mathcal{P}$  is a hereditary  $\mathcal{L}$ -property. Since  $r_{\mathcal{L}} = 2$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(x, y) : p \in S_2(\mathcal{P})\}$ . For each  $0 \leq i \leq q$ , set

$$q_i(x, y) = \{x \neq y\} \cup \{R_i(x, y), R_i(y, x)\} \cup \{\neg R_j(x, y) : j \neq i\} \cup \{\neg R_j(y, x) : j \neq i\}.$$

Then for each  $0 \leq i \leq q$ , let  $p_i(x, y)$  be the unique complete quantifier-free 2-type containing  $q_i(x, y)$ . We leave it to the reader to verify  $S_2(\mathcal{P}) = \{p_i(x, y) : 0 \leq i \leq q\}$ , so  $\mathcal{L}_{\mathcal{P}} = \{R_{p_i}(x, y) : 0 \leq i \leq q\}$ .

**DEFINITION 7.3.1.** Given a complete  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  with domain  $V$ , the *multigraph associated to*  $G$  is  $\Psi(G) := (V, w)$  where for each  $xy \in \binom{V}{2}$ ,  $w(xy) = \max\{j : G \models R_{p_j}(x, y) \vee R_{p_j}(y, x)\}$  (note  $G$  complete implies  $\{j : G \models R_{p_j}(x, y) \vee R_{p_j}(y, x)\} \neq \emptyset$  for all  $xy \in \binom{V}{2}$ ).

An  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  is *downward closed* if for all  $0 \leq i \leq j \leq q$ ,  $G \models \forall x \forall y (R_{p_j}(x, y) \rightarrow R_{p_i}(x, y))$ . Given a multigraph  $(V, w)$  of multiplicity at most  $q$ , let  $\Psi^{-1}(V, w)$  be the  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$  such that for each  $(x, y) \in V^2$ ,  $G \models R_{p_i}(x, y)$  if and only if  $x \neq y$  and  $w(xy) \geq i$ . We leave it to the reader to verify that for any multigraph  $(V, w)$  of multiplicity at most  $q$ ,  $\Psi^{-1}(V, w)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template and  $\Psi(\Psi^{-1}(V, w)) = (V, w)$ .

LEMMA 7.3.2. *Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$ . Then  $\Psi(G) \trianglelefteq_p G$  and for any  $\mathcal{L}$ -structure  $G'$ ,  $G' \trianglelefteq_p G$  implies  $G'$  is a full subgraph of  $\Psi(G)$ . If  $G$  is also downward closed, then for any full subgraph  $G'$  of  $\Psi(G)$ ,  $G' \trianglelefteq_p G$ .*

PROOF. Let  $(V, w) = \Psi(G)$ . By definition of  $\Psi(G)$ , for all  $uv \in \binom{V}{2}$ , if  $w(uv) = i$ , then  $G \models R_{p_i}(u, v) \vee R_{p_i}(v, u)$ . This shows  $Diag^{\Psi(G)}(uv) \in Ch_G(uv)$ , thus  $\Psi(G) \trianglelefteq_p G$ . Suppose  $G' \trianglelefteq_p G$ . Clearly this implies  $G'$  is a multigraph with vertex set  $V$ , say  $G' = (V, w')$ . Fix  $uv \in \binom{V}{2}$  and let  $i = w'(uv)$ . Then  $p_i(x, y) = qftp_{\mathcal{L}}^{G'}(u, v)$ . Since  $G' \trianglelefteq_p G$ , this implies  $G \models R_{p_i}(u, v) \vee R_{p_i}(v, u)$ . Since  $G$  is an  $\mathcal{L}$ -template and  $p_i(x, y) = p_i(y, x)$ , this implies  $G \models R_{p_i}(u, v)$ . By definition of  $\Psi(G)$ ,  $G \models R_{p_i}(u, v)$  implies  $w(uv) \geq i$ . Thus for all  $uv \in \binom{V}{2}$ ,  $w(uv) \geq w'(uv)$  so  $G'$  is a full subgraph of  $\Psi(G)$ .

Suppose now that  $G$  is also downward closed and  $G' = (V, w')$  is a full subgraph of  $\Psi(G) = (V, w)$ . Fix  $uv \in \binom{V}{2}$  and  $i = w'(uv)$ . Since  $G'$  is a full subgraph of  $\Psi(G)$ ,  $w'(uv) = i \leq w(uv)$ . Combining this with the definition of  $\Psi(G)$ , we have  $\max\{j : G \models R_{p_j}(u, v) \vee R_{p_j}(v, u)\} = w(uv) \geq i = w'(uv)$ . Because  $G$  is downward closed, this implies  $G \models R_{p_i}(u, v)$ , which implies that  $p_i(c_u, c_v) \in Ch_G(uv)$ . Since  $qftp_{\mathcal{L}}^{G'}(u, v) = p_i(x, y)$ , this shows  $Diag^{G'}(uv) \in Ch_G(uv)$ . Since this holds for all  $uv \in \binom{V}{2}$ , we've shown  $G' \trianglelefteq_p G$ .  $\square$

COROLLARY 7.3.3. *If  $(V, w)$  is an  $(s, q)$ -graph, then  $\Psi^{-1}(V, w) \in \mathcal{R}(V, \mathcal{P})$  and is downward closed.*

PROOF. We leave it to the reader to verify that the definition of  $\Psi^{-1}$  implies  $\Psi^{-1}(V, w)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. To show  $\Psi^{-1}(V, w)$  is  $\mathcal{P}$ -random, suppose  $G' \trianglelefteq_p \Psi^{-1}(V, w)$ . Then Lemma 7.3.2 implies  $G'$  is a full subgraph of  $\Psi(\Psi^{-1}(V, w)) = (V, w)$ . Since  $(V, w)$  is an  $(s, q)$ -graph, so are all its full subgraphs. Thus  $G' \in \mathcal{P}$ .  $\square$

DEFINITION 7.3.4. If  $G = (V, w)$  is a multigraph, let  $G^+ = (V, w^+)$  where for each  $xy \in \binom{V}{2}$ ,  $w^+(xy) = w(xy) + 1$ .

Observe that  $G$  is an  $(n, s, q)$ -graph if and only if  $G^+$  is an  $(n, s, q + \binom{s}{2})$ -graph.

COROLLARY 7.3.5. *Suppose  $G$  is a finite downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. Then  $sub(G) = P(\Psi(G)^+)$ .*

PROOF. Since  $G$  is downward closed, Lemma 7.3.2 implies the number of full subpatterns of  $G$  is the same as the number of full subgraphs of  $\Psi(G)$ . Let  $\Psi(G) = (V, w)$ . To choose a full subgraph  $(V, w')$  of  $\Psi(G)$ , one must choose for each  $xy \in \binom{V}{2}$  a value  $w'(xy) \in \{0, \dots, w(xy)\}$ . Thus the

number of submultigraphs of  $\Psi(G)$  is

$$\prod_{xy \in \binom{V}{2}} |\{0, 1, \dots, w(xy)\}| = \prod_{xy \in \binom{V}{2}} (w(xy) + 1) = P(\Psi(G)^+).$$

□

LEMMA 7.3.6. *If  $G \in \mathcal{R}([n], \mathcal{P})$ , then there is a downward closed  $G^* \in \mathcal{R}([n], \mathcal{P})$  such that for all  $uv \in \binom{[n]}{2}$ ,  $Ch_G(uv) \subseteq Ch_{G^*}(uv)$  and such that  $sub(G^*) \geq sub(G) \left(\frac{q+1}{q}\right)^{|\Delta|}$ , where*

$$\Delta = \left\{ uv \in \binom{[n]}{2} : Ch_G(uv) \neq Ch_{G^*}(uv) \right\}.$$

PROOF. Define  $G^*$  so that for all  $uv \in \binom{V}{2}$ ,  $G^* \models R_{p_i}(u, v)$  if and only if there is  $j \geq i$  such that  $G \models R_{p_j}(u, v)$ . We leave it to the reader to check that by definition,  $G^*$  is downward closed and for all  $uv \in \binom{V}{2}$ ,  $Ch_G(uv) \subseteq Ch_{G^*}(uv)$ . We now show  $G^*$  is  $\mathcal{P}$ -random. Let  $G' \leq_p G^*$ . By Lemma 7.3.2,  $G'$  is a full subgraph of  $\Psi(G^*)$ . Observe that by definition of  $G^*$  and  $\Psi$ ,  $\Psi(G)$  and  $\Psi(G^*)$  are the same multigraph. Lemma 7.3.2 implies  $\Psi(G) \leq_p G$ , thus since  $G$  is  $\mathcal{P}$ -random,  $\Psi(G)$  is an  $(s, q)$ -graph. Thus  $G'$  is a full subgraph of  $\Psi(G) = \Psi(G^*)$  which is an  $(s, q)$ -graph, implying that  $G'$  is an  $(s, q)$ -graph, that is,  $G' \in \mathcal{P}$ . This shows  $G^*$  is  $\mathcal{P}$ -random.

Suppose  $uv \in \Delta$ . Then  $Ch_G(uv) \subsetneq Ch_{G^*}(uv)$ . Observe that since  $|Ch_{G^*}(uv)| \leq |S_2(\mathcal{P})| = q + 1$ , this implies  $\frac{|Ch_{G^*}(uv)|}{|Ch_G(uv)|} \geq \frac{q+1}{q}$ . Thus the following holds.

$$sub(G^*) = \prod_{uv \in \binom{V}{2}} |Ch_{G^*}(uv)| = sub(G) \prod_{uv \in \Delta} \frac{|Ch_{G^*}(uv)|}{|Ch_G(uv)|} \geq sub(G) \left(\frac{q+1}{q}\right)^{|\Delta|}.$$

□

PROPOSITION 7.3.7. *Suppose  $G$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. Then  $G$  is  $\mathcal{P}$ -random if and only if  $\Psi(G)$  is an  $(s, q)$ -graph if and only if  $\Psi(G)^+$  is an  $(s, q + \binom{s}{2})$ -graph.*

PROOF. Fix  $G$  a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$ . Let  $\Psi(G) = (V, w)$  and  $\Psi(G)^+ = (V, w^+)$  as in Definition 7.3.4. That  $\Psi(G)$  is an  $(s, q)$ -graph if and only if  $\Psi(G)^+$  is an  $(s, q + \binom{s}{2})$ -graph follows from Definition 7.3.4. We now show  $G$  is  $\mathcal{P}$ -random if and only if  $\Psi(G)$  is an  $(s, q)$ -graph. Suppose first  $G$  is  $\mathcal{P}$ -random. Since  $\Psi(G)$  is a full subgraph of itself, Lemma 7.3.2 implies  $\Psi(G) \leq_p G$ . Since  $G$  is  $\mathcal{P}$ -random, this implies  $\Psi(G) \in \mathcal{P}$ , i.e.  $\Psi(G)$  is an  $(s, q)$ -graph. Conversely, suppose  $\Psi(G)$  is an  $(s, q)$ -graph. Then Corollary 7.3.3 implies  $\Psi^{-1}(\Psi(G)) = G$  is  $\mathcal{P}$ -random. □

**7.3.2. Counting.** In this subsection we prove approximate enumeration theorems for  $\mathcal{P}_n$ , for certain cases of  $s$  and  $q$ . Recall that given  $n, s, q$ ,

$$\text{ex}_{\Pi}(n, s, q) = \max\{P(G) : G \in F(n, s, q)\} \quad \text{and} \quad \mathcal{P}(n, s, q) = \{G \in F(n, s, q) : P(G) = \text{ex}_{\Pi}(n, s, q)\},$$

and the elements of  $\mathcal{P}(n, s, q)$  are called *product-extremal*.

LEMMA 7.3.8. *If  $G_{ex} \in \mathcal{R}([n], \mathcal{P})$ , then  $G$  is downward closed.*

PROOF. Suppose not. Then let  $G^* \in \mathcal{R}([n], \mathcal{P})$  be as in Lemma 7.3.6. Since  $G$  is not downward closed,  $G^* \neq G$ , so  $\Delta \neq \emptyset$  and  $\text{sub}(G^*) \geq \text{sub}(G) \binom{q+1}{q} > \text{sub}(G)$ , contradicting that  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ .  $\square$

PROPOSITION 7.3.9. *Suppose  $G$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $[n]$ . Then  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$  if and only if  $\Psi(G)^+ \in \mathcal{P}(n, s, q + \binom{s}{2})$  if and only if  $\Psi(G) \in \mathcal{P}(n, s, q)$ . Consequently,  $\text{ex}(n, \mathcal{P}) = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})$ .*

PROOF. Fix  $G$  a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $[n]$ . It is straightforward to check that by Definition 7.3.4,  $\Psi(G)^+ \in \mathcal{P}(n, s, q + \binom{s}{2})$  if and only if  $\Psi(G) \in \mathcal{P}(n, s, q)$ . So it suffices to show  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$  if and only if  $\Psi(G) \in \mathcal{P}(n, s, q)$ . Suppose  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . Proposition 7.3.7 implies  $\Psi(G) \in F(n, s, q)$ . By Corollary 7.3.5,  $\text{sub}(G) = P(\Psi(G)^+)$ . Suppose  $\Psi(G)$  is not product-extremal. Then there is  $H \in F(n, s, q)$  such that  $P(H) > P(\Psi(G))$ . Clearly this implies  $P(H^+) > P(\Psi(G)^+)$ . By Corollary 7.3.3,  $\Psi^{-1}(H)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. Thus Proposition 7.3.7 and the fact that  $\Psi(\Psi^{-1}(H)) = H \in F(n, s, q)$  imply  $\Psi^{-1}(H)$  is  $\mathcal{P}$ -random. Thus  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$ . By Corollary 7.3.5,  $\text{sub}(\Psi^{-1}(H)) = P(\Psi(\Psi^{-1}(H))^+) = P(H^+) > P(\Psi(G)^+) = \text{sub}(G)$ , but this contradicts that  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ .

Conversely, suppose  $\Psi(G) \in \mathcal{P}(n, s, q)$ . Since  $G$  is downward closed, Proposition 7.3.7 and the fact that  $\Psi^{-1}(\Psi(G)) = G$  imply  $G$  is  $\mathcal{P}$ -random. Suppose now that  $G \notin \mathcal{R}_{ex}([n], \mathcal{P})$ . Then there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  such that  $\text{sub}(G^*) > \text{sub}(G)$ . By Lemma 7.3.6, we may replace  $G^*$  with a downward closed element of  $\mathcal{R}([n], \mathcal{P})$  and maintain the property  $\text{sub}(G^*) > \text{sub}(G)$ . By Corollary 7.3.5, this implies  $P(\Psi(G)^+) = \text{sub}(G) < \text{sub}(G^*) = P(\Psi(G^*)^+)$ . Clearly this implies  $P(\Psi(G)) < P(\Psi(G^*))$ . But now Proposition 7.3.7 implies  $\Psi(G^*) \in F(n, s, q)$ , contradicting that  $\Psi(G)$  is product extremal.

We now show  $\text{ex}(n, \mathcal{P}) = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})$ . Let  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . By definition,  $\text{ex}(n, \mathcal{P}) = \text{sub}(G)$ . By Lemma 7.3.8,  $G$  is downward closed, so by Corollary 7.3.5,  $\text{sub}(G) = P(\Psi(G)^+)$ . We showed

above that  $\Psi(G)^+ \in \mathcal{P}(n, s, q + \binom{s}{2})$ , so by definition,  $P(\Psi(G)^+) = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})$ . This shows  $\text{ex}(n, \mathcal{P}) = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})$ , as desired.  $\square$

The following corollary is an immediate consequence of the definitions of  $\pi(\mathcal{H})$  and  $\text{ex}_{\Pi}(s, q + \binom{s}{2})$  and the fact that  $\text{ex}(n, \mathcal{P}) = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})$ .

**COROLLARY 7.3.10.**  $\pi(\mathcal{P}) = \text{ex}_{\Pi}(s, q + \binom{s}{2})$ .

Applying Theorem 7.7.8 yields the following approximate enumeration theorem.

**THEOREM 7.3.11.**  $|F(n, s, q)| = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})2^{o(n^2)} = \text{ex}_{\Pi}(s, q + \binom{s}{2})^{(1+o(1))\binom{n}{2}}$ .

**PROOF.** Theorem 6.4.4 of Chapter 6, Theorem 7.7.8, Corollary 7.3.10, and Proposition 7.3.9 imply

$$|\mathcal{P}_n| = \pi(\mathcal{P})^{(1+o(1))\binom{n}{2}} = \text{ex}(n, \mathcal{P})2^{o(n^2)} = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})2^{o(n^2)} = \text{ex}_{\Pi}(s, q + \binom{s}{2})^{(1+o(1))\binom{n}{2}}.$$

Since  $\mathcal{P}_n = F(n, s, q)$  this finishes the proof.  $\square$

Combining this with our computations of  $\text{ex}_{\Pi}(n, s, q + \binom{s}{2})$  for various cases in Chapter 5 yields the following approximate enumeration theorems. Recall the definition of  $\gamma$  from Theorem 5.2.7 in Chapter 5.

**COROLLARY 7.3.12.** *Let  $s, a, b, q$  are integers satisfying  $s \geq 2$ ,  $a, b \geq 0$  and  $q = a\binom{s}{2} + b$ .*

- (i) *If  $0 \leq b \leq s - 2$  then  $|F(n, s, q)| = (a + 1)^{\binom{n}{2} + o(n^2)}$ .*
- (ii) *If  $b = \binom{s}{2} - t$  where  $2 \leq t \leq \frac{s}{2}$ , then  $|F(n, s, q)| = a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_{s-t}(n) + o(n^2)}$ .*
- (iii) *If  $(s, q) = (4, 9)$ , then  $|F(n, 4, 9)| = 2^{\gamma n^2 + o(n^2)}$ .*

**PROOF.** If (i) holds, then Theorem 5.2.3 (Extremal) from Chapter 5 implies  $\text{ex}_{\Pi}(s, q + \binom{s}{2}) = a + 1$ , so Theorem 7.3.11 implies  $|F(n, s, q)| = \text{ex}_{\Pi}(s, q + \binom{s}{2})^{(1+o(1))\binom{n}{2}} = (a + 1)^{\binom{n}{2} + o(n^2)}$ . If (ii) holds, then Theorem 5.2.5 (Extremal) implies  $\text{ex}_{\Pi}(n, s, q + \binom{s}{2}) = a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_{s-t}(n)}$ , so Theorem 7.3.11 implies  $|F(n, s, q)| = \text{ex}_{\Pi}(n, s, q + \binom{s}{2})2^{o(n^2)} = a^{\binom{n}{2}} \left(\frac{a+1}{a}\right)^{t_{s-t}(n) + o(n^2)}$ . If (iii) holds, then Theorem 5.2.7 (Extremal) implies  $\text{ex}_{\Pi}(4, 9 + \binom{4}{2}) = \text{ex}_{\Pi}(4, 15) = 2^{\gamma}$ . Thus Theorem 7.3.11 implies

$$|F(n, 4, 9)| = \text{ex}_{\Pi}(4, 15)^{(1+o(1))\binom{n}{2}} = 2^{\gamma n^2 + o(n^2)}.$$

$\square$



**7.3.3. Stability and Approximate Structure.** In this section we prove an approximate structure theorem for  $\mathcal{P}$  for certain values of  $s$  and  $q$ . If  $G = (V, w)$  and  $G' = (V, w')$  are multigraphs, then  $\Delta(G, G') = \{xy \in \binom{V}{2} : w(xy) \neq w'(xy)\}$  and  $G$  and  $G'$  are called  $\delta$ -close in the sense of Chapter 5 if  $|\Delta(G, G')| \leq \delta n^2$  ( $\delta$ -close alone means in the sense of Definition 6.2.6).

**THEOREM 7.3.13.** *If  $(s, q)$  is in case (i) or (ii) of Corollary 7.3.12, then for all  $\delta > 0$  there is an  $\epsilon > 0$  and  $M$  such that for all  $n \geq M$  the following holds. If  $G \in F(n, s, q)$  satisfies  $P(G^+) > \text{ex}_\pi(n, s, q + \binom{s}{2})^{1-\epsilon}$ , then  $G^+$  is  $\delta$ -close in the sense of Chapter 5 to an element of  $\mathcal{P}(n, s, q + \binom{s}{2})$ .*

**PROOF.** This follows immediately from Theorems 5.2.3 (Stability) and 5.2.5 (Stability) applied to  $(s, q')$  where  $q' = q + \binom{s}{2}$ .  $\square$

**LEMMA 7.3.14.** *There is a constant  $C$  such that for all  $\epsilon > 0$  there is  $M$  such that for all  $n \geq M$  the following holds. If  $G \in \mathcal{R}([n], \mathcal{P})$  is not downward closed and  $\text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}$ , then there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed and  $\text{dist}(G, G^*) \leq C\epsilon$ .*

**PROOF.** Assume  $n$  is sufficiently large so that  $\text{ex}(n, \mathcal{P}) \leq \pi(\mathcal{P})^{2n^2}$ . Let  $G^*$  be as in Lemma 7.3.8. Then  $G^* \in \mathcal{R}([n], \mathcal{P})$  is downward closed, and in the notation of Lemma 7.3.8,  $\text{sub}(G^*) \geq \text{sub}(G) \left(\frac{q+1}{q}\right)^{|\Delta|}$ , where

$$\Delta = \{uv \in \binom{[n]}{2} : \text{Ch}_G(uv) \neq \text{Ch}_{G^*}(uv)\}.$$

Observe that because  $G$  and  $G^*$  are  $\mathcal{L}_{\mathcal{P}}$ -templates,  $\Delta = \{uv \in \binom{[n]}{2} : (u, v) \in \text{diff}(G, G^*)\}$ , thus  $|\Delta| = \frac{1}{2}|\text{diff}(G, G^*)|$ . Now observe our assumptions imply

$$\text{ex}(n, \mathcal{P}) \geq \text{sub}(G^*) \geq \text{sub}(G) \left(\frac{q+1}{q}\right)^{|\Delta|} \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}.$$

Rearranging this, we obtain that  $\left(\frac{q+1}{q}\right)^{|\Delta|} \leq \text{ex}(n, \mathcal{P})^\epsilon \leq \pi(\mathcal{P})^{2\epsilon n^2}$ . Taking logs and dividing by  $\log((q+1)/q)$  we obtain that

$$|\Delta| \leq 2\epsilon \log(\pi(\mathcal{P})) / \log((q+1)/q) n^2.$$

Thus if we set  $C = 4\log(\pi(\mathcal{P})) / \log((q+1)/q)$ , this implies  $|\text{diff}(G, G^*)| \leq 2|\Delta| \leq C\epsilon n^2$ , so  $\text{dist}(G, G^*) \leq C\epsilon$ , as desired.  $\square$

**COROLLARY 7.3.15.** *If  $(s, q)$  is in case (i) or (ii) of Corollary 7.3.12, then  $\mathcal{P}$  has a stability theorem.*

**PROOF.** Suppose  $(s, q)$  is in case (i) or (ii) of Corollary 7.3.12. Let  $C$  and  $M$  be as in Lemma 7.3.14. Fix  $\delta > 0$  and choose  $\epsilon_1 > 0$  and  $M_1$  as in Theorem 7.3.13 for  $\delta/4$ . Let  $\epsilon = \min\{\epsilon_1, \delta/(4C)\}$  and

$n \geq \max\{M, M_1\}$ . Suppose  $G \in \mathcal{R}([n], \mathcal{P})$  and  $\text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}$ . We want to show there is  $G' \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$  such that  $\text{dist}(G, G') \leq \delta$ . By Lemma 7.3.14, there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed and  $C\epsilon = \delta/4$ -close to  $G$ .

Proposition 7.3.7 implies  $\Psi(G^*)$  is an  $(s, q)$ -graph and  $\Psi(G^*)^+$  is an  $(s, q + \binom{s}{2})$ -graph. Corollary 7.3.5 implies  $\text{sub}(G^*) = P(\Psi(G^*))^+$ . Thus  $\Psi(G^*)^+$  is an  $(s, q + \binom{s}{2})$ -graph with  $P(\Psi(G^*)^+) > \text{ex}(n, \mathcal{P})^{1-\epsilon} = \text{ex}_{\Pi}(s, q + \binom{s}{2})^{1-\epsilon}$ , where the equality is from Proposition 7.3.9. Theorem 7.3.13 implies  $\Psi(G^*)^+$  is  $\delta/4$ -close in the sense of Chapter 5 to an element of  $\mathcal{P}(n, s, q + \binom{s}{2})$ , say  $H = ([n], w^H)$ . Let  $H' = ([n], w^{H'})$  be such that  $(H')^+ = H$ , i.e. for all  $xy \in \binom{[n]}{2}$ ,  $w^{H'}(xy) = w^H(xy) - 1$ . Clearly Definition 7.3.4 implies  $H' \in \mathcal{P}(n, s, q)$  and  $\Psi(G)$  is  $\delta$ -close to  $H'$  in the sense of Chapter 5. By Proposition 7.3.9,  $\Psi^{-1}(H') \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . We claim  $G^*$  is  $\delta/2$ -close to  $\Psi^{-1}(H')$ . Note that because  $G^*$  is downward closed, if  $\Psi(G^*) = (V, w^{G^*})$ , then

$$\{i : p_i(x, y) \in \text{Ch}_{G^*}(xy)\} = \{0, \dots, w^{G^*}(xy)\}.$$

Similarly, since  $\Psi^{-1}(H')$  is downward closed (by definition of  $\Psi^{-1}$ ), we have

$$\{i : p_i(x, y) \in \text{Ch}_{\Psi^{-1}(H')}(xy)\} = \{0, \dots, w^{H'}(xy)\}.$$

Thus

$$\begin{aligned} \text{diff}(G^*, \Psi^{-1}(H')) &= \left\{ (x, y) \in V^2 : \{i : p_i(x, y) \in \text{Ch}_{G^*}(xy)\} \neq \{i : p_i(x, y) \in \text{Ch}_{\Psi^{-1}(H')}(xy)\} \right\} \\ &= \left\{ (x, y) \in V^2 : w^{G^*}(xy) \neq w^{H'}(xy) \right\}. \end{aligned}$$

Since  $\Psi(G^*)$  and  $H'$  are  $\delta/4$ -close in the sense of Chapter 5,

$$|\Delta(\Psi(G^*), H')| = |\{xy \in \binom{V}{2} : w^{G^*}(xy) \neq w^{H'}(xy)\}| \leq \delta/4n^2.$$

Thus  $|\text{diff}(G^*, \Psi^{-1}(H'))| \leq \delta/2n^2$  and  $G^*$  and  $\Psi^{-1}(H')$  are  $\delta/2$ -close. Combining what we've shown, we have  $\text{dist}(G, \Psi^{-1}(H')) \leq \text{dist}(G, G^*) + \text{dist}(G^*, \Psi^{-1}(H')) \leq \delta$ .  $\square$

Let  $U_a(n)$  be the set of  $G \in F(n, s, q)$  which are full subgraphs of the unique multigraph in  $\mathbb{U}_a(n)$ . Given  $1 \leq t \leq s/2$ , let  $T_{s-t}(n)$  be the set of  $G \in F(n, s, q)$  which are full subgraphs some element of  $\mathbb{T}_{s-t, a}(n)$ .

We can now apply Theorem 6.4.7 to obtain approximate structure theorems in these cases.

COROLLARY 7.3.16. *Let  $s, q, a, b$  be integers such that  $s \geq 3$ ,  $a, b \geq 0$ , and  $q = a \binom{s}{2} + b$ . For all  $\delta > 0$  there exists  $\beta > 0$  and  $M$  such that for all  $n \geq M$ , the following hold.*

(i) *If  $0 \leq b \leq s - 2$  then*

$$(81) \quad \frac{|F(n, s, q) \setminus \mathbb{U}_{1,a}^\delta(n)|}{|F(n, s, q)|} \leq 2^{-\beta n^2}.$$

(ii) *If  $b = \binom{s}{2} - t$  for some  $1 \leq t \leq s/2$ , then*

$$(82) \quad \frac{|F(n, s, q) \setminus \mathbb{T}_{s-t,a}^\delta(n)|}{|F(n, s, q)|} \leq 2^{-\beta n^2}.$$

PROOF. Assume  $(s, q)$  is in cases (i) or (ii) and fix  $\delta > 0$ . By Corollary 7.3.15,  $\mathcal{P}$  has a stability theorem. Thus, Theorem 6.4.7 implies there is  $\beta > 0$  and  $M$  such that for all  $n \geq M$ ,

$$(83) \quad \frac{|F(n, s, q) \setminus E^\delta(n)|}{|F(n, s, q)|} \leq 2^{-\beta n^2},$$

where

$$E(n) = \{G \in F(n, s, q) : G \trianglelefteq_p G' \text{ some } G' \in \mathcal{R}_{ex}([n], \mathcal{P})\},$$

and  $E^\delta(n) = \{G \in F(n, s, q) : \exists G' \in E(n) \text{ such that } \text{dist}(G, G') \leq \delta\}$ . By Proposition 7.3.9,  $G' \in \mathcal{R}_{ex}([n], \mathcal{P})$  if and only if  $\Psi(G') \in \mathcal{P}(n, s, q)$ . Combining this with Proposition 7.2.2 yields that

$$E(n) = \{G \in F(n, s, q) : G \text{ is a full subgraph of some } H \in \mathcal{P}(n, s, q)\}.$$

Then note that given two multigraphs  $G$  and  $G'$  with the same domain,  $\text{dist}(G, G') \leq \delta$  if and only if  $G$  and  $G'$  are  $\delta/2$ -close in the sense of Chapter 5. Thus  $E^\delta(n)$  is equal to the set

$$\{G \in F(n, s, q) : G \text{ is } \delta/2\text{-close in the sense of Chapter 5 to a full subgraph of some } G' \in \mathcal{P}(n, s, q)\}.$$

If we are in case (i), then Theorem 5.2.3 (Stability) implies that if  $n$  is sufficiently large, the elements of  $\mathcal{P}(n, s, q)$  are  $\delta/2$  in the sense of Chapter 5 to the unique element of  $\mathbb{U}_{1,a}(n)$ . Thus  $E^\delta(n) \subseteq \mathbb{U}_{1,a}^\delta(n)$ . Combining this with (83) yields (81). If we are in case (ii), then Theorems 5.2.5 (Extremal) implies  $\mathcal{P}(n, s, q) = T_{s-t,a}(n)$ , so  $E^\delta(n) = T_{s-t,a}^{\delta/2}(n)$ . Combining this with (83) yields (82).  $\square$

## 7.4. Directed Graphs

In this section we consider results by Kühn, Oshtus, Townsend, and Zhao on the asymptotic structure of digraphs omitting transitive tournaments [73]. We would like to point out that [73] also investigates oriented graphs omitting transitive tournaments as well as digraphs and oriented graphs

omitting cycles. Moreover, their results go much further than the ones we state here, proving precise structure and enumeration results in various cases.

**7.4.1. Statements of results from [73].** We begin with some preliminaries on digraphs and statements of results from [73]. A *digraph* is a pair  $(V, E)$  where  $V$  is a set of vertices and  $E \subseteq V^2$  is a set of *directed edges*. A *tournament* on  $k$  vertices is an orientation of the complete graph on  $k$  vertices. In other words, it is a digraph  $(V, E)$  such that  $|V| = k$  and for all  $xy \in \binom{V}{2}$  exactly one of  $(x, y)$  or  $(y, x)$  is in  $E$ . A tournament  $(V, E)$  is called *transitive* if for all  $x, y, z \in V$ ,  $(x, y) \in E$  and  $(y, z) \in E$  implies  $(x, z) \in E$ . Suppose  $G = (V, E)$  and  $G' = (V', E')$  are digraphs. We say  $G'$  is a *subdigraph* of  $G$ , denoted  $G' \subseteq G$ , if  $V' \subseteq V$  and  $E' \subseteq E$ . If  $V = V'$  then  $G$  is a *full digraph* of  $G$ . We say  $G$  and  $G'$  are *isomorphic* if there is a bijection  $f : V \rightarrow V'$  such that for all  $xy \in \binom{V}{2}$ ,  $(x, y) \in E$  if and only if  $(f(x), f(y)) \in E'$ . Given a digraph  $H$ ,  $G$  is *H-free* if there is no  $G' \subseteq G$  with  $G' \cong H$ . For the rest of this section,  $k \geq 2$  is a fixed integer and  $T_{k+1}$  is a fixed transitive tournament on  $k + 1$  vertices. The subject of this section is the following set of digraphs, where  $n \in \mathbb{N}$ .

$$\text{Forb}_{di}(n, T_{k+1}) = \{G = ([n], E) : G \text{ is a } T_{k+1}\text{-free digraph}\}.$$

Recall that  $T_k(n)$  is the set of  $k$ -partite Turán graphs with vertex set  $[n]$  and  $t_k(n)$  is the number of edges in an element of  $T_k(n)$ . Let  $DT_k(n)$  be the set of digraphs which can be obtained from an element of  $T_k(n)$  by replacing all the edges with two directed edges. More precisely,  $DT_k(n)$  is the set of digraphs  $G = ([n], E)$  such that for some  $G' = ([n], E') \in T_k(n)$ ,  $E = \{(x, y) \in [n]^2 : xy \in E'\}$ . Given a digraph  $G = (V, E)$ , set

$$\begin{aligned} f_1(G) &= \{xy \in \binom{V}{2} : \text{exactly one of } (x, y) \text{ or } (y, x) \text{ is in } E\}, \\ f_2(G) &= \{xy \in \binom{V}{2} : (x, y) \text{ and } (y, x) \text{ are in } E\}, \text{ and} \\ e(G) &= f_1(G) + \log_2(3)f_2(G). \end{aligned}$$

Observe that in this notation the number of full subdigraphs of  $G$  is  $2^{e(G)}$ . Let

$$\max(n, T_{k+1}) = \max\{e(G) : G \in \text{Forb}_{di}(n, T_{k+1})\}.$$

This notion is called “ $\text{ex}(n, T_{k+1})$ ” in [73]. We have changed the notation to avoid confusion with Definition 6.4.1. A digraph  $G \in \text{Forb}_{di}(n, T_{k+1})$  is *edge-extremal* if  $e(G) = \max(n, T_{k+1})$ . The following is Lemma 4.1 from [73].

**THEOREM 7.4.1 (Kühn-Oshtus-Townsend-Zhao [73]).** *For all  $n \in \mathbb{N}$ ,  $\max(n, T_{k+1}) = t_k(n) \log_2(3)$  and  $DT_k(n)$  is the set of edge-extremal elements of  $Forb_{di}(n, T_{k+1})$ .*

Given two digraphs  $G$  and  $G'$  with vertex set  $[n]$  and  $\delta > 0$ , write  $G = G' \pm \delta n^2$  to denote that  $G$  can be obtained from  $G'$  by adding or removing at most  $\beta n^2$  directed edges. Given  $n$  and  $\delta$ , let

$$DT_k^\delta(n) = \{G \in Forb_{di}(n, T_{k+1}) : G = G' \pm \delta n^2 \text{ some } G' \in DT_k(n)\} \text{ and}$$

$$\mathbb{DT}_k(n) = \{G \in Forb_{di}(n, T_{k+1}) : G \subseteq G' \text{ some } G' \in DT_k(n)\} \text{ and}$$

$$\mathbb{DT}_k^\delta(n) = \{G \in Forb_{di}(n, T_{k+1}) : G = G' \pm \delta n^2 \text{ some } G' \in \mathbb{DT}_k(n)\}$$

The following is Lemma 4.3 from [73].

**THEOREM 7.4.2 (Kühn-Oshtus-Townsend-Zhao [73]).** *Let  $k \geq 2$ . For all  $\delta > 0$  there is  $\beta > 0$  such that the following holds for all sufficiently large  $n$ . If  $G \in Forb_{di}(n, T_{k+1})$  satisfies  $e(G) \geq \max(n, T_{k+1}) - \epsilon n^2$ , then  $G \in DT_k^\delta(n)$ .*

The following theorem follows from the proof of Lemma 4.5 from [73].

**THEOREM 7.4.3 (Kühn-Oshtus-Townsend-Zhao [73]).** *Let  $k \geq 2$ . For all  $\delta > 0$  there is  $\beta > 0$  such that the following holds for all sufficiently large  $n$ .*

$$\frac{|Forb_{di}(n, T_{k+1}) \setminus \mathbb{DT}_k^\delta(n)|}{|Forb_{di}(n, T_{k+1})|} \leq 2^{-\beta n^2}.$$

The following is Lemma 5.1 in [73]. They in fact prove a much stronger result where the  $o(n^2)$  error is replaced with  $O(n)$ . The result we quote below is only as strong as what we can recover using Chapter 6.

**THEOREM 7.4.4 (Kühn-Oshtus-Townsend-Zhao [73]).** *Let  $k \geq 2$ .  $|Forb_{di}(n, T_{k+1})| = 3^{t_k(n) + o(n^2)}$ .*

The proof is [73] served as one of the main blueprints for the proofs in Chapter 6 and uses the same tools employed there, namely the hypergraph containers theorem, a graph removal lemma, and a stability theorem.

**7.4.2. Preliminaries.** In this subsection we interpret the basic definitions from Chapter 6 in the setting of digraphs. Let  $\mathcal{L} = \{R(x, y)\}$  consist of a single binary relation symbol. We consider digraphs as  $\mathcal{L}$ -structures in the natural way, that is, given  $G = (V, E)$  a digraph,  $G \models R(x, y)$  if

and only if  $(x, y) \in E$ . Let  $\mathcal{P}$  to be the closure of  $Forb_{di}(T_{k+1})$  under isomorphism. Clearly  $\mathcal{P}$  is a hereditary  $\mathcal{L}$ -property. Since  $r_{\mathcal{L}} = 2$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(x, y) : p \in S_2(\mathcal{P})\}$ . Set

- (1)  $q_1(x, y) = \{x \neq y, R(x, y), \neg R(y, x)\}$
- (2)  $q_2(x, y) = \{x \neq y, R(y, x), \neg R(x, y)\}$
- (3)  $q_3(x, y) = \{x \neq y, R(y, x), R(x, y)\}$
- (4)  $q_4(x, y) = \{x \neq y, \neg R(y, x), \neg R(x, y)\}$ .

For each  $i = 1, 2, 3, 4$ , let  $p_i(x, y)$  be the unique complete quantifier-free 2-type extending  $q_i(x, y)$ .

We leave it to the reader to verify that  $S_2(\mathcal{P}) = \{p_i(x, y) : i \in [4]\}$ . Therefore,

$$\mathcal{L}_{\mathcal{P}} = \{R_{p_1}(x, y), R_{p_2}(x, y), R_{p_3}(x, y), R_{p_4}(x, y)\}.$$

An  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  is *downward closed* if  $G \models \forall x \forall y (R_{p_3}(x, y) \leftrightarrow R_{p_1}(x, y) \wedge R_{p_2}(x, y))$  and  $G \models \forall x \forall y (x \neq y \rightarrow R_{p_4}(x, y))$ . The idea is that if  $G$  is downward closed, then for any pair of elements  $xy$  in  $G$ , if “ $(x, y)$  is an edge” is a choice according to  $G$ , then “ $(x, y)$  is not an edge” is also a choice according to  $G$ .

DEFINITION 7.4.5. Suppose  $G$  is a complete  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  with domain  $V$ . The *digraph associated to  $G$*  is  $\Psi(G) := (V, E)$  where  $(x, y) \in E$  and  $(y, x) \notin E$  if and only if

$$G \models R_{p_1}(x, y) \vee R_{p_2}(y, x) \vee R_{p_3}(x, y) \vee R_{p_3}(y, x),$$

and  $(x, y), (y, x) \in E$  if and only if  $G \models R_{p_3}(x, y) \vee R_{p_3}(y, x)$ .

Given a digraph  $(V, E)$ , define  $\Psi^{-1}(V, E)$  to be the  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$  such that for all  $(x, y) \in V^2$ ,  $G \models R_{p_1}(x, y)$  if and only if  $(x, y) \in E$ ,  $G \models R_{p_2}(x, y)$  if and only if  $(y, x) \in E$ ,  $G \models R_{p_3}(x, y)$  if and only if  $(x, y), (y, x) \in E$ , and  $G \models R_{p_4}(x, y)$  if and only if  $x \neq y$ . We leave it to the reader to verify that  $\Psi(\Psi^{-1}(V, E)) = (V, E)$  and that  $\Psi^{-1}(V, E)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template.

LEMMA 7.4.6. *If  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template, then  $\Psi(G) \trianglelefteq_p G$  and  $G' \trianglelefteq_p G$  implies  $G'$  is a full subdigraph of  $\Psi(G)$ . If further,  $G$  is downward closed, then for any full subdigraph  $G'$  of  $\Psi(G)$ ,  $G' \trianglelefteq_p G$ .*

PROOF. We first show that  $\Psi(G) \trianglelefteq_p G$ . Fix  $uv \in \binom{V}{2}$ . We want to show that if  $p(x, y) = qftp_{\mathcal{L}}^{\Psi(G)}(u, v)$ , then  $p(c_u, c_v) \in Ch_G(uv)$ . If  $p(x, y) = p_1(x, y)$ , then  $(u, v) \in E$  and  $(v, u) \notin E$ . by definition of  $\Psi(G)$ , we must have  $G \models R_{p_1}(u, v) \vee R_{p_2}(v, u) \vee R_{p_3}(u, v) \vee R_{p_3}(v, u)$  and  $G \models \neg R_{p_3}(u, v) \wedge \neg R_{p_3}(v, u)$ . Thus  $G \models R_{p_1}(u, v) \vee R_{p_2}(v, u)$ . Since  $p_2(y, x) = p_1(x, y)$  this implies

by definition that  $p_1(c_u, c_v) \in Ch_G(uv)$ , as desired. The case when  $p(x, y) = p_2(x, y)$  follows a symmetric argument. If  $p(x, y) = p_3(x, y)$ , then  $(u, v)$  and  $(v, u)$  are in  $E$ , so by definition of  $\Psi(G)$ ,  $G \models R_{p_3}(u, v) \vee R_{p_3}(v, u)$ . Since  $p_3(x, y) = p_3(y, x)$  this implies  $p_3(c_u, c_v) \in Ch_G(uv)$  as desired. If  $p(x, y) = p_4(x, y)$ , then neither  $(u, v)$  nor  $(v, u)$  are in  $E$ , so by definition of  $\Psi(G)$ ,

$$G \models \neg R_{p_1}(u, v) \wedge \neg R_{p_2}(v, u) \wedge \neg R_{p_3}(u, v) \wedge \neg R_{p_3}(v, u) \wedge \neg R_{p_1}(v, u) \wedge \neg R_{p_2}(u, v).$$

Since  $G$  is complete, this implies  $G \models R_{p_4}(u, v) \vee R_{p_4}(v, u)$ , which implies  $p_4(c_u, c_v) \in Ch_G(uv)$ , as desired.

Suppose now  $G' \preceq_p G$ . It is clear this implies  $G'$  is a digraph with vertex set  $V$ . Let  $G' = (V, E')$  and  $\Psi(G) = (V, E)$ . We want to show  $E' \subseteq E$ . Fix  $(x, y) \in E'$ . Then  $qftp_{\mathcal{L}}^{G'}(x, y) \in \{p_1(x, y), p_3(x, y)\}$ . Since  $G' \preceq_p G$ , this means either  $G \models R_{p_1}(x, y)$  or  $G \models R_{p_3}(x, y)$ . In either case, by definition of  $\Psi(G)$ ,  $(x, y) \in E$ .

Suppose now  $G$  is downward closed and  $G'$  is a full subdigraph of  $\Psi(G)$ . Let  $G' = (V, E')$  and  $\Psi(G) = (V, E)$ . Fix  $xy \in \binom{V}{2}$ . We want to show  $p(x, y) = qftp_{\mathcal{L}}^{G'}(x, y)$ , then  $G \models R_p(x, y)$ . Because  $G$  is downward closed, if  $p(x, y) = p_4(x, y)$ , then we are done since  $x \neq y$  implies  $G \models R_{p_4}(x, y)$ . If  $p(x, y) = p_1(x, y)$ , then  $(x, y) \in E'$ . Since  $G' \subseteq \Psi(G)$ , this implies  $(x, y) \in E$ . By definition of  $\Psi(G)$ , this implies  $G \models R_{p_1}(x, y) \vee R_{p_2}(y, x) \vee R_{p_3}(x, y) \vee R_{p_3}(y, x)$ . Because  $G$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template and  $p_1(x, y) = p_2(y, x)$ , this implies  $G \models R_{p_1}(x, y)$ , as desired. A similar argument takes care of the case when  $p(x, y) = p_2(x, y)$ . Suppose now  $p(x, y) = p_3(x, y)$ . Then  $(x, y), (y, x) \in E'$ . Since  $G' \subseteq \Psi(G)$ , this implies  $(x, y), (y, x) \in E$ . By definition of  $\Psi(G)$ , this implies

$$G \models R_{p_3}(x, y) \vee R_{p_3}(y, x).$$

Since  $p_3(x, y) = p_3(y, x)$  and  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template, this implies  $G \models R_{p_3}(x, y)$ , so we are done.  $\square$

LEMMA 7.4.7. *If  $H = (V, E) \in Forb_{di}(n, T_{k+1})$ , then  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$  and  $\Psi^{-1}(H)$  is downward closed.*

PROOF. We leave it to the reader to verify that by definition,  $\Psi^{-1}(H)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $[n]$ . To show  $\Psi^{-1}(H)$  is  $\mathcal{P}$ -random, let  $G \preceq_p \Psi^{-1}(H)$ . By Lemma 7.4.6, this implies  $G$  is a full subdigraph of  $\Psi(\Psi^{-1}(H)) = H$ . Since  $H$  is  $T_{k+1}$ -free, so is any digraph. Thus  $G \in \mathcal{P}$ . This shows  $\Psi^{-1}(H)$  is  $\mathcal{P}$ -random.  $\square$

COROLLARY 7.4.8. *If  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template. Then  $sub(G) \leq 2^{e(\Psi(G))}$  and equality holds if  $G$  is downward closed.*

PROOF. Since  $G$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template, Lemma 7.4.6 implies  $\text{sub}(G)$  is at most the number of subdigraphs of  $\Psi(G)$ , which is equal to  $2^{e(\Psi(G))}$  by definition of  $e(\Psi(G))$ . If  $G$  is downward closed, then equality holds by Lemma 7.4.7.  $\square$

PROPOSITION 7.4.9. *Suppose  $G$  is a finite downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. Then  $G$  is  $\mathcal{P}$ -random if and only if  $\Psi(G)$  is  $T_{k+1}$ -free.*

PROOF. Suppose  $\Psi(G)$  is not  $T_{k+1}$ -free. Then  $\Psi(G) \notin \mathcal{P}$ . By Lemma 7.4.6,  $\Psi(G) \leq_p G$ , so this implies  $G$  is not  $\mathcal{P}$ -random. Conversely, suppose  $G$  is not  $\mathcal{P}$ -random. Then there is  $G' \leq_p G$  such that  $G' \notin \mathcal{P}$ . In other words  $G'$  is a digraph which is not  $T_{k+1}$ -free. By Lemma 7.4.6,  $G' \subseteq \Psi(G)$  implies  $\Psi(G)$  is not  $T_{k+1}$ -free.  $\square$

**7.4.3. Counting.** In this subsection we give a proof of Theorem 7.4.4 using Theorem 7.4.1 and Chapter 6.

LEMMA 7.4.10. *If  $G \in \mathcal{R}([n], \mathcal{P})$  then there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed with the property that  $\text{sub}(G^*) \geq \text{sub}(G)(4/3)^{\frac{1}{2}|\text{diff}(G, G^*)|}$ .*

PROOF. Suppose not. Let  $G^*$  be an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $[n]$  and satisfying the following.

- $G^* \models \forall x \forall y (x \neq y \rightarrow R_{p_4}(x, y))$ .
- If  $G \models R_{p_3}(x, y)$  then  $G^* \models R_{p_3}(x, y) \wedge R_{p_1}(x, y) \wedge R_{p_2}(x, y)$ .
- If  $G \models R_{p_1}(x, y)$  then  $G^* \models R_{p_1}(x, y)$  and if  $G \models R_{p_2}(x, y)$  then  $G^* \models R_{p_2}(x, y)$ .

We leave it to the reader to verify that  $G^*$  has the property that for all  $(x, y) \in [n]^2$ ,  $\text{Ch}_{G^*}(xy) \supseteq \text{Ch}_G(xy)$ . If  $(x, y) \in \text{diff}(G, G^*)$ , then  $\text{Ch}_{G^*}(xy) \supsetneq \text{Ch}_G(xy)$  and

$$4 = |S_2(\mathcal{P})| \geq |\text{Ch}_{G^*}(xy)| > |\text{Ch}_G(xy)| \geq 1$$

imply  $\frac{|\text{Ch}_{G^*}(xy)|}{|\text{Ch}_G(xy)|} \geq 4/3$ . Thus Corollary 7.4.8 implies

$$\text{sub}(G^*) = \prod_{uv \in \binom{V}{2}} |\text{Ch}_{G^*}(uv)| = \text{sub}(G) \prod_{\{uv: (u,v) \in \text{diff}(G^*, G)\}} \frac{|\text{Ch}_{G^*}(uv)|}{|\text{Ch}_G(uv)|} \geq \text{sub}(G) \left(4/3\right)^{\frac{1}{2}|\text{diff}(G, G^*)|}.$$

We have only left to show that  $G^* \in \mathcal{R}([n], \mathcal{P})$ . Let  $H \leq_p G^*$ . We want to show  $H \in \mathcal{P}$ . By definition of  $G^*$  and  $\Psi$ ,  $\Psi(G)$  is the same digraph as  $\Psi(G^*)$ . By Proposition 7.4.6,  $H$  is a subdigraph of  $\Psi(G) = \Psi(G^*)$ . By Proposition 7.4.6,  $\Psi(G) \leq_p G$ . Since  $G$  is  $\mathcal{P}$ -random, this implies  $\Psi(G) \in \mathcal{P}$ , which implies any subdigraph of  $\Psi(G)$  is in  $\mathcal{P}$ . In particular,  $H \in \mathcal{P}$ .  $\square$



PROPOSITION 7.4.11. *If  $G$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $[n]$ , then  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$  if and only if  $\Psi(G) \in DT_k(n)$ .*

PROOF. Suppose first  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . By definition  $sub(G) = ex(n, \mathcal{P})$ . By Lemma 7.4.10, if  $G$  is not downward closed, then there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed such that  $sub(G^*) \geq sub(G)(4/3)^{1/2|\text{diff}(G^*, G)|}$ . Since then  $G^*$  is downward closed and  $G$  is not downward closed,  $G^* \neq G$  implies  $|\text{diff}(G^*, G)| \geq 1$ . Thus  $sub(G^*) > sub(G)$ , contradicting that  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . Thus  $G$  is downward closed, so Corollary 7.4.8 implies  $sub(G) = 2^{e(\Psi(G))}$ . Suppose  $\Psi(G) \notin DT_k(n)$ . Then Theorem 7.4.1 implies that for any  $H \in DT_k(n)$ ,  $2^{e(H)} > 2^{e(\Psi(G))}$ . By Lemma 7.4.7,  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$  and is downward closed, so Corollary 7.4.8 implies  $sub(\Psi^{-1}(H)) = 2^{e(H)} > 2^{e(\Psi(G))} = sub(G)$ , contradicting that  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ .

Suppose now that  $\Psi(G) \in DT_k(n)$ . Lemma 7.4.7 implies  $\Psi^{-1}(\Psi(G)) = G \in \mathcal{R}([n], \mathcal{P})$  and is downward closed. Suppose towards a contradiction that  $G \notin \mathcal{R}_{ex}([n], \mathcal{P})$ . Then there is  $G' \in \mathcal{R}([n], \mathcal{P})$  such that  $sub(G') > sub(G)$ . By applying Lemma 7.4.10, we may assume  $G'$  is downward closed. By Proposition 7.4.6,  $\Psi(G') \in \mathcal{P}$ . Corollary 7.4.8,  $2^{e(\Psi(G'))} = sub(G') > sub(G) = 2^{e(\Psi(G))}$ , which implies  $e(\Psi(G')) > e(\Psi(G))$ , contradicting that  $\Psi(G)$  is edge extremal (since by Theorem 7.4.1, elements of  $DT_k(n)$  are edge-extremal).  $\square$

COROLLARY 7.4.12.  $ex(n, \mathcal{P}) = 2^{\max(n, T_{k+1})} = 3^{t_k(n)}$ . Consequently  $\pi(\mathcal{P}) = 3^{(1-\frac{1}{k})\frac{1}{2}}$ .

PROOF. Let  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . Then by definition,  $ex(n, \mathcal{P}) = sub(G)$ . Proposition 7.4.11 implies  $\Psi(G) \in DT_k(n)$  and Corollary 7.4.8 implies  $sub(G) = 2^{e(\Psi(G))}$ . Combining these facts with Theorem 7.4.1 implies  $ex(n, \mathcal{P}) = sub(G) = 2^{\max(n, T_{k+1})} = 3^{t_k(n)}$  as desired. By definition of  $\pi(\mathcal{P})$ , this implies  $\pi(\mathcal{P}) = 3^{(1-\frac{1}{k})\frac{1}{2}}$ .  $\square$

**Proof of Theorem 7.4.4.** By Theorem 7.7.8 and Corollary 7.4.12,

$$|\mathcal{P}_n| = |\text{Forb}_{di}(n, T_{k+1})| = ex(n, \mathcal{P})2^{o(n^2)} = 3^{t_k(n)+o(n^2)}.$$

$\square$

**7.4.4. Stability and approximate structure.** In this section we prove Theorem 7.4.3 using Theorem 7.4.2 and Chapter 6.

PROPOSITION 7.4.13.  $\mathcal{P}$  has a stability theorem.

PROOF. Fix  $\delta > 0$ . Choose  $\epsilon$  and  $M$  such that Theorem 7.4.2 implies that for all  $n < M$  and  $G \in \text{Forb}_{di}(n, T_{k+1})$ , if  $e(G) \geq \max(n, T_{k+1})(1 - \epsilon)$ , then  $G \in DT_k^{\delta/2}(n)$ . Suppose now  $G \in \mathcal{R}([n], \mathcal{P})$  satisfies  $\text{sub}(G) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}$ . We want to show there is  $G' \in \mathcal{R}_{ex}([n], \mathcal{P})$  such that  $\text{dist}(G, G') \leq \delta$ . Apply Lemma 7.4.7 to obtain  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed and which satisfies  $\text{sub}(G^*) \geq \text{sub}(G)(4/3)^{\frac{1}{2}|\text{diff}(G^*, G)|}$ . Then our assumptions imply

$$\text{ex}(n, \mathcal{P})^{1-\epsilon} \leq \text{sub}(G)(4/3)^{\frac{1}{2}|\text{diff}(G^*, G)|} \leq \text{sub}(G^*) \leq \text{ex}(n, \mathcal{P}).$$

Rearranging this, we obtain that  $(4/3)^{\frac{1}{2}|\text{diff}(G^*, G)|} \leq \text{ex}(n, \mathcal{P})^\epsilon$ . Assume  $n$  is sufficiently large so that  $\text{ex}(n, \mathcal{P}) \leq \pi(\mathcal{P})^{2n^2}$ . Then we have  $(4/3)^{\frac{1}{2}|\text{diff}(G^*, G)|} \leq \pi(\mathcal{P})^{2\epsilon n^2}$ . Rearranging this implies

$$|\text{diff}(G^*, G)| \leq 4\epsilon\pi(\mathcal{P})n^2 / \log(4/3).$$

Assume  $\epsilon$  was chosen sufficiently small so that  $4\epsilon\pi(\mathcal{P})n^2 / \log(4/3) \leq \delta/2$ , so  $\text{dist}(G, G^*) \leq \delta/2$ .

Then Proposition 7.4.9 implies  $\Psi(G^*) \in \text{Forb}_{di}(n, T_{k+1})$  and Corollary 7.4.8 implies  $2^{e(\Psi(G^*))} = \text{sub}(G^*) \geq \text{ex}(n, \mathcal{P})^{1-\epsilon}$ . By Corollary 7.4.12, this implies  $2^{e(\Psi(G^*))} \geq 2^{\max(n, T_{k+1})(1-\epsilon)}$ , which implies  $e(\Psi(G^*)) \geq \max(n, T_{k+1})(1 - \epsilon)$ . By Theorem 7.4.2, this implies  $\Psi(G^*) \in DT_k^{\delta/2}(n)$ . Let  $H \in DT_k(n)$  be such that  $\Psi(G^*) = H \pm \delta n^2$ . By Proposition 7.4.7,  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$  and is downward closed. Then  $\Psi(\Psi^{-1}(H)) = H \in DT_k(n)$  implies by Proposition 7.4.11 that  $\Psi^{-1}(H) \in \mathcal{R}_{ex}([n], \mathcal{P})$ . We show  $G^*$  and  $\Psi^{-1}(H)$  are  $\delta/2$ -close. Since  $G^*$  and  $\Psi^{-1}(H)$  are downward closed,

$$\text{diff}(G^*, \Psi^{-1}(H)) = \{(x, y) \in V^2 : \text{Ch}_G(xy) \neq \text{Ch}_{\Psi^{-1}(H)}(xy)\} = E(\Psi(G^*))\Delta E(H).$$

Since  $\Psi(G^*) = H \pm \delta n^2$ , this implies  $|\text{diff}(G^*, \Psi^{-1}(H))| \leq |E(\Psi(G^*))\Delta E(H)| \leq \delta/2n^2$ . Combining all this, we have that  $\text{dist}(G, \Psi^{-1}(H)) \leq \text{dist}(G, G^*) + \text{dist}(G^*, \Psi^{-1}(H)) \leq \delta$ . This shows  $G$  is  $\delta$ -close to an element of  $\mathcal{R}_{ex}([n], \mathcal{P})$  so we have shown  $\mathcal{P}$  has a stability theorem.  $\square$

We can now prove Theorem 7.4.2 using theorems from Chapter 6.

**Proof of Theorem 7.4.2.** Proposition 7.4.13 and Theorem 6.4.7 from Chapter 6 imply that for all  $\delta > 0$ , there exist  $\beta > 0$  and  $M$  such that

$$(84) \quad \frac{|\text{Forb}_{di}(n, T_{k+1}) \setminus E^\delta(n, \mathcal{P})|}{|\text{Forb}_{di}(n, T_{k+1})|} \leq 2^{-\beta n^2},$$

where

$$E(n, \mathcal{P}) = \{G \in \text{Forb}_{di}(n, T_{k+1}) : G \leq_p G' \text{ some } G' \in \mathcal{R}_{ex}([n], \mathcal{P})\},$$

and  $E^\delta(n, \mathcal{P}) = \{G \in \text{Forb}_{di}(n, T_{k+1}) : \text{dist}(G, G') \leq \delta \text{ for some } G' \in E(n, \mathcal{P})\}$ . By Proposition 7.4.11,  $G' \in \mathcal{R}_{ex}([n], \mathcal{P})$  if and only if  $\Psi(G') \in DT_{k+1}(n)$  and  $G \preceq_p G'$  if and only if  $G$  is a subdigraph of  $\Psi(G')$ . Thus

$$E(n, \mathcal{P}) = \{G \in \text{Forb}_{di}(n, T_{k+1}) : G \subseteq G' \text{ for some } G' \in DT_k(n)\} = \mathbb{DT}_k(n).$$

Combining this with the fact that  $|\Delta(M, N)| = \frac{1}{2}|\text{diff}(M, N)|$  for any  $M, N \in \text{Forb}_{di}(n, T_{k+1})$  implies  $E^\delta(n, \mathcal{P}) = \mathbb{DT}_k^{\delta/2}(n)$ . Combining this with (84) finishes the proof.

### 7.5. Hereditary properties of hypergraphs

In this section we show that Theorem 7.7.8 agrees with existing enumeration theorems for hereditary properties of hypergraphs. The notation and results in this section are from a paper by Doston and Nagle, [45]. As is pointed out in [45], these results extend those for hereditary properties of graphs from [3, 27] and for families of graphs appearing in [90, 88, 87]. We would like to point out that these results also follow from independent work by Ishigami in [61], which will be considered in detail in Section 7.7 of this chapter.

**7.5.1. Statements of Results from [45].** Given  $k \geq 2$ , a  $k$ -uniform hypergraph is a pair  $G = (V, E)$  where  $V$  is a set of vertices and  $E \subseteq \binom{V}{k}$  is a set of edges. Given two  $k$ -uniform hypergraphs  $G = (V, E)$  and  $G' = (V', E')$ , we say  $G$  is a *subhypergraph* of  $G'$  if  $V \subseteq V'$  and  $E \subseteq E'$ . We say  $G$  is an *induced subhypergraph* of  $G'$  if  $V \subseteq V'$  and  $E = E \cap \binom{V'}{k}$ . A hereditary property of  $k$ -uniform hypergraphs is a nonempty class of finite  $k$ -uniform hypergraphs closed under isomorphism and taking induced subhypergraphs. Given  $\mathcal{P}$  a hereditary property of  $k$ -uniform hypergraphs, the *extremal number* of  $\mathcal{P}$  is

$$\begin{aligned} \max(n, \mathcal{P}) &= \max\{|A| : A \subseteq \binom{[n]}{k} \text{ and there exists } B \subseteq \binom{[n]}{k} \setminus A \text{ such that} \\ &\quad ([n], B \cup A') \in \mathcal{P}_n \text{ for all } A' \subseteq A\} \end{aligned}$$

This notion comes from [45] and is denoted there by “ $ex(n, \mathcal{P})$ ”. We have changed the notation to avoid confusion with Definition 6.4.1. The following is Theorem 1.1 from [45].

**THEOREM 7.5.1 (Doston-Nagle [45]).** *For any  $k \geq 2$  and hereditary property  $\mathcal{P}$  of  $k$ -uniform hypergraphs,*

$$|\mathcal{P}_n| = 2^{\max(n, \mathcal{P}) + o(n^k)}.$$

**7.5.2. Preliminaries.** In this subsection we interpret the basic definitions from Chapter 6 in the setting of  $k$ -uniform hypergraphs. Fix  $k \geq 2$  and let  $\mathcal{L} = \{R(x_1, \dots, x_k)\}$  be the language of  $k$ -uniform hypergraphs. For the rest of this section,  $\mathcal{P}$  is a fixed hereditary property of  $k$ -uniform hypergraphs. Since  $r_{\mathcal{L}} = k$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(\bar{x}) : p \in S_k(\mathcal{P})\}$ . Let  $\bar{x} = (x_1, \dots, x_k)$  and set

$$q_1(\bar{x}) = \{x_i \neq x_j : i \neq j\} \cup \{R(\mu(\bar{x})) : \mu \in Perm(k)\} \cup \{\neg R(\bar{x}') : \cup \bar{x}' \subsetneq \cup \bar{x}\} \text{ and}$$

$$q_2(\bar{x}) = \{x_i \neq x_j : i \neq j\} \cup \{\neg R(\mu(\bar{x})) : \mu \in Perm(k)\} \cup \{\neg R(\bar{x}') : \cup \bar{x}' \subsetneq \cup \bar{x}\}.$$

For  $i = 1, 2$ , let  $p_i(\bar{x})$  be the unique quantifier-free  $k$ -type containing  $q_i(\bar{x})$ . It is straightforward to check that  $S_k(\mathcal{P}) \subseteq \{p_1(\bar{x}), p_2(\bar{x})\}$ . Thus,  $\mathcal{L}_{\mathcal{P}}$  is a nonempty subset of  $\{R_{p_1}(\bar{x}), R_{p_2}(\bar{x})\}$  ( $\mathcal{L}_{\mathcal{P}}$  is nonempty since  $\mathcal{P}$  is nonempty). Observe that  $\mathcal{L}_{\mathcal{P}} = \{R_{p_1}(\bar{x})\}$  if and only if all elements in  $\mathcal{P}$  are complete,  $\mathcal{L}_{\mathcal{P}} = \{R_{p_2}(\bar{x})\}$  if and only if all elements in  $\mathcal{P}$  are independent, and otherwise,  $\mathcal{L}_{\mathcal{P}} = \{R_{p_1}(\bar{x}), R_{p_2}(\bar{x})\}$ .

**7.5.3. Counting.** In this subsection we show that Theorem 7.5.1 is a special case of Theorem 7.7.8.

LEMMA 7.5.2. *For all  $n \geq 2$ ,  $2^{\max(n, \mathcal{P})} = ex(n, \mathcal{P})$ .*

PROOF. By the observations in the preceding subsection,  $\emptyset \neq \mathcal{L}_{\mathcal{P}} \subseteq \{R_{p_1}(\bar{x}), R_{p_2}(\bar{x})\}$ . We first show that  $2^{\max(n, \mathcal{P})} \leq ex(n, \mathcal{P})$ . Let  $A \subseteq \binom{[n]}{k}$  be such that  $|A| = \max(n, \mathcal{P})$  and there is  $B \subseteq \binom{[n]}{k} \setminus A$  such that  $([n], B \cup A') \in \mathcal{P}$  for all  $A' \subseteq A$ . Define an  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  as follows. Let  $dom(G) = [n]$ , and for each  $\bar{a} \in [n]^r \setminus [n]^r$ , set  $G \models R_{p_2}(\bar{a})$ . Suppose now  $\bar{a} \in [n]^r$ . If  $\cup \bar{a} \subseteq B$ , then define  $G \models R_{p_1}(\bar{a}) \wedge \neg R_{p_2}(\bar{a})$ . If  $\cup \bar{a} \in A$ , define  $G \models R_{p_1}(\bar{a}) \wedge R_{p_2}(\bar{a})$ . For all other cases, define  $G \models \neg R_{p_1}(\bar{a}) \wedge R_{p_2}(\bar{a})$ . By construction,  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template. We claim  $G \in \mathcal{R}([n], \mathcal{P})$ . By construction, for each  $e \in \binom{[n]}{k}$  the following holds. If  $e \in B$  then  $Ch_G(e) = \{p_1\}$ . If  $e \in A$  then  $Ch_G(e) = \{p_1, p_2\}$  and for all other  $e$ ,  $Ch_G(e) = \{p_2\}$ . Suppose  $\chi$  is a choice function for  $G$ . Define a hypergraph  $H = ([n], E)$  by setting  $E = \{e \in \binom{[n]}{k} : \chi(e) = p_1\}$ . Because for each  $e$ ,  $\chi(e) \in Ch_G(e)$ , our definitions of  $\chi$  and  $E$  imply  $E \subseteq B \cup A$ . By assumption, any such  $H$  is in  $\mathcal{P}$ . By definition of  $H$ ,  $H \preceq_{\chi} G$ , so we have shown that for all choice functions  $\chi$  for  $G$ , there is  $H \in \mathcal{P}$  with  $H \preceq_{\chi} G$ , that is,  $G \in \mathcal{R}(n, \mathcal{P})$ . Thus  $sub(G) \leq ex(n, \mathcal{P})$ . By Lemma 6.3.19 and definition of  $G$ ,

$$sub(G) = \prod_{e \in \binom{[n]}{k}} |Ch_G(e)| = 2^{|A|} = 2^{\max(n, \mathcal{P})}.$$

This shows  $ex(n, \mathcal{P}) \geq 2^{\max(n, \mathcal{P})}$ . We now show  $ex(n, \mathcal{P}) \leq 2^{\max(n, \mathcal{P})}$ . Let  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$  and set

$$A = \{e \in \binom{[n]}{k} : Ch_G(e) = \{p_1, p_2\}\} \quad \text{and} \quad B = \{e \in \binom{[n]}{k} : Ch_G(e) = \{p_1\}\}.$$

By definition,  $B \subseteq \binom{[n]}{k} \setminus A$ . We claim that for all  $A' \subseteq A$ ,  $H_{A'} := ([n], B \cup A') \in \mathcal{P}_n$ . Because  $G \in \mathcal{R}([n], \mathcal{P})$ , it suffices to show that  $H_{A'} \leq_p G$ . Define  $\chi : \binom{[n]}{k} \rightarrow S_k(\mathcal{P}, C_{[n]})$  as follows.

$$\chi(e) = \begin{cases} p_1 & \text{if } e \in B \cup A' \\ p_2 & \text{if } e \in \binom{[n]}{k} \setminus (B \cup A'). \end{cases}$$

It is straightforward to check that by definition of  $A$  and  $B$ ,  $\chi$  is a choice function for  $G$  and  $H_{A'} \leq_\chi G$ . Thus  $H_{A'} \in \mathcal{P}_n$  for all  $A' \subseteq A$ . This implies by definition that  $\max(n, \mathcal{P}) \geq |A|$ . By Lemma 6.5.3 and definition of  $A$ ,

$$sub(G) = \prod_{e \in \binom{[n]}{k}} |Ch_G(e)| \leq 2^{|A|}.$$

Because  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ , this implies  $ex(n, \mathcal{P}) = sub(G) \leq 2^{|A|} \leq 2^{\max(n, \mathcal{P})}$ . Thus  $ex(n, \mathcal{P}) = 2^{\max(n, \mathcal{P})}$ , as desired.  $\square$

**Proof of Theorem 7.5.1.** By Lemma 7.5.2 and Theorem 7.7.8,  $|\mathcal{P}_n| = ex(n, \mathcal{P})2^{o(n^k)} = 2^{\max(n, \mathcal{P}) + o(n^k)}$ .  $\square$

REMARK 7.5.3. In the setting of graphs,  $2^{\max(n, \mathcal{P})} \approx 2^{(1-\frac{1}{c})\binom{n}{2}}$ , where  $c$  is the coloring number of  $\mathcal{P}$ . By definition,  $ex(n, \mathcal{P}) \approx \pi(\mathcal{P})\binom{n}{2}$ , thus  $\pi(\mathcal{P}) \approx 2^{1-\frac{1}{c}}$ .

## 7.6. Triangle-free Triple Systems

Let  $F$  be the hypergraph with vertex set  $\{1, 2, 3, 4, 5\}$  and edge set  $\{123, 124, 345\}$ , where  $xyz$  denotes the set  $\{x, y, z\}$ . A 3-uniform hypergraph is called *triangle-free* (or  $F$ -free) if it contains no subgraph isomorphic to  $F$ . This section considers the results from [24], [62], [20] about triangle-free 3-uniform hypergraphs. We will use much of the notation from Subsection 7.5 of this chapter.

**7.6.1. Statements of Results from [24], [62], and [20].** A 3-uniform hypergraph  $G = (V, E)$  is called *tripartite* if and only if there is some partition  $U_1, U_2, U_3$  of  $V$  such that  $xyz \in E$  implies  $x, y$ , and  $z$  are all in different parts of the partition. A tripartite 3-uniform hypergraph is called *balanced* if the partition  $U_1, U_2, U_3$  can be chosen to be an equipartition. Given  $n$ , let  $E(n)$  denote the set of balanced tripartite 3-uniform hypergraph on  $[n]$  and let  $e(n)$  be the number of edges in an element of  $E(n)$ .

Let  $ex_{edge}(n, F)$  be the maximal number of edges in an  $F$ -free 3-uniform hypergraph on  $n$ . If  $G$  is an  $F$ -free 3-uniform hypergraph with  $n$  vertices and  $ex_{edge}(n, F)$  edges, we say  $G$  is *edge-extremal*. Let  $F(n)$  be the set of  $F$ -free 3-uniform hypergraphs with vertex set  $[n]$ .

The following is a consequence of the main theorem in [24].

**THEOREM 7.6.1 (Bollobás [24]).** *Suppose  $G = (V, E)$  is a triangle-free 3-uniform hypergraph on  $n$  vertices. Then  $|E| \leq e(n)$ . If  $|E| = e(n)$ , then  $G$  is isomorphic to an element of  $E(n)$ .*

It is observed in [20] that  $e(n) = \lfloor \frac{n}{3} \rfloor \lfloor \frac{n+1}{3} \rfloor \lfloor \frac{n+2}{3} \rfloor = \frac{n^3}{27} + o(n^3)$ . Combining this with a theorem of Nagle and Rödl in [83], we obtain the following theorem.

**THEOREM 7.6.2 (Nagle-Rödl [83], Bollobás [24]).**  $|F(n)| = 2^{e(n)+o(n^2)} = 2^{\frac{n^3}{27}+o(n^3)}$ .

From now on, in this section, we will just say “hypergraph” in place of “3-uniform hypergraph.” Given a hypergraph  $G = (V, E)$  and a partition  $U_1, U_2, U_3$  of  $V$ , a *non-crossing edge* for the partition is an edge  $xyz \in E$  such that for some  $i \in \{1, 2, 3\}$ ,  $|xyz \cap U_i| \geq 2$ . The following is Theorem 5 in [20], which is proved in [62].

**THEOREM 7.6.3 (Keevash-Mubayi [62]).** *For every  $\delta > 0$  there is an  $\epsilon > 0$  and  $M$  such that for all  $n > M$ , if  $G = (V, E)$  is a triangle-free hypergraph with  $|V| = n$  and  $|E| \geq (1 - \epsilon)\frac{n^3}{27}$ , then there is a partition  $U_1, U_2, U_3$  of  $V$  such that  $E$  contains at most  $\delta n^3$  crossing edges with respect to this partition.*

Given an element  $G$  of  $F(n)$ , an optimal partition of  $G = (V, E)$  is a partition  $U_1, U_2, U_3$  of  $[n]$  so that  $E$  contains the minimal number of crossing edges for  $U_1, U_2, U_3$ . Given  $\delta > 0$ , let  $F(n, \delta)$  be the set of  $G \in F(n)$  such that there is an optimal partition for  $G$  with at most  $\delta n^3$  crossing edges. Then Theorem 7.6.3 and a hypergraph regularity lemma are used in [20] to prove the following.

**THEOREM 7.6.4 (Balogh-Mubayi [20]).** *For all  $\delta > 0$  there is  $\beta > 0$  such that for sufficiently large  $n$ , the following holds.*

$$\frac{|F(n) \setminus F(n, \delta)|}{|F(n)|} \leq 2^{-\beta n^3}.$$

Given hypergraphs  $G = (V, E)$  and  $G' = (V, E')$ , let  $\Delta(G, G') = E\Delta E'$ . We leave it as an exercise to check that  $\Delta(G, G') = \{xyz \in \binom{V}{3} : (x, y, z) \in \text{diff}(G, G')\}$ . Thus

$$|\Delta(G, G')| = \frac{1}{3!} \text{dist}(G, G') = \frac{1}{6} \text{dist}(G, G').$$

The arguments for the following corollary are either standard or appear somewhere in [20].

**COROLLARY 7.6.5.** *For all  $\delta > 0$  there is an  $\epsilon > 0$  and  $M$  such that for all  $n > M$ , the following hold. If  $G = ([n], E) \in F(n)$  and  $|E| \geq (1 - \epsilon)\frac{n^3}{27}$ , then  $G$  is  $\delta$ -close to an element of  $E(n)$ .*

**PROOF.** Fix  $\delta > 0$  and choose  $\epsilon_1 \leq \delta^2$  and  $M_1$  so that Theorem 7.6.3 holds for  $\delta^2/2$ . Fix  $n > M$  and  $G = ([n], E) \in F(n)$  such that  $|E| \geq (1 - \epsilon_1)\frac{n^3}{27}$ . By Theorem 7.6.3, there is a partition  $U_1, U_2, U_3$  of  $V$  such that  $E$  contains at most  $(\delta^2/2)n^3$  crossing edges with respect to this partition. Let  $\epsilon_2 = 6\delta$ . Suppose towards a contradiction that for some  $i \in \{1, 2, 3\}$ ,  $|U_i| > n/3 + \epsilon_2 n$ , say  $|U_1| > n/3 + \epsilon_2 n$ . Let  $x = |U_1| - n/3$ . Then we have  $|U_2| + |U_3| \leq 2n/3 - x$ , which implies by the AM-GM inequality that  $|U_2||U_3| \leq (n/3 - x/2)^2$ . So

$$|U_1||U_2||U_3| = (n/3 + x)\left(\frac{n}{3} - \frac{x}{2}\right)^2 = (n/3 + x)(n^2/9 - nx/3 + x^2/4) = \frac{n^3}{27} - \frac{nx^2}{4} + \frac{x^3}{4}.$$

Since  $|U_1| = n/3 + x \leq n$ , we have that  $x \leq 2n/3$ . Thus

$$\frac{n^3}{27} - \frac{nx^2}{4} + \frac{x^3}{4} = \frac{n^3}{27} + \frac{x^2}{4}(x - n) \leq \frac{n^3}{27} + \frac{x^2}{4}(-n/3) = \frac{n^3}{27} - \frac{x^2 n}{12}.$$

Since  $x \geq \epsilon_2 n$ , this implies  $|U_1||U_2||U_3| \leq \frac{n^3}{27} - \frac{\epsilon_2^2 n^3}{12}$ . But now the total number of edges in  $G$  is by assumption at most

$$(\delta^2/2)n^3 + \frac{n^3}{27} - \frac{\epsilon_2^2 n^3}{12} = \frac{n^3}{27} \left(1 + \frac{27\delta^2}{2} - \frac{\epsilon_2^2 27}{12}\right) < (1 - \epsilon_1)\frac{n^3}{27},$$

where the last inequality is because  $\epsilon_2 = 6\delta$  and  $\epsilon_1 \leq \delta^2$ . This contradicts that  $|E| > (1 - \epsilon_1)\frac{n^3}{27}$ .

Thus for each  $i$ ,  $||U_i| - n/3| \leq \epsilon_2 n$ . If  $n$  is sufficiently large, this implies there is an equipartition  $V_1, V_2, V_3$  of  $[n]$  such that for each  $i$ ,  $|V_i \Delta U_i| \leq 2\epsilon_2 n$ . Let  $G' = (V, E')$  be the complete tripartite hypergraph with parts  $V_1, V_2, V_3$ . Consider the following subsets of  $\Delta(G, G')$ .

- Let  $\Delta_1$  be the set of  $e \in \Delta(G, G')$  such that  $e$  contains a vertex in  $V_i \Delta U_i$  for some  $i \in \{1, 2, 3\}$ . For each  $i = 1, 2, 3$ ,  $|V_i \Delta U_i| \leq 2\epsilon_2 n$  so there are at most  $2\epsilon_2 n^2$  edges containing a vertex in  $V_i \Delta U_i$ . Thus  $|\Delta_1| \leq 6\epsilon_2 n^3$ .
- Let  $\Delta_2$  be the set of  $e \in E$  which are crossing edges for  $U_1, U_2, U_3$  which are also crossing edges for  $V_1, V_2, V_3$ . By assumption on the  $U_i$ ,  $|\Delta_2| \leq \delta^2 n^3/2$  edges.
- Let  $\Delta_3$  be the set of  $e \in \binom{[n]}{3}$  which are non-crossing in  $U_1, U_2, U_3$  and non-crossing in  $V_1, V_2, V_3$  but which are not in  $E$ . Since  $G$  is  $F$ -free, we must have that  $|\Delta_3| \leq e(n) - |E|$ . Since  $|E| \geq (1 - \epsilon_1)\frac{n^3}{27}$  and  $e(n) = \frac{n^3}{27} + o(n^3)$ , we may assume  $n$  is sufficiently large so that  $|\Delta_3| \leq |E| - e(n) \leq 2\epsilon_1 n^3/27$ .

We claim  $\Delta(G, G') = \Delta_1 \cup \Delta_2 \cup \Delta_3$ . Indeed, suppose  $e \in \Delta(G, G') \setminus \Delta_1$ . Then either  $e$  is crossing for  $U_1, U_2, U_3$  and for  $V_1, V_2, V_3$  or  $e$  is non-crossing for  $U_1, U_2, U_3$  and for  $V_1, V_2, V_3$ . If  $e$  is crossing for  $U_1, U_2, U_3$  and for  $V_1, V_2, V_3$ , then by definition of  $G'$ ,  $e \in E'$ , so  $e \in \Delta(G, G')$  implies  $e \notin E$ . This shows  $e \in \Delta_3$ . On the other hand, if  $e$  is non-crossing for  $U_1, U_2, U_3$  and for  $V_1, V_2, V_3$ , then by definition of  $G'$ ,  $e \notin E'$ , so  $e \in \Delta(G, G')$  implies  $e \in E$ . This shows  $e \in \Delta_2$ . Thus  $\Delta(G, G') = \Delta_1 \cup \Delta_2 \cup \Delta_3$  and our bounds above for  $|\Delta_i|$  imply the following.

$$|\Delta(G, G')| \leq n^3(\delta^2/2 + 12\epsilon_2 + 2\epsilon_1/27) < 74\delta n^3,$$

where the last inequality is because  $\epsilon_2 = 6\delta$  and  $\epsilon_1 \leq \delta^2/2$ . Thus  $|\text{diff}(G, G')| < 6(74)\delta n^3$ , so  $\text{dist}(G, G') < 6(74)\delta$ . Clearly by scaling the  $\delta$  we start with, we can obtain the conclusion of the corollary.  $\square$

Let  $\mathbb{E}(n) = \{G \in F(n) : G \text{ is a subhypergraph of some } G' \in E(n)\}$ . Given  $\delta > 0$ , let  $\mathbb{E}^\delta(n)$  be the set of  $G \in F(n)$  which are  $\delta$ -close to an element of  $E(n)$ . We will use Corollary 7.6.5 along with the machinery of Chapter 6 to prove the following approximate structure theorem, which can be thought of as a restatement of Theorem 7.6.4.

**THEOREM 7.6.6.** *For all  $\delta > 0$  there is  $\beta > 0$  such that for sufficiently large  $n$ ,*

$$\frac{|F(n) \setminus \mathbb{E}^\delta(n)|}{|F(n)|} \leq 2^{-\beta n^3}.$$

**7.6.2. Preliminaries.** Let  $\mathcal{L} = \{R(x, y, z)\}$  be the language of 3-uniform hypergraph as in Subsection 7.5. Let  $\mathcal{P}$  be the class of all finite triangle-free 3-uniform hypergraphs, considered as  $\mathcal{L}$ -structures. It's clear that  $\mathcal{P}$  is a hereditary  $\mathcal{L}$ -property and  $\mathcal{L}_{\mathcal{P}}$  is the same as  $\mathcal{L}_{\mathcal{P}}$  from Subsection 7.5 with  $k = 3$ , that is  $\mathcal{L}_{\mathcal{P}} = \{p_1(x_1, x_2, x_3), p_2(x_1, x_2, x_3)\}$ , where  $p_1$  is the type saying the  $x_i$  are distinct and contained in a hyperedge, and  $p_2$  is the type saying the  $x_i$  are distinct and not contained in a hyperedge.

**DEFINITION 7.6.7.** Given an  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  with domain  $V$ , the hypergraph associated to  $G$  is  $\Psi(G) := (V, E)$ , where  $E$  is the set of  $A \in \binom{V}{3}$  such that for some enumeration  $\bar{a}$  of  $A$ ,  $G \models p_1(\bar{a})$ .

We say that an  $\mathcal{L}_{\mathcal{P}}$ -template is *downward closed* if  $G \models \forall x \forall y \forall z R_{p_1}(x, y, z) \rightarrow R_{p_2}(x, y, z)$ . Given a hypergraph  $(V, E)$ , define  $\Psi^{-1}(V, E)$  to be the  $\mathcal{L}_{\mathcal{P}}$ -structure  $G$  which has domain  $V$  and for each  $(x, y, z) \in V^3$ ,  $G \models R_{p_1}(x, y, z)$  if and only if  $xyz \in E$ , and  $G \models R_{p_2}(x, y, z)$  if and only if



$x, y, z$  are pairwise distinct. We leave it to the reader to check that for any hypergraph  $(V, E)$ ,  $\Psi(\Psi^{-1}(V, E)) = (V, E)$  and  $\Psi^{-1}(V, w)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template.

LEMMA 7.6.8. *Suppose  $G$  is a complete  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$ . Then  $\Psi(G) \trianglelefteq_p G$  and for all  $G' \trianglelefteq_p G$ ,  $G'$  is a subhypergraph of  $\Psi(G)$ . If  $G$  is also downward closed, then for all  $G'$  a subhypergraph of  $\Psi(G)$ ,  $G' \trianglelefteq_p G$ .*

PROOF. Let  $\Psi(G) = (V, E)$ . We first show  $\Psi(G) \trianglelefteq_p G$ . Let  $uvw \in \binom{V}{3}$  and let  $p(x, y, z) = qftp_{\mathcal{L}}^{\Psi(G)}(u, v, w)$ . We want to show  $G \models R_p(\mu(u, v, w))$  for some  $\mu \in Perm(3)$ . Suppose  $p = p_1$ , so  $uvw \in E$ . Then by definition of  $\Psi(G)$ , there is  $\mu \in Perm(3)$  such that  $G \models R_{p_1}(\mu(u, v, w))$ . Suppose  $p = p_2$ , so  $uvw \notin E$ . Then by definition of  $\Psi(G)$ ,  $G \models \neg R_{p_1}(\mu(u, v, w))$  for all  $\mu \in Perm(3)$ . Since  $G$  is complete, this implies there is some  $\mu \in Perm(3)$  such that  $G \models R_{p_2}(\mu(u, v, w))$ , as desired.

Suppose  $G' = (V, E') \trianglelefteq_p G$  and let  $\Psi(G) = (V, E)$ . We want to show  $E \subseteq E'$ . Let  $uvw \in E'$ . Then  $qftp_{\mathcal{L}}^{G'}(u, v, w) = p_1(x, y, z)$ . Since  $G' \trianglelefteq_p G$ , this implies  $G \models R_{p_1}(\mu(u, v, w))$  for some  $\mu \in Perm(3)$ . By definition of  $\Psi(G)$ , this implies  $uvw \in E$ , as desired. Suppose now that  $G$  is also downward closed. Let  $G' = (V, E')$  be a subhypergraph of  $\Psi(G) = (V, E)$ . Fix  $uvw \in \binom{n}{3}$ . If  $p(x, y, z) = qftp_{\mathcal{L}}^{G'}(u, v, w)$ , we want to show  $G \models R_p(\mu(u, v, w))$  for some  $\mu \in Perm(3)$ . If  $p(x, y, z) = p_1(x, y, z)$ , then  $uvw \in E' \subseteq E$ , which implies by the definition of  $\Psi(G)$  that  $G \models R_{p_1}(\mu(u, v, w))$  for some  $\mu \in Perm(3)$ . Thus  $G \models R_p(\mu(u, v, w))$  for some  $\mu \in Perm(3)$  as desired. Suppose now  $p(x, y, z) = p_2(x, y, z)$ . We want to show  $G \models R_{p_2}(\mu(u, v, w))$  for some  $\mu \in Perm(3)$ . Because  $G$  is complete,  $G \models R_q(\mu(u, v, w))$  for some  $q \in \{p_1, p_2\}$  and  $\mu \in Perm(3)$ . If  $q = p_2$ , then we are done. If  $q = p_1$ , then because  $G$  is downward closed,  $G \models R_{p_1}(\mu(u, v, w))$  implies  $G \models R_{p_2}(\mu(u, v, w))$ . This finishes the proof.  $\square$

COROLLARY 7.6.9. *If  $(V, E) \in \mathcal{P}$ , then  $\Psi^{-1}(V, E) \in \mathcal{R}(V, \mathcal{P})$  and is downward closed.*

PROOF. We leave it to the reader to check that by definition,  $\Psi^{-1}(V, E)$  is a downward closed  $\mathcal{L}_{\mathcal{P}}$ -template. To show  $\Psi^{-1}(V, E)$  is  $\mathcal{P}$ -random, let  $H \trianglelefteq_p \Psi^{-1}(V, E)$ . By Lemma 7.6.8,  $H$  is a subhypergraph of  $\Psi(\Psi^{-1}(V, E)) = (V, E)$ . Since  $(V, E) \in \mathcal{P}$ , this implies  $H \in \mathcal{P}$ .  $\square$

COROLLARY 7.6.10. *Suppose  $G$  is a finite complete downward closed  $\mathcal{L}_{\mathcal{P}}$ -structure. Then  $sub(G) = 2^{e(\Psi(G))}$ .*

PROOF. By Lemma 7.6.8, the full subpatterns of  $G$  are exactly the subhypergraphs of  $\Psi(G)$ . Since the number of subhypergraphs of  $\Psi(G)$  is  $2^{e(\Psi(G))}$ , this finishes the proof.  $\square$

**7.6.3. Counting.** In this section we prove the enumeration theorem, Theorem 7.6.2. We begin with some preliminary results.

LEMMA 7.6.11. *Suppose  $G \in \mathcal{R}([n], \mathcal{P})$  is not downward closed. Then there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed such that  $\text{sub}(G^*) \geq \text{sub}(G)2^{\frac{1}{6}|\text{diff}(G, G^*)|}$ .*

PROOF. Define  $G^*$  to agree with  $G$  everywhere except on  $uvw \in \binom{[n]}{3}$  where  $\text{Ch}_G(uvw) = \{p_1(c_u, c_v, c_w)\}$ . On such  $uvw$ , define  $G^* \models R_{p_1}(\mu(u, v, w)) \wedge R_{p_2}(\mu(u, v, w))$  for all  $\mu \in \text{Perm}(3)$ . We leave it to the reader to verify that  $G^*$  is an  $\mathcal{L}_{\mathcal{P}}$ -template (by definition and because  $G$  is). Observe that for all  $(x, y, z) \in [n]^3$ ,  $\text{qftp}_{\mathcal{L}_{\mathcal{P}}}^G(x, y, z) \neq \text{qftp}_{\mathcal{L}_{\mathcal{P}}}^{G^*}(x, y, z)$  if and only if  $\text{Ch}_G(xyz) \neq \text{Ch}_{G^*}(xyz)$ . Suppose  $(x, y, z) \in \text{diff}(G, G^*)$ . Then  $\text{Ch}_G(xyz) \neq \text{Ch}_{G^*}(xyz)$  and  $\text{Ch}_G(xyz) \subseteq \text{Ch}_{G^*}(xyz)$  implies  $\text{Ch}_G(xyz) \subsetneq \text{Ch}_{G^*}(xyz)$ . Since  $S_3(\mathcal{P})$  contains only two elements, and because  $G$  complete implies  $|\text{Ch}_G(xyz)| \geq 1$ , this implies  $|\text{Ch}_G(xyz)| = 1$  and  $|\text{Ch}_{G^*}(xyz)| = 2$ . Therefore

$$\text{sub}(G^*) = \text{sub}(G) \left( \prod_{\{xyz:(x,y,z) \in \text{diff}(G, G^*)\}} \frac{|\text{Ch}_{G^*}(xyz)|}{|\text{Ch}_G(xyz)|} \right) = \text{sub}(G)2^{\frac{1}{6}|\text{diff}(G, G^*)|}.$$

We have only left to show that  $G^*$  is  $\mathcal{P}$ -random. Suppose  $H \trianglelefteq_p G^*$ . By Proposition 7.6.8,  $H$  is a subhypergraph of  $\Psi(G^*)$ . Observe that by definition of  $G^*$  and  $\Psi$ ,  $\Psi(G^*) = \Psi(G)$ . By Proposition 7.6.8,  $\Psi(G) \trianglelefteq_p G$ , so since  $G$  is  $\mathcal{P}$ -random,  $\Psi(G) = \Psi(G^*) \in \mathcal{P}$ . Then  $H$  a subhypergraph of  $\Psi(G^*) = \Psi(G) \in \mathcal{P}$  implies  $H \in \mathcal{P}$ , so  $G^*$  is  $\mathcal{P}$ -random.  $\square$

PROPOSITION 7.6.12. *For all integers  $n \geq 2$ , the following holds. Suppose  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . Then  $\Psi(G) \in E(n)$ . Consequently,  $\text{ex}(n, \mathcal{P}) = 2^{\text{ex}_{\text{edge}}(n, F)}$ .*

PROOF. Suppose  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . By Proposition 7.6.8,  $\Psi(G) \trianglelefteq_p G$ , so since  $G$  is  $\mathcal{P}$ -random,  $\Psi(G) \in \mathcal{P}_n = F(n)$ . By Lemma 7.6.11, if  $G$  were not downward closed, then there is  $G^* \in \mathcal{R}([n], \mathcal{P})$  which is downward closed and such that  $\text{sub}(G^*) \geq \text{sub}(G)2^{\frac{1}{6}|\text{diff}(G, G^*)|}$ . Since  $G$  not downward closed and  $G^*$  downward closed,  $\text{diff}(G, G^*) \neq \emptyset$  would imply  $\text{sub}(G^*) > \text{sub}(G)$ , contradicting that  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . Thus  $G$  is downward closed, so Corollary 7.6.10 implies  $\text{sub}(G) = 2^{e(\Psi(G))}$ . Suppose towards a contradiction that  $\Psi(G) \notin E(n)$ . Then by Theorem 7.6.1, for any  $H \in E(n)$ ,  $2^{e(H)} > 2^{e(\Psi(G))}$ . By Corollary 7.6.9,  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$  and is downward closed. By Corollary 7.6.1,  $\text{sub}(\Psi^{-1}(H)) = 2^{e(H)} > 2^{e(\Psi(G))} = \text{sub}(G)$ , contradicting that  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ . Thus  $\Psi(G) \in E(n)$ . Consequently, we have shown if  $G \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ , then  $\text{sub}(G) = \text{ex}(n, \mathcal{P}) = 2^{e(\Psi(G))} = 2^{e(n)}$ . By Theorem 7.6.1, this shows  $\text{ex}(n, \mathcal{P}) = 2^{\text{ex}_{\text{edge}}(n, F)}$ .  $\square$

COROLLARY 7.6.13.  $\pi(\mathcal{P}) = 2^{6/27}$ .

PROOF. Recall  $\pi(\mathcal{P}) = ex(n, \mathcal{P})^{1/\binom{n}{3}}$ . By Proposition 7.6.12 and Theorem 7.6.1,

$$ex(n, \mathcal{P}) = 2^{ex_{edge}(n, F)} = 2^{e(n)} = 2^{\frac{n^3}{27} + o(n^3)} = 2^{\frac{6}{27}\binom{n}{3} + o(n^3)}.$$

□

**Proof of Theorem 7.6.2.** Corollary 7.6.13 and Theorem 6.4.4 imply the following.

$$|\mathcal{P}_n| = |F(n)| = \pi(\mathcal{P})^{\binom{n}{3} + o(n^3)} = 2^{\frac{6}{27}\binom{n}{3} + o(n^3)} = 2^{\frac{n^3}{27} + o(n^3)}.$$

□

**7.6.4. Stability and Approximate Structure.** In this subsection we prove  $\mathcal{P}$  has a stability theorem in the sense of Chapter 6 and use this to prove Theorem 7.6.6.

LEMMA 7.6.14. *Suppose  $G$  and  $G'$  are in  $\mathcal{R}([n], \mathcal{P})$  and are downward closed. Then for all  $\delta > 0$ ,  $G$  and  $G'$  are  $\delta$ -close if and only if  $\Psi(G)$  and  $\Psi(G')$  are  $\delta$ -close.*

PROOF. It is straightforward to check that because  $G$  and  $G'$  are both  $\mathcal{L}_{\mathcal{P}}$ -templates, for all  $(x, y, z) \in [n]^3$ ,  $qftp_{\mathcal{L}}^G(x, y, z) = qftp_{\mathcal{L}}^{G'}(x, y, z)$  if and only if  $Ch_G(xyz) = Ch_{G'}(xyz)$ . By definition of  $\Psi$  and because  $G$  and  $G'$  are downward closed, for all  $(x, y, z) \in [n]^3$ ,  $qftp_{\mathcal{L}}^{\Psi(G)}(x, y, z) = qftp_{\mathcal{L}}^{\Psi(G')}(x, y, z)$  if and only if  $Ch_G(xyz) = Ch_{G'}(xyz)$ . This shows  $\text{diff}(G, G') = \text{diff}(\Psi(G), \Psi(G'))$ . Clearly this implies the desired conclusion. □

PROPOSITION 7.6.15.  *$\mathcal{P}$  has a stability theorem.*

PROOF. Fix  $\delta > 0$  and choose  $\epsilon < \delta/4$  sufficiently small so that for sufficiently large  $n$ , the conclusion of Proposition 7.6.5 holds for  $\delta/4$ . Suppose  $G \in \mathcal{R}([n], \mathcal{P})$  satisfies  $sub(G) \geq ex(n, \mathcal{P})^{1-\epsilon}$ . Choose  $G^*$  to be a downward closed element of  $\mathcal{R}([n], \mathcal{P})$  as in Lemma 7.6.11. Then

$$(85) \quad ex(n, \mathcal{P}) \geq sub(G^*) \geq sub(G) 2^{\frac{1}{6}|\text{diff}(G, G^*)|} \geq ex(n, \mathcal{P})^{1-\epsilon} 2^{\frac{1}{6}|\text{diff}(G, G^*)|}.$$

Assume  $n$  is sufficiently large so that  $ex(n, \mathcal{P}) \leq \pi(\mathcal{P})^{2n^3}$ . Rearranging (85), we obtain that  $2^{\frac{1}{6}|\text{diff}(G, G^*)|} \leq ex(n, \mathcal{P})^\epsilon \leq \pi(\mathcal{P})^{2\epsilon n^3}$ . Taking logs and rearranging this, we obtain that  $|\text{diff}(G, G^*)| \leq C\epsilon n^3$  where  $C = 12 \log(\pi(\mathcal{P}))/\log 2$ . Assume we chose  $\epsilon$  sufficiently small so that  $C\epsilon \leq \delta/4$ , so  $\text{dist}(G, G^*) \leq \delta/4$ .

Proposition 7.6.8 implies  $\Psi(G^*) \leq_p G^*$ , so since  $G^*$  is  $\mathcal{P}$ -random,  $\Psi(G^*) \in \mathcal{P}_n$ . Corollary 7.6.10 implies  $sub(G^*) = 2^{e(\Psi(G^*))}$ . Our assumptions imply  $sub(G^*) \geq sub(G) \geq 2^{(1-\epsilon)\frac{6}{27}\binom{n}{3}}$ , so this

implies  $e(\Psi(G^*)) \geq (1 - \epsilon) \frac{6}{27} \binom{n}{3}$ . Proposition 7.6.5 then implies  $\Psi(G^*)$  is  $\delta/2$ -close to an element of  $H \in E(n)$ . By Corollary 7.6.9,  $\Psi^{-1}(H) \in \mathcal{R}([n], \mathcal{P})$  and is a downward closed. Thus Lemma 7.6.14 implies that because  $\Psi(G^*)$  and  $\Psi(\Psi^{-1}(H)) = H$  are  $\delta/2$ -close, so are  $G^*$  and  $\Psi^{-1}(H)$ . By Corollary 7.6.10, Proposition 7.6.12, and because  $H \in E(n)$ ,  $\text{sub}(\Psi^{-1}(H)) = 2^{e(H)} = \text{ex}(n, \mathcal{P})$ . Thus  $\Psi^{-1}(H) \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$  and  $\text{dist}(G, \Psi^{-1}(H)) \leq \text{dist}(G, G^*) + \text{dist}(G^*, \Psi^{-1}(H)) \leq \delta$ . This finishes the proof.  $\square$

**Proof of Theorem 7.6.6.** Fix  $\delta > 0$ . Choose  $\beta > 0$  such that Theorem 6.4.7 holds for  $\delta$ . Proposition 7.6.15 and Theorem 6.4.7 imply that for sufficiently large  $n$ ,

$$\frac{|\mathcal{P}_n \setminus E^\delta(n, \mathcal{P})|}{|\mathcal{P}_n|} \leq 2^{-\beta n^3},$$

where recall  $E(n, \mathcal{P}) = \{G \in \mathcal{P}_n : G \trianglelefteq_p G' \text{ for some } G' \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})\}$ . Thus to finish the proof, it suffices to show that  $\mathbb{E}(n) = E(n, \mathcal{P})$ . By Proposition 7.6.12, if  $G' \in \mathcal{R}_{\text{ex}}([n], \mathcal{P})$ , then  $\Psi(G') \in E(n)$ . Lemma 7.6.12 implies  $G \trianglelefteq_p G'$  if and only if  $G$  is a subhypergraph of  $\Psi(G')$ . Thus  $E(n, \mathcal{P}) = \{G \in \mathcal{P}_n : G \text{ is a subhypergraph of some } G' \in E(n)\}$ . By definition, this shows  $E(n, \mathcal{P}) = \mathbb{E}(n)$ .  $\square$

## 7.7. Hereditary properties of colored hypergraphs

In this section we show that Theorem 6.4.4 agrees with existing enumeration theorems for hereditary properties of colored  $k$ -uniform hypergraphs which were proved by Ishigami in [61]. We include this example as it is the most general enumeration theorem of hereditary properties in the literature (to our knowledge). As is pointed out in [61], these results extend those for hereditary properties of hypergraphs from [45] as well as enumeration results for hereditary graph properties in [3, 27].

**7.7.1. Statements of Results from [61].** The definitions and results in this section are from [61]. Given an integer  $k \geq 2$  and a set  $C$ , a  $C$ -colored  $k$ -uniform hypergraph, also called a  $(k, C)$ -graph, is a pair  $G = (V, H)$ , where  $V$  is a vertex set and  $H : \binom{V}{k} \rightarrow C$  is a function. The set  $C$  is called the set of colors. Given two  $(k, C)$ -graphs  $G = (V, H)$  and  $G' = (V', H')$ ,  $G$  is a *subgraph* of  $G'$  if  $V \subseteq V'$  and for all  $e \in \binom{V}{k}$ ,  $H(e) = H'(e)$ . We say  $G$  and  $G'$  are *isomorphic*, denoted  $G \cong G'$ , if there is a bijection  $f : V \rightarrow V'$  such that for all  $e \in \binom{V}{k}$ ,  $H(e) = H'(f(e))$ . A hereditary property of  $(k, C)$ -graphs is a nonempty class of finite  $(k, C)$ -graphs which is closed under subgraphs and isomorphism. Observe that if  $C$  has only two elements, then  $(k, C)$ -graphs can be seen as  $k$ -uniform hypergraphs.

Assume  $k \geq 2$ ,  $C$  is finite, and  $\mathcal{P}$  is a hereditary property of  $(k, C)$ -graphs. Let  $2^C$  denote the powerset of  $C$ . Then an  $n$ -vertex  $(k, 2^C \setminus \emptyset)$ -graph  $G = (V, H)$  is call  $\mathcal{P}$ -good if and only if  $\mathcal{P}$  contains any  $(k, C)$ -graph  $G' = (V, H')$  with the property that for all  $e \in \binom{V}{k}$ ,  $H'(e) \in H(e)$ . Define

$$\max(n, \mathcal{P}) = \max \left\{ \frac{1}{\binom{n}{k}} \sum_{e \in \binom{[n]}{k}} \log_2 |H(e)| : ([n], H) \text{ is a } \mathcal{P}\text{-good } (k, 2^C \setminus \emptyset)\text{-graph} \right\}$$

This notion comes from [61] and is denoted there by “ $\text{ex}(n, \mathcal{P})$ ” in [61]. We have changed the notation to avoid confusion with Definition 6.2.6. The following is Theorem 1.1 from [61].

**THEOREM 7.7.1 (Ishigami [61]).** *Suppose  $k \geq 2$  is a fixed finite integer and  $C$  is a fixed finite set. If  $\mathcal{P}$  is a hereditary property of  $(k, C)$ -graphs then*

$$|\mathcal{P}_n| = 2^{(\max(n, \mathcal{P}) + o(1)) \binom{n}{k}}.$$

The goal of this section is to preprove Theorem 7.7.1 using the machinery of Chapter 6.

**7.7.2. Preliminaries.** In this subsection we interpret the basic definitions from Chapter 6 in the setting of colored hypergraphs. Fix  $k \geq 2$ , a finite set  $C$ , and a hereditary property of  $(k, C)$ -graphs  $\mathcal{P}$ . Assume that for all  $c \in C$ ,  $\mathcal{P}$  contains structures  $G = (V, H)$  such that there is  $e \in \binom{V}{k}$  with  $H(e) = c$  (otherwise replace  $C$  with a smaller set). Let  $\mathcal{L} = \{E_c(x_1, \dots, x_k) : c \in C\}$  consist of a single  $k$ -ary relation symbol for every color  $c \in C$ . We will consider  $(k, C)$ -graphs as  $\mathcal{L}$ -structures in the obvious way, namely, give an  $(k, C)$ -graph  $G = (V, H)$ ,  $\bar{a} \in V^k$ , and  $c \in C$ , define  $G \models E_c(\bar{a})$  if and only if  $|\cup \bar{a}| = k$  and  $c \in H(\cup \bar{a})$ . Since  $r_{\mathcal{L}} = k$ ,  $\mathcal{L}_{\mathcal{P}} = \{R_p(x_1, \dots, x_k) : p \in S_k(\mathcal{P})\}$ . For each  $c \in C$ , define  $q_c(x_1, \dots, x_k)$  to be the following set of formulas, where  $\bar{x} = (x_1, \dots, x_k)$ :

$$\begin{aligned} & \{x_i \neq x_j : i \neq j\} \cup \{E_c(\mu(\bar{x})) : \mu \in \text{Perm}(k)\} \cup \\ & \{\neg R_{c'}(\mu(\bar{x})) : c' \neq c \in C, \mu \in \text{Perm}(k)\} \cup \{\neg R(\bar{x}') : \cup \bar{x}' \subsetneq \bar{x}\}. \end{aligned}$$

Let  $p_c(\bar{x})$  be the unique quantifier-free  $\mathcal{L}$ -type containing  $q_c(\bar{x})$ . We leave it to the reader to verify that  $S_k(\mathcal{P}) = \{p_c(\bar{x}) : c \in C\}$  (note this uses the assumption that  $\mathcal{P}$  contains structures with edges colored by  $c$  for each  $c \in C$ ). The following is an important definition which allows us to translate Ishigami’s result to our setting.

**DEFINITION 7.7.2.** Suppose  $G$  is a complete  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$ . The  $(k, 2^C \setminus \{\emptyset\})$ -graph associated to  $G$  is  $\Psi(G) := (V, H)$ , where for each  $e \in \binom{V}{k}$ ,

$$H(e) = \{c \in C : \text{for some enumeration } \bar{e} \text{ of } e, G \models R_{p_c}(\bar{e})\}.$$

Given a  $(k, 2^C \setminus \{\emptyset\})$ -graph  $(V, H)$ , let  $\Psi^{-1}(V, H)$  be the  $\mathcal{L}_{\mathcal{P}}$ -structure with domain  $V$  such that for each  $\bar{a} \in V^k$  and  $c \in C$ ,  $G \models R_{p_c}(\bar{a})$  if and only if  $|\cup \bar{a}| = k$  and  $c \in H(\cup \bar{a})$ . We leave the following observations to the reader.

**OBSERVATION 7.7.3.** *Suppose  $(V, H)$  is a  $(k, 2^C \setminus \{\emptyset\})$ -graph. Then*

- (a)  $\Psi(\Psi^{-1}(V, H)) = (V, H)$  and
- (b)  $\Psi^{-1}(V, H)$  is an  $\mathcal{L}_{\mathcal{P}}$ -template.

**PROPOSITION 7.7.4.** *Suppose  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$  and  $(V, H) = \Psi(G)$ . Then for any  $(k, C)$ -graph  $M = (V, H')$ ,  $M \trianglelefteq_p G$  if and only if for all  $e \in \binom{V}{k}$ ,  $H'(e) \in H(e)$ .*

**PROOF.** Suppose  $M = (V, H') \trianglelefteq_p G$ . We want to show that for all  $e \in \binom{V}{2}$ ,  $H'(e) \in H(e)$ . Fix  $e \in \binom{V}{k}$  and let  $c = H'(e)$ . Note this means  $qftp^M(\bar{e}) = p_c(\bar{x})$  where  $\bar{e}$  is any enumeration of  $e$ . Then  $M \trianglelefteq_p G$  implies there is an enumeration  $\bar{e}$  of  $e$  such that  $G \models R_{p_c}(\bar{e})$ . By definition of  $\Psi(G)$ ,  $c \in H(e)$ . Thus  $H'(e) \in H(e)$ . Conversely, suppose  $M = (V, H')$  and for all  $e \in \binom{V}{k}$ ,  $H'(e) \in H(e)$ . We define a choice function  $\chi$  for  $G$ . Fix  $e \in \binom{V}{k}$  and let  $c = H'(e)$ . Then by assumption,  $c \in H(e)$ , which implies by definition of  $\Psi$ ,  $G \models R_{p_c}(\bar{e})$  for some enumeration  $\bar{e}$  of  $e$ . Define  $\chi(e) = p_c(c_{\bar{e}})$ . Then by construction,  $\chi \in Ch(G)$  and  $M \trianglelefteq_{\chi} G$ , thus  $M \trianglelefteq_p G$ .  $\square$

**PROPOSITION 7.7.5.** *If  $V$  is a finite set,  $G$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$ , and  $\Psi(G) = (V, H)$ . Then  $sub(G) = \prod_{e \in \binom{V}{k}} |H(e)|$ .*

**PROOF.** By Proposition 7.7.4,  $sub(G) = |\{M : M \trianglelefteq_p G\}|$  is the same as the number of  $(k, C)$ -graphs  $(V, H')$  with the property that for all  $e \in \binom{V}{k}$ ,  $H'(e) \in H(e)$ . Clearly the number of such  $(k, C)$ -graphs is  $\prod_{e \in \binom{V}{k}} |H(e)|$ .  $\square$

**PROPOSITION 7.7.6.** *Let  $V$  be a set. If  $(V, H)$  is a  $\mathcal{P}$ -good  $(k, 2^C \setminus \{\emptyset\})$ -graph, then  $\Psi^{-1}(V, H) \in \mathcal{R}(V, \mathcal{P})$ . If  $G \in \mathcal{R}(V, \mathcal{P})$ , then  $\Psi(G)$  is a  $\mathcal{P}$ -good  $(k, 2^C \setminus \{\emptyset\})$ -graph.*

**PROOF.** Suppose first  $(V, H)$  is a  $\mathcal{P}$ -good  $(k, 2^C \setminus \{\emptyset\})$ -graph. By part (a) of Observation 7.7.3,  $\Psi^{-1}(V, H)$  is an  $\mathcal{L}_{\mathcal{P}}$ -template with domain  $V$ . To show  $\Psi^{-1}(V, H)$  is  $\mathcal{P}$ -random, let  $M = (V, H') \trianglelefteq_p \Psi^{-1}(V, H)$ . We want to show  $M \in \mathcal{P}$ . By part (b) of Observation 7.7.3,  $\Psi(\Psi^{-1}(V, H)) = (V, H)$ . Therefore, Proposition 7.7.4 implies for all  $e \in \binom{V}{k}$ ,  $H'(e) \in H(e)$ . Since  $(V, H)$  is  $\mathcal{P}$ -good, this implies  $M \in \mathcal{P}$ . Thus  $\Psi^{-1}(V, H) \in \mathcal{R}(V, \mathcal{P})$ .

Suppose  $G \in \mathcal{R}(V, \mathcal{P})$ . Let  $\Psi(G) = (V, H)$ . To show  $(V, H)$  is a  $\mathcal{P}$ -good  $(k, 2^C \setminus \{\emptyset\})$ -graph, let  $M = (V, H')$  be such that for all  $e \in \binom{V}{k}$ ,  $H'(e) \in H(e)$ . By Proposition 7.7.4,  $M \preceq_p G$ . Since  $G$  is  $\mathcal{P}$ -random,  $M \in \mathcal{P}$ .  $\square$

COROLLARY 7.7.7. *For all  $n$ ,  $ex(n, \mathcal{P}) = 2^{\max(n, \mathcal{P})} \binom{n}{k}$ .*

PROOF. Fix  $n$ . We first show  $ex(n, \mathcal{P}) \leq 2^{\max(n, \mathcal{P})} \binom{n}{k}$ . Suppose  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$  and  $\Psi(G) = ([n], H)$ . By Proposition 7.7.6,  $\Psi(G)$  is a  $\mathcal{P}$ -good  $(k, 2^C \setminus \{\emptyset\})$ -graph, so by definition,  $\max(n, \mathcal{P}) \geq \frac{1}{\binom{n}{k}} \sum_{e \in \binom{[n]}{k}} \log_2 |H(e)|$ . Therefore

$$2^{\max(n, \mathcal{P})} \binom{n}{k} \geq 2^{\sum_{e \in \binom{[n]}{k}} \log_2 |H(e)|} = \prod_{e \in \binom{[n]}{k}} |H(e)| = sub(G),$$

where the last equality is by Proposition 7.7.5. This shows  $2^{\max(n, \mathcal{P})} \binom{n}{k} \geq sub(G) = ex(n, \mathcal{P})$ , where the equality is because  $G \in \mathcal{R}_{ex}([n], \mathcal{P})$ . We now show  $ex(n, \mathcal{P}) \geq 2^{\max(n, \mathcal{P})} \binom{n}{k}$ . Choose  $G' = ([n], H)$  a  $\mathcal{P}$ -good  $(k, 2^C \setminus \{\emptyset\})$ -graph such that  $\max(n, \mathcal{P}) = \frac{1}{\binom{n}{k}} \sum_{e \in \binom{[n]}{k}} \log_2 |H(e)|$ . Let  $G = \Psi^{-1}(G')$ . Proposition 7.7.6 implies  $G \in \mathcal{R}([n], \mathcal{P})$ , so  $sub(G) \leq ex(n, \mathcal{P})$ . By Proposition 7.7.5,  $sub(G) = \prod_{e \in \binom{[n]}{k}} |H(e)|$ . Thus we have

$$\prod_{e \in \binom{[n]}{k}} |H(e)| = 2^{\sum_{e \in \binom{[n]}{k}} \log_2 |H(e)|} = 2^{\max(n, \mathcal{P})} \binom{n}{k},$$

where the last equality is by choice of  $G'$ . Therefore  $2^{\max(n, \mathcal{P})} \binom{n}{k} = sub(G) \leq ex(n, \mathcal{P})$ , as desired.  $\square$

We now give a restatement of Theorem 6.4.4 which will be convenient for us.

THEOREM 7.7.8 (Restatement of Theorem 6.4.4). *Suppose  $\mathcal{L}$  is a finite relational language with  $r = r_{\mathcal{L}} \geq 2$  and  $\mathcal{H}$  is a hereditary  $\mathcal{L}$ -property. Then  $|\mathcal{H}_n| = ex(n, \mathcal{H}) 2^{o(n^r)}$ .*

PROOF. By Theorem 6.4.4, it suffices to show

$$ex(n, \mathcal{H}) 2^{o(n^r)} = \begin{cases} \pi(\mathcal{H}) \binom{n}{r}^{o(n^r)} & \text{if } \pi(\mathcal{H}) > 1. \\ 2^{o(n^r)} & \text{if } \pi(\mathcal{H}) = 1. \end{cases}$$

This is obvious by definition of  $\pi(\mathcal{H}) = \lim_{n \rightarrow \infty} ex(n, \mathcal{H})^{1/\binom{n}{r}}$ .  $\square$

We now see that Theorem 7.7.1 follows easily from Theorem 7.7.8.

**Proof of Theorem 7.7.1.** Suppose  $\mathcal{P}$  is a hereditary property of  $(k, C)$ -graphs. Theorem 7.7.8 and Corollary 7.7.7 imply  $|\mathcal{P}_n| = \text{ex}(n, \mathcal{P})2^{o(n^k)} = 2^{\max(n, \mathcal{P})\binom{n}{k} + o(n^k)}$ .



## An Application of Model Theoretic Ramsey Theory

The work in this chapter is joint with M. Malliaris and appears in [76]. Recently Chudnovsky, Kim, Oum, and Seymour established that any prime graph contains one of a short list of induced prime subgraphs [39]. A module of a graph  $G = (V, E)$  is a set of vertices  $X \subseteq V$  such that any vertex  $v \in V \setminus X$  is either connected or non-connected to all vertices in  $X$ . Prime graphs are graphs which contain no non-trivial modules. The interest in prime graphs arises from questions around so-called modular decompositions of graphs, as well as the fact that the celebrated Erdős-Hajnal conjecture reduces to the case where the omitted graph is prime.

In this chapter, we re-prove the main theorem of [39] making use of model-theoretic ingredients, in a way that improves the bounds and offers a different structural perspective on the graphs in question. Our background aim is to exemplify the usefulness of model-theoretic ideas in proofs in finite combinatorics. This approach complements that of [75], where certain indicators of complexity which had been identified by people working in combinatorics coincided with model theoretic dividing lines, so could be characterized by means of model theory.

The model-theoretic contribution of the present argument may be described as follows. The proof of [39] proceeds by means of several cases, sketched in section 2 below, and applies Ramsey's theorem as a main tool. In [75] it was shown that Ramsey's theorem works much better when the graph is so-called stable, a finitization of an important structural property identified by model theory (for history, see the introduction to [75] or the original source [94]). Our approach, then, is essentially to reconfigure the proof of [39] so that the procedure for extracting the given configurations is different depending on the degree of stability of the graph, and can take advantage of this additional structural information.

We believe this approach raises some interesting questions about model theory's potential contribution to calibrating arguments about finite objects. We have not tried to construct examples showing the bound we obtain is optimal, in part because we believe that a further development of what might be called 'model-theoretic Ramsey theory' in the spirit of [75] may, in general, allow for even finer calibrations in the finite setting. At the same time, it is important to add that model theory

works here to amplify the combinatorial analysis rather than to replace it. Already in the present argument, the contribution of combinatorics in e.g. identifying definitions such as ‘module’ (which is much stronger than, if in some sense analogous to, the model-theoretic notion of an indiscernible sequence) and in isolating the original collection of induced configurations appears essential. It is the interaction of these ideas and perspectives which to us seems most interesting.

Complementing this approach, the chapter concludes with the proof of an infinite analogue of Theorem 8.2.1 which implies the finite version, but without explicit bounds.

### 8.1. Definitions and notation

In this section we state relevant definitions and notation, most of which, but not all, is from [39]. Given a set  $X$ , let  $\binom{X}{2} = \{Y \subseteq X : |Y| = 2\}$ . A graph is a pair  $(V, E)$  where  $V$  is a set of vertices and  $E \subseteq \binom{V}{2}$  is a set of edges. Unless otherwise stated, all of the following definitions and notation apply to both infinite and finite graphs. Given a graph  $G$ , we write  $xy$  as shorthand for the edge  $\{x, y\}$ . We will often write  $V(G) = V$  and  $E(G) = E$ . A set of vertices  $X$  inside a graph is called a *module* if every vertex outside of  $X$  is adjacent to every vertex in  $X$  or non-adjacent to every vertex in  $X$ . A module  $X$  of a graph  $G$  is called *trivial* if  $|X| = 1$  or  $X = V(G)$ . A graph  $G$  is called *prime* if it has no non-trivial modules. We say a set of vertices  $X$  is *independent* if every pair of vertices in  $X$  is non-adjacent, and we say  $X$  is *complete* if every pair of vertices in  $X$  is adjacent. We say a vertex  $v$  is *mixed* on a subset  $X \subseteq V$  if there are  $x, y \in X$  such that  $vx \in E$  and  $vy \notin E$ . Given a graph  $G = (V, E)$ , the *complement* of  $G$ , denoted  $\overline{G}$ , is the graph with vertex set  $V$  and edge set  $\binom{V}{2} \setminus E$ . Given two graphs  $G$  and  $H$ , we will say  $G$  “contains a copy of  $H$ ” to mean there is an induced subgraph of  $G$  which is isomorphic to  $H$ .

We now introduce important structural configurations which will appear throughout the chapter. Fix an integer  $n \geq 1$ .

- A *half-graph* of height  $n$  is a graph with  $2n$  vertices  $a_1, \dots, a_n, b_1, \dots, b_n$  such that  $a_i$  is adjacent to  $b_j$  if and only if  $i \leq j$ .
- The *bipartite half-graph* of height  $n$ ,  $H_n$ , is a graph with  $2n$  vertices  $a_1, \dots, a_n, b_1, \dots, b_n$  such that  $a_i$  is adjacent to  $b_j$  if and only if  $i \leq j$  and such that  $\{a_1, \dots, a_n\}$  and  $\{b_1, \dots, b_n\}$  are independent sets.
- The *half split graph* of height  $n$ ,  $H'_n$ , is a graph with  $2n$  vertices  $a_1, \dots, a_n, b_1, \dots, b_n$  such that  $a_i$  is adjacent to  $b_j$  if and only if  $i \leq j$  and such that  $\{a_1, \dots, a_n\}$  is an independent set and

$\{b_1, \dots, b_n\}$  is a complete set (a graph is a *split graph* if its vertices can be partitioned into a complete set and an independent set).

- Let  $H'_{n,I}$  be the graph obtained from  $H'_n$  by adding a new vertex adjacent to  $a_1, \dots, a_n$  (and no others). Let  $H_n^*$  be the graph obtained from  $H'_n$  by adding a new vertex adjacent to  $a_1$  (and no others).
- The *thin spider* with  $n$  legs is a graph with  $2n$  vertices  $a_1, \dots, a_n, b_1, \dots, b_n$  such that  $\{a_1, \dots, a_n\}$  is an independent set,  $\{b_1, \dots, b_n\}$  is a complete set, and  $a_i$  is adjacent to  $b_j$  if and only if  $i = j$ . The *thick spider* with  $n$  legs is the compliment of the thin spider with  $n$  legs. In particular, it is a graph with  $2n$  vertices  $a_1, \dots, a_n, b_1, \dots, b_n$  such that  $\{a_1, \dots, a_n\}$  is an independent set,  $\{b_1, \dots, b_n\}$  is a complete set, and  $a_i$  is adjacent to  $b_j$  if and only if  $i \neq j$ . A *spider* is a thin spider or a thick spider.
- A sequence of distinct vertices  $v_0, \dots, v_m$  in a graph  $G$  is called a *chain* from a set  $I \subseteq V(G)$  to  $v_m$  if  $m \geq 2$  is an integer,  $v_0, v_1 \in I$ ,  $v_2, \dots, v_m \notin I$ , and for all  $i > 0$ ,  $v_{i-1}$  is either the unique neighbor or the unique non-neighbor of  $v_i$  in  $\{v_0, \dots, v_{i-1}\}$ . The *length* of a chain  $v_0, \dots, v_m$  is  $m$ .

Given an integer  $m \geq 1$ ,  $K_m$  denotes the complete graph on  $m$ . Given integers  $m, n$ ,  $K_{m,n}$  denotes the complete bipartite graph with parts of sizes  $m$  and  $n$ , that is, the graph with  $m + n$  vertices  $\{a_1, \dots, a_m, b_1, \dots, b_n\}$  such that  $\{a_1, \dots, a_m\}$  and  $\{b_1, \dots, b_n\}$  are independent and  $a_i$  is adjacent to  $b_j$  for each  $1 \leq i \leq m$  and  $1 \leq j \leq n$ . Given a graph  $G = (V, E)$ , the *line graph* of  $G$  is the graph  $G'$  which has vertex set  $V(G') = E(G)$  and edge set consisting of pairs of elements  $e_1 \neq e_2 \in E(G)$  such that  $e_1 \cap e_2 \neq \emptyset$ . Given an integer  $m$ , a *path* of length  $m$  is a set  $v_0, \dots, v_m$  vertices such that  $v_i$  is adjacent to  $v_j$  if and only if  $j = i + 1$  or  $i = j + 1$ . The  *$m$ -subdivision* of a graph  $G$  is the graph obtained from  $G$  by replacing every edge in  $G$  with an induced path of length  $m + 1$ . A perfect matching of height  $n$  is the disjoint union  $n$  edges, that is, a graph with  $2n$  vertices  $\{a_1, \dots, a_n, b_1, \dots, b_n\}$  such that  $\{a_1, \dots, a_n\}$  and  $\{b_1, \dots, b_n\}$  are independent and  $a_i$  is adjacent to  $b_j$  if and only if  $i = j$ .

Note that in all of these definitions except that of a chain and of an  $m$ -subdivision, it makes sense to replace  $m$  and  $n$  by any cardinals  $\lambda$  and  $\mu$ . In section 6, we will wish to discuss versions of some of these configurations where  $m$  or  $n$  is replaced by an infinite cardinal. In those cases, we will use the same notation as laid out in this section.

### 8.2. Outline of proof of main theorem from [39]

In this section we give an outline of the proof of Theorem 8.2.1 presented in [39]. We do this to allow for comparison to the proofs we present in sections 8.4 and 8.5. Our outline consists of the statements of the propositions from [39] which form the main steps in their proof, then a flow chart illustrating the structure of the proof. We think this outline is sufficient for understanding the global structure of the proof. For more details we direct the reader to the original chapter [39]. Throughout  $R(n_1, \dots, n_k)$  denotes the smallest integer  $m$  such for that any coloring of the edges of  $K_m$  with  $k$ , there is complete graph on  $n_i$  vertices in color  $i$  for some  $1 \leq i \leq k$ .

**THEOREM 8.2.1** (Theorem 1.2 of [39]). *For every integer  $n \geq 3$  there is  $N$  such that every prime graph with at least  $N$  vertices contains one of the following graphs or their compliments as an induced subgraph.*

- (1) *The 1-subdivision of  $K_{1,n}$  (denoted by  $K_{1,n}^{(1)}$ ).*
- (2) *The line graph of  $K_{2,n}$ .*
- (3) *The thin spider with  $n$  legs.*
- (4) *The bipartite half-graph of height  $n$ .*
- (5) *The graph  $H'_{n,I}$ .*
- (6) *the graph  $H_n^*$ .*
- (7) *A prime graph induced by a chain of length  $n$ .*

We will use the following fact from [39].

**PROPOSITION 8.2.2** (Corollary 2.3 from [39]). *Let  $t > 3$ . Every chain of length  $t$  contains a chain of length  $t - 1$  inducing a prime subgraph.*

The following are the propositions which form the main steps of the proof of Theorem 8.2.1 in [39].

**PROPOSITION 8.2.3** (Proposition 3.1 from [39]). *For all integers  $n, n_1, n_2 > 0$ , there is  $N = f(n, n_1, n_2)$  such that every prime graph with an  $N$ -vertex independent set contains an induced subgraph isomorphic to*

- (1) *a spider with  $n$  legs,*
- (2)  *$\overline{L(K_{2,n})}$ ,*
- (3) *the bipartite half-graph of height  $n$ ,*
- (4) *the disjoint union of  $n_1$  copies of  $K_2$ , denoted  $n_1 K_2$  (i.e. an induced matching of size  $n_1$ ), or*

(5) the half split graph of height  $n_2$ .

Specifically,  $f(n, n_1, n_2) = 2^{M+1}$  where  $M = R(n_1 + n, 2n - 1, n + n_2, n + n_2 - 1)$ .

PROPOSITION 8.2.4 (Proposition 4.1 from [39]). *Let  $t \geq 2$  and  $n, n'$  be positive integers. Let  $h(n, n', 2) = n$  and*

$$h(n, n', i) = (n - 1)R(n, n, n, n, n, n, n', n', h(n, n', i - 1)) + 1$$

for an integer  $i > 2$ . Let  $v$  be a vertex of a graph  $G$  and let  $M$  be an induced matching of  $G$  consisting of  $h(n, n', t)$  edges not incident with  $v$ . If for each edge  $e = xy$  in  $M$ , there is a chain of length at most  $t$  from  $\{x, y\}$  to  $v$ , then  $G$  has an induced subgraph isomorphic to one of the following:

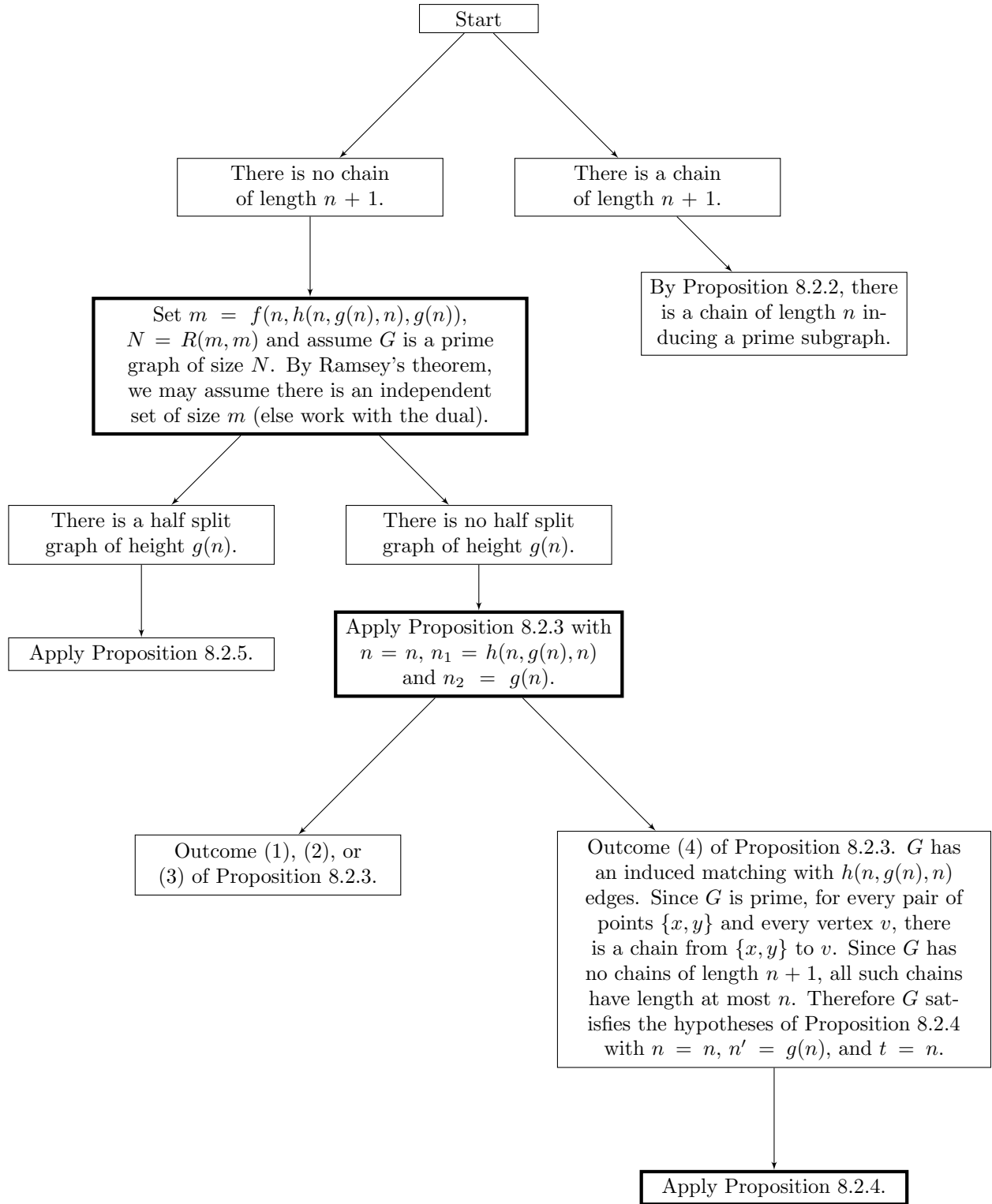
- (1)  $K_{1,n}^{(1)}$ ,
- (2) the bipartite half-graph of height  $n$ ,
- (3)  $\overline{L(K_{2,n})}$ ,
- (4) a spider with  $n$  legs, or
- (5) the half split graph of height  $n'$ .

PROPOSITION 8.2.5 (Proposition 5.1 of [39]). *For every positive integer  $n$ , there exists*

$$N = g(n) = 4^{n-2}(n + 1) + 2(n - 2) + 1$$

such that every prime graph having a half split graph of height at least  $N$  as an induced subgraph contains a chain of length  $n + 1$  or an induced subgraph isomorphic to one of  $H'_{n,I}$ ,  $H_n^*$ ,  $\overline{H_n^*}$ .

In the flow chart below, the bold boxes denote steps which involve Ramsey's theorem. A box with no descendants indicates that the conclusion of the theorem is satisfied in that case. In this chart, the functions  $f$ ,  $h$ , and  $g$  are from Propositions 8.2.3, 8.2.4, and 8.2.5 respectively.



For the rest of the chapter, given  $n \geq 2$ , let  $N_{8.2.1} = N_{8.2.1}(n)$  be the bound obtained for Theorem 8.2.1 in [39], that is,  $N_{8.2.1}(n) = R(m, m)$  where  $m = f(n, h(n, g(n), n), g(n))$ .

REMARK 8.2.6. *Note this proof shows the following: a prime graph  $G$  with an independent set of size  $m$  and no chain of length  $n + 1$  satisfies the conclusion of the theorem.*

### 8.3. Tree Lemma

In this section we prove a key lemma, Theorem 8.3.6, which allows us to improve the bounds in Theorem 8.2.1. This lemma is [75] Theorem 3.5 tailored to the specific setting of graphs. [75] Theorem 3.5 handles arbitrary finite sets of formulas, and uses model-theoretic tools such as types and  $R$ -rank. The bounds there are computed in terms of several associated constants, including the VC-dimension which was used to bound the branching of the trees. For the purposes of the present argument, we give here a streamlined proof for the special case of graphs written with graph theorists in mind. Corollary 8.3.7 gives the bound in this case.

We now state relevant versions of definitions and lemmas from [75].

Recall that a *tree* is a partial order  $(P, \trianglelefteq)$  such that for each  $p \in P$ , the set  $\{q \in P : p \triangleleft q\}$  is a well-order under  $\trianglelefteq$ . Given an integer  $n \geq 2$ , define

$$2^{<n} = \bigcup_{i=0}^{n-1} \{0, 1\}^i,$$

where  $\{0, 1\}^0 = \langle \rangle$  is the *empty string*, and for  $i > 0$ ,  $\{0, 1\}^i$  is the usual cartesian product. This set has a natural tree structure given by  $\eta \trianglelefteq \eta'$  if and only if  $\eta = \langle \rangle$  or  $\eta$  is an initial segment of  $\eta'$ . We will write  $\eta \triangleleft \eta'$  to denote that  $\eta \trianglelefteq \eta'$  and  $\eta \neq \eta'$ . Given  $\eta \in \{0, 1\}^i$ , let  $|\eta| = i$  denote *length* of  $\eta$  (the length of the empty string  $\langle \rangle$  is 0). A main idea in the proof of Theorem 8.3.6 is to take a graph  $G = (V, E)$ , and arrange  $G$  into a tree by indexing its vertex set with elements of  $2^{<n}$ . Suppose  $G = (V, E)$  is a graph, and we have an indexing  $V = \{a_\eta : \eta \in X\}$  of the vertices of  $G$  by some  $X \subseteq 2^{<n}$ . Given  $\eta \in X$ , we will say the *height* of  $a_\eta$ , denoted  $ht(a_\eta)$  is  $|\eta|$ . A *branch* is a set of the form  $\{a_\eta : \eta \in Y\}$  where  $Y$  is a maximal collection of comparable elements in  $X$ . The *length* of a branch is its cardinality. Given  $\eta, \eta' \in 2^{<n}$  and elements  $a_\eta, a_{\eta'}$  indexed by  $\eta$  and  $\eta'$ , we say  $a_\eta$  and  $a_{\eta'}$  *lie along the same branch* if  $\eta \trianglelefteq \eta'$  or  $\eta' \trianglelefteq \eta$ . If  $\eta \triangleleft \eta'$ , we say  $a_\eta$  *precedes*  $a_{\eta'}$ . Given  $\eta = \langle \eta_1, \dots, \eta_i \rangle \in \{0, 1\}^i$ , set  $\eta \wedge 0 = \langle \eta_1, \dots, \eta_i, 0 \rangle$  and  $\eta \wedge 1 = \langle \eta_1, \dots, \eta_i, 1 \rangle$ . If  $x = a_{\eta \wedge 0}$  or  $x = a_{\eta \wedge 1}$ , then we say  $a_\eta$  is the *immediate predecessor* of  $x$  and write  $pred(x) = a_\eta$ . We will also write  $a_\eta \wedge i$  to mean  $a_{\eta \wedge i}$ . Given  $j \in \{0, 1\}$  and  $i \geq 1$ , let  $j^i$  denote the element of  $\{0, 1\}^i$  which has every coordinate equal to  $j$ .

DEFINITION 8.3.1. Given a graph  $G = (V, E)$  on  $n$  vertices and  $A \subseteq 2^{<n}$ , we say that an indexing  $V = \{a_\eta : \eta \in A\}$  of  $V$  by the elements of  $A$  is a *type tree*, if for each  $\eta \in A$  the following holds.

- If  $\eta \wedge 0 \in A$ , then  $a_{\eta \wedge 0}$  is non-adjacent to  $a_\eta$ . If  $\eta \wedge 1 \in A$ , then  $a_{\eta \wedge 1}$  is adjacent to  $a_\eta$ .
- If  $\eta \wedge 0$  and  $\eta \wedge 1$  are both in  $A$ , then for all  $\eta' \triangleleft \eta$ ,  $a_{\eta \wedge 1}$  is adjacent to  $a_{\eta'}$  if and only if  $a_{\eta \wedge 0}$  is adjacent to  $a_{\eta'}$ .

This notion of type tree is a special case of the model theoretic notion of a type tree. We believe for the purposes of this chapter it is better to deal only with this special version for graphs. For the general definition, see [94]. Given a graph  $(V, E)$  and  $v \in V$ , let  $N(v) = \{w \in V : vw \in E\}$  be the *neighborhood* of  $v$ .

LEMMA 8.3.2. *Every finite graph  $G = (V, E)$  can be arranged into a type tree.*

PROOF. Suppose  $|V| = n$ . We arrange the vertices of  $G$  into a type tree indexed by a subset of  $2^{<n}$ .

- Stage 1: Choose any element of  $G$  to be  $a_\emptyset$ , and set  $A_0 = \{a_\emptyset\}$ . Set  $X_1 = N(a_\emptyset)$  and  $X_0 = V \setminus (\{a_\emptyset\} \cup N(a_\emptyset))$ . Note  $X_1, X_0$  partition  $V \setminus A_0$ .
- Stage  $m + 1$ . Suppose we've defined elements in the tree up to height  $m \geq 0$  and for each  $0 \leq i \leq m$ ,  $A_i$  is the set vertices of height  $i$ . Suppose further that we have a collection of sets of vertices  $\{X_{\eta \wedge i} : \eta \in A_m, i \in \{0, 1\}\}$  which partition  $V \setminus \bigcup_{i=1}^m A_i$  and such that for each  $\eta \in A_m$ ,  $X_{\eta \wedge 1} \subseteq N(a_\eta)$  and  $X_{\eta \wedge 0} \subseteq V \setminus (N(a_\eta) \cup \{a_\eta\})$ . Then for each  $\eta \in A_m$  and  $i \in \{0, 1\}$ , if  $X_{\eta \wedge i} \neq \emptyset$ , choose  $a_{\eta \wedge i}$  to be any element of  $X_{\eta \wedge i}$ . Define  $A_{m+1}$  to be the set of these  $a_{\eta \wedge i}$ . Now for each  $a_\nu \in A_{m+1}$  and  $i \in \{0, 1\}$ , set

$$X_{\nu \wedge 1} = N(a_\nu) \cap X_\nu \text{ and}$$

$$X_{\nu \wedge 0} = (V \setminus (N(a_\nu) \cup \{a_\nu\})) \cap X_\nu.$$

By assumption,  $\{X_\nu : \nu \in A_{m+1}\}$  is a partition of  $V \setminus \bigcup_{i=1}^m A_i$ , and by construction, for each  $\nu \in A_{m+1}$ ,  $\{X_{\nu \wedge 1}, X_{\nu \wedge 0}\}$  is a partition of  $X_\nu \setminus A_{m+1}$ . Therefore,  $\{X_{\nu \wedge i} : \nu \in A_{m+1}, i \in \{0, 1\}\}$  is a partition of  $V \setminus \bigcup_{i=1}^{m+1} A_i$ .

All elements of  $V$  will be chosen after at most  $n$  steps. So we obtain an indexing of  $V$  by a subset of  $2^{<n}$  which is a type tree by construction.  $\square$

DEFINITION 8.3.3. Suppose  $G = (V, E)$  is a finite graph.



- (1) The *tree rank* of  $G$ , denoted  $t(G)$ , is the largest integer  $t$  such that there is a subset  $V' \subseteq V$  and an indexing  $V' = \{a_\eta : \eta \in 2^{<t}\}$  which is a type tree (i.e.  $V'$  is a full binary type tree of height  $n$ ).
- (2) The *tree height* of  $G$ , denoted  $h(G)$ , is the smallest integer  $h$  such that every indexing of  $V$  which is a type tree has a branch of length  $h$ .

LEMMA 8.3.4. *Suppose  $t, h$  are integers, and  $G = (V, E)$  is a finite graph with tree rank  $t$  and tree height  $h$ . Then  $G$  contains a complete or independent set of size  $\max\{t, h/2\}$ .*

PROOF. By definition of tree rank, there is  $V' \subseteq V$  and an indexing  $V' = \{a_\eta : \eta \in 2^{<t}\}$  which is a type tree. Then by definition of a standard type tree,  $I_1 = \{a_{<}, a_0, \dots, a_{0^{t-1}}\}$  is an independent set of size  $t$ . On the other hand, by definition of tree height and Lemma 8.3.2, there is an indexing  $V = \{a_\eta : \eta \in B\}$  of  $V$  by a subset  $B \subseteq 2^{<n}$  which is a standard type tree and which contains a branch  $J$  with length  $h$ . Let  $a_\tau$  be the last element of  $J$  and note  $h = ht(a_\tau)$ . If  $|N(a_\tau) \cap J| \geq \frac{|J|}{2}$ , set  $I_2 = N(a_\tau) \cap J$ . Otherwise set  $I_2 = (V \setminus N(a_\tau)) \cap J$ . In either case,  $|I_2| \geq |J|/2 = h/2$ . We now show that  $I_2$  is complete or independent. Suppose  $x$  and  $y$  are elements of  $I_2$ . By definition of  $I_2$ ,  $a_\tau$  is adjacent to  $x$  if and only if  $a_\tau$  is adjacent to  $y$ . Note  $x$  and  $y$  lie along the same branch, so without loss of generality we may assume  $x$  precedes  $y$ . By construction,  $a_\tau$  is adjacent to  $x$  if and only if  $y$  is adjacent to  $x$ . So if  $I_2 = N(a_\tau) \cap J$ ,  $I_2$  must be a complete set, and if  $I_2 = (V \setminus N(a_\tau)) \cap J$ ,  $I_2$  must be an independent set. We've now shown  $G$  contains a complete or independent set of size  $\max\{|I_1|, |I_2|\} \geq \max\{t, h/2\}$ .  $\square$

DEFINITION 8.3.5. Suppose  $G = (V, E)$  is a graph,  $A \subseteq 2^{<n}$ , and  $V = \{a_\eta : \eta \in A\}$  is a type tree.

- (1) Given an element  $a_\eta \in V$ , we say there is a *full binary tree of height  $k$  below  $a_\eta$*  if the following holds. There is a set  $V' \subseteq \{a_\sigma : a_\eta \subseteq a_\sigma\}$  and a bijection  $f : V' \rightarrow 2^{<k}$  with the property that  $a_\sigma$  precedes  $a_{\sigma'}$  in  $V'$  if and only if  $f(a_\sigma) \triangleleft f(a_{\sigma'})$  in  $2^{<k}$ .
- (2) The *tree rank* of an element  $a_\eta \in V$ , denoted  $t(a_\eta)$ , is the largest  $k$  such that there is a full binary tree of height  $k$  below  $a_\eta$ .

THEOREM 8.3.6. *Suppose  $n \geq 2$  is an integer and  $G = (V, E)$  is a graph of size  $n$ . Then*

$$h(G) \geq \frac{(n/t(G))^{\frac{1}{t(G)+1}}}{2}.$$

PROOF. Suppose  $A \subseteq 2^{<n}$  and  $V = \{a_\eta : \eta \in A\}$  of  $V$  is a type tree. Let  $h$  be the length of the longest branch in this tree, and let  $t = \max\{t(a_\eta) : \eta \in A\}$ . Note  $t \leq t(G)$ . Given a fixed  $\ell$  and  $s$ ,

set

$$Z_\ell^s = \{a_\eta \in V : t(a_\eta) = s, ht(a_\eta) = \ell\}$$

$$X_\ell^s = \{a_\eta \in Z_\ell^s : t(p(a_\eta)) = s\}, \text{ and}$$

$$Y_\ell^s = \{a_\eta \in Z_\ell^s : t(p(a_\eta)) = s + 1\}.$$

Let  $N_\ell^s = |Z_\ell^s|$ ,  $x_\ell^s = |X_\ell^s|$  and  $y_\ell^s = |Y_\ell^s|$ . Then note that that for each  $s$  and  $\ell$ ,  $N_\ell^s = x_\ell^s + y_\ell^s$ , and  $n = \sum_{\ell=0}^h \sum_{s=0}^t N_\ell^s$ . We claim the following facts hold.

- (i) For all  $s \leq t$  and  $\ell$ ,  $x_{\ell+1}^s \leq N_\ell^s$ .
- (ii) For all  $s < t$  and all  $\ell$ ,  $y_{\ell+1}^s \leq 2N_\ell^{s+1}$ .
- (iii) For all  $s < t$  and all  $\ell$ ,  $N_{\ell+1}^s \leq N_\ell^s + 2N_\ell^{s+1}$ .
- (iv) For all  $1 \leq s \leq t$ ,  $N_0^{t-s} = 0$ .
- (v) For all  $\ell$ ,  $N_\ell^t \leq 1$ .
- (vi) For all  $0 \leq s \leq t$ ,  $N_1^{t-s} \leq 2$ .

Item (i) holds by definition. Item (ii) follows because every element has at most 2 successors. Item (iii) follows directly from (i), (ii) and the fact that for all  $s$  and  $\ell$ ,  $N_\ell^s = x_\ell^s + y_\ell^s$ . Item (iv) follows from the fact that the only element of height 0 is  $a_{<}$ , which has height  $t$ . Item (v) follows from the fact that if for some  $\ell$ , if  $N_\ell^t \geq 2$ , then we would have  $t(a_{\langle \rangle}) \geq t + 1$ . Item (vi) is because the tree is binary, so the second level can have at most two elements.

We now show that for each  $0 \leq s \leq t$  and  $0 \leq \ell < h$ ,  $N_{\ell+1}^{t-s} \leq (2(\ell + 1))^s$ . If  $s = 0$  this follows immediately from (v).

Case  $s = 1$ : We want to show for all  $0 \leq \ell < h$ ,  $N_{\ell+1}^{t-1} \leq (2(\ell + 1))^s$ . The case where  $\ell = 0$  is done by (vi). Let  $\ell > 0$  and suppose by induction  $N_\ell^{t-1} \leq 2\ell$ . By (iii), (v) and our induction hypothesis,

$$N_{\ell+1}^{t-1} \leq N_\ell^{t-1} + 2N_\ell^t \leq 2\ell + 2 = 2(\ell + 1).$$

Case  $s > 1$ : Suppose by induction that for all  $0 \leq s' < s$ , the following holds: for all  $0 \leq \ell < h$ ,  $N_{\ell+1}^{t-s'} \leq (2(\ell + 1))^{s'}$ . We want to show that for all  $0 \leq \ell < h$ ,  $N_{\ell+1}^{t-s} \leq (2(\ell + 1))^s$ . The case  $\ell = 0$  is done by (vi). Let  $\ell > 0$  and suppose by induction that for all  $0 \leq \ell' < \ell$ ,  $N_{\ell'+1}^{t-s} \leq (2(\ell' + 1))^s$ . Then by (iii) and our induction hypothesis,

$$N_{\ell+1}^{t-s} \leq N_\ell^{t-s} + 2N_\ell^{t-s+1} \leq (2\ell)^s + 2(2\ell)^{s-1} = (2\ell)^s \left( \frac{\ell+1}{\ell} \right) \leq (2(\ell + 1))^s.$$

Therefore, for all  $0 \leq \ell < h$ ,

$$N_{\ell+1} \leq \sum_{0 \leq s \leq t} N_{\ell+1}^s \leq \sum_{0 \leq s \leq t} (2(\ell+1))^s \leq t(2(\ell+1))^t \leq t(2h)^t.$$

This implies that

$$n = N_0 + \sum_{0 \leq \ell < h} N_{\ell+1} \leq 1 + \sum_{0 \leq \ell < h} t(2h)^t \leq t(2h)^{t+1}$$

Rearranging this we obtain that

$$\frac{(n/t)^{\frac{1}{t+1}}}{2} \leq h.$$

Since  $t \leq t(G)$  this implies  $\frac{(n/t(G))^{\frac{1}{t(G)+1}}}{2} \leq h$ . This finishes the proof.  $\square$

Combining Theorem 8.3.6 and Lemma 8.3.4 immediately implies the following.

**COROLLARY 8.3.7.** *Suppose  $G = (V, E)$  is a graph with tree rank  $t$  and  $n$  vertices. Then  $G$  contains a complete or independent set of size at least  $\frac{(n/t)^{\frac{1}{t+1}}}{4}$ .*

#### 8.4. Finitary proof leveraging Theorem 8.3.6

The following is an adaptation of Proposition 3.1 [39].

**PROPOSITION 8.4.1.** *Suppose  $G = (V, E)$  has tree height  $t \geq R(n_1, n, n, n_2)$  witnessed by  $T \subseteq V$  and the indexing  $T = \{a_\eta : \eta \in 2^{<t}\}$  which is a type tree. Then  $G[T]$  contains one of the following as an induced subgraph.*

- (i) a thin spider with  $n$  legs,
- (ii) the bipartite half-graph of height  $n$ ,
- (iii) the disjoint union of  $n_1$  copies of  $K_2$ , denoted by  $n_1 K_2$ , or
- (iv) the half split graph of height  $n_2$ .

**PROOF.** Consider the sets  $A = \{a_{<}, a_0, \dots, a_{0^{t-1}}\}$  and  $B = \{a_1, a_{01}, \dots, a_{0^{t-1} \wedge 1}\}$ . Rename the elements of  $A$  and  $B$  so that  $\langle a_{<}, a_0, \dots, a_{0^{t-1}} \rangle = \langle x_1, x_2, \dots, x_t \rangle$  and  $\langle a_1, a_{01}, \dots, a_{0^{t-1} \wedge 1} \rangle = \langle y_1, y_2, \dots, y_t \rangle$ . Note that by definition of a standard type tree and our choice of  $A$ , we have the following.

- $A$  is an independent set.
- For each  $i \in [t]$ ,  $x_i y_i \in E$ .
- For each  $i < j$ ,  $x_i y_j \notin E$ .

We now define a coloring of the edges of the complete graph with vertex set  $[t]$  with colors  $(a, b) \in \{0, 1\}^2$ . Given  $i < j \in [t]$ , define the color  $(a, b)$  of the edge  $ij$  as follows. Set  $a = 1$  if and only if  $x_j y_i \in E$  and  $b = 1$  if and only if  $y_i y_j \in E$ . By Ramsey's theorem, there is a subset  $I \subseteq [t]$  such that all the edges of  $I$  have the same color  $(a, b)$  and the following holds.

$$|I| = \begin{cases} n_1 & \text{if } (a, b) = (0, 0) \\ n & \text{if } (a, b) = (0, 1) \\ n & \text{if } (a, b) = (1, 0) \\ n_2 & \text{if } (a, b) = (1, 1) \end{cases}$$

Set  $Z = \{x_i : i \in I\} \cup \{y_i : i \in I\}$ . Then if  $(a, b) = (0, 0)$ ,  $G[Z]$  forms an induced copy of  $n_1 K_2$ . If  $(a, b) = (0, 1)$ , then  $G[Z]$  forms an induced copy of a thin spider with  $n$  legs. If  $(a, b) = (1, 0)$ , then  $G[Z]$  forms an induced copy of a bipartite half-graph of height  $n$ . Finally if  $(a, b) = (1, 1)$ , then  $G[Z]$  forms an induced copy of the half split graph of height  $n_2$ .  $\square$

REMARK 8.4.2. (1) *In the proof of Proposition 8.4.1, we could also have built our configuration over a complete set by instead taking  $A = \{a_{<}, a_1, a_{11}, \dots, a_{1^{t-1}}\}$  and  $B = \{a_0, a_{10}, \dots, a_{1^{t-1} \wedge 0}\}$ .*

(2) *If we don't care whether we build over complete or empty sets, then what Proposition 8.4.1 uses is the length of the longest "straight path" through the tree consisting of nodes with two children, which is at least the tree rank.*

COROLLARY 8.4.3. *Suppose  $G$  is a prime graph with tree height  $t \geq R(h(n, g(n), n), n, n, g(n))$ . Then  $G$  contains one of the following or the compliment of one of the following as an induced subgraph.*

- (1) *The 1-subdivision of  $K_{1,n}$  (denoted by  $K_{1,n}^{(1)}$ ).*
- (2) *The line graph of  $K_{2,n}$  (denoted by  $L(K_{2,n})$ ).*
- (3) *The thin spider with  $n$  legs.*
- (4) *The bipartite half-graph of height  $n$ .*
- (5) *The graph  $H'_{n,I}$ .*
- (6) *the graph  $H_n^*$ .*
- (7) *A prime graph induced by a chain of length  $n$ .*

PROOF. If  $G$  contains a chain of length  $n + 1$ , we are done. So assume this is not the case. Apply Proposition 8.4.1 with  $n_1 = h(n, g(n), n)$  and  $n_2 = g(n)$ . In outcomes 8.4.1.(i) and 8.4.1.(ii), we are done. If  $G$  contains a half split graph of height  $g(n)$  apply Proposition 8.2.5 to obtain  $H'_{n,I}$  or

$H_n^*$ . So assume now  $G$  contains no half split graph of height  $g(n)$ . The only possible outcome left is 8.4.1.(iii), i.e., that  $G$  contains an induced matching with  $n_1 = h(n, g(n), n)$  edges. Combining this with our assumptions that  $G$  is prime, contains no chains of length  $n + 1$ , and contains no half split graph of height  $g(n)$ , we have that Proposition 8.2.4 implies  $G$  contains a copy of  $K_{1,n}^{(1)}$ , the bipartite half-graph of height  $n$ ,  $\overline{L(K_{2,n})}$ , or a spider with  $n$  legs. This finishes the proof.  $\square$

We now prove Theorem 8.2.1 with a value for  $N$  which is asymptotically much smaller than  $N_{8.2.1}$ .

**THEOREM 8.4.4.** *Let  $n \geq 2$  and recall*

$$m = f(n, h(n, g(n), n), g(n)) = 2^{R(n+h(n,g(n),n), 2n-1, n+g(n), n+g(n)-1)+1}.$$

*Suppose*

$$N = R(h(n, g(n), n), n, n, g(n))(5m)^{R(h(n,g(n),n), n, n, g(n))+1},$$

*and  $G$  is a prime graph with at least  $N$  vertices. Then the conclusion of Theorem 8.2.1 holds.*

*Moreover, for large  $n$ ,*

$$N \ll R(m, m) = N_{8.2.1}.$$

**PROOF.** Suppose  $G$  is a prime graph with at least  $N$  vertices. Suppose first that the tree height,  $t = t(G)$  is at least  $R(h(n, g(n), n), n, n, g(n))$ . Then Corollary 8.4.3 implies  $G$  contains one of the desired configurations, so we are done. Assume now that  $t \leq R(h(n, g(n), n), n, n, g(n))$ . Remark 8.2.6 and Proposition 8.2.2 imply that that if  $G$  contains a complete or independent set of size  $m$  then the conclusion of Theorem 8.2.1 holds. We show  $G$  contains a complete or independent set of size  $m$ . By Corollary 8.3.7,  $G$  contains a complete or independent independent set  $I$  such that  $|I| \geq \frac{(N/t)^{\frac{1}{t+1}} - 2}{4}$ , so it suffices to show that  $\frac{(N/t)^{\frac{1}{t+1}}}{4} \geq m$ . By definition of  $N$  and our assumption on  $t$ ,  $N \geq t(5m)^{t+1}$ . This implies  $\frac{(N/t)^{\frac{1}{t+1}}}{4} \geq \frac{5m}{4} \geq m$ . This finishes the proof that the conclusion of Theorem 8.2.1 holds. We've now left to show that  $N \ll N_{8.2.1}$ . Let  $x = R(h(n, g(n), n), n, n, g(n))$ . Then we want to show that large  $n$ ,  $x(5m)^{x+1} \ll R(m, m)$ . Note that  $x \leq \log_2 m$  and recall that by [98], as long as  $m \geq 2$ ,  $R(m, m) \geq (\sqrt{2})^m$ . Combining these facts, we have that the following holds for large  $m$  (equivalently, for large  $n$ ).

$$x(5m)^{x+1} \leq (\log_2 m)(5m)^{2 \log_2 m + 1} \ll (\sqrt{2})^m \leq R(m, m).$$

$\square$

REMARK 8.4.5. *The theorem uses the fact that any graph  $G$  contains a complete or independent set of size  $\max\{t(G), h(G)/2\}$ , the inverse relationship between  $t(G)$  and  $h(G)$  from Theorem 8.3.6, and the fact that a binary type tree contains the building blocks of the desired configurations. These ingredients, i.e. Theorem 8.4.1, Lemma 8.3.4, and Theorem 8.3.6, hold for arbitrary graphs.*

### 8.5. An infinitary proof

In this section we prove an analogue of Theorem 8.2.1 in the infinite setting, and show it implies the finite version, although without the explicit bounds. Throughout this section we work in the first-order language of graphs,  $\mathcal{L} = \{E(x, y)\}$ , and employ standard model theoretic notation. Given sets  $A$  and  $B$ , we will write  $AB$  as shorthand for  $A \cup B$ , and given a tuple of elements  $\bar{a}$ , we will often write  $\bar{a}$  to mean the set of elements in the tuple. The following proposition is proved in [39] in the setting of finite graphs, but the proof presented there also holds in the setting of infinite graphs. Given an integer  $n$ , we will write  $R(n)$  to mean  $R(n, n)$ .

PROPOSITION 8.5.1 (Proposition 2.1 in [39]). *Suppose  $G$  is a graph and  $I \subseteq V(G)$  is a set with at least two vertices, and suppose  $v \in V(G) \setminus I$ . Then  $G$  has a chain from  $I$  to  $v$  if and only if all modules containing  $I$  as a subset contain  $v$ .*

A useful and straightforward corollary of this is the following.

COROLLARY 8.5.2. *A graph  $G = (V, E)$  is prime if and only if for every set of pairwise distinct vertices  $\{x_1, x_2, x_3\} \subseteq V$ , there is chain from  $\{x_1, x_2\}$  to  $x_3$  in  $G$ .*

PROOF. Suppose  $G = (V, E)$  is a prime graph and  $x_1, x_2, x_3 \in V$  are pairwise distinct vertices. Suppose there is no chain from  $\{x_1, x_2\}$  to  $x_3$ . Then by Proposition , there is a module  $I$  containing  $\{x_1, x_2\}$  as a subset and not containing  $v$ . But now  $I$  is a nontrivial module, contradicting that  $G$  is prime.

Conversely, suppose for every set  $\{x_1, x_2, x_3\} \subseteq V$  of pairwise distinct vertices, there is chain from  $\{x_1, x_2\}$  to  $x_3$  in  $G$ . We show that any module  $I$  in  $G$  is either a singleton or all of  $V$ . Suppose by contradiction  $I$  is a module which is neither a singleton, nor all of  $V$ . Then there are  $x_1 \neq x_2 \in I$  and  $x_3 \in V \setminus I$ . By assumption there is a chain from  $\{x_1, x_2\}$  to  $x_3$ , so Proposition 8.5 implies that every module containing  $\{x_1, x_2\}$  also contains  $x_3$ . In particular,  $x_3 \in I$ , a contradiction.  $\square$

DEFINITION 8.5.3. Fix an integer  $n \geq 1$ .

- (1) Let  $\phi_n(x, y, z)$  be the formula saying that there exists a chain of length at most  $n$  from  $\{x, y\}$  to  $z$ .
- (2) Let  $\psi_n$  be the sentence saying that for any pairwise distinct  $x_1, x_2, x_3$ , there is a chain of length at most  $n$  from  $\{x_1, x_2\}$  to  $x_3$ , i.e. the sentence

$$\forall x_1 x_2 x_3 \left( \left( \bigwedge_{1 \leq i \neq j \leq 3} x_i \neq x_j \right) \rightarrow \phi_n(x_1, x_2, x_3) \right).$$

- (3) Let  $\sigma_n$  be the sentence saying that there exists a copy of  $H_n$  or a copy of  $\overline{H_n}$  as an induced subgraph.
- (4) Let  $\theta_n$  be the sentence saying there exists a copy of  $H'_{n,I}$ ,  $H_n^*$  or  $\overline{H_n^*}$ .
- (5) Let  $\rho_n$  be the sentence which says that one of the following or the compliment of one of the following appears as induced subgraph:  $K_{1,n}^{(1)}$ ,  $L(K_{2,n})$ , a spider with  $n$  legs.

Given  $k \geq 1$ , we will call a graph  $G$   $k$ -edge-stable if  $G$  omits all half-graphs of height  $k$ . We will call  $G$  edge-stable when it is  $k$ -edge stable for some  $k$  (equivalently, when its edge relation is a stable formula). Call a subset of  $I$  of  $G$  *edge indiscernible* if it is indiscernible with respect to the edge relation. We remark that Proposition 8.2.5 applies in the case of an infinite prime graph as well as a finite one, via exactly the same proof as in [39]. Given a formula  $\phi$ , we let  $\phi^1 = \phi$  and  $\phi^0 = \neg\phi$ . We now recall a definition and claim from [75].

DEFINITION 8.5.4. Given  $\ell \geq 2$ , let  $\Delta_\ell = \{E(x_0, x_1)\} \cup \{\phi_{\ell,m}^i : m \leq \ell, i \in \{0, 1\}\}$ , where

$$\phi_{\ell,m}^i = \phi_{\ell,m}^i(x_0, \dots, x_{\ell-1}) = \exists y \left( \bigwedge_{j < \ell} E(x_j, y)^{\text{if } i=0} \wedge \bigwedge_{m \leq j \leq \ell} E(x_j, y)^{\text{if } i=1} \right).$$

CLAIM 8.5.5 (Claim 3.2 of [75]). *Suppose  $G$  is an  $\ell$ -edge stable graph. Suppose  $m \geq 4\ell$  and  $\langle a_i : i < \alpha \rangle$  is a  $\Delta_\ell$ -indiscernible sequence in  $G$ , and  $b \in G$ . Then either  $|\{i : E(a_i, b)\}| < 2\ell$  or  $|\{i : \neg E(a_i, b)\}| < 2\ell$ .*

PROPOSITION 8.5.6. *For any integer  $n \geq 1$ , any infinite graph satisfying  $\psi_n \wedge \neg\sigma_n \wedge \neg\theta_n$  is prime, edge-stable, and contains one of the following or the compliment of one of the following as an induced subgraph.*

- (1) A spider with  $\omega$  many legs,
- (2)  $L(K_{2,\omega})$ ,
- (3) A perfect matching of length  $\omega$ .

PROOF. Since  $G \models \psi_n$ , Corollary 8.5.2 implies  $G$  is prime. Set  $\ell = R(R(g(n)))$ . We show  $G$  is  $\ell$ -edge-stable. Suppose by contradiction  $G$  contains a half-graph  $a_1b_1, \dots, a_\ell b_\ell$  so that  $E(a_i, b_j)$  if and only if  $i \leq j$ . By Ramsey's theorem, there is a complete or independent set  $A \subseteq \{a_1, \dots, a_\ell\}$  such that  $|A| = R(g(n))$ . By reindexing, assume  $A = \{a_1, \dots, a_{R(g(n))}\}$ . Applying Ramsey's theorem again, we have that there is a complete or independent set  $B' \subseteq \{b_1, \dots, b_{R(g(n))}\}$  such that  $|B'| = g(n)$ . By reindexing, assume  $B' = \{b_1, \dots, b_{g(n)}\}$ . Then  $a_1b_1, \dots, a_{g(n)}b_{g(n)}$  forms an induced copy of  $H_{g(n)}, \overline{H_{g(n)}}$ , or a half split graph of height  $g(n)$ . Since  $G \models \neg\sigma_n$ , it must contain a half split graph of height  $g(n)$ . By Proposition 8.2.5,  $G$  contains an induced copy of  $H'_{n,I}, H_n^*$ , or  $\overline{H_n^*}$ , contradicting that  $G \models \neg\theta_n$ . Therefore  $G$  is  $\ell$ -edge-stable.

By Ramsey's theorem there is an infinite  $\Delta_\ell$ -indiscernible sequence  $I = \{c_i : i < \omega\}$  in  $G$ . Note  $I$  is a complete or independent set. Without loss of generality, assume it is independent (otherwise we obtain the compliments everything that follows). Claim 8.5.5 implies that for all  $b \notin I$ , either  $|\{c_i : E(b, c_i)\}| \leq 2\ell$  or  $|\{c_i : \neg E(b, c_i)\}| \leq 2\ell$ . Given  $b \notin I$ , set

$$f(b) = \begin{cases} 1 & \text{if } |\{c_i : E(b, c_i)\}| \leq 2\ell \\ 0 & \text{if } |\{c_i : \neg E(b, c_i)\}| \leq 2\ell, \end{cases}$$

and set  $S_b = \{c_i : E(b, c_i)^{f(b)}\}$ . We construct two sequences  $J_1 = \{a_i : i < \omega\}$  and  $J_2 = \{b_i : i < \omega\}$  along with a sequence of sets  $\{A_i : i < \omega\}$  with the following properties.

- For each  $k < \omega$ ,  $b_k \notin Ib_1 \dots b_{k-1}$  and  $a_k \in S_{b_k}$ ,
- for each  $i, j < \omega$ ,  $E(b_i, a_j)^{f(b_i)} \Leftrightarrow i = j$ ,
- $I \supseteq A_1 \supseteq A_2 \supseteq \dots$  and for each  $k < \omega$ ,  $|A_k| = \omega$ ,
- For each  $j \leq k < \omega$ ,  $A_k \cap S_{b_j} = \emptyset$ .

Step 0: Since  $I$  is not a module, there is a vertex  $b_1$  which is mixed on  $I$ . Note that since  $I$  is edge-indiscernible, we must have that  $b_1 \notin I$ . Choose  $a_1 \in S_{b_1}$  and set  $A_1 = I \setminus S_{b_1}$ . Note that since  $|I| = \omega$  and  $a_1S_{b_1}$  is finite,  $|A_1| = \omega$ .

Step  $k$ : Suppose now we've constructed  $b_1a_1, \dots, b_{k-1}a_{k-1}$ , and  $A_1, \dots, A_{k-1}$  satisfying the desired hypotheses. Since  $A_{k-1}$  is not a module, there is  $b_k$  which is mixed on  $A_{k-1}$ . In other words,  $A_{k-1} \cap S_{b_k} \neq \emptyset$ . Since  $I$  is edge-indiscernible,  $b_k$  is not in  $I$ . For each  $j < k$ ,  $A_{k-1} \cap S_{b_j} = \emptyset$  implies  $b_j$  is not mixed on  $A_{k-1}$ . Therefore  $b_k \notin \{b_1 \dots b_{k-1}\}$ . Choose  $a_k \in S_{b_k} \cap A_{k-1}$  and set  $A_k = A_{k-1} \setminus a_kS_{b_k}$ . Note that by our induction hypothesis,  $|A_{k-1}| = \omega$  and by definition  $a_kS_{b_k}$  is finite, so  $|A_k| = \omega$ . This completes the construction.



By Ramsey's theorem, there are infinite subsequences  $I_1 = (a'_i)_{i < \omega} \subseteq (a_i)_{i < \omega}$  and  $I_2 = (b'_i)_{i < \omega} \subseteq (b_i)_{i < \omega}$  such that  $I_1 I_2 = (a'_i b'_i)_{i < \omega}$  is edge-indiscernible. If  $I_2$  is a complete set and  $f(b'_1) = 0$ , then  $I_1 I_2$  is a thick spider with  $\omega$  many legs. If  $I_2$  is a complete set and  $f(b'_1) = 1$ , then  $I_1 I_2$  is a thin spider with  $\omega$  many legs. If  $I_2$  is an independent set and  $f(b'_1) = 0$ , then  $I_1 I_2$  forms a copy of  $\overline{L(K_{2,\omega})}$ . Therefore we are left with the case when  $I_2$  is an independent set and  $f(b'_1) = 1$ . In this case  $I_1 I_2$  forms a perfect matching of length  $\omega$ .  $\square$

The following argument is an infinitary version of the argument used to prove Proposition 8.2.4 in [39].

PROPOSITION 8.5.7. *Suppose  $G$  is an infinite, prime, edge-stable graph satisfying  $\psi_n$  and suppose  $M$  is an infinite perfect matching in  $G$ . Then  $G$  contains one of the following or the complement of one of the following as an induced subgraph.*

- (1)  $K_{1,\omega}^{(1)}$ ,
- (2)  $L(K_{2,\omega})$ ,
- (3) A spider with  $\omega$ -many legs.

PROOF. Suppose  $G$  is an infinite, prime, edge-stable graph satisfying  $\psi_n$  and suppose  $M$  is an infinite perfect matching  $M$  in  $G$ . Since  $M$  is not prime,  $V(G) \setminus V(M) \neq \emptyset$ . Since  $G$  is prime and satisfies  $\psi_n$ , Corollary 8.5.2 implies that for every  $v \in V(G) \setminus V(M)$  there is an integer  $t(v) \leq n$  such that there is a chain of length less than or equal to  $t(v)$  from  $v$  to  $e$  for infinitely many  $e \in M$ . Set  $t = t(M) = \min\{t(v) : v \in V(G) \setminus V(M)\}$ . We show by induction on  $2 \leq t \leq n$  that the conclusion of the proposition is true.

Fix  $v \in V$  such that  $t(v) = t$  and an infinite  $M' \subseteq M$  such that there is a chain of length at most  $t$  from  $v$  to  $e$  for every  $e \in M'$ . Suppose first that  $t = 2$ . Then  $vM'$  is isomorphic to  $K_{1,\omega}^{(1)}$  and we are done. Assume now  $2 < t \leq n$  and suppose by induction that for all  $2 \leq t' < t$ , if  $G$  contains an infinite perfect matching  $M''$  with  $t(M'') = t'$ , then the conclusion of the proposition holds. Enumerate  $M' = \{x_i y_i : i < \omega\}$  and delete the edges  $e \in M'$  on which  $v$  is mixed. Since  $t > 2$ , we have deleted only finitely many elements of  $M'$ . For each  $i < \omega$  choose a chain  $C_{x_i y_i} = \{v_0, v_1, \dots, v_n\}$  from  $x_i y_i$  to  $v$  (so  $\{x_i, y_i\} = \{v_0, v_1\}$ ). Set  $z_i = v_2$ .

Note by assumption,  $v$  is not mixed on any  $x_i y_i$ , so  $z_i \neq v$ , and since  $M'$  is a matching,  $z_i \notin M'$ . By Ramsey's theorem, the sequence  $(x_i y_i z_i)_{i < \omega}$  contains an infinite indiscernible sequence  $(x'_i y'_i z'_i)_{i < \omega}$ . Since  $t > 2$ , we must have that for each  $i < \omega$ ,  $z'_i$  is not mixed on  $x'_j y'_j$  for all  $j \neq i$ , so in

particular,  $E(z'_1, x'_2) \equiv E(z'_1, y'_2)$ . Since  $G$  is edge-stable, we have that  $E(z'_1 x'_2) \equiv E(z'_2 x'_1)$  and  $E(z'_1 y'_2) \equiv E(z'_2 y'_1)$ . Combining all of this, we have

$$E(z'_2 x'_1) \equiv E(z'_1 x'_2) \equiv E(z'_1 y'_2) \equiv E(z'_2 y'_1).$$

By relabeling if necessary, we may assume  $E(z'_1 y'_1)$  and  $\neg E(z'_1, x'_1)$ . By indiscernibility and our assumptions, the type of  $(x'_i y'_i z'_i)_{i < \omega}$  depends only on  $E(z'_1, x'_2)$  and  $E(z'_1, z'_2)$ . Suppose first that  $E(z'_1, z'_2)$ , so  $(z'_i)_{i < \omega}$  is a complete set. If  $E(z'_1, x'_2)$ , then  $(z'_i, x'_i)_{i < \omega}$  is a thick spider with  $\omega$  many legs. If  $\neg E(z'_1, x'_2)$ , then  $(z'_i, y'_i)_{i < \omega}$  is a thin spider with  $\omega$  many legs.

Suppose now that  $\neg E(z'_1, z'_2)$ , so  $(z'_i)_{i < \omega}$  is an independent set. If  $E(z'_1, x'_2)$ , then  $(z'_i, x'_i)_{i < \omega}$  is a copy of  $\overline{L(K_{2,\omega})}$ . If  $\neg E(z'_1, x'_2)$ , then  $M'' := (z'_i, y'_i)_{i < \omega}$  is an infinite perfect matching. In this case, we now have that for each  $i < \omega$ ,  $C_{x'_i y'_i} \setminus \{x'_i\}$  is a chain of length at most  $t - 1$  from  $\{z'_i, y'_i\}$  to  $v$ , that is  $t(M'') = t - 1$ . By our induction hypothesis,  $G$  satisfies the conclusion of the proposition.  $\square$

We now prove a version of Theorem 8.2.1 for infinite graphs, then use it to prove Theorem 8.2.1.

**THEOREM 8.5.8.** *An infinite prime graph  $G$  contains one of the following.*

- (1) Copies of  $H_n, \overline{H_n}, H_n^*, \overline{H_n^*}, H'_{n,I}$ , or  $\overline{H'_{n,I}}$  for arbitrarily large finite  $n$ ,
- (2) Prime graphs induced by arbitrarily long finite chains,
- (3)  $K_{1,\omega}^{(1)}$  or its compliment,
- (4)  $L(K_{2,\omega})$  or its compliment,
- (5) A spider with  $\omega$  many legs.

**PROOF.** Suppose  $G$  is an infinite prime graph which fails 1 and 2. Since  $G$  is prime but fails 2, Proposition 8.2.2 implies  $G$  does not contain arbitrarily long finite chains. Thus there is  $n_1 \in \mathbb{N}$  such that  $G \models \psi_{n_1}$ . Since  $G$  fails 1, there is  $n_2$  such that  $G$  contains no copy of  $H_{n_2}, H_{n_2}^*, \overline{H_{n_2}^*}$ , or  $H'_{n_2,I}$ . Let  $n_3 = \max\{n_1, n_2\}$ , then  $G$  is prime and satisfies  $\phi_{n_3} \wedge \neg \sigma_{n_3} \wedge \neg \theta_{n_3}$ . Applying Corollary 8.5.6, we have that either  $G$  satisfies 5 or 4, or  $G$  contains an induced perfect matching of length  $\omega$ . If  $G$  contains an induced perfect matching of length  $\omega$ , Proposition 8.5.7 implies  $G$  satisfies 3, 4, or 5.  $\square$

**Proof of Theorem 8.2.1.** Fix  $n \geq 1$ . By definition, any finite prime graph  $G$  satisfying  $\sigma_n$  or  $\theta_n$  contains one of the desired configurations. If a finite prime graph  $G$  of size at least 3 satisfies  $\neg \psi_n$ , then  $G$  contains three distinct points  $x, y, z$  such that there is no chain of length less than or equal to  $n$  from  $\{x, y\}$  to  $z$ . Corollary 8.5.2 implies that there is some chain from  $\{x, y\}$  to  $z$ . Therefore

there is a chain  $v_0, \dots, v_t$  of length  $t \geq n + 1$  from  $\{x, y\}$  to  $z$ . Since initial sequences of chains are chains,  $v_0, \dots, v_{n+1}$  is a chain of length  $n + 1$ . By Proposition 8.2.2,  $G$  contains a chain of length  $n$  inducing a prime subgraph. So if  $G$  has size at least 3 and satisfies  $\sigma_n \vee \theta_n \vee \neg\psi_n$ , we are done.

We now show there is  $N$  such that any finite prime graph of size at least  $N$  satisfying  $\neg\sigma_n \wedge \neg\theta_{g(n)} \wedge \psi_n$  must also satisfy  $\rho_n$ . This combined with the above finishes the proof. Suppose by contradiction that no such  $N$  exists. Then there are arbitrarily large finite graphs which satisfy  $\neg\sigma_n \wedge \neg\theta_n \wedge \psi_n \wedge \neg\rho_n$ , so by compactness there is an infinite graph  $G$  satisfying  $\neg\sigma_n \wedge \neg\theta_n \wedge \psi_n \wedge \neg\rho_n$ . By Proposition 8.5.6,  $G$  is edge-stable and contains an infinite perfect matching. But then Proposition 8.5.7 clearly implies  $G \models \rho_n$ , a contradiction.  $\square$

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## Education

- 2012-2016 **Ph.D. in Pure Mathematics**, *University of Illinois Chicago (UIC)*, Chicago, IL.  
advisors: David Marker and Dhruv Mubayi
- 2010 - 2012 **MS in Pure Mathematics**, *UIC*, Chicago, IL.
- 2006 - 2010 **BS in Mathematics**, *University of Chicago*, Chicago, IL.

## Research Interests

**Interactions between model theory and finite combinatorics, logical 0-1 laws, extremal combinatorics, finite Ramsey theory.**

## Papers and Preprints

- Model theoretic properties of the Urysohn sphere**, with Gabriel Conant, *Annals of Pure and Applied Logic*, 167, (2016), no. 1, 49-72.  
URL: <http://www.sciencedirect.com/science/article/pii/S0168007215000986>
- Discrete metric spaces: structure, enumeration, and 0-1 laws**, *arXiv:1502.01212v2*, with Dhruv Mubayi, accepted *Journal of Symbolic Logic*.
- On unavoidable induced subgraphs in large prime graphs**, *arXiv:1511.02544*, with M. Malliaris, submitted.

## Awards

- Spring 2015 **Louise Hay Award**, *UIC*, Chicago, IL.  
Awarded to a math graduate student for academic excellence.
- Spring 2014, Sum. 2015 **Research Training Grant Pre-doctoral Fellow**, *UIC*, Chicago, IL.
- 2006-2010 **Deans List**, *University of Chicago*, Chicago, IL.
- 2010 **Graduated with General Honors**, *University of Chicago*, Chicago, IL.

## Semesters

- Spring 2014 **Program Associate**, *Semester Program on Model Theory, Arithmetic Geometry and Number Theory*, Mathematical Sciences Research Institute (MSRI), Berkeley, CA.

## Conference Talks

- May 2016 **SIU Mathematics Conference**, *Southern Illinois University*, Carbondale, IL.
- Nov. 2015 **Homogeneous Structures Workshop**, *Banff International Research Station*, Banff, AB.
- Oct. 2015 **AMS Central Fall Sectional Meeting 2015: special session in model theory**, *Loyola University*, Chicago, IL.

- Oct. 2015 **AMS Central Fall Sectional Meeting 2015: special session on topics in graph theory, hypergraphs and set systems**, *Loyola University*, Chicago, IL.
- August 2015 **Logic and Random Graphs**, *Lorentz Center*, Leiden, Netherlands.
- August 2015 **Logic Colloquium 2015**, *University of Helsinki*, Helsinki, Finland.
- July 2015 **Random Structures and Algorithms**, *Carnegie Melon University*, Pittsburgh, PA.
- March 2015 **Lake Michigan Combinatorics Workshop**, *Notre Dame University*, Notre Dame, IN.
- March 2015 **Midwest Women in Mathematics Symposium**, *Dominican University*, Chicago, IL.

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## Seminar Talks

- Dec. 2015 **Model Theory Seminar**, *CUNY*, New York, NY.
- Dec. 2015 **Logic Seminar**, *Harvard University*, Cambridge, MA.
- Dec. 2015 **Connecticut Logic Seminar**, *Wesleyan University*, Middleton, CT.
- Nov. 2015 **Algorithms, Combinatorics, and Optimization Seminar**, *Carnegie Mellon University*, Pittsburgh, PA.
- Oct. 2015 **Logic Seminar**, *The Ohio State University*, Columbus, OH.
- Oct. 2015 **Logic Seminar**, *McMaster University*, Hamilton, ON, Canada.
- Oct. 2015 **Logic Seminar**, *University of Illinois Urbana-Champaign (UIUC)*, Urbana, IL.
- Oct. 2015 **Logic Seminar**, *UIC*, Chicago, IL.
- Sept. 2015 **Model theory seminar**, *University of Notre Dame*, Notre Dame, IN.
- May 2015 **Graph Theory Seminar**, *UIUC*, Urbana, IL.
- Feb. 2014 **Model Theory Seminar**, *University of California at Berkeley*, Berkeley, CA.
- Sept. 2013 **Logic Seminar**, *UIC*, Chicago, IL.

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## Service

- 2012-present **Coorganizer, UIC Chapter of Association for Women in Mathematics**, Chicago, IL.  
Organized events including the following (see [www.math.uic.edu/AWM](http://www.math.uic.edu/AWM) for more).
  - *Women in mathematics: reflections and history from female mathematicians.*  
Panel for university community consisting of prominent women mathematicians and historians of women in math.
  - *Finding pioneering women in American mathematics: the why, the how, and the what.*  
Lecture for math department at UIC by historian Jeanne LaDuke.
- 2013-present **Coorganizer of the Louise Hay Logic Seminar**, *UIC*, Chicago, IL.
- 2014-2015 **Coorganizer of Graduate Combinatorics Seminar**, *UIC*, Chicago, IL.
- Spring 2014 **Coorganizer of Finite Structures Seminar**, *MSRI*, Berkeley, CA.
- 2012-2013 **Copresident of the Mathematics Graduate Student Organization**, *UIC*, Chicago, IL.  
Organized weekly departmental tea and social events for graduate students

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## Teaching Experience

### Lecturer

- Fall 2013 **Math 210: Calculus 3**, *UIC*, Chicago, IL.

Summers 2012, 2013 **Math 090: Intermediate Algebra**, *UIC*, Chicago, IL.  
*As part of Summer Enrichment Workshop, a program for incoming freshman to take free math classes the summer before starting college.*

### Teaching Assistant

Summer 2016 **Math 417: Complex Analysis and Applications**, *UIC*, Chicago, IL.  
Fall 2015 **Math 075: Beginning Algebra**, *UIC*, Chicago, IL.  
Fall 2012, Sum. 2014 -Spring 2015 **Math 180: Calculus I**, *UIC*, Chicago, IL.  
Spring 2012, Spring 2013 **Math 181: Calculus II**, *UIC*, Chicago, IL.  
Fall 2011 **Math 165: Business Calculus**, *UIC*, Chicago, IL.  
Sum. 2011 **Math 417: Complex Analysis and Applications**, *UIC*, Chicago, IL.  
Spring 2011 **Math 090: Intermediate Algebra**, *UIC*, Chicago, IL.  
Fall 2010 **Math 210: Calculus III**, *University of Chicago*, Chicago, IL.  
Fall 2008 -Spring 2010 **Math 131-132: Elementary Functions and Calculus I-II**, *University of Chicago*, Chicago, IL.  
Summer 2009 **Algebra I**, *SESAME program, University of Chicago*, Chicago, IL.  
*As part of summer REU, TA-ed course for Chicago Public School teachers aimed at certifying them to teach high school algebra.*

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### Workshops and Conferences

Nov. 2015 **Homogeneous Structures Workshop**, *Banff International Research Station*, Banff, AB.  
Oct. 2015 **AMS Central Fall Sectional Meeting**, *Loyola University*, Chicago, IL.  
Sept. 2015 **Logic and Random Graphs**, *Lorentz Center*, Leiden, Netherlands.  
Aug. 2015 **Logic Colloquium**, *University of Helsinki*, Helsinki, Finland.  
July 2015 **Random Structures and Algorithms**, *Carnegie Mellon University*, Pittsburgh, PA.  
June 2015 **CanaDAM**, *University of Saskatchewan*, Saskatoon, SK.  
Spring 2015 **Midwest Women in Mathematics Symposium**, *Dominican University*, Chicago, IL.  
Spring 2015 **Lake Michigan Combinatorics Workshop**, *Notre Dame University*, Notre Dame, IN.  
Fall 2014 **Association for Symbolic Logic Winter Meeting (with Joint Math Meeting)**, San Antonio, TX.  
May 2014 **Model Theory in Geometry and Arithmetic**, *MSRI*, Berkeley, CA.  
Feb. 2014 **Connections for Women: Model Theory, Arithmetic Geometry, and Number Theory**, *MSRI*, Berkeley, CA.  
Feb. 2014 **Introductory Workshop: Model Theory, Arithmetic Geometry, and Number Theory**, *MSRI*, Berkeley, CA.  
Oct. 2013 **Differential Algebra Day**, *Notre Dame University*, Notre Dame, IN.  
Oct. 2013 **Workshop on Homogeneous Structures**, *Hausdorff Institute*, Bonn, Germany.  
June 2013 **Model Theory Meeting**, Ravello, Italy.  
May 2013 **Carol Wood Retirement Conference**, *Wesleyan University*, Middletown, CT.

- July 2012 **Summer Graduate School - Model Theory**, *MSRI*, Berkeley, CA.
- April 2012 **Association for Symbolic Logic Meeting**, *University of Wisconsin*, Madison, WI.
- Oct. 2011 **Mid-Atlantic Mathematical Logic Conference**, *Rutgers University*, Piscataway, NJ.
- Sept. 2011 **Ward Henson Retirement Conference**, *UIUC*, Urbana, IL.
- Sum. 2009 **Research Experience for Undergraduates**, *University of Chicago*, Chicago, IL.
- Sum. 2008 **Research Experience for Undergraduates**, *University of Chicago*, Chicago, IL.