The Nature of Quantum Truth:

Logic, Set Theory, & Mathematics in the Context of Quantum Theory

BY

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THESIS

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to S.B.,

for everything.

in particular, your unwavering love and support, and our penguin,

as well as the 'kim's whale,' and everything that we have to look forward to...

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ABP	atomic bisection property
iff	if and only if
EA	exchange axiom
GLB	greatest lower bound
LHS	left hand side
LUB	least upper bound
max	maximum
min	minimum
NBG	von Neumann-Bernays-Gödel
OL	ortholattice
OML	orthomodular lattice
RHS	right hand side
RZFC	Reduced Zermel-Fraenkel with choice
SP	superposition principle
wff	well-formed formula
wlog	without loss of generality
wrt	with respect to

LIST OF ABBREVIATIONS (Continued)

wts	want to show
ZFC	Zermelo-Fraenkel with choice

SUMMARY

The purpose of this dissertation is to construct a radically new type of mathematics whose underlying logic differs from the ordinary classical logic used in standard mathematics, and which we feel may be more natural for applications in quantum mechanics. Specifically, we begin by constructing a first order quantum logic, the development of which closely parallels that of ordinary (classical) first order logic — the essential differences are in the nature of the logical axioms, which, in our construction, are motivated by quantum theory. After showing that the axiomatic first order logic we develop is *sound* and *complete* (with respect to a particular class of models), this logic is then used as a foundation on which to build (axiomatic) mathematical systems — and we refer to the resulting new mathematics as "quantum mathematics." As noted above, the hope is that this form of mathematics is more natural than classical mathematics for the description of quantum systems, and will enable us to address some foundational aspects of quantum theory which are still troublesome — e.g. the measurement problem — as well as possibly even inform our thinking about quantum gravity.

After constructing the underlying logic, we investigate properties of several mathematical systems — e.g. axiom systems for abstract algebras, group theory, linear algebra, etc. — in the presence of this quantum logic. In the process, we demonstrate that the resulting quantum mathematical systems have some strange, but very interesting features, which indicates a richness in the structure of mathematics that is classically inaccessible. Moreover, some of these features do indeed suggest possible applications to foundational questions in quantum theory.

We continue our investigation of quantum mathematics by constructing an axiomatic quantum set theory, which we show satisfies certain desirable criteria. Ultimately, we hope that such a set theory will lead to a foundation for quantum mathematics in a sense which parallels the foundational role of classical set theory in classical mathematics. One immediate application of the quantum set theory we develop is to provide a foundation on which to construct *quantum natural numbers*, which are the quantum analog of the classical counting numbers. It turns out that in a special class of models, there exists a 1-1 correspondence between the quantum natural numbers and bounded observables in quantum theory whose eigenvalues are (ordinary) natural numbers. This 1-1 correspondence is remarkably satisfying, and not only gives us great confidence in our quantum set theory, but indicates the naturalness of such models for quantum theory itself. We go on to develop a Peano-like arithmetic for these new "numbers," as well as consider some of its consequences. Finally, we conclude by summarizing our results, and discussing directions for future work.

CHAPTER 1

INTRODUCTION

Reasoning and 'what constitutes a valid argument' have been of interest to humankind since at least the time of Aristotle, who was the first to organize patterns of argument into logical forms. Additionally, the idea of axiomatizing a mathematical theory and deducing theorems from the axioms dates back to Euclid. However, it is only relatively recently that any attempt was made to use symbolic logic to actually formalize mathematics — the idea that axiomatic mathematics could be considered as an extended system of formal logic originated with Frege, and the first attempts to systematically carry out such a program were initiated by Whitehead and Russell in their famous *Principia Mathematica*. This approach to mathematics — i.e. the formulation of a mathematical theory as a logical system to which further axioms specific to the mathematics are appended — has been in use for just over a century, and the ability to use first order logic in this manner has revolutionized the way we "do mathematics." It is only through this process that mathematics has achieved the level of rigor with which we are familiar.

Contemporaneous with these developments in logic and axiomatic theories were two others, each of which had significant implications for modes of thought at that time. The first of these is Cantor's development of (what is known today as) *naive* set theory, along with his related construction of the infinite cardinal numbers and their associated arithmetic — subsequent work led to the development of *axiomatic* set theory, which ultimately enabled set theory to play a foundational role in mathematics (alongside first order logic). The second of these is the development of quantum theory, which was necessitated by experimental evidence of the breakdown of the laws of Newtonian mechanics at microscopic scales. With the advent of quantum theory and the developing understanding of its consequences, came the realization that perhaps nature was suggesting a new type of logic which effectively accounted for the strange new behaviors and properties of quantum systems. The publication of Birkhoff and von Neumann's seminal 1936 paper marked the inception of *quantum logic*, which ultimately branched off in directions which were abstractions of the logic that they originally envisioned. Although such branches of quantum logic have less of a direct connection to physical systems themselves or measurement related questions about them, they are potentially useful in a more abstract sense, as they suggest important applications within the field of mathematical logic.

Motivated by these historical developments, the work[1] described in this document is the construction of a quantum logic, and the beginning of an investigation into the properties of (axiomatic) mathematical systems which have this logic as their foundation, as well as a consideration of some implications of this new *quantum mathematics* for quantum theory.[2]

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[1]The work described in this thesis was done in collaboration with Richard DeJonghe and Tom Imbo.

[2]We note that we are not the first to consider the possibility of using a first order quantum logic as the underlying logic for mathematical systems — the initial contributions to quantum mathematics were made by Dunn (9) and Takeuti (22), and their powerful papers both appeared around 1980. Although these papers initially generated some interest within the community, the follow-up to their work has been somewhat minimal (due, in part, to the inherent difficulty of the subject), and the systematic study of quantum mathematics has essentially lain fallow for approximately 30 years.

More specifically, we begin by constructing a first order quantum logic,[1] the development of which closely parallels that of ordinary (classical) first order logic — the essential differences are in the nature of the logical axioms, which are motivated by quantum theory.

After describing the deductive system (i.e. axioms and inference rules) for the logic, which we denote by $\mathcal{Q}(\mathcal{L})$, we go on to develop a semantics (i.e. a model theory) for $\mathcal{Q}(\mathcal{L})$, and prove *soundness* and *completeness* theorems for our deductive system relative to this semantics.

It turns out that we can obtain an axiomatization of classical logic from $\mathcal{Q}(\mathcal{L})$ by simply adding another axiom, from which it follows that our quantum logic is *sub-classical* — i.e. every theorem of $\mathcal{Q}(\mathcal{L})$ will also be a theorem of classical logic. As such, every model of classical logic will still be a model of the quantum logic $\mathcal{Q}(\mathcal{L})$ — that is, $\mathcal{Q}(\mathcal{L})$ doesn't eliminate any classical models, but does allow for more models than are allowed classically. The differences from classical logic do not end here. We go on to consider several (axiomatic) mathematical systems — e.g. axiom systems for set theory, group theory, linear algebra, etc. — using the first order quantum logic as the underlying logic, and in the process demonstrate that quantum mathematics has some strange, but very interesting features. One example of such an intriguing feature is that certain axiomatizations of mathematical systems which are equivalent in classical mathematics (in the sense that they have exactly the same theorems) are not necessarily equivalent in quantum mathematics. This suggests a richness in the structure of mathematics which is classically inaccessible. Additionally, in our discussion of quantum mathematics we will

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[1]Although our first order logic is based on the quantum logics of Dunn (9) and Dishkant (10), the work discussed here goes well beyond either of their constructions of quantum logic.

encounter models which are extremely natural from the point of view of quantum theory. We go on to show that certain classical properties no longer hold in these natural models, as well as consider what these models are trying to teach us about quantum mechanics. In particular, we give an initial (albeit very brief) analysis of the status of the Schrödinger and von Neumann equations (as quantum mechanical axioms) in the context of quantum mathematics.

Another aspect of this work is an initial foray into quantum set theory — that is, we construct an axiomatic set theory based on the quantum logic $\mathcal{Q}(\mathcal{L})$, which we hope will ultimately lead to a foundation for quantum mathematics in a sense which parallels the foundational role of classical set theory in classical mathematics.[1] However, as quantum mathematics and the quantum set theory are both in their infancy, our immediate goals with regard to quantum set theory are much less lofty — in particular, we put forth two modest goals which, we believe, are respectable minimal criteria any attempt at quantum set theory should satisfy. First, recalling that quantum logic is sub-classical (and therefore includes all of classical logic and mathematics as a special case), we expect that quantum set theory should be a generalization of classical set theory, and in particular, those models of our quantum set theory with the standard bivalent truth values should give rise to all such models of classical set theory. Second, we expect that quantum set theory should *at least* be powerful enough to develop a notion of a 'natural number,' as well as an arithmetic for these numbers, which we again expect to reduce to classi-

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[1]Such an attempt was first made by Gaisi Takeuti (22), but the resulting quantum set theory is very difficult to work with, as has been noted by Takeuti himself. Our construction has the advantage that it is not only more intuitive, but more tractable as well.

cal arithmetic when the truth values are (the standard) bivalent truth values used in classical logic. We demonstrate that our quantum set theory meets these criteria, and moreover, we go on to show that in a special class of models, there exists a 1-1 correspondence between the new *quantum natural numbers* and bounded observables in quantum theory whose eigenvalues are (ordinary) natural numbers. This 1-1 correspondence is remarkably satisfying, and gives us great confidence in our quantum set theory, as well as its possible future applications in quantum mechanics. We then go on to study the arithmetic of the new "numbers" in these natural models, as well as consider some consequences of the arithmetic.

All in all, we feel that the work described in this document provides strong evidence that quantum mathematics has a richness and complexity worthy of further investigation. Moreover, we believe that a systematic study of axiomatic quantum theory based on quantum mathematics is warranted, and that this framework has the potential for enabling us to address foundational questions in quantum theory such as, e.g., the measurement problem, as well as for possibly informing our thinking about quantum gravity and associated issues surrounding the interpretation of the "collapse" of the wave function.

Finally, although this document is written assuming that the reader is familiar with classical first order logic and set theory, as well as has some background or familiarity with axiomatic mathematics, the appendices are intended to offer some assistance. Additionally, the relevant concepts and definitions for the theory of orthomodular lattices (which are necessary for the model theory of the quantum logic $\mathcal{Q}(\mathcal{L})$) are developed in appendix A.

CHAPTER 2

FIRST-ORDER QUANTUM LOGIC

2.1 Introduction

Mathematics enables organization and computation within quantitative science, and logic provides a foundation for mathematics. However, while classical physical systems have an internal logic which is very closely affiliated with the usual first order logic used to do mathematics, quantum theory has an intrinsic logic of its own which is different from that of classical physics. This motivates an investigation of mathematics based on (some form of) quantum logic, and one may wonder whether it is more natural to use such a "quantum mathematics" as the appropriate framework via which to describe quantum theory. To this end, we consider several mathematical systems — e.g. axiomatizations for set theory, group theory, linear algebra, etc. — using the first order quantum logic developed in this chapter as the underlying logic, as well as discuss some consequences of this for quantum theory.

After showing that the axiomatic first order logic that we develop is sound and complete (with respect to a particular class of models), we demonstrate that mathematics based on this quantum logic has some strange, but very interesting features, some of which may have applications to foundational aspects of quantum theory such as, e.g., the unification of unitary evolutions (which govern the dynamics of closed systems) and measurement evolutions.

2.1.1 Overview

In Section 2.2 we define the syntax as well as set out the basic axioms and inference rules for our logic, which is based on work by Dunn (9) and Dishkant (10). This discussion closely parallels the development for any first order predicate logic (see, e.g. (11) or (23)) — the essential differences are in the nature of the logical axioms, which, in our construction, are motivated by quantum theory. In Section 2.3 we present a semantics for our quantum logic. Here we demonstrate soundness and completeness for the semantics relative to the deductive system of Section 2.2.2 via a method developed by Dishkant (10) (although we will actually prove a stronger result than in (10)). In Section 2.4, we discuss some important and interesting features of mathematics based on this logic, and finally, in Section 2.5, we make some comments concerning the construction of models. We conclude and summarize the discussion in Section 2.6.

This chapter is somewhat terse and technical, and is written assuming the reader has some background knowledge of standard first order logic. The material is essential to the discussions in the subsequent chapters in the sense that without laying the framework for the deductive system and model theory for the first order quantum logic, the applications to specific mathematical systems cannot be made formally correct or rigorous. However, even without understanding all of the technical machinery described here, the reader can get the jist of the following chapters from a basic understanding of the gross features of what follows. To this end, the reader interested only in the qualitative features of the logic and a general overview of the details developed in this chapter is referred to Sections 2.2.1 and 2.2.2 for basic set-up and notation, as well as Sections 2.4 and 2.6 for a relatively qualitative discussion of our results.

2.2 Syntax & Deductive System

In this section we set out to define precisely what we mean by quantum mathematical systems. The discussion here will formalize both the object language (in which we can make mathematical statements) as well as important portions of the metalanguage (in which we make statements about mathematical statements).

2.2.1 The Object Language

We begin by defining the basic symbols in our object language. First, we define a set of logical symbols $\mathcal{B}_S := \{\wedge, \sim, \forall\}$ (representing logical 'and', 'not' and 'for all', respectively), along with an infinite set of (individual) variables $\mathcal{B}_V$, as well as auxiliary symbols $\mathcal{B}_A$ consisting of (left and right) parentheses and commas. In the sequel we reserve the letters x, y, z to stand for arbitrary variables (so they will be 'metavariables' — i.e. variables in the metalanguage which stand for arbitrary elements of $\mathcal{B}_V$). We then define our *basic symbols* to consist of the set $\mathcal{B} := \mathcal{B}_S \cup \mathcal{B}_V \cup \mathcal{B}_A$, which is the set of symbols which are common to any mathematical structure we consider.

A *language* $\langle \mathcal{L}, \alpha \rangle$ is then defined to be a set of "operation symbols" $\mathcal{L}^F$, along with a set of "predicate symbols" $\mathcal{L}^P$ (where we assume that $\mathcal{L}^P \neq \varnothing$), with $\mathcal{L} := \mathcal{L}^P \cup \mathcal{L}^F$ and $\mathcal{L} \cap \mathcal{B} = \varnothing$, along with a function $\alpha : \mathcal{L} \to \mathbb{N}$ (where we let $\mathbb{N} := \{0, 1, 2, \ldots\}$ denote the natural numbers).

For $f \in \mathcal{L}^F$ we call $\alpha(f)$ the *arity of* f, and say that f is an n*-ary operation.*[1] For $P \in \mathcal{L}^P$ we call $\alpha(P)$ the *arity of* P, and say that P is an n*-ary predicate.* As in ordinary first order mathematical languages, the elements of $\mathcal{L}^F$ will represent mathematical operations, while the elements of $\mathcal{L}^P$ effectively define properties of mathematical objects.

An example of a possible language $\langle \mathcal{L}_G, \alpha \rangle$ with which to construct axioms for groups is built up from $\mathcal{L}_G^P := \{\approx\}$ and $\mathcal{L}_G^F := \{e, \cdot, {}^{-1}\}$, with $\alpha(\approx) := 2$, $\alpha(e) := 0$, $\alpha({}^{-1}) := 1$, and $\alpha(\cdot) := 2$, where '$\approx$' is interpreted as the predicate "equality," 'e' is interpreted as the "identity element," '${}^{-1}$' is interpreted as the "inverse," and '$\cdot$' is interpreted as the "binary operation" for the group. Clearly, each different mathematical structure — such as sets, monoids, groups, lattices, etc. — will have a language associated with it; however, this language need not be unique. Additionally, for notational convenience, we will often refer to a language $\langle \mathcal{L}, \alpha \rangle$ simply as $\mathcal{L}$, taking the function α as implicit. In the sequel, let $\mathcal{L}$ denote any fixed language.

We define an $\mathcal{L}$*-term* inductively in the standard way — a string of symbols t is an $\mathcal{L}$*-term* if and only if either $t = x$ for $x \in \mathcal{B}_V$, or $t = f(t_1, \ldots, t_{\alpha(f)})$ for each t_i an $\mathcal{L}$-term and $f \in \mathcal{L}^F$.[2] Continuing our example in the language $\mathcal{L}_G$, both $(e \cdot x)$ and $((y \cdot z) \cdot e)$ with $x, y, z \in \mathcal{B}_V$ would be $\mathcal{L}_G$-terms, while $(e \approx e)$ would not.

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[1] A 0-ary operation is called a constant.

[2] Note that for a binary operation such as '$\cdot$', and terms t, u, we will often commit the standard abuse of notation and write $(t \cdot u)$ instead of $\cdot(t, u)$. Similarly, for the binary predicate $\approx$, we will write $t \approx u$ instead of $\approx (t, u)$.

The *well-formed formulas of* $\mathcal{L}$ (or $\mathcal{L}$-*wffs*) are then constructed inductively in the standard way using the symbols in $\mathcal{L} \cup \mathcal{B}$ — namely a string of symbols s is an $\mathcal{L}$-wff if and only if s is of the following form

1. $P(t_1, t_2, ...t_{\alpha(P)})$ for $\mathcal{L}$-terms t_i and $P \in \mathcal{L}^P$ (such $\mathcal{L}$-wffs are called *atomic*)

2. $\sim A$ for A an $\mathcal{L}$-wff

3. $(A \wedge B)$ for A, B both $\mathcal{L}$-wffs

4. $(\forall x)(A)$ for $x \in \mathcal{B}_V$ and A an $\mathcal{L}$-wff.

Returning to the language $\mathcal{L}_G$, both the strings

$$e \approx e \quad \text{and} \quad (x \approx e \ \wedge \ e \approx e)$$

would be $\mathcal{L}_G$-wffs, while $(\forall y)(y \cdot e)$ would not. Using the usual notion of free and bound variables, for an $\mathcal{L}$-wff B (respectively, $\mathcal{L}$-term t), we write $B(x)$ (respectively, $t(x)$) to indicate that the only free variable occurring in B (respectively, t) is x. Then, for any $\mathcal{L}$-term t and $\mathcal{L}$-wff $B(x)$, we write $B(t)$ to represent the formula B with all free occurrences of x replaced by t (and similarly for multiple free variables). Finally, we define an $\mathcal{L}$-*sentence* to be an $\mathcal{L}$-wff with no free variables.

We now introduce some notation which will be useful in the sequel — for $\mathcal{L}$-wffs A and B, we define[1]

$$A \vee B := \sim (\sim A \wedge \sim B), \quad (\exists x)(B) := \sim (\forall x)(\sim B), \quad A \to B := \sim A \vee (A \wedge B),$$

and then define $A \leftrightarrow B := (A \to B) \wedge (B \to A)$. Also, for some fixed $\mathcal{L}$-wff A, define $\perp := A \wedge (\sim A)$, and $\top := \sim \perp$. To reduce notational clutter, we take $\sim$ to bind tighter than $\wedge, \vee$, which bind tighter than $\to$, which, in turn, binds tighter than $\exists, \forall$. We warn the reader that we may occasionally either omit or add parentheses to make things clearer — ideally this will only reduce confusion and never further it.

2.2.2 Formal Deduction

Now that we have defined our object language, we proceed to construct a formal deductive system for quantum logic.[2] We first define our *quantum logical axioms for* $\mathcal{L}$ to be the set of axiom schema (Q1) — (Q6) below (meaning the collection of formulas (Q1) — (Q6) with A, B replaced by *every* $\mathcal{L}$-wff and t by *every* $\mathcal{L}$-term), which we denote by $\mathcal{Q}_A(\mathcal{L})$.

(Q1) $A \to \top \wedge A$

(Q2) $(\sim\sim A) \to A$ and $A \to (\sim\sim A)$

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[1] Readers familiar with orthomodular lattices will recognize '→' as the infamous 'Sasaki hook'.

[2] All of the axioms and inference rules that follow are chosen with an eye toward a completeness theorem for a particular class of models — namely, those whose "truth value algebra" is an orthomodular lattice. These models, in turn, are motivated by quantum theory (see e.g., the seminal work of Birkhoff and von Neumann (2)).

(Q3) $A \wedge B \rightarrow A$ and $A \wedge B \rightarrow B$

(Q4) $[A \wedge (A \rightarrow B))] \rightarrow B$

(Q5) $(A \wedge \sim A) \rightarrow B$

(Q6) $(\forall x)B(x) \rightarrow B(t)$

By an $\mathcal{L}$-*rule* we mean a pair (Γ, A) where Γ is a finite set of $\mathcal{L}$-wffs and A is an $\mathcal{L}$-wff. For $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ we will write '$\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n \Longrightarrow A$' to denote the $\mathcal{L}$-rule (Γ, A). We then define our *quantum rules of inference* to be the schema[1]

(R1) $A \rightarrow B,\ B \rightarrow C \implies A \rightarrow C$

(R2) $A \rightarrow B \implies \sim B \rightarrow \sim A$

(R3) $A \rightarrow B,\ A \rightarrow C \implies A \rightarrow (B \wedge C)$

(R4) $A \rightarrow B(z) \implies A \rightarrow (\forall x)B(x)$

(R5) $A,\ A \rightarrow B \implies B.$

In (R4) we assume z does not occur free in A. We denote the set of all quantum rules of inference for the language $\mathcal{L}$ by $\mathcal{Q}_R(\mathcal{L})$, and define the quantum logic $\mathcal{Q}(\mathcal{L})$ as $\mathcal{Q}(\mathcal{L}) \coloneqq \mathcal{Q}_A(\mathcal{L}) \cup \mathcal{Q}_R(\mathcal{L})$.

In the case in which $\mathcal{L}^P$ includes "equality" $\approx$, we define (E1) — (E3) (listed below for $x, y, z \in \mathcal{B}_V$) to be the *equality axioms for* $\mathcal{L}$, which we collectively denote $\mathcal{E}(\mathcal{L})$, and which

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[1]Analogues of our (Q1) — (Q6) and (R1) — (R4) appear in Dunn (9) numbered 1–10, and he refers to this system as OM#. It is important to point out that we have transferred his 'relational logic' approach into a more standard treatment of axioms and derivability.

ensure that the predicate $\approx$ behaves like "equality." When the only predicate in $\mathcal{L}^P$ is "equality," then we will say that $\mathcal{L}$ is an *equational language*.

(E1) Reflexivity: $(\forall x)[x \approx x]$

(E2) Symmetry: $(\forall x)(\forall y)[(x \approx y) \to (y \approx x)]$

(E3) Weak Transitivity: $(\forall x)(\forall y)(\forall z)[[(x \approx y) \wedge (y \approx z)] \to (x \approx z)]]$

We also define an alternate form of transitivity below which will be useful for us in some contexts.[1]

(E3′) Strong Transitivity: $(\forall x)(\forall y)(\forall z)[(x \approx y) \to [(y \approx z) \to (x \approx z)]]$.

In the framework of classical logic, (E3) and (E3′) are logically equivalent (as we will show in the following section). However, as we will see, (E3′) is strictly stronger in the quantum logic $\mathcal{Q}(\mathcal{L})$. Also in classical logic, it is customary to enforce the following *substitution axioms for* $\mathcal{L}$, which consist of an axiom for each $f \in \mathcal{L}$ and $m \in \{1, \ldots, \alpha(f)\}$, where $x, y, z_1, \ldots z_{\alpha(f)} \in \mathcal{B}_V$.

(Sub) Substitution of the operation f in the m^{th} slot for the predicate P:

$$(\forall x)(\forall y)(\forall z_1)\cdots(\forall z_{\alpha(f)})\big(P(x, y, \ldots) \to$$
$$[P(f(z_1, \ldots, z_{m-1}, x, z_{m+1}, \ldots, z_{\alpha(f)}), f(z_1, \ldots, z_{m-1}, y, z_{m+1}, \ldots, z_{\alpha(f)}), \ldots)]\big).$$

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[1] Note, however, that (E3′) is not part of $\mathcal{E}(\mathcal{L})$ — that is, we do not require (E3′) as an axiom for languages with "equality" $\approx$.

In particular, note that for "equality" $\approx$, this becomes

$$(\forall x)(\forall y)(\forall z_1)\cdots(\forall z_{\alpha(f)})\big((x \approx y) \to$$
$$[f(z_1,\ldots,z_{m-1},x,z_{m+1},\ldots,z_{\alpha(f)}) \approx f(z_1,\ldots,z_{m-1},y,z_{m+1},\ldots,z_{\alpha(f)})]\big).$$

We will say that an operation f *satisfies substitution for the predicate* P if (Sub) is satisfied for every "slot" of f. In particular, for a language $\mathcal{L}$ with "equality" $\approx$, if the $\mathcal{L}$-wff schema (Sub) holds in $\hat{A}$ for "equality" and for some operation $f \in \mathcal{L}$ in every "slot," we say that $\hat{A}$ *has substitution for* f *with respect to* $\approx$. As we will see in Sections 3.4 and 3.5, some very natural constructions in quantum mathematics fail to satisfy (Sub) for at least some of their operations.

Now that we have established the axioms for our quantum logic, we can define formal deductions. For any set of $\mathcal{L}$-wffs Γ, an $\mathcal{L}$-wff A is said to be *derivable from* Γ (in symbols, $\Gamma \vdash A$) if there exists a sequence $A_1, A_2, \ldots, A_n$ (called a *formal argument*) such that $\gamma_n = A$ and for each $i < n$, either $A_i \in \Gamma \cup \mathcal{Q}_A(\mathcal{L}) \cup \mathcal{E}(\mathcal{L})$ (or simply $A_i \in \Gamma \cup \mathcal{Q}_A(\mathcal{L})$ if $\approx \notin \mathcal{L}^P$) or there exists a subset $\Gamma_0 \subseteq \{\gamma_1, \ldots \gamma_{i-1}\}$ such that $\Gamma_0 \Longrightarrow A_i$ by some $\mathcal{L}$-rule (R1) — (R5). The statement 'A is derivable from Γ' informally means that one can construct a proof of A from the set of statements Γ. We then extend the notion of derivable to sets of $\mathcal{L}$-wffs in the natural way — a set Γ' of $\mathcal{L}$-wffs is derivable from Γ if every $\gamma \in \Gamma'$ is derivable from Γ. Given a set of $\mathcal{L}$-wffs Γ, and two $\mathcal{L}$-wffs A, B, we will write $\Gamma, A \vdash B$ to mean $\Gamma \cup \{A\} \vdash B$, and $A \vdash B$ to mean $\{A\} \vdash B$. Also, we will write $\vdash A$ to mean $\varnothing \vdash A$. It is easy to see that this deduction system is *monotonic* — i.e. for two sets of $\mathcal{L}$-wffs Γ, Γ' with $\Gamma \subseteq \Gamma'$, and any $\mathcal{L}$-wff A, whenever $\Gamma \vdash A$ then also $\Gamma' \vdash A$.

Another important notion is that of logical equivalence — for $\mathcal{L}$-wffs A and B and any set of $\mathcal{L}$-wffs Γ, we say that A and B are *logically equivalent with respect to* Γ if $\Gamma \vdash A \leftrightarrow B$ (or simply *logically equivalent* if $\Gamma = \varnothing$). We present examples utilizing these concepts in Section 2.2.3. For any set of $\mathcal{L}$-wffs Γ, the *quantum theorems of* Γ (denoted $\mathcal{T}(\Gamma)$) are defined to be the set of all $\mathcal{L}$-wffs derivable from $\Gamma \cup \mathcal{E}(\mathcal{L})$ if "equality" $\approx$ is a predicate in the language; otherwise, $\mathcal{T}(\Gamma)$ is defined to be the set of all $\mathcal{L}$-wffs derivable from Γ. One simple example is that the axiom schema (E3) is derivable from (E3$'$) for any language $\mathcal{L}$ with "equality" $\approx$ and any set of $\mathcal{L}$-wffs Γ (see Section 2.2.3).

We conclude our discussion of the syntax by defining a *mathematical system* (or *M-system*) to be a pair $(\mathcal{L}, \mathcal{A})$ where $\mathcal{L}$ is a language, and $\mathcal{A}$ is a set of $\mathcal{L}$-wffs (which are effectively the set of mathematical axioms for the M-system). Furthermore, for a given M-system $(\mathcal{L}, \mathcal{A})$ and any $\mathcal{L}$-wff A, we say that A is a *theorem of* $(\mathcal{L}, \mathcal{A})$ (or just a "theorem of $\mathcal{A}$" if $\mathcal{L}$ is clear from the context) if $A \in \mathcal{T}(\mathcal{A})$. For example, recalling the language $\mathcal{L}_G$ defined above, one possibility for $\mathcal{A}_G$ is the following set of axioms (with $x, y, z \in \mathcal{B}_V$).

(G1) $(\forall x)(\forall y)(\forall z)[(x \cdot y) \cdot z \approx x \cdot (y \cdot z)]$

(G2) $(\forall x)[e \cdot x \approx x]$ and $(\forall x)[x \cdot e \approx x]$

(G3) $(\forall x)[x \cdot x^{-1} \approx e]$ and $(\forall x)[x^{-1} \cdot x \approx e]$

(G4) $(\forall x)(\forall y)[x \approx y \to x^{-1} \approx y^{-1}]$

(G5) $(\forall x)(\forall y)(\forall z)[x \approx y \to x \cdot z \approx y \cdot z]$ and $(\forall x)(\forall y)(\forall z)[x \approx y \to z \cdot x \approx z \cdot y]$

Note that (G4) and (G5) are just (Sub) for '$^{-1}$' and '$\cdot$', respectively.

2.2.3 Some Simple Formal Arguments

In this section we will provide some examples of the syntax discussed above. In what follows, assume that $\mathcal{L}$ is some fixed language, and recall the definition of the Sasaki hook, namely $A \to B :=\sim A \vee (A \wedge B)$

Lemma 2.1. Let A be any $\mathcal{L}$-wff. Then $\vdash A \to A$.

Proof. We construct the following formal argument —

$s_1 := A \to \top \wedge A$ (Q1)

$s_2 := \top \wedge A \to A$ (by (Q3))

$s_3 := A \wedge B \to A$ (by (R1) from s_1 and s_2)

□

Lemma 2.2. Let A, B, C be $\mathcal{L}$-wffs. Then

$$[A \to (B \to C)] \vdash [(A \wedge B) \to C]. \tag{2.1}$$

Proof.

$$s_1 \coloneqq (A \wedge B) \;\rightarrow\; A \qquad \text{(by (Q3))}$$

$$s_2 \coloneqq (A \wedge B) \;\rightarrow\; B \qquad \text{(by (Q3))}$$

$$s_3 \coloneqq A \;\rightarrow\; (B \rightarrow C) \qquad \text{(by assumption)}$$

$$s_4 \coloneqq (A \wedge B) \rightarrow (B \rightarrow C) \qquad \text{(by (R1) from } s_1 \text{ and } s_3\text{)}$$

□

$$s_5 \coloneqq (A \wedge B) \rightarrow B \wedge (B \rightarrow C) \qquad \text{(by (R3) from } s_2 \text{ and } s_4\text{)}$$

$$s_6 \coloneqq [B \wedge (B \rightarrow C)] \;\rightarrow\; C \qquad \text{(by (Q4))}$$

$$s_7 \coloneqq (A \wedge B) \;\rightarrow\; C \qquad \text{(by (R1) from } s_5 \text{ and } s_6\text{)}$$

Corollary 2.3. (E3) is derivable from (E3′).

Proof. Let $A = t \approx u$, $B = u \approx v$, and $C = t \approx v$ in the above lemma, where t, u, v are any $\mathcal{L}$-terms. Note that the axiom schema

$$[(t \approx u) \wedge (u \approx v)] \rightarrow (t \approx v)$$

is an equivalent way of stating the axiom (E3) (and similarly for (E3′)). □

Typically we will be more informal in demonstrating that an $\mathcal{L}$-wff is derivable.

Lemma 2.4. Let A, B, C be $\mathcal{L}$-wffs, and Γ be any set of $\mathcal{L}$-wffs. Then

1. $\vdash \top$
2. $\top \rightarrow A \vdash A$
3. $A \vdash B \vee A$
4. $A \vdash \top \rightarrow A$
5. $A, B \vdash A \wedge B$

6. If $\Gamma \vdash A \leftrightarrow B$ then $\Gamma, A \vdash B$ and $\Gamma, B \vdash A$.

7. $\vdash A \wedge B \leftrightarrow B \wedge A$

8. $\vdash A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

9. $\vdash \sim\sim A \leftrightarrow A$.

Proof.

1. Let $Z := (\sim\sim A) \to A$ for some fixed $\mathcal{L}$-wff A, so we have Z by (Q2). Then $\bot \to \sim Z$ by (Q5). (R2) then gives $\sim\sim Z \to \top$. But $Z \to \sim\sim Z$ by (Q2), so (R1) yields $Z \to \top$. Modus ponens (R5) then gives $\top$.

2. From (1) above, we have $\top$. By assumption we have $\top \to A$, and so (R5) yields A.

3. First $(\sim B \wedge \sim A) \to \sim A$ by (Q3). This then yields $\sim\sim A \to B \vee A$ by (R2). By (Q2) and (R1) we then obtain $A \to B \vee A$. Using the assumption A and (R5) then gives $B \vee A$.

4. (Q1) gives $A \to \top \wedge A$. By assumption we have A, so (R5) gives $\top \wedge A$, and so by (3) above we have $\sim \top \vee (\top \wedge A) = \top \to A$.

5. By (4) above, we have $\top \to A$ and $\top \to B$. Then by (R3), this gives $\top \to (A \wedge B)$, which by (2) above gives $A \wedge B$.

6. First assume Γ, A. From Γ we have $A \leftrightarrow B$, and then (Q3) and (R5) yield $A \to B$. By assumption we have A, and so (R5) gives B. Assuming Γ, B, the proof works similarly.

7. We have $A \wedge B \to B$, as well as $A \wedge B \to A$ by (Q3). (R3) then gives $A \wedge B \to B \wedge A$. In similar fashion, we have $B \wedge A \to A \wedge B$. Then (5) above yields $A \leftrightarrow B$.

8. We have $A \wedge (B \wedge C) \to A$ and $A \wedge (B \wedge C) \to (B \wedge C)$ by (Q3). Then by (Q3) again we have $B \wedge C \to B$ and $B \wedge C \to C$, so (R1) gives $A \wedge (B \wedge C) \to B$ and $A \wedge (B \wedge C) \to C$. Then, using (R3) we obtain $A \wedge (B \wedge C) \to (A \wedge B)$ and (R3) again yields $A \wedge (B \wedge C) \to (A \wedge B) \wedge C$. A similar argument yields the other arrow, and (5) above gives the desired conclusion.

9. This follows trivially from (Q2) and (5) above.

□

Lemma 2.5. Let A, B, C, D be $\mathcal{L}$-wffs. Then $(A \vee B) \to (C \vee D)$ is derivable from $A \to C$ and $B \to D$ — that is

$$A \to C,\ B \to D \vdash (A \vee B) \to (C \vee D).$$

Proof. First, $\sim C \to \sim A$ and $\sim D \to \sim B$ by (R2), and then $(\sim C \wedge \sim D) \to \sim A$ and $(\sim C \wedge \sim D) \to \sim B$ by (Q3) and (R1). Then $(\sim C \wedge \sim D) \to (\sim A \wedge \sim B)$ by (R3) and then, by (R2) and the definition of $\vee$, this gives $(A \vee B) \to (C \vee D)$. □

In classical logic there is a clear connection between derivability ($\vdash$) and implication ($\to$) in the form of the 'deduction theorem' which states, for any set of $\mathcal{L}$-wffs Γ and $\mathcal{L}$-wffs A and B, that $\Gamma, A \vdash B$ if and only if $\Gamma \vdash A \to B$. We note that nothing of the sort holds for our quantum logic $\mathcal{Q}(\mathcal{L})$ — not even for the case $\Gamma = \varnothing$.

We conclude this section with an extremely useful theorem whose proof is similar to that for the classical first order predicate calculus.

Theorem 1. *(Replacement)* Let $(\mathcal{L}, \mathcal{A})$ be any M-system, and let A and B be $\mathcal{L}$-wffs which are logically equivalent with respect to $\mathcal{A}$. Further let C_A and C_B be $\mathcal{L}$-wffs which are identical except that one instance of A in C_A is replaced by B in C_B. Then C_A and C_B are logically equivalent with respect to $\mathcal{A}$.

Proof. Assume that A and B are logically equivalent with respect to $\mathcal{A}$, so that $\mathcal{A} \vdash A \leftrightarrow B$. We proceed by induction on the construction of $\mathcal{L}$-wffs, so let φ_A and φ_B be $\mathcal{L}$-wffs such as in the hypothesis with $\mathcal{A} \vdash \varphi_A \leftrightarrow \varphi_B$. Then $\varphi_A \rightarrow \varphi_B$ by (Q3), and so $\sim \varphi_B \rightarrow \sim \varphi_A$ by (R2). Similarly we have $\sim \varphi_A \rightarrow \sim \varphi_B$, and so Lemma 2.4 (5) gives $\sim \varphi_A \leftrightarrow \sim \varphi_B$. Let D be some other $\mathcal{L}$-wff. $\varphi_A \wedge D \rightarrow \varphi_A$ and $\varphi_A \rightarrow D$ by (Q3), and so $\varphi_A \wedge D \rightarrow \varphi_B$ by (R1) and the inductive hypothesis, and then (R3) gives $\varphi_A \wedge D \rightarrow \varphi_B \wedge D$. Similarly, we have $\varphi_B \wedge D \rightarrow \varphi_A \wedge D$, and so Lemma 2.4 (5) gives $\varphi_A \wedge D \leftrightarrow \varphi_B \wedge D$. By (7) in Lemma 2.4, we have $D \wedge \varphi_A \leftrightarrow D \wedge \varphi_B$. Finally, assume all free variables in φ_A, φ_B are listed $(x, y_1, \ldots, y_n)$. Then by (Q6),

$$\varphi_A(x, y_1, \ldots, y_n) \rightarrow (\forall x)(\varphi_A(x, y_1, \ldots, y_n)),$$

and also $\varphi_B \rightarrow \varphi_A$ by inductive hypothesis and (Q3). (R1) then gives

$$\varphi_B(x, y_1, \ldots, y_n) \rightarrow (\forall x)(\varphi_A(x, y_1, \ldots, y_n)).$$

(Q6) gives

$$(\forall x)(\varphi_B(x, y_1, \ldots, y_n)) \rightarrow \varphi_B(x, y_1, \ldots, y_n),$$

and so (R1) again gives

$$(\forall x)(\varphi_B(x, y_1, \ldots, y_n)) \to (\forall x)(\varphi_A(x, y_1, \ldots, y_n)).$$

The other expression

$$(\forall x)(\varphi_A(x, y_1, \ldots, y_n)) \to (\forall x)(\varphi_B(x, y_1, \ldots, y_n))$$

is derived similarly, so Lemma 2.4 (5) gives the desired conclusion. □

Of course, using induction this can be extended to any number of replacements of A by B.

2.3 Quantum Models

In order to set up our discussion of models of mathematical systems whose underlying logic is $\mathcal{Q}(\mathcal{L})$, we begin with some preliminary definitions and concepts.

For any set A and any language $\mathcal{L}$, a *truth function on* A is a set of maps

$$\{[\![P]\!] : A^{\alpha(P)} \to L \mid P \in \mathcal{L}^P\},$$

where L is a complete orthomodular lattice. For $a_1, a_2, ...a_{\alpha(P)} \in A$ — so that $(a_1, a_2, ...a_{\alpha(P)}) \in A^{\alpha(P)}$ — we have that $[\![P]\!] : A^{\alpha(P)} \to L$ is such that

$$[\![P]\!]((a_1, a_2, ...a_{\alpha(P)})) := [\![P(a_1, a_2, ...a_{\alpha(P)})]\!].$$

In a language $\mathcal{L}$ with "equality" $\approx$, we further require that (for all $a, b \in A$)[1]

$$[\![a \approx b]\!] = 1 \quad \text{if and only if} \quad a = b.$$

In the notion of "model" that we'll define below, $[\![P(a,b,...)]\!]$ can informally be interpreted as the "truth value" of the atomic $\mathcal{L}$-wff $P(a,b,...)$.

For any language $\mathcal{L}$, an *$\mathcal{L}$-structure* $\hat{A}$ is a sequence

$$\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$$

consisting of

1. A non-empty set[2] A (called the *underlying set* of $\hat{A}$) where the variables are interpreted.
2. A complete orthomodular lattice L (called the *truth value algebra of* $\hat{A}$).
3. A truth function $\{[\![P]\!]\}_{P \in \mathcal{L}^P}$ on A (called the *truth function for* $\hat{A}$), where $\bigcup_{P \in \mathcal{L}^P} Im([\![P]\!])$ generates L as a complete ortholattice.[3]

---

[1]Note that in this statement, equality is used in three distinct ways. $\approx$ denotes the predicate "equality", while the first use of "=" denotes equality in L and the second use of "=" denotes equality in the meta-language.

[2]Note that more generally we can develop the $\mathcal{L}$-structures to be based on *classes* instead of *sets*.

[3]$Im([\![P]\!])$ denotes the image of the set A under the map $[\![P]\!]$. We require L to be generated by $\bigcup_{P \in \mathcal{L}^P} Im([\![P]\!])$ to avoid odd cases where L is non-Boolean but $Im([\![P]\!])$ only generates a Boolean sub-algebra of L. Such cases are, in some sense, classical. See Section 2.3.1 for a brief discussion.

4. A list F_A which consists of, for every $f \in \mathcal{L}^F$, one $\alpha(f)$-ary function $\hat{f} : A^{\alpha(f)} \to A$ (where $\hat{f}$ is called the *interpretation of* f *in* $\hat{A}$).[1]

We sometimes also say that $\hat{A}$ is *based on* A to mean that A is the underlying set for $\hat{A}$.

By adding each element of a set A as a (constant) symbol to our language $\mathcal{L}$, we obtain a new (extended) language $\mathcal{L}_A$. We then define, for any $\mathcal{L}$-wff $B(y_1, \ldots, y_n)$ an *evaluation of* B to be the $\mathcal{L}_A$-term $B(a_1, \ldots, a_n)$ where $a_1, \ldots, a_n \in A$. An *evaluated* $\mathcal{L}$*-wff* is then defined to be an evaluation of any $\mathcal{L}$-wff.[2] For any $\mathcal{L}_A$-term t with no free variables, define the evaluation of t (denoted here by $\tilde{t}$) inductively by

1. $\tilde{a} := a$ for all $a \in A$.
2. $\tilde{f}(t_1, \ldots, t_{\alpha(f)}) := \hat{f}(\tilde{t_1}, \ldots, \tilde{t}_{\alpha(f)})$

We note that the evaluation of t is clearly an element of A, and $A \subseteq \mathcal{L}_A$. Now, for each $P \in \mathcal{L}^P$, we can then extend $[\![P]\!]$ to a map on all evaluated $\mathcal{L}$-wffs (which we still refer to as a truth function, and which we will still denote by $[\![P]\!]$). We define the extension of $[\![P]\!]$ inductively on the set of all evaluated $\mathcal{L}$-wffs by (where B and C are $\mathcal{L}$-wffs, $t_i(y_1, \ldots, y_n)$ is an $\mathcal{L}$-term, and $a_1, \ldots, a_n \in A$),

1. $[\![P(t_1(y_1, \ldots, y_n), t_2(y_1, \ldots, y_n), ...)]\!] := [\![P(\tilde{t}_1(a_1, \ldots, a_n), \tilde{t}_2(a_1, \ldots, a_n), ...)]\!]$.
2. $[\![\sim B]\!] := \neg[\![B]\!]$.

---

[1] In the following sections we will frequently use the same symbol for an operation in a given language and its interpretation in the relevant $\mathcal{L}$-structure.

[2] Note that any $\mathcal{L}$-wff with no free variables is vacuously an evaluated $\mathcal{L}$-wff.

3. $[\![B \wedge C]\!] := [\![B]\!] \wedge [\![C]\!]$.

4. $[\![(\forall x)B(x)]\!] := \bigwedge_{a \in A} [\![B(a)]\!]$.

For an evaluated $\mathcal{L}$-wff B, we call $[\![B]\!]$ the *truth value of* B. Note that this inductive definition is 'top down' which is best illustrated by an example —

$$[\![(\forall x)[(\forall y)(x \approx y)]]\!] = \bigwedge_{a \in A} [\![(\forall y)(a \approx y)]\!] = \bigwedge_{a \in A} \Big(\bigwedge_{b \in A} [\![a \approx b]\!] \Big).$$

Let $\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure. We say that an $\mathcal{L}$-wff $B(y_1, \ldots, y_n)$ *holds in* $\hat{A}$ if

$$[\![B(a_1, \ldots, a_n)]\!] = 1$$

for any $a_1, \ldots, a_n \in A$, and we will sometimes denote this by $\hat{A} \models B(y_1, \ldots y_n)$. Note that the expression above for the truth value of $B(a_1, \ldots a_n)$ is completely equivalent to

$$[\![(\forall y_1)(\forall y_2) \cdots (\forall y_n) B(y_1, \ldots, y_n)]\!] = 1,$$

i.e. taking the "for all" in the metalanguage is equivalent to using '$\forall$' in the object language when it comes to determining if a given $\mathcal{L}$-wff holds in a given structure.

Finally, we conclude this section with a couple of simple lemmas which are useful computationally.

Lemma 2.6. Let $\mathcal{L}$ be a language, let $\hat{A}$ be an $\mathcal{L}$-structure with underlying set A, and let $A(x)$ be an $\mathcal{L}$-wff (with $x \in \mathcal{B}_V$). Then $(\forall x)A(x)$ holds in $\hat{A}$ iff $A(a)$ holds in $\hat{A}$ for every $a \in A$ (i.e. $\hat{A} \vDash (\forall x)A(x)$ iff $\hat{A} \vDash A(a)$ for every $a \in A$).

Proof. First, letting $\{[\![P]\!]\}$ be the truth function of $\hat{A}$, we assume that $A(a)$ holds in $\hat{A}$ for every $a \in A$, i.e. that for any $a \in A$, we have $[\![A(a)]\!] = 1$. But then we have

$$[\![(\forall x)A(x)]\!] = \bigwedge_{a\in A} [\![A(a)]\!] = \bigwedge_{a\in A} 1 = 1,$$

so that $(\forall x)A(x)$ holds in $\hat{A}$.

Conversely, assume $(\forall x)A(x)$ holds in $\hat{A}$. Then we have

$$1 = [\![(\forall x)A(x)]\!] = \bigwedge_{a\in A} [\![A(a)]\!],$$

which means that $[\![A(a)]\!] = 1$ for all $a \in A$. □

Note that this result is easily extended to an arbitrary number of variables.

Lemma 2.7. Let $\mathcal{L}$ be a language, and let $\hat{A}$ be an $\mathcal{L}$-structure with truth function $\{[\![P]\!]\}$ and underlying set A. Further let $a, b, a_1, \dots a_n \in A$ be such that

$$[\![P(a_1, \dots, a_{m-1}, a, a_{m+1}, a_{\alpha(P)})]\!] = [\![P(a_1, \dots, a_{m-1}, b, a_{m+1}, a_{\alpha(P)})]\!]$$

for every $P \in \mathcal{L}^P$ and for every $m \in \{1, \dots, \alpha(P)\}$. Then for any $\mathcal{L}$-wff $\psi(x, x_1, \dots, x_n)$ (with $x \in \mathcal{B}_V$), we have that $[\![\psi(a, a_1, \dots, a_n)]\!] = [\![\psi(b, a_1, \dots, a_n)]\!]$.

Proof. The proof is by induction on the construction of $\mathcal{L}$wffs. By assumption, the result holds for the atomic $\mathcal{L}$-wffs. Assuming the result holds for some $\mathcal{L}$-wff ψ, we have

$$[\![\neg\psi(a, a_1, \ldots, a_n)]\!] = \neg[\![\psi(a, a_1, \ldots, a_n)]\!] = \neg[\![\psi(b, a_1, \ldots, a_n)]\!] = [\![\neg\psi(b, a_1, \ldots, a_n)]\!].$$

Next, assuming the result holds for ψ_1 and ψ_2 we have

$$\begin{aligned}
[\![(\psi_1 \wedge \psi_2)(a, a_1, \ldots, a_n)]\!] &= [\![\psi_1(a, a_1, \ldots, a_n) \wedge \psi(a, a_1, \ldots, a_n)]\!] \\
&= [\![\psi_1(a, a_1, \ldots, a_n)]\!] \wedge [\![\psi_2(a, a_1, \ldots, a_n)]\!] \\
&= [\![\psi_1(b, a_1, \ldots, a_n)]\!] \wedge [\![\psi_2(b, a_1, \ldots, a_n)]\!] \\
&= [\![\psi_1(b, a_1, \ldots, a_n) \wedge \psi_1(b, a_1, \ldots, a_n)]\!] \\
&= [\![(\psi_1 \wedge \psi_2)(b, a_1, \ldots, a_n)]\!].
\end{aligned}$$

Finally, assume the result holds for some $\psi(x, x_1, \ldots, x_n, y)$. Then

$$\begin{aligned}
[\![(\forall y)\psi(a, a_1, \ldots, a_n, y)]\!] &= \bigwedge_{c \in A} [\![\psi(a, a_1, \ldots, a_n, c)]\!] \\
&= \bigwedge_{c \in A} [\![\psi(b, a_1, \ldots, a_n, c)]\!] \\
&= [\![(\forall y)\psi(b, a_1, \ldots, a_n, y)]\!].
\end{aligned}$$

This completes the induction. $\square$

2.3.1 Models

Finally, we are in a position to define a model. Let $(\mathcal{L}, \mathcal{A})$ be an M-system, and let $\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure. Then, in a language $\mathcal{L}$ with "equality" $\approx$, we say that $\hat{A}$ is a *model for* $(\mathcal{L}, \mathcal{A})$ (or simply a model for $\mathcal{A}$ if $\mathcal{L}$ is clear from the context) if every $\mathcal{L}$-wff in $\mathcal{Q}_A(\mathcal{L}) \cup \mathcal{E}(\mathcal{L}) \cup \mathcal{A}$ holds in $\hat{A}$. (If there is no notion of equality in the language $\mathcal{L}$, we say that $\hat{A}$ is a *model for* $(\mathcal{L}, \mathcal{A})$ if every $\mathcal{L}$-wff in $\mathcal{Q}_A(\mathcal{L}) \cup \mathcal{A}$ holds in $\hat{A}$.)

We call a model *standard* if L is a Boolean algebra, otherwise it is said to be *non-standard*. In the usual approach to classical logic, the only admissible models are ones in which $L = \mathbf{2}$, where $\mathbf{2}$ is the two element Boolean algebra. From this we see that our notion of a standard model goes beyond models typically considered in classical logic. However, it is well-known that an $\mathcal{L}$-wff φ holds in all models where L is a Boolean algebra if and only if φ is true in all models with $L = \mathbf{2}$, and so in this sense all standard models behave classically, which is why, for purposes of this discussion, we will focus on non-standard models.[1]

2.3.2 Soundness and Completeness

Having defined a semantics for our language, the first things to look for are soundness and completeness theorems. It is straightforward to see that our quantum axioms and rules of inference are sound, i.e. every $\mathcal{L}$-wff which is derivable from $\mathcal{A}$ holds in every model of $\mathcal{A}$. Theorems 2 and 3 below make this explicit.

---

[1]For a discussion of Boolean-valued models and their connection to set theory and forcing, see (7).

We would also like a completeness theorem stating that for any language $\mathcal{L}$ and any set of $\mathcal{L}$-wffs $\mathcal{A}$, if an $\mathcal{L}$-wff φ holds in every model of $\mathcal{A}$, then it is derivable from $\mathcal{A}$. Following Dishkant (10), we prove such a completeness theorem; for the details, see Lemmas 2.9 – 2.13, Corollary 2.8 and Theorem 4 below, as well as the associated discussion. More strongly, we have actually been able to demonstrate completeness using only models whose associated truth value algebras are *irreducible*, as illustrated by (Lemma 2.14 and) Theorem 5 below.

Theorem 2. Soundness: Let $(\mathcal{L}, \mathcal{A})$ be an M-system. The axioms (Q1) – (Q6) are true in any model $\hat{W} := (W, L, \{[\![P]\!]\}, F_W)$ of $(\mathcal{L}, \mathcal{A})$.

Proof.

(Q1). $A \to (T \wedge A)$.

We have that

$$[\![A \to (T \wedge A)]\!] = [\![A]\!] \to [\![T \wedge A]\!] = [\![A]\!] \to ([\![T]\!] \wedge [\![A]\!]).$$

Taking $[\![A]\!] := x$, where $x \in L$, we have that $[\![A]\!] \to ([\![T]\!] \wedge [\![A]\!])$ becomes

$$\neg x \vee (x \wedge (1 \wedge x)) = \neg x \vee (1 \wedge x) = \neg x \vee x = 1.$$

Thus, we have that $[\![A \to (T \wedge A)]\!] = 1$ in any model.

(Q2). $\sim\sim A \to A$ and $A \to \sim\sim A$.

We have that $[\![\sim\sim A]\!] = \neg\neg[\![A]\!]$, and so, $[\![\sim\sim A \to A]\!]$ and $[\![A \to \sim\sim A]\!]$ both become

$[\![A]\!] \to [\![A]\!]$. Now, taking $[\![A]\!] := x$, where $x \in L$, we have that $\neg\neg[\![A]\!] = [\![A]\!]$ since the law of double negation holds in any OML. Thus, since $\leq$ is reflexive, we have that $x \to x = 1$, where we have used the fact that $p \to q = 1$ if and only if $p \leq q$. Thus, we see that

$$[\![\sim\sim A \to A]\!] = [\![A \to \sim\sim A]\!] = 1.$$

(Q3). $A \wedge B \to A$ and $A \wedge B \to B$.

We have that

$$[\![A \wedge B \to A]\!] = [\![A \wedge B]\!] \to [\![A]\!] = [\![A]\!] \wedge [\![B]\!] \to [\![A]\!].$$

Taking $[\![A]\!] := x$ and $[\![B]\!] := y$, where $x, y \in L$, we have that $[\![A]\!] \wedge [\![B]\!] \to [\![A]\!]$ becomes

$$x \wedge y \to x.$$

Since $x \wedge y \leq x$ for any $x, y \in L$, we have that $x \wedge y \to x = 1$ (since $p \to q = 1$ if and only if $p \leq q$). Thus, we see that

$$[\![A \wedge B \to A]\!] = 1.$$

Similarly,

$$[\![A \wedge B \to B]\!] = 1.$$

(Q4). $[A \wedge (A \to B)] \to B$.

We have that

$$[\![(A \wedge (A \to B)) \to B]\!] = [\![A \wedge (A \to B)]\!] \to [\![B]\!]$$

$$= [[\![A]\!] \wedge [\![A \to B]\!]] \to [\![B]\!] = [[\![A]\!] \wedge ([\![A]\!] \to [\![B]\!])] \to [\![B]\!].$$

Taking $[\![A]\!] := x$ and $[\![B]\!] := y$, where $x, y \in L$, we have that $[[\![A]\!] \wedge ([\![A]\!] \to [\![B]\!])] \to [\![B]\!]$ becomes

$$[x \wedge (x \to y)] \to y.$$

Now, $x \wedge (x \to y) = x \wedge (\neg x \vee (x \wedge y)) = x \wedge y$, where the last equality follows from the fact that L is an OML. Since $x \wedge y \leq y$ for all $x, y \in L$, we have that $x \wedge y \to y = 1$ (since $p \to q = 1$ if and only if $p \leq q$). Thus, we have that

$$[[\![A]\!] \wedge ([\![A]\!] \to [\![B]\!])] \to [\![B]\!] = 1.$$

(Q5). $A \wedge \sim A \to B$.

We have that

$$[\![A \wedge \sim A \to B]\!] = [\![A \wedge \sim A]\!] \to [\![B]\!] = ([\![A]\!] \wedge [\![\sim A]\!]) \to [\![B]\!]$$

$$= ([\![A]\!] \wedge \neg [\![A]\!]) \to [\![B]\!].$$

Taking $[\![A]\!] := x$ and $[\![B]\!] := y$, where $x, y \in L$, we have that $([\![A]\!] \wedge \neg[\![A]\!]) \to [\![B]\!]$ becomes

$$(x \wedge \neg x) \to y = 0 \to y = 1,$$

where the last equality follows from the fact that $0 \leq y$ for all $y \in L$ as well as the fact that $p \to q = 1$ if and only if $p \leq q$. Thus, we have that

$$([\![A]\!] \wedge \neg[\![A]\!]) \to [\![B]\!] = 1.$$

(Q6). $(\forall x)B(x) \to B(y)$.

We have that (for $a, b \in W$)

$$[\![(\forall x)B(x) \to B(y)]\!] = [\![(\forall x)B(x)]\!] \to [\![B(b)]\!] = \bigwedge_{a\in W} [\![B(a)]\!] \to [\![B(b)]\!].$$

However, since $\bigwedge_{a\in W}[\![B(a)]\!]$ runs over all $a \in W$, we have that

$$\bigwedge_{a\in W} [\![B(a)]\!] \leq [\![B(b)]\!]$$

for any $b \in W$. And so, since $p \to q = 1$ if and only if $p \leq q$, it follows that

$$\bigwedge_{a\in W} [\![B(a)]\!] \to [\![B(b)]\!] = 1.$$

$\square$

Theorem 3. Let $(\mathcal{L}, \mathcal{A})$ be an M-system. The set of $\mathcal{L}$-wffs that are true in any model $\hat{W} := (W, L, \{[\![P]\!]\}, F_W)$ is closed under the rules of inference (R1) — (R5).

Proof.

(R1). $A \to B$ and $B \to C \Longrightarrow A \to C$.

Let $x, y, z \in L$, and assume that $x \to y = 1$ and $y \to z = 1$. Using that $p \to q = 1$ if and only if $p \le q$, these assumptions give that $x \le y$ and $y \le z$, respectively. Since $\le$ is transitive in an OML, we have that $x \le y$ and $y \le z$ give that $x \le z$. Again using that $p \to q = 1$ if and only if $p \le q$, we obtain that $x \to z = 1$.

(R2). $A \to B \Longrightarrow \sim B \to \sim A$.

Let $x, y \in L$, and assume that $x \to y = 1$. Using that $p \to q = 1$ if and only if $p \le q$, this gives that $x \le y$. In an ortholattice, whenever $x \le y$, we have $\neg y \le \neg x$; thus, again using that $p \to q = 1$ if and only if $p \le q$, we have that $\neg y \to \neg x = 1$.

(R3). $A \to B$ and $A \to C \Longrightarrow A \to (B \wedge C)$.

Let $x, y, z \in L$, and assume that $x \to y = 1$ and $x \to z = 1$. Using that $p \to q = 1$ if and only if $p \le q$, these assumptions give that $x \le y$ and $x \le z$, respectively. By definition of the GLB, we have that $x \le y$ and $x \le z$ imply that $x \le y \wedge z$. Again using that $p \to q = 1$ if and only if $p \le q$, the latter result is equivalent to $x \to (y \wedge z) = 1$.

(R4). $A \to B(z) \Longrightarrow A \to (\forall x)B(x)$.

Assume that $[\![A \to B(a)]\!] = 1$ for any $a \in W$. (Note that $[\![A \to B(a)]\!] = [\![A]\!] \to [\![B(a)]\!]$.) Now, using that $p \to q = 1$ if and only if $p \le q$, $[\![A]\!] \to [\![B(a)]\!] = 1$ gives that $[\![A]\!] \le$

$[\![B(a)]\!]$ for any $a \in W$. As such, however, it follows (by definition of the GLB) that $[\![A]\!] \leq \bigwedge_{a \in W} [\![B(a)]\!]$. Again using that $p \to q = 1$ if and only if $p \leq q$, this gives that $[\![A]\!] \to \bigwedge_{a \in W} [\![B(a)]\!] = 1$. However, we have that

$$[\![A]\!] \to \bigwedge_{a \in W} [\![B(a)]\!] = [\![A \to (\forall x) B(x)]\!],$$

which shows that $[\![A \to (\forall x) B(x)]\!] = 1$.

(R5). A and $A \to B \Longrightarrow B$.

Let $x, y \in L$, and assume that $x = 1$ and $x \to y = 1$. Using that $p \to q = 1$ if and only if $p \leq q$, this gives that $x \leq y$, and thus that $y = 1$.

□

Corollary 2.8. Consistency: If $\Gamma \vdash A$, then A is true in any model of Γ.

Proof. This result follows from Theorems 2 and 3. □

Let F denote the free algebra over the extended language which is obtained by considering '$\forall$' as a unary operation in the algebra of $\mathcal{L}$-wffs. Also, recall that for $\mathcal{L}$-wffs A and B and any set of $\mathcal{L}$-wffs Γ, we say that A and B are logically equivalent with respect to Γ if $\Gamma \vdash A \leftrightarrow B$. In what follows, we will write $A \simeq_\Gamma B$ to denote logical equivalence of A and B with respect to Γ.

Lemma 2.9. $\simeq_\Gamma$ is a congruence on F.

Proof.

Reflexivity: Since $\vdash A \to A$ follows from (Q1), we have that $\vdash A \leftrightarrow A$. Since $\mathcal{Q}(\mathcal{L})$ is monotonic, we have that $\Gamma \vdash A \leftrightarrow A$, which shows that $A \simeq_\Gamma A$.

Symmetry: Assume that $A \simeq_\Gamma B$; as such, we have that $\Gamma \vdash A \leftrightarrow B$, which is symmetric by construction — thus, we have that $\Gamma \vdash B \leftrightarrow A$, so that $B \simeq_\Gamma A$.

Transitivity: Assume that $A \simeq_\Gamma B$ and $B \simeq_\Gamma C$; these give that $\Gamma \vdash A \leftrightarrow B$ and $\Gamma \vdash B \leftrightarrow C$, respectively. However, using *replacement* (i.e. Theorem 1) in the latter result (given the former), it follows that $\Gamma \vdash A \leftrightarrow C$, and thus that $A \simeq_\Gamma C$.

$\sim$-congruence: Assume that $A \simeq_\Gamma B$. This gives that $\Gamma \vdash A \leftrightarrow B$. Using *replacement* (i.e. Theorem 1), we obtain that $\Gamma \vdash \sim A \leftrightarrow \sim B$, from which it follows that $\sim A \simeq_\Gamma \sim B$.

$\wedge$-congruence: Assume that $A \simeq_\Gamma B$ and $C \simeq_\Gamma D$; these give that $\Gamma \vdash A \leftrightarrow B$ and $\Gamma \vdash C \leftrightarrow D$, respectively. By (Q3) we have that $A \wedge C \to A$, so we have that $\Gamma \vdash A \wedge C \to A$. Since $\Gamma \vdash A \to B$ follows from the assumptions, (R1) gives that $\Gamma \vdash A \wedge C \to B$. Also, from (Q3) we have that $A \wedge C \to C$, which gives that $\Gamma \vdash A \wedge C \to A$; and by assumption, $\Gamma \vdash C \to D$, so (R1) gives that $\Gamma \vdash A \wedge C \to D$. Now, by (R3), $\Gamma \vdash A \wedge C \to B$ and $\Gamma \vdash A \wedge C \to D$ give that $\Gamma \vdash A \wedge C \to B \wedge D$. Similarly, we obtain $\Gamma \vdash B \wedge D \to A \wedge C$. Together these results give that $\Gamma \vdash B \wedge D \leftrightarrow A \wedge C$, or equivalently $A \wedge C \simeq_\Gamma B \wedge D$.

$\square$

Lemma 2.10. The quotient algebra $F/\simeq_\Gamma$ (called the Lindenbaum–Tarski algebra or LTA associated with Γ) is an orthomodular lattice.

Proof. In what follows, let $[A]$ denote the equivalence class of the $\mathcal{L}$-wff A under the congruence $\simeq_\Gamma$. Note that since $\simeq_\Gamma$ is a congruence (Lemma 2.9), we have that $[\sim A] = \neg[A]$ and $[A \wedge B] = [A] \wedge [B]$.

- wts: $[A] \wedge [B] = [B] \wedge [A]$.

  We have that $[A] \wedge [B] = [A \wedge B]$. By (Q3), we have $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$; together these give that $\vdash A \wedge B \to B \wedge A$ by (R3). Similarly, we have $\vdash B \wedge A \to A \wedge B$. And so, $\vdash A \wedge B \leftrightarrow B \wedge A$, which gives that $[A \wedge B] = [B \wedge A]$. Thus, we have that

  $$[A] \wedge [B] = [A \wedge B] = [B \wedge A] = [B] \wedge [A].$$

- wts: $[A] \wedge ([B] \wedge [C]) = ([A] \wedge [B]) \wedge [C]$.

  We have that $[A] \wedge ([B] \wedge [C]) = [A] \wedge [B \wedge C] = [A \wedge (B \wedge C)]$. By (Q3) we have $\vdash A \wedge (B \wedge C) \to A$ and $\vdash A \wedge (B \wedge C) \to B \wedge C$. Again by (Q3), we have that $\vdash B \wedge C \to B$ and $\vdash B \wedge C \to C$. Using the latter results along with (R1) and the fact that $\vdash A \wedge (B \wedge C) \to B \wedge C$, we obtain $\vdash A \wedge (B \wedge C) \to B$ and $\vdash A \wedge (B \wedge C) \to C$. Now, by (R3) $\vdash A \wedge (B \wedge C) \to A$ and $\vdash A \wedge (B \wedge C) \to B$ give that $\vdash A \wedge (B \wedge C) \to A \wedge B$. And, this result, along with the previously obtained result $\vdash A \wedge (B \wedge C) \to C$ give that $\vdash A \wedge (B \wedge C) \to (A \wedge B) \wedge C$. Following a similar procedure, we obtain that $\vdash (A \wedge B) \wedge C \to A \wedge (B \wedge C)$ so that $\vdash (A \wedge B) \wedge C \leftrightarrow A \wedge (B \wedge C)$; equivalently, $[(A \wedge B) \wedge C] = [A \wedge (B \wedge C)]$. Thus, we have that

  $$[A] \wedge ([B] \wedge [C]) = [(A \wedge B) \wedge C] = [A \wedge (B \wedge C)] = ([A] \wedge [B]) \wedge [C].$$

- wts: $\neg\neg[A] = [A]$.

  We have that $\neg\neg[A] = \neg[\sim A] = [\sim\sim A]$. By (Q2), we have that $\vdash \sim\sim A \to A$ and $\vdash A \to \sim\sim A$, which together give that $A \simeq_\Gamma \sim\sim A$ (or equivalently, $[\sim\sim A] = [A]$). Thus, we have that

$$\neg\neg[A] = \neg[\sim A] = [\sim\sim A] = [A].$$

- wts: $[A] \wedge ([B] \wedge \neg[B]) = [B] \wedge \neg[B]$.

  We have that $[A] \wedge ([B] \wedge \neg[B]) = [A] \wedge ([B] \wedge [\sim B]) = [A] \wedge [B \wedge \sim B] = [A \wedge (B \wedge \sim B)]$. By (Q3) we have that $\vdash A \wedge (B \wedge \sim B) \to B \wedge \sim B$, and by (Q5) we have that $\vdash B \wedge \sim B \to A \wedge (B \wedge \sim B)$. Together these results give that $A \wedge (B \wedge \sim B) \simeq_\Gamma B \wedge \sim B$ (or equivalently, $[A \wedge (B \wedge \sim B)] = [B \wedge \sim B]$). Thus, we have that

$$[A] \wedge ([B] \wedge \neg[B]) = [A] \wedge ([B] \wedge [\sim B]) = [A] \wedge [B \wedge \sim B] = [A \wedge (B \wedge \sim B)]$$

$$= [B \wedge \sim B] = [B] \wedge [\sim B] = [B] \wedge \neg[B].$$

- wts: $[A] \wedge ([A] \vee [B]) = [A]$.

  We have that $[A] \wedge ([A] \vee [B]) = [A] \wedge [A \vee B] = [A \wedge (A \vee B)]$. By (Q3) we have that $\vdash A \wedge (A \vee B) \to A$. Also, by (Q3), (R2), (Q2) and (R1), we have that $\vdash A \to A \vee B$, while (Q1) gives that $\vdash A \to A$. Using these two results, (R3) gives that $\vdash A \to A \wedge (A \vee B)$.

And so, we see that $A \wedge (A \vee B) \simeq_\Gamma A$ (or equivalently, $[A \wedge (A \vee B)] = [A]$). Thus, we have that

$$[A] \wedge ([A] \vee [B]) = [A] \wedge [A \vee B] = [A \wedge (A \vee B)] = [A].$$

- wts: $[A] \wedge [B] = [A] \wedge (\neg[A] \vee ([A] \wedge [B]))$.

  We have that $[A] \wedge (\neg[A] \vee ([A] \wedge [B])) = [A] \wedge ([\sim A] \vee [A \wedge B]) = [A] \wedge [\sim A \vee (A \wedge B)] = [A \wedge (\sim A \vee (A \wedge B))] = [A \wedge (A \to B)]$, where the last equality follows from the definition of $\to$. By (Q4) we have that $\vdash (A \wedge (A \to B)) \to B$, while by (Q3), $\vdash (A \wedge (A \to B)) \to A$; together these results give that $\vdash (A \wedge (A \to B)) \to A \wedge B$ by (R3). Now, by (Q3), (R2), (Q2) and (R1), we have that $\vdash (A \wedge B) \to (\sim A \vee (A \wedge B))$; also by (Q3), we have that $\vdash A \wedge B \to A$. Together these results give that $\vdash A \wedge B \to (A \wedge (\sim A \vee (A \wedge B)))$, or equivalently, (by definition of $\to$) $\vdash A \wedge B \to (A \wedge (A \to B))$. As such, we have that $A \wedge B \simeq_\Gamma (A \wedge (A \to B))$ (or equivalently, $[A \wedge B] = [A \wedge (A \to B)]$). Thus, we have that

$$[A] \wedge (\neg[A] \vee ([A] \wedge [B])) = [A] \wedge ([\sim A] \vee [A \wedge B]) = [A] \wedge [\sim A \vee (A \wedge B)]$$

$$= [A \wedge (\sim A \vee (A \wedge B))] = [A \wedge (A \to B)] = [A \wedge B] = [A] \wedge [B].$$

□

Let $f : F \to F/ \simeq_\Gamma$ be the natural homomorphism from the free algebra F to the quotient algebra.

Lemma 2.11. If $\Gamma \vdash A \to B$, then $f(A) \leq f(B)$.

Proof. Assume that $\Gamma \vdash A \to B$. Note that we also have $\Gamma \vdash B \to B$ (from (Q1), (Q3) and (R1)). Together, these give that $\Gamma \vdash A \vee B \to B$, using (R2), (R3) and the definition of $\vee$. Now, we also have that $X \to X \vee Y$ (from (Q3), (R2), (Q2) and (R1)), so we have $\Gamma \vdash B \to A \vee B$. Thus, we see that we have $\Gamma \vdash B \leftrightarrow A \vee B$, or that $B \simeq_\Gamma A \vee B$. From this, we have that $f(A) \vee f(B) = f(A \vee B) = f(B)$, or $f(A) \leq f(B)$ □

Lemma 2.12. Consider $\hat{W} = (W, F/\simeq_\Gamma, \{\tilde{f}\}, F_W)$, where W is the free algebra over $\mathcal{L}$ (i.e. the set of all possible $\mathcal{L}$-wffs not involving '$\forall$'), and $\tilde{f}$ is the restriction of $f : F \to F/\simeq_\Gamma$ to atomic $\mathcal{L}$-wffs. Then $\hat{W}$ is a model of Γ.

Proof. We first note that by Lemma 2.10 we have that $F/\simeq_\Gamma$ is an orthomodular lattice. (Note that although we haven't shown that $F/\simeq_\Gamma$ is a complete lattice, we don't actually need for the OM lattice in the model to be complete — we only need that the GLBs which are necessary to evaluate the quantified statements exist.)

Now, we also need to show that

$$f((\forall x)B(x)) = \bigwedge_{a \in W} f(B(a)).$$

That is, we wts that for any variable y and any assignment of that variable to an element $a \in W$, the evaluation of $B(y)$ is such that $f((\forall x)B(x)) \leq B(a)$; we also wts that if $\beta \leq f(B(a))$, then $\beta \leq f((\forall x)B(x))$. Now, by Lemma 2.11, if $\Gamma \vdash X \to Y$, then $f(X) \leq f(Y)$. So, we wts that $\Gamma \vdash (\forall x)B(x) \to B(y)$ and if $\Gamma \vdash \beta \to B(y)$, then $\Gamma \vdash \beta \to (\forall x)B(x)$. Wrt the latter, we note that by (R4), we have that $A \to B(z) \Longrightarrow A \to (\forall x)B(x)$. So, if $\Gamma \vdash A \to B(z)$ for any variable

z, then (R4) gives that $\Gamma \vdash A \to (\forall x)B(x)$. Now, by (Q6), we have that $(\forall x)B(x) \to B(y)$, so $\Gamma \vdash (\forall x)B(x) \to B(y)$.

Finally, since the above discussion demonstrates that $\hat{W}$ is a model, we have that consistency (Corollary 2.8) gives that $\tilde{f}(A) = 1$ whenever $\Gamma \vdash A$. □

Lemma 2.13. If $f(A) = 1$, then $\Gamma \vdash A$.

Proof. If $f(A) = 1$, then we have that $A \simeq_\Gamma T$. By axiom (Q5), we have that $X \wedge \sim X \to Y$, and by (R2) this gives that $\sim Y \to (X \vee \sim X)$ (using (Q2), (R1) and the definition of $\vee$). Thus, we have that $A \to (B \vee \sim B)$, or that $\Gamma \vdash A \to (B \vee \sim B)$. Thus, $f(A) \leq f(B \vee \sim B)$. However, since $f(A) = 1$ by assumption, we see that $f(B \vee \sim B) = 1$ as well, giving that $B \vee \sim B \simeq_\Gamma T$. And, since $\simeq_\Gamma$ is a congruence, it is transitive, so $B \vee \sim B \simeq_\Gamma A$. As such, we have that $\Gamma \vdash (B \vee \sim B) \to A$. Also, we have that $\vdash B \vee \sim B$, so it follows that $\Gamma \vdash B \vee \sim B$. Using these, (R5) gives that $\Gamma, B \vee \sim B \vdash A$, which is the same as $\Gamma \vdash A$ (since $\vdash B \vee \sim B$). □

Theorem 4. Completeness: If an $\mathcal{L}$-wff A is true in any model of Γ, then $\Gamma \vdash A$.

Proof. By Lemma 2.12, $\hat{W}$ is a model of Γ. Since $f(A) = 1$ by assumption, Lemma 2.13 gives that $\Gamma \vdash A$. □

The theorem above demonstrates that our deductive system is complete with respect to our model theoretic semantics. We now prove a simple lemma which is useful for proving completeness with respect to models associated with *irreducible* truth value algebras.

Lemma 2.14. Let Γ be some set of $\mathcal{L}$-wffs, and let A be the underlying set of a model of Γ, with truth function $\{[\![P]\!]\}$ into truth value algebra $L_1 \times L_2$, where $\bigcup_{P \in \mathcal{L}^P} Im([\![P]\!])$ generates

$L_1 \times L_2$. Let $p_1 : L_1 \times L_2 \to L_1$ be the natural projection map onto L_1. Then replacing $L_1 \times L_2$ with L_1, and $[\![P]\!]$ with $p_1 \circ [\![P]\!]$ for each $P \in \mathcal{L}^P$, yields a new model of Γ.

Proof. For any $\gamma \in \Gamma$, $[\![\gamma]\!] = 1$, and so $p_1 \circ [\![\gamma]\!] = 1$. Also, since p_1 is surjective and $\bigcup_{P \in \mathcal{L}^P} Im([\![P]\!])$ generates $L_1 \times L_2$, we clearly have that $\bigcup_{P \in \mathcal{L}^P} Im(p_1 \circ [\![P]\!])$ generates L_1. It remains to show that L_1 is complete enough to evaluate the appropriate '$\forall$' statements, so consider an $\mathcal{L}$-wff $B(x)$, and let $(\alpha, \beta) = \bigwedge_{a \in A} [\![B(a)]\!]$, i.e. (α, β) is the greatest lower bound of the set of all $[\![B(a)]\!] =: (\alpha_a, \beta_a)$, where $a \in A$. Hence $\alpha \leq \alpha_a$ for all $a \in A$, so α is a lower bound for the set of all $\alpha_a = p_1 \circ [\![B(a)]\!]$. Also, for any $\alpha' \in L_1$ which is a lower bound for all the $p_1 \circ [\![B(a)]\!]$, we have that $(\alpha', 0)$ is a lower bound for all the $[\![B(a)]\!]$ ($(\alpha', 0)$ is in the sub-algebra generated by $\bigcup_{P \in \mathcal{L}^P} Im([\![P]\!])$), and so by definition $(\alpha', 0) \leq (\alpha, \beta)$, and so $\alpha' \leq \alpha$, showing that

$$\alpha = \bigwedge_{a \in A} p_1 \circ [\![B(a)]\!] = p_1 \circ [\![(\forall x)(B(x))]\!],$$

and so L_1 is complete enough to serve as a model. □

Theorem 5. Let Γ be a set of $\mathcal{L}$-wffs, and φ an $\mathcal{L}$-wff. Then $[\![\varphi]\!] = 1$ in every model of Γ if $[\![\varphi]\!] = 1$ in every model of Γ with irreducible truth value algebra.

Proof. We prove the contrapositive, so assume that there is some model of Γ in which $[\![\varphi]\!] \neq 1$. By the completeness theorem (i.e. Theorem 4), we must have that $[\![\varphi]\!] \neq 1$ in the Lindenbaum-Tarski algebra associated with Γ — i.e. $\Gamma \nvdash \top \leftrightarrow \varphi$. Now define $\mathcal{A}$ to consist of exactly those sets of axioms A in which (i) $\Gamma \subseteq A$, and (ii) $[\![\varphi]\!]^A \neq 1$, where $\{[\![P]\!]^A\}$ is the truth function which maps $\mathcal{L}$-wffs to the Lindenbaum-Tarski algebra associated with A.

$\mathcal{A}$ is partially ordered by inclusion, and we will use Zorn's Lemma to prove that $\mathcal{A}$ contains some maximal element M. Let $\mathcal{C} \subseteq \mathcal{A}$ be linearly ordered under inclusion, and define $U := \bigcup \mathcal{C}$. Clearly U is an upper bound for $\mathcal{C}$ under inclusion, so it suffices to show that $U \in \mathcal{A}$. Clearly $\Gamma \subseteq U$. We prove property (ii) by contradiction, so assume $[\![\varphi]\!]^U = 1$, i.e. $U \vdash \top \leftrightarrow \varphi$. The proof of $\top \leftrightarrow \varphi$ is finite by definition, and so may only contain a finite number of wffs $\varphi_1, \varphi_2, \dots, \varphi_n \in U$. Each $\varphi_i \in A_i$ for some $A_i \in \mathcal{C}$. Since $\mathcal{C}$ is linearly ordered by inclusion, we must have that $\bigcup_{i=1}^n A_i = A_j$ for some $j \in \{1, \dots, n\}$, and hence $A_j \vdash \top \leftrightarrow \varphi$, so $[\![\varphi]\!]^{A_j} = 1$, which is a contradiction since $A_j \in \mathcal{A}$. Hence $U \in \mathcal{A}$, and since $\mathcal{C}$ was a generic chain in $\mathcal{A}$, Zorn's Lemma gives that $\mathcal{A}$ has a maximal element Ω.

We claim that the Lindenbaum-Tarski algebra L associated with Ω is irreducible. Assume not, so that $L \simeq L_1 \times L_2$. By Lemma 2.14 above, this produces two new models of Ω with corresponding truth value algebras L_1 and L_2, respectively (and truth functions $\{[\![P]\!]^{(1)}\}$ and $\{[\![P]\!]^{(2)}\}$, respectively). Since $[\![\varphi]\!]^\Omega \neq 1$, we must have (wlog) $[\![\varphi]\!]^{(1)} \neq 1$. Let ψ be some wff such that $[\![\psi]\!]^{(1)} = 1$ and $[\![\psi]\!]^{(2)} = 0$. We cannot have $\Omega \cup \{\psi\} \vdash \top \leftrightarrow \varphi$, since then we would have $[\![\varphi]\!]^{(1)} = 1$. This means that $\Omega \cup \{\psi\} \in \mathcal{A}$, and also $\Omega \subseteq \Omega \cup \{\psi\}$, which contradicts the maximality of Ω in $\mathcal{A}$. Hence L must be irreducible, and so we have constructed an irreducible model in which $[\![\varphi]\!]^\Omega \neq 1$, which establishes the contrapositive. $\square$

And so, this theorem shows that without loss of generality, we can restrict ourselves to models for which the associated truth value algebra is *irreducible*.

2.4 Qualitative Properties of Quantum Mathematics

We begin by defining a property of $\mathcal{L}$-wffs which plays an important role in the quantum logic $\mathcal{Q}(\mathcal{L})$ — namely, for $\mathcal{L}$-wffs A and B, we define the $\mathcal{L}$-wff $A \,\widetilde{\mathcal{C}}\, B$ by

$$A \,\widetilde{\mathcal{C}}\, B := \Big(A \rightarrow [(A \wedge B) \vee (A \wedge \sim B)]\Big) \wedge \Big([(A \wedge B) \vee (A \wedge \sim B)] \rightarrow A\Big), \tag{2.2}$$

and for a given M-system $(\mathcal{L}, \mathcal{A})$, we say that A is compatible with B if $A \,\widetilde{\mathcal{C}}\, B$ is a theorem of $\mathcal{A}$. Note that whenever $A \,\widetilde{\mathcal{C}}\, B$, we also have $A \,\widetilde{\mathcal{C}}\, (\sim B)$. And, although it is not obvious from the definition above, it is straightforward to show that the compatibility relation is symmetric in $\mathcal{Q}(\mathcal{L})$ (i.e. $A \,\widetilde{\mathcal{C}}\, B$ is logically equivalent (with respect to $\varnothing$) to $B \,\widetilde{\mathcal{C}}\, A$); as such, if A is compatible with B, we will simply say that A and B are *compatible.* Now, we note that a set of axioms of classical logic is derivable from the quantum logic $\mathcal{Q}(\mathcal{L})$ described above by adding the property that all $\mathcal{L}$-wffs are compatible — namely, we could add

(CL) $A \,\widetilde{\mathcal{C}}\, B$

as an axiom schema to $\mathcal{Q}(\mathcal{L})$. For convenience we denote $\mathcal{Q}(\mathcal{L}) \cup \{(\text{CL})\}$ (where (CL) implicitly represents an axiom schema) by $\mathcal{C}(\mathcal{L})$. Additionally, for an M-system $(\mathcal{L}, \mathcal{A})$ we will say that an $\mathcal{L}$-wff is *classically derivable* if it is derivable from $\mathcal{A} \cup \{(\text{CL})\}$. Although a simpler set of axiom schema and inference rules can be used to describe classical logic (e.g. as in (11)), the one given by $\mathcal{C}(\mathcal{L})$ is equivalent to these.

Since an axiomatization of classical logic can be obtained from $\mathcal{Q}(\mathcal{L})$ by simply adding another axiom, it is easy to see that the quantum logic $\mathcal{Q}(\mathcal{L})$ is sub-classical — i.e. every theorem of $\mathcal{Q}(\mathcal{L})$ will also be a theorem of classical logic. As such, every model of classical logic will still be a model of the quantum logic $\mathcal{Q}(\mathcal{L})$ — that is, $\mathcal{Q}(\mathcal{L})$ doesn't eliminate any classical models, but does allow for more models than are allowed classically (as will be discussed below). One consequence of this (along with the fact that a set of axioms is consistent if and only if there exists a model of those axioms(23)) is that any M-system $(\mathcal{L}, \mathcal{A})$ which is *consistent* in the presence of classical logic is still consistent when the underlying logic is $\mathcal{Q}(\mathcal{L})$. Another consequence of the fact that the quantum logic $\mathcal{Q}(\mathcal{L})$ is sub-classical is that any set of axioms which are *independent* when the underlying logic is classical remain so in the presence of the quantum logic $\mathcal{Q}(\mathcal{L})$. However, we note that a statement which is provable from a set of axioms in the presence of classical logic may no longer be provable when the underlying logic is sub-classical as in the case of the quantum logic $\mathcal{Q}(\mathcal{L})$.

We next note several features of quantum mathematics which are of particular interest; we mention them here and discuss each of the points further in the context of the relevant examples in the subsequent chapter.

To begin with, one may expect that whenever the quantum logic $\mathcal{Q}(\mathcal{L})$ is used in place of classical logic, any M-system $(\mathcal{L}, \mathcal{A})$ will admit non-standard models. A little thought shows that this is clearly not the case, as $\mathcal{A}$ may contain some "purely logical" axiom schema (such as, e.g. (CL) above) which can be seen to be directly responsible for the classical behavior of the associated M-system. However, what is more surprising is that, even in cases in which $\mathcal{A}$ seems to have "purely mathematical" content, it still might be the case that an M-system admits no non-standard models, as Dunn (9) was the first to show. We will call an M-system $(\mathcal{L}, \mathcal{A})$ (or simply $\mathcal{A}$ if $\mathcal{L}$ is clear from the context) *inherently classical* if $\mathcal{A} \cup \mathcal{Q}_A(\mathcal{L}) \vdash$ (CL). For such M-systems, any theorem of $\mathcal{A}$ which is classically derivable is also a quantum theorem, from which it follows that inherently classical M-systems do not admit any non-standard models. Thus, for such axiom systems $\mathcal{A}$ there is no difference, as far as the mathematics is concerned, between using the quantum logic $\mathcal{Q}(\mathcal{L})$ or classical logic as the underlying logic. It turns out that when a set of $\mathcal{L}$-wffs $\mathcal{A}$ satisfies certain properties (which will be discussed in Sections 3.2 and 3.3), the axiom system $\mathcal{A}$ is inherently classical.[1] However, we note that inherently classical M-systems are the exception rather than the rule.

Now, for any given area of mathematics, there are often alternative but equivalent formulations of the axioms (as a set of $\mathcal{L}$-wffs) in the presence of classical logic, where by equivalent presentations for a language $\mathcal{L}$, we mean that two sets of $\mathcal{L}$-wffs $\mathcal{A}$ and $\mathcal{A}'$ have exactly the same theorems — i.e. $\mathcal{T}(\mathcal{A}) = \mathcal{T}(\mathcal{A}')$. When $\mathcal{Q}(\mathcal{L})$ (instead of, e.g. $\mathcal{C}(\mathcal{L})$) is used for the underlying logic, these equivalent classical presentations may no longer be equivalent.[2] Indeed, examples of this splitting phenomenon are given in Sections 3.2 and 3.3. This sensitivity to the choice

[1] We note that a general characterization of inherently classical M-systems is still lacking. In the sections mentioned above, we provide classes of examples of inherently classical M-systems.

[2] This result does not depend on the specific choice of axiomatization for classical logic — any equivalent set of axioms and inference rules for classical logic can be used.

of classically equivalent mathematical axioms is another novel feature of the quantum logic $\mathcal{Q}(\mathcal{L})$ which enables us to see a richness in the structure of mathematics which is completely inaccessible using classical logic.

We now distinguish between two classes of models. Let $(\mathcal{L}, \mathcal{A})$ be an M-system, and let $\hat{A} = (A, L, \{[\![P]\!]\}, F_A)$ be a model for $\mathcal{A}$. If there is an $\mathcal{L}$-structure $\hat{B} = (A, L', \{[\![P]\!]'\}, F_A)$ which is a standard model for $\mathcal{A}$ such that $L' = \mathbf{2}$ (the two element Boolean algebra), then we say that $\hat{A}$ is *conservative.* Essentially, a model is conservative if each n-ary operation $f \in \mathcal{L}$ has the same interpretation in $\hat{A}$ as it does in some classical model $\hat{B}$ over the same set — that is, (for $a, b \in A$) only the definition of $[\![P(a, b, ...)]\!]$ (for each $P \in \mathcal{L}^P$) differs from $[\![P(a, b, ...)]\!]'$. Clearly every standard model is conservative. A model which is not conservative is said to be *non-conservative.* Examples of non-standard conservative models can be found in Sections 3.3, 3.4, and 3.5, as well as in Chapter 5; an example of a non-conservative model can be found in Section 3.3. Note that non-conservative models allow for the n-ary operations $f \in \mathcal{L}$ to be more general than can be obtained classically. However, non-standard conservative models are still interesting in that they (at least in principle) allow for different mathematical properties to hold compared to the associated bivalent model.

Finally, we note that there are certain orthomodular lattices which are very natural with respect to quantum theory, and as such, these are natural candidates for the truth value algebras in models of our quantum M-systems. For example, despite the wide variation in what is considered quantum logic, there is relatively little disagreement that projection[1] (or subspace) lattices are relevant structures; such lattices seem to be suggested strongly by quantum mechanics, and are of particular interest due to the fact that they can be empirically motivated. When considering the relevant mathematics which utilizes these natural structures as truth value algebras of models, we find that certain classical properties no longer hold (see Sections 3.4 and 3.5 for a discussion). While this fact in and of itself may not be surprising, it is striking that some of the properties which no longer hold are ones as fundamental as (E3′) and (Sub). Although the tendency would simply be to discard any models in which such intuitive properties fail, we hesitate to do so given the naturalness of these models with respect to the quantum theory — instead we want to see what these models are trying to teach us.

2.5 Construction of Models

In the following chapters we will be constructing explicit models of various axiom systems, and so it will behoove us to examine conditions under which a given $\mathcal{L}$-structure is a model of some M-system $(\mathcal{L}, \mathcal{A})$.

Now in any $\mathcal{L}$-structure $(A, L, \{[\![P]\!]\}, F_A)$, every $\mathcal{L}$-wff in $\mathcal{Q}_A(\mathcal{L})$ automatically holds, since L is a complete orthomodular lattice, and we have shown that $\mathcal{Q}(\mathcal{L})$ is sound. In languages $\mathcal{L}$

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[1]See Section A.4.3 for a brief discussion of these structures.

with "equality" $\approx$, (E1) is also automatically satisfied due to the requirement on $[\![\approx]\!]$ that (for all $a, b \in A$)

$$[\![a \approx b]\!] = 1 \quad \text{if and only if} \quad a = b.$$

It is also straightforward to see that (E2) will hold if and only if $[\![\approx]\!]$ is a symmetric function (where by symmetric, we mean $[\![a \approx b]\!] = [\![b \approx a]\!]$ for all $a, b \in A$). Hence (in languages $\mathcal{L}$ with "equality" $\approx$), when attempting to determine if an $\mathcal{L}$-structure with a symmetric truth function $[\![\approx]\!]$ is a model for the M-system $(\mathcal{L}, \mathcal{A})$, it suffices to check only axiom schema (E3) and $\mathcal{A}$. In order to see that (E3) is satisfied, one only needs to check that the following inequality holds in L (for all $a, b, c \in A$).

$$[\![a \approx b]\!] \wedge [\![b \approx c]\!] \leq [\![a \approx c]\!]. \tag{2.3}$$

Given some model $\hat{A}$ of an M-system $(\mathcal{L}, \mathcal{A})$, we can construct other (distinct) models from it, using the following theorem.

Theorem 6. Let $(\mathcal{L}, \mathcal{A})$ be an M-system with model $\hat{A} = (A, L, \{[\![P]\!]'\}, F_A)$, and let $\hat{A}_0 = (A, L_0, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure, where (for n-ary predicate $P \in \mathcal{L}^P$ and $a_1, \ldots a_n \in A$)

$$[\![P(a_1, \ldots a_n)]\!]' = 1 \quad \text{iff} \quad [\![P(a_1, \ldots a_n)]\!] = 1.$$

If B is an $\mathcal{L}$-wff which holds in $\hat{A}$ and which can be written as to not contain the symbol '$\sim$' (using $\vee$ and $\exists$ is still allowed), then B holds in $\hat{A}_0$.

Proof. The proof is by induction on the definition of the extension of $[\![P]\!]'$ (for each $P \in \mathcal{L}^P$) to all evaluated $\mathcal{L}$-wffs. First, we consider the base case of an atomic $\mathcal{L}$-wff (for any $P \in \mathcal{L}^P$ with arity n) $P(t_1(y_1, \dots y_k), \dots t_n(y_1, \dots y_k))$ such that $\hat{A} \vDash P(t_1, \dots t_n)$. Then, for any $a_1, \dots, a_k \in A$, we have that $[\![P(t_1(a_1, \dots a_k), \dots t_n(a_1, \dots a_k))]\!]' = 1$ since $P(t_1, \dots t_n)$ holds in $\hat{A}$. Then, by assumption, we must have that $[\![P(t_1(a_1, \dots a_k), \dots t_n(a_1, \dots a_k))]\!] = 1$ for any $a_1, \dots a_k \in A$, from which it follows that $\hat{A}_0 \vDash P(t_1, \dots t_n)$.

For the inductive step, first assume that for $\mathcal{L}$-wffs B and C that $\vdash B$ and $\vdash C$ hold in $\hat{A}_0$ — i.e. assume that $\hat{A} \vDash B$ and $\hat{A} \vDash C$. Then for any evaluations $\tilde{B}$ of B and $\tilde{C}$ of C, $[\![\tilde{B}]\!]' = [\![\tilde{C}]\!]' = 1$. Then $[\![\tilde{B} \wedge \tilde{C}]\!]' = [\![\tilde{B}]\!]' \wedge [\![\tilde{C}]\!]' = 1$, so $\vdash B \wedge C$ holds in $\hat{A}_0$ (and similarly for $\vee$). Next, assume that $\vdash B(x, y_1, \dots, y_n))$ holds in $\hat{A}$, so $[\![B(a, a_1, \dots, a_n)]\!]' = 1$ for all $a, a_1, \dots, a_n \in A$. Then we have

$$[\![(\forall x)B(x, a_1, \dots, a_n)]\!]' = \bigwedge_{a \in A} [\![B(a, a_1, \dots, a_n)]\!]' = \bigwedge_{a \in A} 1 = 1$$

so that $B(x, y_1, \dots, y_n)$ holds in $\hat{A}_0$. For the final case, assume that $\vdash B(x, y_1, \dots, y_n)$ holds in $\hat{A}$. Then for any $a_1, \dots, a_n \in A$ we have

$$[\![(\exists x)B(x, a_1, \dots, a_n)]\!]' = \neg \bigwedge_{a \in A} \neg [\![B(a, a_1, \dots, a_n)]\!]' = \bigvee_{a \in A} [\![B(a, a_1, \dots, a_n)]\!]' = 1,$$

so that $(\exists x)(B)$ holds in $\hat{A}$. This completes the induction. $\square$

Thus, if there is a known model $\hat{A}$ of an M-system $(\mathcal{L}, \mathcal{A})$, and from that model we change only the truth value algebra L and the way that the predicates $P \in \mathcal{L}^P$ are valuated in L, so as to form a new $\mathcal{L}$-structure $\hat{A}_0$, then any $\mathcal{L}$-wff B (which doesn't contain logical negation ~)

which holds in (the original model) $\hat{A}$ will also hold in the new $\mathcal{L}$-structure $\hat{A}_0$. This result will be most useful when the initial model $\hat{A}$ is standard (and especially so when $\hat{A}$ is a model we are familiar with from classical mathematics). More specifically, whenever trying to construct a non-standard conservative model of $(\mathcal{L}, \mathcal{A})$ from, e.g. a standard model, Theorem 6 guarantees that any axiom in $\mathcal{A}$ which does not contain logical negation will hold in the $\mathcal{L}$-structure $\hat{A}_0$. However, any axiom of $\mathcal{A}$ which does contain logical negation needs to be checked explicitly to verify that the $\mathcal{L}$-structure $\hat{A}_0$ is indeed a model.

2.5.1 Classically Equivalent Axiom Systems

In Section 2.4 we mentioned that certain sets of axioms which are equivalent classically may no longer be equivalent in the presence of the quantum logic $\mathcal{Q}(\mathcal{L})$. In this section we show that for an M-system $(\mathcal{L}, \mathcal{A})$, one can always find such classically equivalent sets of axioms in any language $\mathcal{L}$ which is such that $\approx \in \mathcal{L}^P$.

Let $\mathcal{L}$ be any language and let $\varphi, \hat{\varphi}$ be $\mathcal{L}$-wffs. We will say that $\hat{\varphi}$ is a *reduction of* φ if we have that φ and $\hat{\varphi}$ are logically equivalent in the presence of classical logic. Further, for a set of $\mathcal{L}$-wffs Γ and a set of $\mathcal{L}$-wffs Γ' for which $\gamma' \in \Gamma'$ implies that γ' is a reduction of γ for some $\gamma \in \Gamma$, we say that Γ' is a *reduction of* Γ. Additionally, if A is an $\mathcal{L}$-wff such that (CL) $\vdash A$, we say that A is a *classical tautology*, and if, further, A is not derivable from $\varnothing$, we say that A is a *strictly classical tautology*. Then, for any $\mathcal{L}$-wffs φ, A such that A is a strictly classical tautology, the choice $\hat{\varphi} := A \to \varphi$ is an example of a reduction of φ.

Given an M-system $(\mathcal{L}, \mathcal{A})$ and a reduction $\mathcal{A}'$ of $\mathcal{A}$ it is trivial to see that the M-system $(\mathcal{L}, \mathcal{A}')$ is equivalent classically to $(\mathcal{L}, \mathcal{A})$. Whether or not a given reduction of some axiom system is still equivalent in quantum logic is, in principle, a difficult question. However, for a specific class of reductions, we can state some general facts.

For $\mathcal{L}$-wffs A, B, define the $\mathcal{L}$-wff $c(A, B)$ by

$$c(A, B) := [(A \wedge B) \vee (A \wedge \sim B)] \vee [(\sim A \wedge B) \vee (\sim A \wedge \sim B)]. \tag{2.4}$$

It well known from orthomodular lattice theory (see (21) p. 26) that the expression in equation 2.4, when interpreted as a lattice polynomial in a truth value algebra L, has the property that $c(x, y) = 1$ if and only if $x = (x \wedge y) \vee (x \wedge \neg y)$ — that is, when x and y are *compatible* elements of L. This means that, for a given M-system $(\mathcal{L}, \mathcal{A})$ and any model $\hat{A} = (A, L, \{[\![P]\!]\}, F_A)$ thereof, that $[\![c(A, B)]\!] = 1$ for every evaluated $\mathcal{L}$-wffs A, B if and only if L is Boolean (since any orthomodular lattice in which all elements are compatible is necessarily Boolean). Then completeness gives that $\mathrm{CL} \vdash c(A, B)$ for any two $\mathcal{L}$-wffs A and B, so that $c(A, B)$ is always a classical tautology. Whether or not it is strictly classical depends upon the specific A and B — for example, in a language $\mathcal{L}$ with "equality" $\approx$, taking $A := x \approx y$ and $B := y \approx z$ gives that $c(A, B)$ is strictly classical, while $c(A, A)$ is not strictly classical.

In MO_n (for $n \geq 2$)[1] we have that $c(v_i, v_j) = 1$ if $i = j$ and 0 if $i \neq j$. Now, still considering a language $\mathcal{L}$ with "equality" $\approx$, we define the $\mathcal{L}$-wff

$$K := (\forall x)(\forall y)(\forall z)\cdots(\forall w)\big(c(x \approx y, z \approx w)\big).$$

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[1]See Section A.10 in appendix A for a description of these modular ortholattices.

Note that K is clearly a strictly classical tautology, and so, using this fact we can define a reduction of $\mathcal{A}'$ as follows —

$$\mathcal{A}' := \{K \to A \mid A \in \mathcal{A}\}.$$

Define $\mathbf{4} := \{0, 1, 2, 3\}$, and consider the M-system $(\mathcal{L}, \mathcal{A})$, where $\mathcal{L}$ is equational (i.e. $\mathcal{L}^P = \{\approx\}$). For any $f \in \mathcal{L}^F$, define $\hat{f} : \mathbf{4}^{\alpha(f)} \to \mathbf{4}$ to be the constant function taking everything to 0, and define $[\![\approx]\!]$ to be the unique symmetric truth function which satisfies $[\![0 \approx 1]\!] = v_1$, $[\![2 \approx 3]\!] = v_2$, and $[\![i \approx j]\!] = 0$ for all other $i \neq j$. Then we define the $\mathcal{L}$-structure $\hat{\mathbf{4}} := (\mathbf{4}, \mathrm{MO}_2, \{[\![\approx]\!]\}, \langle \hat{f} \rangle_{f \in \mathcal{L}})$, and claim that $\hat{\mathbf{4}}$ is a model for $\mathcal{A}'$. As per the discussion above, since $[\![a \approx b]\!] \wedge [\![b \approx c]\!] \leq [\![a \approx c]\!]$ holds for all $a, b, c \in \mathbf{4}$, we have (E3) and (E1) and (E2) hold since $[\![\approx]\!]$ is a symmetric truth function. Finally, it is easy to verify that $[\![K]\!] = 0$, so that for any $A \in \mathcal{A}$, $[\![K \to A]\!] = 0 \to [\![A]\!] = 1$, so that $\mathcal{A}'$ holds in $\hat{\mathbf{4}}$. This shows that for any equational language $\mathcal{L}$, there is always an axiomatization of a classical M-system which allows non-standard models using $\mathcal{Q}(\mathcal{L})$ as the underlying logic.

The above reduction was clearly very artificial, but reductions can be very natural in some contexts. For example, if we have some binary operation $* \in \mathcal{L}^F$, where $\mathcal{L}$ is a language with "equality" $\approx$, a reduction of substitution such as

$$c(x \approx y, z \approx w) \to [(x \approx y \wedge z \approx w) \to x * z \approx y * w]$$

seems extremely natural in the context of quantum logic.

2.5.2 Classicality Operators

In this section, we discuss two other natural classes of $\mathcal{L}$-wffs which can be useful for reducing axioms in the context of certain classes of models — in particular, those whose associated truth value algebras are *irreducible.* We note, however, that such $\mathcal{L}$-wff schema are only well-defined for languages $\mathcal{L}$ such that $|\mathcal{L}^P| < \infty$.

The first of these $\mathcal{L}$-wff schema $\mathbf{C}(\psi)$ is designed to evaluate to 1 or *true* (in any model) in which $[\![\psi]\!]$ is in the *center* of the truth value algebra. The motivation for $\mathbf{C}(\psi)$ is essentially the fact that the center of an orthomodular lattice L forms a Boolean sub-algebra of L (see Theorem 33), which essentially captures the classical behavior within L. We now construct the $\mathcal{L}$-wff schema $\mathbf{C}(\psi)$. Let $\mathcal{L}^P = \{P_1, \ldots, P_n\}$, and for any $\mathcal{L}$–wff ψ define

$$\begin{aligned}\mathbf{C}(\psi) &:= (\forall s_1^1)\cdots(\forall s_{\alpha(P_1)}^1)[\varphi_{P_1(s_1,\ldots,s_{\alpha(P_1)})}(\psi) \to \psi] \wedge \cdots \\ &\wedge (\forall s_1^n)\cdots(\forall s_{\alpha(P_n)}^n)[\varphi_{P_n(s_1,\ldots,s_{\alpha(P_n)})}(\psi) \to \psi)],\end{aligned} \tag{2.5}$$

where $\varphi_x(y) = x \wedge (\neg x \vee y)$ is the Sasaki projection.

Lemma 2.15. Let $\mathcal{L}$ be a language with a finite number of predicates, let $(A, L, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure, and let ψ be an evaluated $\mathcal{L}$-wff. Then $[\![\mathbf{C}(\psi)]\!] = 1$ if and only if $[\![\psi]\!]$ is in the center of L.

Proof. By definition of $\mathbf{C}(\psi)$, and since $[\![\mathbf{C}(\psi)]\!] = 1$ iff every term in the meet equals one (recalling that '$\forall$' statements evaluate to meets in the truth value algebras), we have that $[\![\mathbf{C}(\psi)]\!] = 1$ iff

$$\varphi_{[\![P(a_1,\ldots,a_{\alpha(P)})]\!]}([\![\psi]\!]) \to [\![\psi]\!] = 1$$

for every predicate P in $\mathcal{L}$, and for every $a_1, \ldots, a_{\alpha(P)} \in L$. But by Lemma A.12, the above statement holds iff $\varphi_{[\![P(a_1,\ldots a_{\alpha(P)})]\!]}([\![\psi]\!]) \leq [\![\psi]\!]$. By Lemma A.11, this is true iff

$$[\![P(a_1, \ldots, a_{\alpha(P)})]\!] \; \mathcal{C} \; [\![\psi]\!]$$

for every $a_1, \ldots, a_{\alpha(P)} \in L$ and $P \in \mathcal{L}^P$ (where $x \; \mathcal{C} \; y$ denotes compatibility of x and y in L). Clearly if $[\![\psi]\!]$ is in the center of L, the previous statement is satisfied. Conversely, since by the definition of an $\mathcal{L}$-structure the set of all $[\![P(a_1, \ldots a_{\alpha(P)})]\!]$ generate L, by Theorem 34, $[\![\psi]\!]$ must be in the center of L. $\square$

We now introduce the second $\mathcal{L}$-wff schema $\mathbf{T}(\psi)$ which, is designed to evaluate to 1 or *true* (in any model) in which $[\![\psi]\!] = 1$ and evaluate to 0 otherwise, whenever two simple $\mathcal{L}$-wffs are also satisfied (in any model). The motivation for $\mathbf{T}(\psi)$ essentially comes from the desire to be able to reduce $\mathcal{L}$-wffs involving the existential quantifier and capture certain classical behaviors of this quantifier in the process (more will be said regarding this in Section 2.5.3 below). In order to define $\mathbf{T}(\psi)$, let $\mathcal{L}^P = \{P_1, \ldots, P_n\}$, and then for any $\mathcal{L}$–wff ψ, we have

$$\begin{aligned} \mathbf{T}(\psi) := (\forall s_1^1)\cdots(\forall s_{\alpha(P_1)}^1)(P_1(s_1, \ldots, s_{\alpha(P_1)}) \to \psi) \wedge \cdots \\ \wedge (\forall s_1^n)\cdots(\forall s_{\alpha(P_n)}^n)(P_n(s_1, \ldots, s_{\alpha(P_n)}) \to \psi)]. \end{aligned} \tag{2.6}$$

The following lemmas regarding the $\mathcal{L}$-wff schema $\mathbf{T}(\psi)$ are useful.

Lemma 2.16. Let $\mathcal{L}$ be a language with a finite number of predicates, and let $(A, L, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure. Further let ψ be any evaluated $\mathcal{L}$-wff such that $[\![\psi]\!] = 1$. Then $[\![\mathbf{T}(\psi)]\!] = 1$.

Proof. Assuming $[\![\chi]\!] = 1$, then $[\![\mathbf{T}(\chi)]\!] = 1$ trivially by Lemma A.12, which gives that in any OML L with $a \in L$, we have $(a \to 1) = 1$. □

Lemma 2.17. Let $\mathcal{L}$ be a language with a finite number of predicates, let $\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure, and let ψ be any $\mathcal{L}$-wff. Further assume that there exists some $a_1, \ldots, a_n \in A$ and some predicate P with arity n such that $[\![P(a_1, \ldots, a_n)]\!] = 1$. Then $[\![\mathbf{T}(\psi) \to \psi]\!] = 1$.

Proof. Let ψ be any $\mathcal{L}$-wff. By Lemma A.12, we only need to show that $[\![\mathbf{T}(\psi)]\!] \le [\![\psi]\!]$. But by the definition of $\mathbf{T}$ and Lemma A.12 we have

$$[\![\mathbf{T}(\psi)]\!] \le \bigwedge_{a_1,\ldots,a_n \in A} \big([\![P(a_1, \ldots a_n)]\!] \to [\![\psi]\!]\big) \le \big(1 \to [\![\psi]\!]\big) = [\![\psi]\!].$$

□

We then have the following result.

Lemma 2.18. Let $\mathcal{L}$ be a language with a finite number of predicates, and let $\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$ be an $\mathcal{L}$-structure such that L is irreducible, and (for all $\mathcal{L}$-wffs ψ)

(i) $[\![\mathbf{C}(\mathbf{T}(\psi))]\!] = 1$;

(ii) $[\![\mathbf{T}(\psi) \to \psi]\!] = 1$.

Then for any evaluated $\mathcal{L}$-wff χ,

$$[\![\mathbf{T}(\chi)]\!] = \begin{cases} 1 & \text{if } [\![\chi]\!] = 1 \\ 0 & \text{if } [\![\chi]\!] \neq 1 \end{cases} \tag{2.7}$$

Proof. First, assume $[\![\chi]\!] = 1$. Then $[\![\mathbf{T}(\chi)]\!] = 1$ by Lemma 2.16. Next, assume $[\![\chi]\!] \neq 1$. By (i) above, $[\![\mathbf{C}(\mathbf{T}(\chi))]\!] = 1$, and so by Lemma 2.15, $[\![\mathbf{T}(\chi))]\!]$ is in the center of L. Since L is irreducible, the center of L is just $\{0, 1\}$. Then by (ii) above, $[\![\mathbf{T}(\chi)]\!] \to [\![\chi]\!] = 1$, so that

$$[\![\mathbf{T}(\chi)]\!] \leq [\![\chi]\!] \neq 1$$

by Lemma A.12, and hence we must have that $[\![\mathbf{T}(\chi)]\!] = 0$. □

The following section illustrates one particular (and important) application of $\mathcal{L}$-wff schema $\mathbf{T}(\psi)$.

2.5.3 Reduction & Existential Quantifiers

As mentioned previously, we would like to use the $\mathcal{L}$-wff schema $\mathbf{T}(\psi)$ defined above to reduce $\mathcal{L}$-wffs involving the existential quantifier $\exists$. (Recall that $(\exists x)(B) := \sim (\forall x)(\sim B)$.) Now, for a model $\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$, and an $\mathcal{L}$-wff $(\exists x)\chi(x)$, we have that

$$[\![(\exists x)\chi(x)]\!] = \bigvee_{a \in A} [\![\chi(a)]\!].$$

From this, we see that we can have $[\![(\exists x)\chi(x)]\!] = 1$ without $[\![\chi(a)]\!] = 1$ for any $a \in A$ — that is, even if $[\![(\exists x)\chi(x)]\!] = 1$, it does not follow that there really exists an $a \in A$ such that $[\![\chi(a)]\!] = 1$. And so, the quantifier $\exists$ is much weaker in $\mathcal{Q}(\mathcal{L})$ than in classical logic. However, the $\mathcal{L}$-wff $(\exists x)\chi(x)$ can be reduced to

$$(\exists x)\mathbf{T}(\chi(x)),$$

so that in any model $\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$ in which both $\mathbf{C}(\mathbf{T}(\chi))$ and $\mathbf{T}(\chi) \to \chi$ hold, we have

$$[\![(\exists x)\mathbf{T}(\chi(x))]\!] = 1,$$

which actually guarantees that the existence of some $a \in A$ such that $[\![\chi(a)]\!] = 1$. And so we see that reducing any axiom involving '$\exists$' in this way enables us to retain the full power of $\mathcal{L}$-wffs which involve the existential quantifier $\exists$ that we are used to from classical logic.

2.6 Summary and Conclusion

In this chapter, we have defined, for any first order language $\mathcal{L}$, the quantum logic $\mathcal{Q}(\mathcal{L})$, which consists of the axioms (Q1) — (Q6) and inference rules (R1) — (R5). We then went on to define notions of formal deduction and derivability in $\mathcal{Q}(\mathcal{L})$. (Recall that for an $\mathcal{L}$-wff A and set of $\mathcal{L}$-wffs Γ, the statement 'A is derivable from Γ' informally means that one can construct a proof of A from the set of statements Γ.)

Following this discussion, we defined an M-system (mathematical system) $(\mathcal{L}, \mathcal{A})$ to be a language $\mathcal{L}$, along with a set of $\mathcal{L}$-wffs $\mathcal{A}$ (which is effectively the set of mathematical axioms). We then described a model theory for an arbitrary M-system $(\mathcal{L}, \mathcal{A})$. We did this by constructing $\mathcal{L}$-structures

$$\hat{A} := (A, L, \{[\![P]\!]\}, F_A)$$

for $(\mathcal{L}, \mathcal{A})$ consisting of (i) an underlying set A in which variables are interpreted, (ii) a truth value algebra L, which is a complete orthomodular lattice, (iii) for each predicate $P \in \mathcal{L}^P$, a map $[\![P]\!]$ which assigns truth values (in L) to the atomic sentences, with $\bigcup_{P \in \mathcal{L}^P} [\![P]\!]$ generates L, and (iv) for every $f \in \mathcal{L}^F$, an interpretation of f in $\hat{A}$. Then, an $\mathcal{L}$-structure $\hat{A}$ is a model for $(\mathcal{L}, \mathcal{A})$ if all of the axioms $\mathcal{A}$ hold in $\hat{A}$. (If $\approx \in \mathcal{L}^P$, then we must also have that the axioms $\mathcal{E}(\mathcal{L})$ are also satisfied in order for $\hat{A}$ to be a model of the M-system.) Note that for simplicity, one can, when considering *conservative* models, effectively think of the associated truth value algebra L along with the truth function for L (which defines how the atomic $\mathcal{L}$-wffs get sent to elements of L) *as* the model. Finally, we note that we have demonstrated soundness and completeness for our semantics relative to our formal deductive system.

We now go on to apply this formalism to specific mathematical structures — that is, we will consider several mathematical systems using $\mathcal{Q}(\mathcal{L})$ as the underlying logic, and we will illustrate some interesting features of such quantum mathematical systems.

CHAPTER 3

QUANTUM MATHEMATICS

3.1 Introduction and Overview

In this chapter, we consider applications of the quantum logic $\mathcal{Q}(\mathcal{L})$ described in Chapter 2. In particular, we consider models of M-systems associated with groups, monoids, orthomodular and Boolean lattices, as well as Hilbert spaces and their operator algebras, in order to demonstrate some of the basic features of mathematical systems in the presence of quantum logic.[1] As noted previously, when considering models which are conservative, we can dispense with most of the formality, as in such cases it suffices to think of the truth value algebra L (along with the truth function for L that defines how the atomic $\mathcal{L}$-wffs get evaluated) *as* the model, keeping everything else in the background.

We reiterate that the quantum logic $\mathcal{Q}(\mathcal{L})$ is sub-classical — i.e. every theorem of $\mathcal{Q}(\mathcal{L})$ will also be a theorem of classical logic (but not vice versa). As such, every model of a classical M-system will still be a model when the underlying classical logic is replaced by $\mathcal{Q}(\mathcal{L})$, but there will, in general, be more models than are allowed classically, and we will see several examples of such *non-standard models* in this chapter.

[1] In what follows, we will often use the phrase 'an M-system $(\mathcal{L}, \mathcal{A})$ associated with, e.g., groups' to refer to an M-system $(\mathcal{L}, \mathcal{A})$ whose axioms $\mathcal{A}$ give rise, in the presence of classical logic, to models which are groups.

Also, as noted in Section 2.4, there are (for any given area of mathematics) often alternative but classically equivalent formulations of the axioms, and when $\mathcal{Q}(\mathcal{L})$ is used for the underlying logic (instead of classical logic), these equivalent classical presentations may no longer be equivalent. This sensitivity to the choice of classically equivalent mathematical axioms is an interesting feature of the quantum logic $\mathcal{Q}(\mathcal{L})$ which demonstrates a richness in the structure of mathematics which is classically inaccessible.

Additionally, one may expect that whenever the quantum logic $\mathcal{Q}(\mathcal{L})$ is used in place of classical logic, any M-system $(\mathcal{L}, \mathcal{A})$ will admit non-standard models. However, this is not always the case, and examples of inherently classical M-systems will be discussed in this chapter. Also, examples of conservative and non-conservative models (defined in Section 2.4) will be discussed as well.

Finally, we note that there are certain orthomodular lattices which are very natural with respect to quantum theory, and which are of particular interest due to the fact that they can be empirically motivated. In this chapter, we examine models which have such lattices as their truth value algebras, and we find that certain classical properties no longer hold in these models. Although the tendency would be simply to discard any models in which such intuitive properties fail, we hesitate to do so given the naturalness of these models with respect to the quantum theory — instead want to see what these models are trying to teach us.

We begin in Section 3.2, where we consider a particular presentation of the axioms for the successor fragment of Peano arithmetic, and show that the M-system associated with these axioms is inherently classical. Then, in Section 3.3, we show a similar result for any axiomatization of abstract algebras which satisfy certain properties. We go on to give some examples which demonstrate that the aforementioned properties are indeed necessary to establish that these M-systems are inherently classical. Finally, we demonstrate an example of a non-conservative model.

In Section 3.4, we consider quantum lattice theory, where we discuss a particular class of non-standard models for an M-system associated with axioms for orthomodular lattices, and note some interesting results with regard to substitution (Sub). We then go on to show that for a class of M-systems associated with an axiomatization of Boolean algebras, if strong transitivity (E3$'$) is satisfied, then there exist no non-standard models — i.e. these M-systems are inherently classical. We also provide examples of M-systems whose models are quantum Boolean algebras which do admit non-standard models; in such cases, only the (required) axiom (E3) — and not (E3$'$) — is satisfied.

Finally, in Section 3.5, we examine M-systems for axiomatizations of both Hilbert spaces and their operator algebras in the presence of our quantum logic $\mathcal{Q}(\mathcal{L})$. We will show that a natural class of conservative models present themselves for both classes of M-systems; and moreover, that the Hilbert space models are related to those for the operator algebras in a very natural way. Additionally, when the truth function for any model of an operator algebra is restricted to the sub-algebra of projection operators, we show that we recover exactly the truth function for the natural class of models for the axiomatization of orthomodular lattices discussed in Section 3.4.

3.2 Quantum Arithmetic

In this section, we briefly consider quantum arithmetic — much more will be said in Chapter 5 regarding a particular quantum arithmetic motivated by the quantum set theory which will be developed in Chapter 4. Here, we begin by discussing the result that for an M-system associated with the successor fragment of Peano arithmetic (for a particular presentation of the axioms), there exist no non-standard models — i.e. such an M-system is inherently classical. It follows from this that there do not exist any non-standard models for the full Peano arithmetic (built upon these successor fragment axioms) since any model for the full Peano arithmetic is a model for the successor fragment when restricted appropriately. We present an alternative formulation of the successor theory axioms which is equivalent to the original (discussed below) when the underlying logic is classical (11).[1] For this alternative presentation of the axioms, a class of non-standard models will be provided in Section 5.3, thereby demonstrating the splitting phenomenon discussed previously.

We begin with an M-system $(\mathcal{L}_{SF}, \mathcal{A}_{SF})$ associated with the successor fragment of Peano arithmetic. We define the language $\langle \mathcal{L}_{SF}, \alpha \rangle$, where $\mathcal{L}_{SF}^{P} \coloneqq \{\approx\}$ and $\mathcal{L}_{SF}^{F} \coloneqq \{0, {}'\}$, with $\alpha(\approx) \coloneqq 2$, $\alpha(0) \coloneqq 0$ and $\alpha({}') \coloneqq 1$; and where '$\approx$' is interpreted as the predicate "equality," '0' is interpreted

[1] We note that although the two sets of successor fragment axioms which are discussed in this section are equivalent in the presence of classical logic, when the arithmetic axioms due to Peano are appended to each of the distinct sets of the successor fragment axioms, the resulting arithmetics are *not* equivalent (even in the presence of classical logic). True Peano arithmetic refers to the standard arithmetic axioms appended to the successor fragment axioms (S1) — (S4) below. The alternative axiomatization of the successor fragment ultimately leads to a weaker arithmetic than Peano arithmetic.

as the "zero element," '$'$' is interpreted as the "successor function." By $\mathcal{A}_{SF}$ we denote the set of axioms (S1) — (S3) and the axiom schema (S4) below (where $\psi(x)$ is any $\mathcal{L}_{SF}$-wff for any $x \in \mathcal{B}_V$).

(S1) $(\forall x)[\sim (x' \approx 0)]$

(S2) $(\forall x)(\forall y)[(x \approx y) \to (x' \approx y')]$

(S3) $(\forall x)(\forall y)[(x' \approx y') \to (x \approx y)]$

(S4) $\big(\psi(0) \wedge (\forall x)[\psi(x) \to \psi(x')]\big) \to (\forall y)\psi(y)$

Note that axiom schema (S2) is just (Sub) for the unary operation '$'$'.

Theorem 7. Any two $\mathcal{L}_{SF}$-wffs are compatible.

This result[1] was first noticed and proved by Dunn (9). It follows immediately from this theorem (as noted in (9)) that, for any $\mathcal{L}_{SF}$-wff A, we have that $\vdash A$ if and only if A is classically derivable in Peano arithmetic. In addition to the axioms (Q1) — (Q6) and the quantum inference rules (R1) — (R5), the proof of this theorem requires (all of) the axioms (S1) — (S4) along with axioms (E1) and (E2); in particular, we note that axiom (E3) does not play a role in the proof.

Now, using the same language $\mathcal{L}_{SF}$, an alternative set of axioms for the successor theory of Peano arithmetic consists of (S1) — (S3) above, along with

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[1]Dunn actually considers a Gentzen style deduction system which has a slightly different syntax where he treats $\vdash$ as part of the object language. His proof can be translated into our approach by treating the relation $A \vdash B$ as the $\mathcal{L}$-wff $A \to B$.

(S$_\star$) $(\forall x)[\sim(x \approx 0) \rightarrow [(\exists y)[\sim(x \approx y')]]$

and

(S$_\infty$) $(\forall x)[\sim(x \approx x')]$, $(\forall x)[\sim(x \approx x'')]$, $\ldots$

where (S$_\infty$) denotes an infinite sequence of axioms. We will use $\widetilde{\mathcal{A}}_{SF}$ to refer to the set of axioms consisting of (S1) — (S3), (S$_\star$), and (S$_\infty$).

As mentioned above, we will describe a class of non-standard models for $\widetilde{\mathcal{A}}_{SF}$ in Chapter 5, the existence of which demonstrates the splitting phenomenon for equivalent classical presentations of the M-systems. That is, when the underlying logic is classical, $\mathcal{A}_{SF}$ and $\widetilde{\mathcal{A}}_{SF}$ are equivalent formulations (11) of the axioms for the successor fragment of Peano arithmetic since any theorem of $\mathcal{A}_{SF}$ is also a theorem of $\widetilde{\mathcal{A}}_{SF}$, and vice versa; however, when $\mathcal{Q}(\mathcal{L}_{SF})$ is the underlying logic, $\mathcal{A}_{SF}$ and $\widetilde{\mathcal{A}}_{SF}$ are no longer equivalent presentations of the axioms since Theorem 7 holds for $\mathcal{A}_{SF}$, while there does not exist such a theorem for $\widetilde{\mathcal{A}}_{SF}$ (as evidenced by the class of non-standard models which will be discussed in Chapter 5).

3.3 Quantum Algebras

As discussed in the previous section, Dunn's Theorem (i.e. Theorem 7) states that (for certain presentations of the axioms), the theorems of arithmetic under the quantum logic $\mathcal{Q}(\mathcal{L})$ are exactly the same as those under classical logic. In this section, we prove a similar theorem for M-systems associated with any abstract algebra which satisfies both strong transitivity (E3$'$) and substitution (Sub), as well as possesses some cancellative 'binary operation'; we assume that the language $\mathcal{L}$ associated with these M-systems is such that $\mathcal{L}^P = \{\approx\}$ — i.e. we assume that $\mathcal{L}$ is an equational language. We then demonstrate both a non-standard (conservative) model of an M-system associated with groups — which has the cancellative property and which does not satisfy strong transitivity (E3′) — as well as a non-standard (conservative) model of an M-system associated with monoids — which satisfies strong transitivity (E3′), but lacks cancellativity — showing that both the cancellation property and strong transitivity (E3′) are indeed necessary for the aforementioned theorem. Finally, we demonstrate an example of a non-conservative model of an M-system associated with monoids.

3.3.1 Some Abstract Algebras with only Standard Models

Fix an M-system $(\mathcal{L}, \mathcal{A})$. We first define what we mean for an $\mathcal{L}$-term to be cancellative.[1] Define the following two $\mathcal{L}$-wff schema (for $t(x,y)$ an $\mathcal{L}$-term)

(LC) $t(x,y) \approx t(x,z) \to y \approx z$

(RC) $t(x,z) \approx t(y,z) \to x \approx y$

If, for a given $t(x,y)$, both (LC) and (RC) are derivable from $\mathcal{A}$, we say that t is *cancellative in* $(\mathcal{L}, \mathcal{A})$.

We now prove some simple lemmas.

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[1] Although we only consider terms with two free variables for simplicity, we could easily generalize our notion of cancellativity to terms with an arbitrary number of free variables.

Lemma 3.1. Let $(\mathcal{L}, \mathcal{A})$ be an M-system which satisfies substitution (Sub) for every $f \in \mathcal{L}^F$. Then, for any $\mathcal{L}$-term $t(y_1, \ldots, y_n)$,

$$\mathcal{A} \vdash (y_1 \approx z_1) \wedge \cdots \wedge (y_n \approx z_n) \rightarrow t(y_1, \ldots, y_n) \approx t(z_1, \ldots, z_n). \tag{3.1}$$

Proof. Since every operation in $\mathcal{L}$ satisfies (Sub), the statement is established by a simple induction on the construction of $\mathcal{L}$-terms. □

If equation 3.1 holds for an M-system $(\mathcal{L}, \mathcal{A})$ and an $\mathcal{L}$-term $t(y_1, \ldots, y_n)$, we will say that *t satisfies substitution in* $(\mathcal{L}, \mathcal{A})$.

Lemma 3.2. Let $\mathcal{L}$ be any language and let A and B both be $\mathcal{L}$-wffs. Then $\vdash A \widetilde{\mathcal{C}} B$ if and only if $\vdash A \wedge (\sim A \vee B)) \rightarrow B$.

Proof. First we assume $A \widetilde{\mathcal{C}} B$. Then $A \rightarrow [\sim (\sim A \vee B) \vee (A \wedge B]$ by (Q3), (R1), replacement (i.e. Theorem 1), and Lemma 2.4. Then

$$[A \wedge (\sim A \vee B)] \rightarrow [(\sim A \vee B) \wedge (\sim (\sim A \vee B) \vee (A \wedge B))]$$

by (Q3) and (R3). Since $A \wedge B \rightarrow B$ by (Q3), and $B \rightarrow (\sim A \vee B) \wedge B$ by (Q3), (R2), (Q1) and (R3), we have that

$$[A \wedge (\sim A \vee B)] \rightarrow [(\sim A \vee B) \wedge (\sim A \vee B) \rightarrow B)]$$

by (R1) and Lemma 2.5. Then, (R1) and (Q4) then give the desired conclusion.

Next, assume $A \wedge (\sim A \vee B)) \to B$. By (Q4), (R2), and (Q2) we have

$$A \to \Big((A \wedge \sim B) \vee [\sim (A \wedge \sim B) \wedge (A \vee (A \wedge \sim B))]\Big),$$

and then by (Q3), (Q2), and (R1) this gives $A \to (A \wedge \sim B) \vee [(\sim A \vee B) \wedge A]$. Our assumption also gives $A \wedge (\sim A \vee B) \to (A \wedge B)$ by (Q3) and (R3), so that by (R1), (Q3) and (R3) we have $A \to (A \wedge \sim B) \vee (A \wedge B)$, from which it follows (since the other implication is trivial and holds in any orthomodular lattice) that $A \widetilde{\mathcal{C}} B$. $\square$

As stated in the beginning of this section, the proof of our main theorem requires (E3′). The following lemma is the reason.

Lemma 3.3. Let $\mathcal{L}$ be an equational language, let $(\mathcal{L}, \mathcal{A})$ be a M-system such that $\mathcal{A} \vdash$(E3′), and let t, u, v be $\mathcal{L}$-terms. Then

$$\mathcal{A} \vdash (t \approx u) \ \widetilde{\mathcal{C}} \ (t \approx v).$$

Proof. Let $A := (t \approx u)$, $B := (t \approx v)$ and $C := (u \approx v)$. From (E3′), and (Q3), along with (E2) and replacement (Theorem 1), we have $(B \wedge C) \to (B \to A)$. Then by (R1) and Lemma 2.5, we have $(B \to C) \to (B \to A)$. Also by (E3′) and (E2), we have $A \to (B \to C)$, and hence by (R1) we have $A \to (B \to A)$. Then by (R2) and (Q2) we have $B \wedge (\sim B \vee \sim A) \to \sim A$, which gives $B \ \widetilde{\mathcal{C}} \ \sim A$ by Lemma 3.2. Recalling the discussion of Section 2.4, this means that $A \ \widetilde{\mathcal{C}} \ B$. $\square$

Lemma 3.4. Let $\mathcal{L}$ be an equational language, and let $(\mathcal{L}, \mathcal{A})$ be a M-system such that

$$\mathcal{A} \vdash (t \approx u) \ \widetilde{\mathcal{C}} \ (v \approx w)$$

for all $\mathcal{L}$-terms t, u, v, w. Then the axiom schema (CL) is derivable from $\mathcal{A}$.

Proof. The hypothesis of this lemma is that all atomic $\mathcal{L}$-wffs are compatible. The proof of the conclusion is an induction on the formation of $\mathcal{L}$-terms. The '$\sim$' and '$\wedge$' steps are essentially a transcription of similar statements for orthomodular lattices (see Kalmbach (21)). The induction on the quantified statements appears in Dunn (9). □

We are now ready to prove our main theorems. Note that the following two theorems, although similar, are for different types of M-systems — i.e. there exists no M-system to which both Theorem 8 and Theorem 9 are both applicable. In particular, Theorem 8 applies to M-systems which satisfy strong transitivity and have some cancellative term which satisfies substitution, while Theorem 9 is relevant for M-systems which satisfy strong transitivity and has a particular type of constant term.

Theorem 8. Let $\mathcal{L}$ be an equational language, let $(\mathcal{L}, \mathcal{A})$ be a M-system such that $\mathcal{A} \vdash$ (E3′), and also let there exist an $\mathcal{L}$-term $t(x, y)$ which is cancellative and which satisfies (Sub) for every $f \in \mathcal{L}^F$ for the predicate $\approx$. Then (CL) is derivable from $\mathcal{A}$ — i.e. such an M-system is inherently classical.

Proof. By Lemma 3.4, it suffices to prove that any two atomic $\mathcal{L}$-wffs $u_1 \approx v_1$ and $u_2 \approx v_2$ are compatible. Then by Lemma 3.1 and since $t(x, y)$ is cancellative, it follows that

$$u_1 \approx v_1 \leftrightarrow t(u_1, v_2) \approx t(v_1, v_2) \quad \text{and} \quad u_2 \approx v_2 \leftrightarrow t(v_1, u_2) \approx t(v_1, v_2).$$

Further using modus ponens (R5) we have that $u_1 \approx v_1$ is logically equivalent to $t(u_1, v_2) \approx t(v_1, v_2)$ and $u_2 \approx v_2$ is logically equivalent to $t(v_1, u_2) \approx t(v_1, v_2)$ (with respect to $\mathcal{A}$).

Then, by Lemma 3.3, we have that

$$[t(u_1, v_2) \approx t(v_1, v_2)] \; \widetilde{\mathcal{C}} \; [t(v_1, u_2) \approx t(v_1, v_2)].$$

Using replacement (i.e. Theorem 1), this gives that $(u_1 \approx v_1) \; \widetilde{\mathcal{C}} \; (u_2 \approx v_2)$. Since this holds for arbitrary $\mathcal{L}$-terms u_1, u_2, v_1, v_2, the conclusion is established. □

Theorem 9. Let $\mathcal{L}$ be an equational language, let $(\mathcal{L}, \mathcal{A})$ be a M-system such that $\mathcal{A} \vdash$ (E3′), and also let there exist an $\mathcal{L}$-term u with no free variables and an $\mathcal{L}$-term $t(x, y)$ such that

$$\mathcal{A} \vdash x \approx y \to t(x, y) \approx u \quad \text{and} \quad \mathcal{A} \vdash t(x, y) \approx u \to x \approx y.$$

Then (CL) is derivable from $\mathcal{A}$ — i.e. such an M-system is inherently classical.

Proof. As in the previous theorem, it suffices to show that any two atomic $\mathcal{L}$-wffs are compatible with respect to $\mathcal{A}$. Let $u_1 \approx v_1$ and $u_2 \approx v_2$ be arbitrary atomic $\mathcal{L}$-wffs. By Lemma 3.3,

$$t(u_1, v_1) \approx u \; \widetilde{\mathcal{C}} \; t(u_2, v_2) \approx u.$$

But by assumption and (R5), we have that $u_1 \approx v_1$ is logically equivalent to $t(u_1, v_1) \approx u$ and $u_2 \approx v_2$ is logically equivalent to $t(u_2, v_2) \approx u$. Hence replacement (i.e. Theorem 1) yields that

$$u_1 \approx v_1 \; \widetilde{\mathcal{C}} \; u_2 \approx v_2.$$

□

And so, we see by Theorem 8 above, that any M-system $(\mathcal{L}, \mathcal{A})$ associated with abstract algebras (whose language $\mathcal{L}$ is equational) which satisfies both strong transitivity (E3′) and substitution, as well as possesses some cancellative 'binary operation' is inherently classical, and so, by definition, will not admit *any* non-standard models. Similarly, by Theorem 9 above, any M-system $(\mathcal{L}, \mathcal{A})$ associated with abstract algebras (whose language $\mathcal{L}$ is equational) which satisfies strong transitivity (E3′), as well as possesses some $\mathcal{L}$-term with no free variables is inherently classical.

3.3.2 A Non-Standard Conservative Model of an M-system Associated with Groups

We define the language $\langle \mathcal{L}_G, \alpha \rangle$ such that $\mathcal{L}_G^P := \{\approx\}$ and $\mathcal{L}_G^F := \{e, \cdot, ^{-1}\}$, with $\alpha(\approx) := 2$, $\alpha(e) := 0$, $\alpha(^{-1}) := 1$, and $\alpha(\cdot) := 2$; and where '$\approx$' is interpreted as the predicate "equality," 'e' is interpreted as the "identity element," '${}^{-1}$' is interpreted as the "inverse," and '$\cdot$' is interpreted as the "binary operation" for a quantum group. Further, let $\mathcal{A}_G$ be the following set of axioms (with $x, y, z \in \mathcal{B}_V$).

(G1) $(\forall x)(\forall y)(\forall z)[(x \cdot y) \cdot z \approx x \cdot (y \cdot z)]$

(G2) $(\forall x)[e \cdot x \approx x]$ and $(\forall x)[x \cdot e \approx x]$

(G3) $(\forall x)[x \cdot x^{-1} \approx e]$ and $(\forall x)[x^{-1} \cdot x \approx e]$

(G4) $(\forall x)(\forall y)[x \approx y \to x^{-1} \approx y^{-1}]$

(G5) $(\forall x)(\forall y)(\forall z)[x \approx y \to x \cdot z \approx y \cdot z]$ and $(\forall x)(\forall y)(\forall z)[x \approx y \to z \cdot x \approx z \cdot y]$

Note that (G4) and (G5) are just (Sub) for '${}^{-1}$' and '$\cdot$', respectively.

We define the $\mathcal{L}_G$-structure $\hat{K} = (\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{MO}_3, \{[\![\approx]\!]\}, \{00, +, -\})$, where the operations are given their usual interpretation in $\mathbb{Z}_2 \times \mathbb{Z}_2$, and we write ij as shorthand for (i, j). Then $[\![\approx]\!]$ is defined to be the unique symmetric truth function which satisfies

$$[\![00 \approx 01]\!] = [\![10 \approx 11]\!] := v_1, \quad [\![00 \approx 10]\!] = [\![01 \approx 11]\!] := v_2, \quad [\![00 \approx 11]\!] = [\![01 \approx 10]\!] := v_3.$$

Now, since the underlying set for $\hat{K}$ is the (classical) group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and since we give the operations their usual interpretation in $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have that the $\mathcal{L}_G$-structure $\hat{K}' = (\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbf{2}, \{[\![\approx]\!]\}, \{00, +, -\})$ is a *standard* model for the M-system $(\mathcal{L}_G, \mathcal{A}_G)$; thus, by Theorem 6, we have that (G1) — (G3) hold in $\hat{K}$. It remains to verify transitivity (E3) along with (G4) and (G5) in order to show that the $\mathcal{L}_G$-structure $\hat{K}$ is a model for the M-system $(\mathcal{L}_G, \mathcal{A}_G)$. Given the above definition for $[\![\approx]\!]$, one can easily verify by brute force computation that these schema hold in $\hat{K}$, so that $\hat{K}$ is, indeed, a model for $\mathcal{A}_G$, which is clearly non-standard and conservative.

Given Theorem 8, it is clear that (E3$'$) cannot hold in this model. To see that it fails, consider the evaluated $\mathcal{L}_G$-wff

$$[\![(01 \approx 10) \rightarrow [(10 \approx 11) \rightarrow (01 \approx 11)]]\!].$$

We see that $[\![01 \approx 10]\!] = v_3$, while

$$[\![(10 \approx 11) \rightarrow (01 \approx 11)]\!] = \neg v_1 \vee (v_1 \wedge v_2) = \neg v_1,$$

and $v_3 \nleq \neg v_1$ in MO_3, so $v_3 \rightarrow \neg v_1 \neq 1$, which shows that (E3$'$) does not, in fact, hold.

As such, we see that the existence of the non-standard conservative model $\hat{K}$ of the M-system $(\mathcal{L}_G, \mathcal{A}_G)$ (whose terms are cancellative — i.e. $\mathcal{A}_G \vdash$ (LC) and $\mathcal{A}_G \vdash$ (RC)) demonstrates that strong transitivity (E3$'$) is *necessary* for Theorem 8 to hold.

3.3.3 A Non-Standard Conservative Model of an M-system Associated with Monoids

We define the language $\langle \mathcal{L}_{Mon}, \alpha \rangle$ such that $\mathcal{L}^P_{Mon} := \{\approx\}$ and $\mathcal{L}^F_{Mon} := \{e, \cdot\}$, with $\alpha(\approx) := 2$, $\alpha(e) := 0$ and $\alpha(\cdot) := 2$; and where '$\approx$' is interpreted as the predicate "equality," 'e' is interpreted as the "identity element," and '$\cdot$' is interpreted as the "binary operation" for a quantum monoid. Further, let $\mathcal{A}_{Mon}$ be the following set of axioms (with $x, y, z \in \mathcal{B}_V$).

(M1) $(\forall x)(\forall y)(\forall z)[(x \cdot y) \cdot z \approx x \cdot (y \cdot z)]$

(M2) $(\forall x)[e \cdot x \approx x]$ and $(\forall x)[x \cdot e \approx x]$

(M3) $(\forall x)(\forall y)(\forall z)[x \approx y \to x \cdot z \approx y \cdot z]$ and $(\forall x)(\forall y)(\forall z)[x \approx y \to z \cdot x \approx z \cdot y]$

Note that these are the axioms $\mathcal{A}_G$ from the previous section which do not refer to '${}^{-1}$' (except that here these axioms consist of $\mathcal{L}_{Mon}$-wffs instead of $\mathcal{L}_G$-wffs).

We now define what will turn out to be a class of (conservative) models for the M-system $(\mathcal{L}_{Mon}, \mathcal{A}_{Mon})$. Let $n \in \{1, 2, \ldots\}$ and define $\bar{n} \coloneqq \{1, 2, \ldots, n\}$. Then for $n \geq 3$, define the $\mathcal{L}_{Mon}$-structure

$$\hat{n} \coloneqq (\bar{n}, \mathrm{MO}_2, \{[\![\approx]\!]\}, \{1, *\}),$$

where we let $[\![\approx]\!]$ be defined by (for all $i, j \in \bar{n}$)

$$[\![i \approx j]\!] \coloneqq \begin{cases} v_1 & \text{if } \{i, j\} = \{1, 2\} \\ v_2 & \text{if } \{i, j\} = \{n, n-1\} \\ \delta_{ij} & \text{otherwise,} \end{cases}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Clearly, $[\![\approx]\!]$ is a symmetric truth function by construction. The "identity operation" e is defined in the obvious way, and $* : \bar{n}^2 \to \bar{n}$ is defined by $i * j \coloneqq \max(i, j)$. It is straightforward to verify that $\hat{n}$ is indeed a model for $\mathcal{A}_{Mon}$. Also, (E3$'$) corresponds to the inequality

$$[\![a \approx b]\!] \leq \neg [\![a \approx c]\!] \vee \big([\![a \approx c]\!] \wedge [\![b \approx c]\!]\big) \tag{3.2}$$

holding in MO_2 for all $a, b, c \in \bar{n}$, and the reader will find it straightforward to check that for $n \geq 4$, the inequality in (equation 3.2) is indeed satisfied, so that (E3′) holds in $\hat{n}$ for all $n \geq 4$.

This class of examples demonstrates that without cancellativity, one may find non-standard conservative models which satisfy (E3′), so that (E3′) alone cannot guarantee that an M-system is inherently classical — i.e. cancellativity is *necessary* for Theorem 8 to hold.

3.3.4 A Non-Conservative Model of an M-system Associated with Monoids

We define the language $\langle \tilde{\mathcal{L}}_{Mon}, \alpha \rangle$ such that $\tilde{\mathcal{L}}^P_{Mon} := \{\approx\}$ and $\tilde{\mathcal{L}}^F_{Mon} := \{\cdot\}$, with $\alpha(\approx) := 2$ and $\alpha(\cdot) := 2$; and where, as for $\mathcal{L}_{Mon}$, '$\approx$' is interpreted as the predicate "equality" and '$\cdot$' is interpreted as the "binary operation" for a quantum monoid. Then define a set $\tilde{\mathcal{A}}_{Mon}$ of alternate axioms for monoids — i.e. let $\tilde{\mathcal{A}}_{Mon}$ be the following axioms (with $x, y, z \in \mathcal{B}_V$).

$$\text{(M2}'\text{)} \quad (\exists x)\Big((\forall y)[(x \cdot y \approx y) \wedge (y \cdot x \approx y)]\Big)$$

along with axioms (M1) and (M3) from the previous section (with these as $\tilde{\mathcal{L}}_{Mon}$-wffs instead of $\mathcal{L}_{Mon}$-wffs). In this presentation of the monoid axioms, we have incorporated the identity element by a 'there exists' statement, rather than treating it as a constant (0-ary) operation. In the presence of the schema (CL), these two presentations of axiom systems for monoids have exactly the same models; however without this schema we will exhibit a model of $(\tilde{\mathcal{L}}_{Mon}, \tilde{\mathcal{A}}_{Mon})$ that has an underlying set with a binary operation which can never be a model of $(\mathcal{L}_{Mon}, \mathcal{A}_{Mon})$, and as such, this model is non-conservative — that is, the M-system $(\tilde{\mathcal{L}}_{Mon}, \tilde{\mathcal{A}}_{Mon})$ admits models which have interpretations of the binary operation which are *not* allowed in *any* standard model.

We now construct such a non-conservative model of $(\tilde{\mathcal{L}}_{Mon}, \tilde{\mathcal{A}}_{Mon})$. Let $A = \{a, b, c\}$ and consider the $\tilde{\mathcal{L}}_{Mon}$-structure $\hat{A} = (A, \mathrm{MO}_2, \{[\![\approx]\!]\}, \{\cdot\})$, where $[\![\approx]\!]$ is the unique symmetric truth function from A to MO_2 satisfying $[\![a \approx b]\!] := v_1$, $[\![a \approx c]\!] := v_2$, and $[\![b \approx c]\!] := 0$. The operation $\cdot : A^2 \to A$ is given by the following multiplication table

$\cdot$	a	b	c
a	a	a	a
b	a	b	a
c	a	a	c

By inspection, one can see that there is no (classical) identity element, which immediately shows that this $\tilde{\mathcal{L}}_{Mon}$-structure cannot possibly be a conservative model. Since $(A, \cdot)$ forms a (classical) semi-group, Theorem 6 gives us immediately that (M1) holds in $\hat{A}$. That the remainder of the axioms in $\tilde{\mathcal{A}}_{Mon}$ do indeed hold in $\hat{A}$ is straightforward, but mildly tedious, to check.

3.4 Quantum Lattice Theory

In this section, we consider quantum lattice theory. We begin by discussing a particular class of non-standard models for M-systems associated with orthomodular lattices, and we note some interesting results with regard to substitution (Sub). We then go on to show that for a class of M-systems associated with Boolean algebras, if strong transitivity (E3′) is satisfied, then such M-systems are inherently classical. We also provide examples of M-systems for Boolean algebras which do admit non-standard models; in such cases, only the (required) axiom (E3) is satisfied.

3.4.1 Quantum Orthomodular Lattices

We begin by discussing an M-system $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$ associated with orthomodular lattices. We define the language $\langle \mathcal{L}_{OML}, \alpha \rangle$ such that $\mathcal{L}^P_{OML} := \{\approx\}$ and $\mathcal{L}^F_{OML} := \{0, \neg, \wedge\}$, with $\alpha(\approx) := 2$, $\alpha(0) := 0$, $\alpha(\neg) := 1$, and $\alpha(\wedge) := 2$; and where '$\approx$' is interpreted as the predicate "equality," '0' is interpreted as the "bottom element," '$\neg$' is interpreted as the "negation," and '$\wedge$' is interpreted as the "meet" or GLB in the models (or quantum orthomodular lattices). Further, let $\mathcal{A}_{OML}$ denote the set of axioms (OL1) — (OL6), and (OM) below (with $x, y, z \in \mathcal{B}_V$).

(OL1) $(\forall x)(\forall y)[x \wedge y \approx y \wedge x]$

(OL2) $(\forall x)(\forall y)(\forall z)[(x \wedge y) \wedge z \approx x \wedge (y \wedge z)]$

(OL3) $(\forall x)[x \approx \neg\neg x]$

(OL4) $(\forall x)(\forall y)[x \vee y \approx \neg(\neg x \wedge \neg y)]$

(OL5) $(\forall x)(\forall y)[x \approx x \wedge (x \vee y)]$

(OL6) $(\forall x)[x \wedge 0 \approx 0]$

(OM) $(\forall x)(\forall y)[x \wedge y \approx x \wedge (\neg x \vee (x \wedge y))]$

Although we still refer to $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$ as an M-system associated with orthomodular lattices, we note that $\mathcal{A}_{OML}$ is not actually equivalent (in the presence of classical logic) to any known axiomatization for orthomodular lattices. This lack of equivalence is due entirely to the fact that axioms enforcing substitution (Sub) to hold for '$\neg$' and '$\wedge$' have not been included in

$\mathcal{A}_{OML}$. The reason for this omission will become apparent after we see the behavior of (Sub) in the natural class of models discussed below.

We now describe a class of non-standard (conservative) models for the M-system $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$. Let L be a complete orthomodular lattice, and let $\hat{A} := (L, L, \{[\![\approx]\!]\}, \{0, \neg, \wedge\})$ be an $\mathcal{L}_{OML}$-structure in which '0', '$\neg$', and '$\wedge$' are given their standard interpretations. Notice that for these $\mathcal{L}_{OML}$-structures, the set of truth values is the same as the underlying set for $\hat{A}$. Also, let $[\![\approx]\!]$ be the negation of the *symmetric difference* in L — i.e.

$$[\![a \approx b]\!] := (a \wedge b) \vee (\neg a \wedge \neg b) = (a \to b) \wedge (b \to a),$$

where "$\to$" is the Sasaki hook — i.e. $a \to b := \neg a \vee (a \wedge b)$, where $a \vee b := \neg(\neg a \wedge \neg b)$.

Since we can immediately see that $[\![\approx]\!]$ is indeed a symmetric truth function, we have that axioms (E1) and (E2) are satisfied; additionally, from (8), we know that the negation of the symmetric difference in an orthomodular lattice satisfies the inequality equation 2.3. Thus, since L is an orthomodular lattice, we have that (E3) is satisfied, and hence, the $\mathcal{L}_{OML}$-structure $\hat{A}$ defined above is a model for $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$.

However, this class of (conservative) models for $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$ does not, in general, satisfy (E3$'$). To see that this is so, consider the orthomodular lattice defined by two Boolean cubes with a common atom and the same top and bottom elements — that is, let the lattice generated by α, β, γ form a Boolean cube, and let γ, δ, ϵ form another Boolean cube, where the top elements of each Boolean cube are identified with one another, and the bottom elements of each Boolean cube are also identified with one another. Then, using the definition of $[\![\approx]\!]$ given above, we have that $[\![\alpha \approx \gamma]\!] = \beta$, $[\![\gamma \approx \epsilon]\!] = \delta$ and $[\![\alpha \approx \epsilon]\!] = \gamma$, so that

$$[\![\alpha \approx \gamma]\!] \nleq [\![\gamma \approx \epsilon]\!] \to [\![\alpha \approx \epsilon]\!],$$

which shows that strong transitivity (E3$'$) is not satisfied. Thus, there exist orthomodular lattices such that the $\mathcal{L}_{OML}$-structure $\hat{A}$ defined above is a model for $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$, but (E3$'$) is not satisfied.

Now, we also note that a model $\hat{A}$ (in the class of models for $(\mathcal{L}_{OML}, \mathcal{A}_{OML})$ described above) has substitution (Sub) for the operation '$\neg$' — i.e. we have that

$$[\![(\forall x)(\forall y)(x \approx y \to \neg x \approx \neg y)]\!] = 1$$

holds in all models. To see that this is so, note that (for any a, b in the underlying set L for $\hat{A}$)

$$[\![\neg a \approx \neg b]\!] = (\neg a \wedge \neg b) \vee (\neg a \wedge \neg b) = [\![a \approx b]\!],$$

where we have used the fact that the law of double negation (i.e. $\neg\neg a = a$) holds in any orthomodular lattice, as well as the definition of '$\vee$' in terms of '$\neg$,' and '$\wedge$' and the fact that '$\wedge$' is commutative in any orthomodular lattice.

However, $\hat{A}$ does not have substitution (Sub) for the operation '$\wedge$' — i.e. in general

$$[\![a \approx b]\!] \not\leq [\![(a \wedge c) \approx (b \wedge c)]\!].$$

To see that this is so, consider the example from above (which was used to illustrate that (E3′) does not hold generally) — in particular, note that

$$[\![\neg\alpha \approx 1]\!] = \neg\alpha \quad \text{and} \quad [\![\neg\alpha \wedge \neg\epsilon \approx 1 \wedge \neg\epsilon]\!] = [\![\gamma \approx \neg\epsilon]\!] = \neg\delta.$$

And since $\neg\alpha \not\leq \neg\delta$, we see that substitution for '$\wedge$' does not hold in general. Note, however, that if $a \; \widetilde{\mathcal{C}} \; b$, $a \; \widetilde{\mathcal{C}} \; c$ and $b \; \widetilde{\mathcal{C}} \; c$, then substitution (Sub) for '$\wedge$' does hold generally.

The lack of substitution (Sub) for the operation '$\wedge$' with respect to "equality" $\approx$ in this extremely natural class of models suggests that this is, perhaps, natural behavior for "quantum equality" — this is to say that the lack of substitution (Sub) should be seen not as a failure of a classical property that *should* hold in a model, but rather as a manifestation of the true behavior[1] of "equality" in mathematics based on quantum logic. Thus, substitution (Sub) for '$\wedge$' with respect to "equality" $\approx$ (and even the property of strong transitivity (E3′) for "equality" $\approx$) — as intuitive as they are — are *very* classical properties, and are *not* something that

[1]Although it is not impossible to find (non-standard) models in which substitution and/or strong transitivity of equality hold, we expect that such models will be few and far between — i.e. "most" non-standard models will not satisfy these properties. Moreover, these properties do not, in general, seem to hold in the most natural (from the point of view of quantum theory) classes of models.

quantum mathematics seems to want in general.[1]

3.4.2 Quantum Boolean Algebras

In what follows, we discuss an M-system $(\mathcal{L}_{BA}, \mathcal{A}_{BA})$ associated with Boolean algebras. We define the language $\langle \mathcal{L}_{BA}, \alpha \rangle$ such that $\mathcal{L}^P_{BA} := \{\approx\}$ and $\mathcal{L}^F_{BA} := \{0, \neg, \wedge\}$, with $\alpha(\approx) := 2$, $\alpha(0) := 0$, $\alpha(\neg) := 1$, and $\alpha(\wedge) := 2$, which we note is the same language as $\mathcal{L}_{OML}$, and where (as noted for $\langle \mathcal{L}_{OML}, \alpha \rangle$) '$\approx$' is interpreted as the predicate "equality," '0' is interpreted as the "bottom element," '$\neg$' is interpreted as the "negation," and '$\wedge$' is interpreted as the "meet" or GLB in the model (or quantum Boolean algebra). Further, let $\mathcal{A}_{BA}$ denote the set of axioms (OL1) — (OL6) from above, along with (BA), (BS1) and (BS2) below (with $x, y, z \in \mathcal{B}_V$).

(BA) $(\forall x)(\forall y)[x \approx (x \wedge y) \vee (x \wedge \neg y)]$

(BS1) $(\forall x)(\forall y)[x \approx y \to \neg x \approx \neg y]$

(BS2) $(\forall x)(\forall y)(\forall z)[x \approx y \to (x \wedge z) \approx (y \wedge z)]$

Note that (BS1) and (BS2) are just substitution (Sub) for '$\neg$' and '$\wedge$', respectively.[2]

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[1]Of course, it is possible that a reduced form of (Sub) holds in these models.

[2]Although $\mathcal{A}_{BA}$ is not the usual presentation of the axioms for Boolean algebras, $\mathcal{A}_{BA}$ is equivalent to the standard axiomatization of Boolean algebras in the presence of classical logic.

Now, we want to show that in the presence of (E3′), the M-system $(\mathcal{L}_{BA}, \mathcal{A}_{BA})$ is inherently classical, but that if (E3′) is not satisfied, then the M-system $(\mathcal{L}_{BA}, \mathcal{A}_{BA})$ does admit non-standard models.

Theorem 10. There exist no non-standard models for $(\mathcal{L}_{BA}, \mathcal{A}_{BA} \cup \{(\text{E3}')\})$.

Proof. In order to show this, we make use of the well-known fact that any Boolean algebra is isomorphic to some commutative ring with identity, all of whose elements are idempotent. Let the term with two free variables $x + y$ be defined by

$$x + y := (x \wedge \neg y) \vee (\neg x \wedge y), \tag{3.3}$$

and note that such a term is cancellative. Also note that by Lemma 3.1, the M-system $(\mathcal{L}_{BA}, \mathcal{A}_{BA})$ has substitution (Sub) for the terms $x + y$ (i.e. such terms satisfy equation 3.1) since the axiom schema (BS1) and (BS2) give us substitution (Sub) for $\neg$ and $\wedge$, respectively. And so, if we impose strong transitivity of equality (E3′), then by Theorem 8, we have that there exist no non-standard models for Boolean algebras even when the underlying logic is $\mathcal{Q}(\mathcal{L}_{BA})$. □

However, if (E3′) is not satisfied and only the (required) transitivity of equality (E3) is satisfied, then there do exist non-standard models. In order to see this, consider the following $\mathcal{L}_{BA}$-structure $\hat{A} := (A, L, \{[\![\approx]\!]\}, \{0, \wedge, \neg\})$, where '0', '$\wedge$', and '$\neg$' are given their usual interpretations, $A := \{1, 0, a, \neg a\}$ is the free Boolean algebra on one generator (i.e. the Boolean diamond), and L is $\text{MO}_2 := \{1, 0, v_1, v_2, \neg v_1, \neg v_2\}$. Further, let $[\![\approx]\!]$ be defined as follows — take

$[\![i \approx i]\!] := 1$ for all $i \in A$, take $[\![i \approx j]\!] = [\![j \approx i]\!]$ for all $i, j \in A$, and take $[\![i \approx \neg i]\!] := 0$ for all $i \in A$; also let $[\![a \approx 0]\!] = [\![\neg a \approx 1]\!] := v_1$ and $[\![a \approx 1]\!] = [\![\neg a \approx 0]\!] := v_2$.

We see immediately that $[\![\approx]\!]$ is indeed a symmetric truth function; notice also that by Theorem 6, $\hat{A}$ will automatically satisfy the axioms (OL1) — (OL6) and (BA) since the underlying set A for $\hat{A}$ is a (classical) Boolean algebra (and since we give each $f \in \mathcal{L}^F_{BA}$ its usual interpretation). Additionally, it is easy to verify (by simply checking all cases) that axiom (E3) is satisfied; similarly, it is straightforward to check that axioms (BS1) and (BS2) are also satisfied. As such, we have that the $\mathcal{L}_{BA}$-structure $\hat{A} := (A, L, \{[\![\approx]\!]\}, \{0, \wedge, \neg\})$ is indeed a non-standard conservative model for the M-system $(\mathcal{L}_{BA}, \mathcal{A}_{BA})$.

3.5 Quantum Linear Algebra

It is fitting that we finish our examination of mathematics with quantum logic by focusing on the area of mathematics responsible for the inception of quantum logic (2). In this section, we examine M-systems associated with Hilbert spaces as well as M-systems associated with their operator algebras.[1] For M-systems associated with both Hilbert spaces and operator algebras, we will see that a natural class of conservative models present themselves. Moreover, the Hilbert space models are related to those of operator algebras in a very natural way. For the case of Hilbert spaces, these natural models satisfy substitution for all their operations. This is

[1]In order to keep the discussion simple, we assume that everything in sight is topologically well-behaved. We note that we have not yet considered the topological axioms for the M-systems associated with either Hilbert spaces or their operator algebras. The discussion here is focused on the algebraic properties (which, of course, follow from the algebraic axioms), and as such we are actually just considering M-systems for axiomatizations of vector spaces and linear algebra. However, we still refer to the models for these sets of axioms as Hilbert spaces and operator algebras, respectively.

not the case, however, for the operation of multiplication in the operator algebra models. Furthermore, we show that the operator algebra models yield precisely the symmetric difference models discussed above (see Section 3.4) when they are restricted to the lattice of projection operators. Finally, we discuss the possibility of modifying the von Neumann equation within the framework of quantum linear algebra to possibly allow for classes of quantum evolutions beyond what is allowed in standard quantum mechanics.

We note that we will only be considering conservative models in this section, and as such, we will be considerably more informal in our discussion — conservativity allows us to dispense with much of the technicality of the model theory. For M-systems associated with both Hilbert spaces and their operator algebras, we allow any choice of axioms such that (conservativity of the models along with) Theorem 6 guarantees they are satisfied. Additionally, we take the truth value algebra to be the lattice of closed linear subspaces of $\mathcal{H}$ (which we denote by $L_{\mathcal{H}}$), and we essentially think of $L_{\mathcal{H}}$ (along with a map which assigns, to each atomic sentence, some element of $L_{\mathcal{H}}$) as the model.

3.5.1 Hilbert Spaces

For concreteness, we take the language and axioms for vector spaces as in (23), which we will refer to as $\mathcal{L}_{VS}$ and $\mathcal{A}_{VS}$, respectively. (Note that implicitly the language $\mathcal{L}_{VS}$ is such that $\mathcal{L}_{VS}^{P} = \{\approx\}$.) As mentioned above, the specific choice of axioms is not relevant here (since we will only consider conservative models), provided that we choose our axioms such that Theorem 6 guarantees they are satisfied. We note that we will work over the field $\mathbb{C}$ for concreteness.

The non-standard (conservative) models we construct below will be based on (separable) Hilbert spaces $\mathcal{H}$. We will use the Dirac bra-ket notation for the inner product in $\mathcal{H}$, and for any $A \subseteq \mathcal{H}$, we define

$$A^{\perp} := \{\ |\psi\rangle \in \mathcal{H} \mid \langle\psi|\phi\rangle = 0 \text{ for all } |\phi\rangle \in A\ \}.$$

That is, $A^{\perp} \subseteq \mathcal{H}$ is the closed subspace consisting of vectors which are orthogonal to every vector in the original set $A \subseteq \mathcal{H}$. Additionally, for any $|\psi\rangle \in \mathcal{H}$, we define $|\psi\rangle^{\perp} := \{|\psi\rangle\}^{\perp}$, which is the (closed) subspace of $\mathcal{H}$ which is orthogonal to the vector $|\psi\rangle \in \mathcal{H}$. Further, we let $|0\rangle$ denote the zero vector in $\mathcal{H}$.

Consider the $\mathcal{L}_{VS}$-structure $\hat{\mathcal{H}} := (\mathcal{H}, L_{\mathcal{H}}, \{[\![\approx]\!]\}, F_{\mathcal{H}})$, where $L_{\mathcal{H}}$ is the lattice[1] of closed linear subspaces of $\mathcal{H}$, and let each operation $f \in \mathcal{L}_{VS}^{F}$ have its usual interpretation; additionally, define $[\![\approx]\!] : \mathcal{H}^2 \to L_{\mathcal{H}}$ by

$$[\![|\psi\rangle \approx |\phi\rangle]\!] := (|\psi\rangle - |\phi\rangle)^{\perp}.$$

We see immediately that $[\![\approx]\!]$ is, by construction, a manifestly symmetric truth function, and as such, both (E1) and (E2) are satisfied. Also, as mentioned above, we assume that the axioms $\mathcal{A}_{VS}$ are chosen such that Theorem 6 guarantees that they are satisfied. Hence, it remains to examine (E3) to see that $\hat{\mathcal{H}}$ is indeed a model for the M-system $(\mathcal{L}_{VS}, \mathcal{A}_{VS})$. That is, we need to show that

$$\big(|\psi\rangle - |\phi\rangle\big)^{\perp} \cap \big(|\phi\rangle - |\chi\rangle\big)^{\perp} \subseteq \big(|\psi\rangle - |\chi\rangle\big)^{\perp}. \tag{3.4}$$

---

[1]This lattice is complete and orthomodular, and is actually a modular ortholattice in the finite dimensional case. See Section A.4.3 for a description of the subspace/projection lattices of a Hilbert space.

To see that this is satisfied, suppose that $|\eta\rangle \in \big(|\psi\rangle - |\phi\rangle\big)^{\perp} \cap [|\phi\rangle - |\chi\rangle]^{\perp}$; this gives that $\langle\eta|\psi\rangle - \langle\eta|\phi\rangle = 0$ and $\langle\eta|\phi\rangle - \langle\eta|\chi\rangle = 0$, from which it follows that $\langle\eta|\psi\rangle = \langle\eta|\chi\rangle$. However, this shows that $|\eta\rangle \in \big(|\psi\rangle - |\chi\rangle\big)^{\perp}$, and since $|\eta\rangle$ was generic, equation 3.4 holds.

Furthermore, it is easy to see that

$$[\![|\psi\rangle \approx |\phi\rangle]\!] = [\![\lambda|\psi\rangle \approx \lambda|\phi\rangle]\!] \quad \text{for all } \lambda \in \mathbb{C},$$

as well as that

$$[\![|\psi\rangle \approx |\phi\rangle]\!] = [\![\big(|\psi\rangle + |\chi\rangle\big) \approx \big(|\phi\rangle + |\chi\rangle\big)]\!] \quad \text{for all } |\chi\rangle \in \mathcal{H}.$$

As such, we have that both (LC) and (RC) hold for the binary operation '+', as well as that substitution is satisfied for vector addition and scalar multiplication. We note, however, that (E3′) does not hold.

We conclude this section by noting that in the class of non-standard (conservative) models $\hat{\mathcal{H}}$ discussed above, the Schrödinger equation appears as a vector equation given by (where H is the Hamiltonian operator, which we take to be bounded for simplicity, and we set $\hbar = 1$)

$$i\frac{d}{dt}|\psi\rangle \approx H|\psi\rangle. \tag{3.5}$$

Taking equation 3.5 as an axiom of quantum theory, we necessarily have that $[\![i\frac{d}{dt}|\psi(t)\rangle \approx H|\psi(t)\rangle]\!] = \mathcal{H}$ (where $\mathcal{H}$ is the top element of $L_{\mathcal{H}}$); however, for any $|\chi\rangle, |\eta\rangle \in \mathcal{H}$, we have that $[\![|\chi\rangle \approx |\eta\rangle]\!] = \mathcal{H}$ if and only if $|\chi\rangle$ and $|\eta\rangle$ are actually the *same* vector (since $[\![\approx]\!]$ is a truth function). From this, it is clear that in the class of models $\hat{\mathcal{H}}$ described above, all usual unitary quantum dynamics are retained, but equation 3.5 cannot allow for any new dynamics (even though vector equations can, in general, be assigned truth values other than the standard $\{0,1\}$).

3.5.2 Operator Algebras

There are a variety of languages and axioms for operator algebras (for example, those of B^*-algebras, C^*-algebras, von Neumann algebras, etc.). As in the previous section, any axiomatization for which Theorem 6 guarantees that all the axioms hold will suffice for our purposes. Let $\mathcal{A}_{OA}$ be one such axiomatization and $\mathcal{L}_{OA}$ the corresponding equational language (i.e. $\mathcal{L}_{OA}^P = \{\approx\}$).[1]

We use the bounded linear operators on any separable Hilbert space to construct a model for the M-system $(\mathcal{L}_{OA}, \mathcal{A}_{OA})$. Again, for a separable Hilbert space $\mathcal{H}$, let $L_{\mathcal{H}}$ be the lattice of closed linear subspaces of $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. Also, let each operation $f \in \mathcal{L}_{OA}^F$ have its standard interpretation. Then two natural choices giving symmetric truth functions $[\![\approx]\!] : \mathcal{B}(\mathcal{H})^2 \to L_{\mathcal{H}}$ and $[\![\approx]\!]' : \mathcal{B}(\mathcal{H})^2 \to L_{\mathcal{H}}$ are as follows:

(i) $[\![A \approx B]\!] := \ker(A^\dagger - B^\dagger)$,

(ii) $[\![A \approx B]\!]' := \ker(A - B)$,

---

[1]Although we will refer to $(\mathcal{L}_{OA}, \mathcal{A}_{OA})$ as an M-system associated with operator algebras, we note that $\mathcal{A}_{OA}$ is not actually equivalent (in the presence of classical logic) to any known axiomatization for operator algebras, as standard axiomatizations include axioms enforcing substitution (Sub) to hold in any model, and $\mathcal{A}_{OA}$ does not have these axioms.

where $A^\dagger$ denotes the Hermitian conjugate of A. While $[\![\approx]\!]'$ may seem to be the more natural choice at first, it will be enlightening at this point to connect back to the models $\hat{\mathcal{H}}$ discussed in the previous section. Although either choice for a truth function yields an interesting class of models, it turns out that it is actually $[\![\approx]\!]$ above which is related to the truth function for models of Hilbert spaces in a nice way, as we now show.

In standard (classical) mathematics, for $A, B \in \mathcal{B}(\mathcal{H})$, the equation $A \approx B$ simply means that $A|\psi\rangle \approx B|\psi\rangle$ for all $|\psi\rangle \in \mathcal{H}$. Now for a separable Hilbert space $\mathcal{H}$, consider the non-standard model $\hat{\mathcal{H}}$ defined in the previous section. Then[1]

$$\begin{aligned}
[\![(\forall|\psi\rangle)(A|\psi\rangle \approx B|\psi\rangle)]\!] &= \bigwedge_{|\psi\rangle\in\mathcal{H}} [\![A|\psi\rangle \approx B|\psi\rangle]\!] = \bigcap_{|\psi\rangle\in\mathcal{H}} ((A-B)|\psi\rangle)^\perp \\
&= \{|\phi\rangle \in \mathcal{H} \mid \langle\phi|(A-B)|\psi\rangle = 0 \text{ for all } |\psi\rangle \in \mathcal{H}\ \} \\
&= \{|\phi\rangle \in \mathcal{H} \mid (A-B)^\dagger|\phi\rangle = 0\ \} = \ker(A^\dagger - B^\dagger) \\
&= [\![A \approx B]\!]. \qquad (3.6)
\end{aligned}$$

Given the naturalness of $[\![\approx]\!]$ with regards to our previous vector space models, we define $\hat{O}_{\mathcal{H}} := (\mathcal{B}(\mathcal{H}), L_{\mathcal{H}}, \{[\![\approx]\!]\}, F_{\mathcal{B}(\mathcal{H})})$ to be the $\mathcal{L}_{OA}$-structures which, as it turns out, are indeed models (i.e. quantum operator algebras).

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[1] In this expression, we use the same notation for the truth function associated with models for distinct M-systems — that is, for the expression, $[\![(\forall|\psi\rangle)(A|\psi\rangle \approx B|\psi\rangle)]\!]$, we have that $[\![\approx]\!]$ is the truth function from the class of models $\hat{\mathcal{H}}$ defined above in Section 3.5.1, while for the expression $[\![A \approx B]\!]$, we have that $[\![\approx]\!]$ is the truth function (i) above from the class of models $\hat{\mathcal{O}}$ below.

Now the lattice of closed subspaces of $\mathcal{H}$ is isomorphic to the lattice of projection operators, and so we can use $\hat{O}_{\mathcal{H}}$ to construct a model of $L_{\mathcal{H}}$ in the following way. Define $\kappa : L_{\mathcal{H}}^2 \to L_{\mathcal{H}}$ to be (for $V, W \subseteq \mathcal{H}$ both closed linear subspaces) $\kappa(V, W) := [\![P_V \approx P_W]\!]$, where P_V, P_W are the projection operators onto V, W respectively. Then define the $\mathcal{L}_{OM}$-structure $\hat{L}_{\mathcal{H}} := (L_{\mathcal{H}}, L_{\mathcal{H}}, \{\kappa\}, \{\{|0\rangle\}, {}^{\perp}, \cap\})$. As we will show below, this structure is precisely the symmetric difference model of the previous section[1] — this is clear for all aspects of $\hat{L}_{\mathcal{H}}$ except for the truth function κ, which we now discuss.

For $V, W \in L_{\mathcal{H}}$ with associated (Hermitian) projectors P_V and P_W, we have $\kappa(V, W) = \ker(P_V - P_W)$. Now $|\psi\rangle \in \ker(P_V - P_W)$ if and only if $P_V|\psi\rangle = P_W|\psi\rangle$. First, assume $|\psi\rangle \in (V \cap W) \vee (V^{\perp} \cap W^{\perp})$, so that $|\psi\rangle = |\phi\rangle + |\chi\rangle$ with $|\psi\rangle \in (V \cap W)$ and $|\chi\rangle \in (V^{\perp} \cap W^{\perp})$. Then

$$P_V|\psi\rangle = P_V|\phi\rangle + P_V|\chi\rangle = |\phi\rangle = P_W|\phi\rangle + P_W|\chi\rangle = P_W|\psi\rangle, \tag{3.7}$$

and so we have $[\![V \approx W]\!]^{SD} \subseteq \kappa(V, W)$, where we let $[\![\approx]\!]^{SD}$ denote the truth function in the symmetric difference models discussed in Section 3.4.1 (in order to distinguish it from the truth functions used in this section). Now consider $|\psi\rangle \in \kappa(V, W)$, so that $P_V|\psi\rangle = P_W|\psi\rangle \in V \cap W$.

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[1]Recall that the symmetric difference models have a truth function defined by (letting $[\![\approx]\!]$ used in Section 3.4.1 now be denoted by $[\![\approx]\!]^{SD}$ in order to distinguish it from the truth functions used in this section)

$$[\![V \approx W]\!]^{SD} = (V \wedge W) \vee (V^{\perp} \wedge W^{\perp}),$$

where for subspaces, ${}^{\perp}$ denotes the negation and '$P \vee Q$' is the closed linear span of P and Q.

Then, for I the identity operator in $\mathcal{B}(\mathcal{H})$, since $P_{V^\perp} = I - P_V$ and $P_{W^\perp} = I - P_W$, we have $P_{V^\perp}|\psi\rangle = P_{W^\perp}|\psi\rangle \in (V^\perp \cap W^\perp)$ and

$$|\psi\rangle = P_V|\psi\rangle + P_{V^\perp}|\psi\rangle \in (V \cap W) \vee (V^\perp \cap W^\perp), \tag{3.8}$$

so $|\psi\rangle \in [\![V \approx W]\!]^{SD}$. Hence, we have both inclusions, which gives that $[\![V \approx W]\!]^{SD} = \kappa(V, W)$ for all $V, W \in L_{\mathcal{H}}$, showing that $\mathcal{L}_{OM}$-structure $\hat{L}_{\mathcal{H}}$ just described is indeed the symmetric difference model. We cannot help but believe that the natural and smooth interplay between the symmetric difference models of Section 3.4.1, the Hilbert space model of the previous section, and the operator algebra model of this section hints at a profound significance for these classes of models with respect to quantum theory.

Now, in quantum theory, the usual way of expressing the time-evolution of quantum states is via time-dependent density operators which satisfy the von Neumann equation $\frac{d}{dt}\rho = -i[H, \rho]$. In the class of non-standard (conservative) models $\hat{\mathcal{O}}$ discussed above, von Neumann equation is given by

(vN) $\quad \frac{d}{dt}\rho \approx -i[H, \rho]$.

A similar argument to the one given above for the Schrödinger equation tells us that if (vN) is taken as an axiom of quantum mechanics, we again have only the standard unitary dynamics of quantum mechanics.

Now suppose that we wanted to somehow generalize the von Neumann equation in the framework of quantum logic to allow for more than the standard unitary evolutions as solutions. For this we have a natural tool at our disposal, namely that of reducing the axiom (vN) in some way. Define $B\rho := \frac{d}{dt}\rho + i[H,\rho]$, where we note that since operator algebras satisfy (Sub) for '+', the $\mathcal{L}_{OA}$-wffs $B\rho \approx 0$ and $\frac{d}{dt}\rho \approx -i[H,\rho]$ are logically equivalent. It seems fairly natural to attempt a reduction utilizing $c(B\rho \approx 0, A)$ for some $\mathcal{L}_{OA}$-wff A — i.e. something like

(vN′) $\quad c(B\rho \approx 0, A) \rightarrow B\rho.$

There is no reason to suspect *a priori*, that for an appropriate choice of an $\mathcal{L}_{OA}$-wff A, we have *only* standard unitary dynamics for all states ρ. The challenge is to come up with a reduction which does indeed allow for additional dynamical evolutions, and we are currently investigating such reductions, along with alternative possibilities for reductions of (vN) which can, in principle, allow for certain types of additional dynamics in quantum theory. The ultimate goal is to be able to unify ordinary time-evolution of closed quantum systems and measurement evolutions, thereby resolving the infamous measurement problem of quantum theory.

3.6 Conclusion

In this chapter we have considered a wide variety of examples of M-systems based on the quantum logic $\mathcal{Q}(\mathcal{L})$. In particular, examples of inherently classical M-systems have been given, as have examples of conservative and non-conservative models. Additionally, we have encountered extremely natural classes of models, as well as examined relationships between some of them. Moreover, we have demonstrated that certain classical properties no longer hold in these natural models, and have begun to consider what these models are trying to suggest to us about quantum theory.

In future work we would like to further explore consequences of the quantum logic $\mathcal{Q}(\mathcal{L})$ for quantum theory — in particular, we would like to systematically examine (axiomatic) quantum mechanics built on the relevant quantum mathematics. In addition, we would also like to continue to examine the different properties and features of a variety of M-systems in the presence of the quantum logic $\mathcal{Q}(\mathcal{L})$.

CHAPTER 4

QUANTUM SET THEORY

4.1 Introduction

As a branch of mathematics, axiomatic set theory is on very different footing than other areas of mathematics. One reason for this is that set theoretic concepts are actually necessary for precisely defining certain (mathematical) concepts. More strongly, axiomatic set theory, along with first order classical logic, is capable of providing a foundation for all of modern mathematics. Another aspect of axiomatic set theory which sets it apart from other areas of mathematics is the model theory associated with it. One of the major goals of axiomatic set theory is to precisely capture a particular intended model, which we often, in practice, actually conflate with set theory itself.[1]

In what follows, we construct a *quantum set theory* — that is, an axiomatic set theory based on the quantum logic $\mathcal{Q}(\mathcal{L})$ developed previously in Chapter 2 — which we hope will ultimately lead to a foundation for quantum mathematics in a sense which parallels the foundational role of classical set theory in classical mathematics. However, as quantum mathematics and quantum set theory are both in their infancy, our immediate goals with regard to quantum set theory are much less lofty — in particular, we put forth two modest goals which, we believe, are a

[1]This differs significantly from the general goals of an axiomatic theory, which are essentially to unify a plethora of interesting examples (which in the axiomatic approach, become distinct models).

respectable minimal criteria any attempt at quantum set theory should satisfy. First, recalling that quantum logic is sub-classical (and therefore includes all of classical logic and mathematics as a special case), we expect that quantum set theory must be a generalization of classical set theory, and in particular, those models of our quantum set theory with the standard bivalent truth values should reduce to models of classical set theory. Second, we expect that quantum set theory should *at least* be powerful enough to develop a notion of a 'natural number,' as well as a quantum arithmetic for these numbers, which we again expect to reduce to classical arithmetic when the truth values are (the standard) $\{0, 1\}$.

With regard to the quantum set theory described below, we note that, analogous to the (historical) development of classical axiomatic set theory, our quantum set theory is motivated by a seemingly natural class of models, and it is these intended models which we have in mind as we set forth our axioms for quantum set theory. Additionally, we note that other attempts at developing a quantum set theory have already been made — Gaisi Takeuti's (22) work on quantum set theory precedes ours by several decades. Takeuti's construction of a quantum set theory also has specific intended models which are a generalization of "Boolean-valued models" for classical set theory. This leads to a much larger (and possibly richer) quantum set theory relative to our construction.[1] The resulting set theory is very difficult to work with, as has been noted by Takeuti himself. The advantage that our quantum set theory has over Takeuti's

[1]However, the semantics for Takeuti's quantum set theory is *very* different from ours.

is that it is not only more intuitive, but is more tractable as well, as the development below will illustrate.

4.1.1 Overview

In Section 4.2 we discuss the classical ZFC axioms, and give a brief description of their content; then in Section 4.2.1 we give a brief description of the classical universe of sets — i.e. the intended model of the ZFC axioms. Following this, we develop and discuss the axioms for our quantum set theory in Section 4.3, and show that in the presence of classical logic, they're equivalent to the standard ZFC axioms. We then go on, in Section 4.4 to describe candidates for the "quantum universe" of sets (Section 4.4.1), as well as discuss the intended class of models for our quantum set theory. Finally, in Section 4.4.2, we prove that the intended class of models does satisfy the axioms of our quantum set theory — i.e. they are indeed models.

4.2 Classical Axiomatic Set Theory

In this section, we consider the Zermelo-Fraenkel axioms along with the axiom of *choice* (or ZFC for short) for classical set theory,[1] as well as give a brief qualitative description of the axioms; for a more detailed discussion of the ZFC axioms for (classical) set theory, the reader is referred to, e.g. Enderton (12). As in standard ZF set theory, we will have only two *primitive notions* — namely that of a set and "membership." In doing so, we have that the

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[1]Although there exist other well-known axiomatizations of classical set theory — e.g. that of von Neumann, Bernays and Gödel (or the NBG axiomatization) — we will not be concerned with these alternative sets of axioms here.

only members (or elements) of sets are sets, and as such, the development of the set theory is relatively streamlined.

We note that we will not be concerned here with discussing or resolving any of the issues related to the foundational role of classical set theory. Rather, we will simply accept classical (axiomatic) set theory and the intended model of the *classical universe* of sets as a foundation on which to build our quantum set theory.

This being said, we define the language $\mathcal{L}_{set} = \mathcal{L}^P_{set} := \{\in\}$, where $\alpha(\in) := 2$ — that is, '$\in$' is a binary predicate which is interpreted as the "membership" relation. However, for notational convenience, we introduce the following *defined* binary predicates — i.e.[1]

$$(x = y) := (\forall u)(u \in x \leftrightarrow u \in y), \tag{4.1}$$

and

$$x \subseteq y := (\forall u)(u \in x \rightarrow u \in y). \tag{4.2}$$

Additionally, we define

$$(x \neq y) := \neg(x = y) \quad \text{and} \quad x \notin y := \neg(x \in y). \tag{4.3}$$

---

[1]Note that here and in what follows, we move to a more standard notation, using '=' to denote the (defined) predicate "equality" in $\mathcal{L}_{set}$ instead of '≈' as in the previous chapters.

Now, since in what (immediately) follows we will be considering *classical* set theory, we note that for any $\mathcal{L}_{set}$-wff $\psi(x)$, a statement of the form $(\exists x)(\psi(x))$ holding in a model implies the *existence* of some set A in the model such that $\psi(A)$ holds — as such, a statement of the form

$$(\exists x)(\forall u)(u \in x \leftrightarrow \psi(u))$$

which holds in a given model yields a set in that model whose elements are precisely those a's which satisfy $\psi(a)$, and any two sets so defined will be equal by the definition of equality in equation 4.1 above. Given this, and assuming the existence of *some* model of the ZFC axioms presented below, we can define notation to refer to any sets whose existence is guaranteed in any model by the axioms; for convenience, we now do so for certain sets.

Empty Set: We define $\varnothing$ (using ZFC3) to be the set satisfying (for any choice of set x) $(\forall u)(u \in \varnothing \leftrightarrow u \in x \wedge u \neq u)$.

Pairs and Singletons: For any two sets x and y, we define $\{x, y\}$ to be the set satisfying $(\forall u)(u \in \{x, y\} \leftrightarrow u = x \vee u = y)$, and define $\{x\} := \{x, x\}$. These exist by ZFC2 below.

Intersection: For any two sets x and y, we define (using ZFC3) $x \cap y$ to be the set satisfying $(\forall u)(u \in x \cap y \leftrightarrow u \in x \wedge u \in y)$.

Union: For any set x, we define $\bigcup x$ to be the set which satisfies the statement $(\forall u)(u \in \bigcup x \leftrightarrow (\exists z)(u \in z \wedge z \in x))$. This set exists by ZFC4 below. Then for any two sets x and y we define $x \cup y := \bigcup\{x, y\}$.

Power Set: For any set x, we define (using ZFC5) $\mathcal{P}(x)$ to be the set satisfying

$(\forall u)(u \in \mathcal{P}(x) \leftrightarrow u \subseteq x)$.

Set Builder Notation: For any sets x and y and any $\mathcal{L}_{set}$-wff ψ, we define $\{u \in x \ : \ \psi(u, y)\}$ to be that set containing exactly those elements of u for which $\psi(u, y)$ is true. This set exists by ZFC3 below.

Using this notation, we now list the ZFC axioms for classical set theory.

ZFC1 *Extensionality:* $(\forall x)(\forall y)[x = y \rightarrow (\forall z)(x \in z \leftrightarrow y \in z)]$.

ZFC2 *Pairing:* $(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow u = x \vee u = y)$.

ZFC3 *Separation Schema:* For ψ any $\mathcal{L}_{set}$-wff,

$$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u, y)).$$

ZFC4 *Union:* $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))$.

ZFC5 *Power Set:* $(\forall x)(\exists y)(\forall u)(u \in y \leftrightarrow u \subseteq x)$.

ZFC6 *Infinity:* $(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \rightarrow y \cup \{y\} \in x))$.

ZFC7 *Replacement Schema:* For ψ any $\mathcal{L}_{set}$-wff,

$$[(\forall x)(\forall y)(\forall z)[(\psi(x, y) \wedge \psi(x, z)) \rightarrow y = z]]$$
$$\rightarrow (\forall x)(\exists z)(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y, u))].$$

ZFC8 *Regularity:* $(\forall x)(x \neq \varnothing \rightarrow (\exists y)(y \in x \wedge y \cap x = \varnothing))$.

ZFC9 *Choice:*

$$(\forall z)\big([(\forall x)(\forall y)(x \in z \rightarrow x \neq \varnothing) \wedge (x \in z \wedge y \in z \wedge x \neq y \rightarrow x \cap y = \varnothing)]$$

$$\rightarrow (\exists s)(\forall t)[t \in z \rightarrow (\exists u)(s \cap t = \{u\})]\big).$$

We now provide a brief qualitative description of the ZFC axioms listed above. We begin by noting that ordinarily the role of the extensionality axiom is to dictate that *sets are determined by their members.* However, recalling that equality '=' is a defined predicate rather than a predicate in the language $\mathcal{L}_{set}$, the role of extensionality in this presentation of the axioms is simply to provide a basis upon which to *prove* that equality (as defined in equation 4.1) satisfies substitution. (Since this result will be useful, it is marked off as Lemma 4.1 at the end of this section.)

The pairing axiom states that, given any two sets, one can form a new set whose members are exactly the two original sets. The axiom schema of separation states that for any $\mathcal{L}_{set}$-wff ψ and any *class* Z, there exists some *set* u which contains all and only elements of Z which have the property specified by the $\mathcal{L}_{set}$-wff ψ. As for the union, power set, and infinity axioms, we have that the union axiom essentially states that given a collection of sets, one can obtain a new set whose members are exactly the members of the original collection, while the power set axiom states that for any set, the set of all subsets of that set forms a set itself; and the infinity axiom stipulates the existence of at least one infinite set.

We next consider the replacement axiom schema and the regularity axiom, both of which are necessary for essentially technical reasons. The replacement axiom has important proof-theoretic consequences and is necessary for the construction of the higher cardinals (see (14)), while the regularity axiom prevents some counter-intuitive behavior such as sets being elements of themselves (see, e.g., Lemma 4.2).

Finally, the axiom of choice essentially states that for any collection of sets, it is possible to form a new set containing exactly one element from each set in the original collection.[1]

We conclude this section with two useful lemmas; note that the first of these is proven in quantum logic since the proof is the same as in classical logic.

Lemma 4.1. In the language $\mathcal{L}_{set}$, ZFC1 implies that equality '=' as defined in equation 4.1 above satisfies the substitution property, i.e. we have both

$$\text{ZFC1} \vdash (\forall x)(\forall y)[x = y \to (\forall z)(x \in z \leftrightarrow y \in z)]$$

and

$$\text{ZFC1} \vdash (\forall x)(\forall y)[x = y \to (\forall z)(z \in x \leftrightarrow z \in y)].$$

Proof. The second statement follows from the definition of '=' using (Q3) and (R5), and the first is simply a restatement of axiom ZFC1. □

[1]For a brief discussion of the axiom of choice, see appendix B.4, and for a lengthy and in-depth investigation of this controversial axiom, see (18).

Lemma 4.2. In the language $\mathcal{L}_{set}$, the sentence $(\forall x)(x \notin x)$ is classically derivable from the ZFC axioms.

Proof. We know that $\{x\} \neq \varnothing$ since $x \in \{x\}$. Then by ZFC8, we have that there is some $y \in \{x\}$ such that $y \cap \{x\} = \varnothing$. But by definition of the singleton, we know that $y \in \{x\}$ means that $y = x$, and hence $x \cap \{x\} = \varnothing$, so that for any $z \in x$, we know that $z \notin \{x\}$, i.e. that $z \neq x$. Hence $x \notin x$. $\square$

4.2.1 The Classical Universe of Sets

As mentioned previously, one of the major objectives of axiomatic set theory is to precisely capture a particular intended model, which we often, in practice, actually conflate with set theory itself. It is this intended model which we now describe — that is, we construct the *classical universe of sets.*[1]

The construction typically begins with the empty set $\varnothing$, which, for notational purposes is defined to be V_0, and then proceeds inductively — that is, $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, where α runs over all the successor ordinals, and $V_\beta := \bigcup_{\gamma \in \beta} V_\gamma$, where β runs over all the limit ordinals.[2]

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[1]The reader is referred to Enderton (12) or Halmos (15) for a discussion of the relevant concepts, etc. from naive set theory.

[2]Note that this procedure actually requires *transfinite induction.* The reader unfamiliar with ordinal numbers and transfinite induction is referred to Section B.3 of the appendix.

More precisely, let **Ord** be the class of ordinal numbers, and define

$$
\begin{aligned}
V_{\mathbf{0}} &:= \varnothing \\
V_{\alpha+1} &:= \mathcal{P}(V_\alpha) \\
V_\alpha &:= \bigcup_{\beta \in \alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}
\end{aligned}
$$

Then, we have that the classical universe of sets $\mathcal{V}$ is given by

$$\mathcal{V} := \bigcup_{\alpha \in \mathbf{Ord}} V_\alpha \quad \text{where the union is understood to give rise to a proper class.} \tag{4.4}$$

We note that the principle of transfinite induction (see Theorem 40) and the fact that (by Lemma 42) every ordinal is well-ordered by the membership relation, essentially guarantee that each V_α (for $\alpha \in \mathbf{Ord}$) is, in fact, well-defined.

The truth function for this model is characterized by actual "membership" in the V_α's — that is,

$$[\![x \in y]\!] := \begin{cases} 1, & \text{if } x \text{ is a member of } y \\ 0, & \text{if } x \text{ is a } \textit{not} \text{ member of } y. \end{cases}$$

We conclude this section by noting that the entire apparatus of transfinite recursion is actually *built* on set theory, and as such, it is not possible to prove the *existence* of the classical universe — i.e. the intended model of the ZFC axioms — without *assuming* the consistency of the ZFC axioms.[1] Moreover, we are only able to proceed with the construction of models of our quantum set theory if we assume the existence of the classical universe, and thus, assume consistency of the classical ZFC axioms.

4.3 Quantum Axiomatic Set Theory

In this section, we define a reduced version of the ZFC axioms which we will take as the axioms for our quantum set theory. As above, we take the language $\mathcal{L}_{set} = \mathcal{L}_{set}^{P} := \{\in\}$, where $\alpha(\in) := 2$, and the binary predicate '$\in$' is interpreted as the "membership" relation. Also, we continue to use the '$=$','$\neq$', '$\notin$', and '$\subseteq$' as defined previously in equation 4.1 — equation 4.3. We further define

$$(x \doteq y) := (\forall z)(x \in z \leftrightarrow y \in z). \tag{4.5}$$

Additionally, for a given variable y, we will abuse notation slightly, defining

$$\begin{aligned}(x \in y^*) &:= (\exists z)(z \doteq y \wedge x \in z)\\ (y^* \in z) &:= (y \in z).\end{aligned} \tag{4.6}$$

Note that these definitions effectively enable us to treat '$\cdot^*$' as if it were a unary function symbol.[2]

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[1]Of course, a set of axioms is, by definition, consistent if and only if there exists a model for those axioms.

[2]See Definition B.4 in the appendix.

Finally, the formula schema $\mathbf{C}(\psi)$ and $\mathbf{T}(\psi)$ (defined in Section 2.5.2) become, for an $\mathcal{L}_{set}$-wff ψ,

$$\mathbf{C}(\psi) = (\forall s)(\forall t)(\varphi_{s\epsilon t}(\psi) \to \psi) \tag{4.7}$$

and

$$\mathbf{T}(\psi) = (\forall s)(\forall t)(s \in t \to \psi), \tag{4.8}$$

where $\varphi_x(y) = x \vee (\neg x \wedge y)$ denotes the Sasaki projection. Also, recall that $[\![(\exists x)\mathbf{T}(\psi(x))]\!] = 1$ in a model will guarantee (in the presence of other axioms) the actual existence of an a such that $[\![\psi(a)]\!] = 1$ in that model, assuming that the associated truth value algebra is irreducible.[1]

Now, paralleling the discussion in Section 4.2, we define notation to refer to certain sets whose existence will be guaranteed in any model of the quantum set theory axioms RZFC1 — RZFC12 below.

Empty Set: We define (using RZFC3) $\varnothing$ to be the quantum set satisfying (for any choice of set x)

$(\forall u)(u \in \varnothing \leftrightarrow u \neq u \wedge u \in x)$.

Singletons: For any set x, we define $\{x\}$ to be the quantum set satisfying

$(\forall u)(u \in \{x\} \leftrightarrow u^* = x^*)$. This exists by axiom RZFC2.

---

[1]Recall that wlog, we can restrict the truth value algebras to those which are irreducible by the completeness Theorem 5.

Intersection: For any two sets x and y, we define $x \cap y$ to be the quantum set satisfying

$(\forall u)(u \in x \cap y \leftrightarrow u \in x \wedge u^* \in y^*)$, which exists by RZFC3.

Pairwise Union: For any two sets x and y, we define $x \cup y$ to be the quantum set which satisfies

$(\forall u)(u \in x \cup y \leftrightarrow u \in x \vee u \in y)$, which exists by RZFC10.

Set Union: For any set x, we define $\bigcup x$ to be the quantum set which satisfies the statement

$(\forall u)(u \in \bigcup x \leftrightarrow (\exists z)(u \in z \wedge z \in x))$, This set exists by axiom RZFC4.

Power Set: For any set x, we define $\mathcal{P}(x)$ to be the quantum set satisfying

$(\forall u)(u \in \mathcal{P}(x) \leftrightarrow u^* \subseteq x)$, which exists by RZFC5.

Set Builder Notation: For any sets x and y and any $\mathcal{L}_{set}$-wff ψ, we define $\{u \in x \ : \ \psi(u^*, y)\}$

to be that quantum set which satisfies $u \in x \leftrightarrow u \in z \wedge \psi(u^*, y)$, which exists by RZFC3.

Using this notation, we now list the axioms for our quantum set theory — namely, a reduced version of the ZFC axioms which we collectively refer to as the RZFC axioms.

(RZFC1) *Extensionality:* $(\forall x)(\forall y)[\mathbf{T}(x = y) \to (\forall z)(x \in z \leftrightarrow y \in z)]$.

(RZFC2) *Singleton:* $(\forall x)(\exists z)\mathbf{T}\big((\forall u)(u \in z \leftrightarrow u^* = x^*\big)$.

(RZFC3) *Separation Schema:* For ψ any $\mathcal{L}_{set}$-wff,

$$(\forall x)(\forall y)(\exists z)\mathbf{T}\big[(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u^*, y))\big].$$

(RZFC4) *Union:* $(\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))\big]$.

(RZFC5) *Power Set:* $(\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow u^* \subseteq x)\big]$.

(RZFC6) *Infinity:* $(\exists x)\mathbf{T}(\varnothing \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))$.

(RZFC7) *Replacement Schema:* For ψ any $\mathcal{L}_{set}$-wff,

$$[(\forall x)(\forall y)(\forall z)[(\psi(x,y) \wedge \psi(x,z)) \to y = z]]$$
$$\to (\forall x)(\exists z)\mathbf{T}\big[(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y^*, u^*))\big].$$

(RZFC8) *Regularity:* $(\forall x)(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing))$.

(RZFC9) *Choice:*

$$(\forall z)\big([(\forall x)(\forall y)(x \in z \to x \neq \varnothing) \wedge (x \in z \wedge y \in z \wedge x \neq y \to x \cap y = \varnothing)]$$
$$\to (\exists s)\mathbf{T}\big[(\forall t)[t \in z \to (\exists u)(s \cap t = \{u\})]\big]\big).$$

(RZFC10) *Pairwise Union:* $(\forall x)(\forall y)(\exists z)\mathbf{T}[(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)]$.

(RZFC11) $*$*-classicality:* $(\forall x)[(\forall u)\ \mathbf{C}(u \in x) \to x = x^*]$.

(RZFC12) $\mathbf{T}$-*normality:* For ψ any $\mathcal{L}_{set}$-wff, $\mathbf{C}(\mathbf{T}(\psi)) \wedge [\mathbf{T}(\psi) \to \psi]$.

Note that in the RZFC axioms above, the classical pairing axiom from the ZFC axioms has been replaced by the singleton and pairwise union axioms. This change is motivated by the fact that, as we will see in Section 4.4,

$$\bigcup(\{x\} \cup \{y\}) \neq x \cup y$$

in general.

In the lemma below, we demonstrate that equality '=' (as defined in equation 4.1 above) has some properties that we would expect equality to have.

Lemma 4.3. In the language $\mathcal{L}_{set}$, we have that

(i) $\vdash (\forall x)(x = x)$

(ii) $\vdash (\forall x)(\forall y)(x = y \to y = x)$

(iii) $\vdash (\forall x)(\forall y)(\forall z)(x = y \wedge y = z \to x = z)$

(iv) $\vdash (\forall x)(\forall y)[x = y \to (\forall z)(z \in x \leftrightarrow z \in y)]$

(v) RZFC1 $\vdash (\forall x)(\forall y)[\mathbf{T}(x = y) \to (\forall z)(x \in z \leftrightarrow y \in z)]$

Proof. Number (iv) above is already established in Lemma 4.1, and (v) follows trivially, since the statement to be proved *is* RZFC1. To utilize the full power of orthomodular lattice theory, we will prove the remaining items above using the completeness theorem for the quantum logic $\mathcal{Q}(\mathcal{L})$ described in Chapter 2, so that we only need to examine each of the above sentences in an arbitrary $\mathcal{L}_{set}$-structure — to this end let $\hat{A}$ be an $\mathcal{L}_{set}$-structure with truth valuation $[\![\in]\!]$ and underlying class A.[1]

First, considering (i), for any $a \in A$, we have

$$[\![a = a]\!] = [\![(\forall u)(u \in a \leftrightarrow u \in a)]\!] = \bigwedge_{a \in A} \big([\![u \in a]\!] \leftrightarrow [\![u \in a]\!]\big) = 1,$$

[1]We urge the reader to pay careful attention to the '=' which is a defined predicate in $\mathcal{L}_{set}$ and the '=' which represents equality in the truth value algebra of the model.

where the last equality is by Lemma A.12. Then (i) holds by Lemma 2.6 (and the completeness theorem).

Considering (ii), for any $a, b \in A$, we have

$$[\![a = b]\!] = \bigwedge_{c \in A} \Big([\![c \in a]\!] \leftrightarrow [\![c \in b]\!] \Big) = [\![b = a]\!],$$

where the final equality holds by Lemma A.12, so that we have

$$[\![a = b \to b = a]\!] = \Big([\![a = b]\!] \to [\![b = a]\!] \Big) = 1$$

by the same lemma, and so (ii) holds by Lemma 2.6 and completeness as well.

Moving on to (iii), for any $a, b, c \in A$, we have

$$\begin{aligned}
\Big([\![a = b]\!] \wedge [\![b = c]\!] \Big) &= \Big[\bigwedge_{d \in A} \Big([\![d \in a]\!] \leftrightarrow [\![d \in b]\!] \Big) \wedge \bigwedge_{e \in A} \Big([\![e \in b]\!] \leftrightarrow [\![e \in c]\!] \Big) \Big] \\
&= \bigwedge_{f \in A} \Big[\Big([\![f \in a]\!] \leftrightarrow [\![f \in b]\!] \Big) \wedge \Big([\![f \in b]\!] \leftrightarrow [\![f \in c]\!] \Big) \Big] \\
&\leq \bigwedge_{f \in A} \Big([\![f \in a]\!] \leftrightarrow [\![f \in c]\!] \Big) = [\![a = c]\!],
\end{aligned}$$

where the second line is obtained by definition of the greatest lower bound, and the inequality follows from Lemma A.12. Then (iii) holds by Lemma 2.6 and completeness. $\square$

We note that (i)–(iii) in the above lemma are just the standard equality axioms (E1) — (E3), while (iv)-(v) in the above lemma are substitution axioms for the predicate '$\in$'.

We now show that the RZFC axioms are, in the presence of classical logic, equivalent to the ZFC axioms listed in Section 4.2 above.

Theorem 11. The RZFC axioms presented above are a reduction of the ZFC axioms presented in Section 4.2.

Proof. We need to show that the classical and reduced ZFC axioms are equivalent in the presence of the schema (CL) (i.e. using classical logic). RZFC12 is a tautology of classical logic (so is automatically implied by ZFC). RZFC8 is unchanged from the classical axiomatization (so the reduced versions imply the classical versions and vice versa). RZFC1, RZFC3-RZFC7 and RZFC9 now include the '**T**' operator, but this is simply the identity (up to logical equivalence) in classical logic, and so we can ignore this operator for the purposes of proving equivalence.

Axioms RZFC3, RZFC5, and RZFC7 also replace some instances of $x, y, \ldots$ with $x^*, y^*, \ldots$. Clearly, if RZFC11 holds in a model with the standard bivalent truth values, then, since $\mathbf{C}(\psi)$ is satisfied in any such model, we have $x = x^*$, and hence the classical versions of these axioms are logically equivalent to the reduced ones by the substitution property of '=' (see Lemma 4.1). Hence, the RZFC axioms automatically implies the classical axioms ZFC1, ZFC3, ZFC5, and ZFC7 — moreover, if we demonstrate that RZFC11 is implied by the classical ZFC axioms, we obtain the reduced axioms RZFC3, RZFC5, and RZFC7 for free. Hence we only need to show that (in classical logic) ZFC2 follows from the RZFC axioms, as well as that the reduced axioms RZFC2, RZFC10 and RZFC11 follow from the classical ZFC axioms (and the axiom schema (CL)).

The reduced axioms RZFC2 and RZFC10 follow trivially from the classical pairing and union, since we have the classical $\{x\}$ as well as the classical union $x \cup y$. Now, we show that the ZFC axioms of Section 4.2 imply RZFC11 ($*$-classicality). By definition of '$=$', '$u \in x^*$', and '$\doteq$', we have

$$\begin{aligned}(x = x^*) &\leftrightarrow (\forall u)(u \in x \leftrightarrow u \in x^*)\\ &\leftrightarrow (\forall u)[u \in x \leftrightarrow (\exists z)(z \doteq x \wedge u \in z)]\\ &\leftrightarrow (\forall u)[u \in x \leftrightarrow (\exists z)((\forall s)(z \in s \leftrightarrow x \in s) \wedge u \in z)]. \end{aligned} \tag{4.9}$$

Hence, we only need show the double implication in the final line. If $u \in x$, then taking $z = x$, we see that the RHS of the implication in the bottom line holds. Conversely, if the RHS of said implication holds, then by the (classical) pairing axiom, there is some z with[1] $z \in \{x\}$ and $u \in z$, so since $z = x$ this implies $u \in x$. This establishes the double implication, so we see that $x = x^*$ so that RZFC11 holds.

Next, we assume the RZFC axioms, and wish to show that ZFC2 follows. But for arbitrary sets x, y we have the singleton sets[2] $\{x\}$, $\{y\}$ by RZFC2, and hence we have the set $\{x\} \cup \{y\}$

[1]Here $\{x\}$ is the singleton existing by virtue of the *classical* axioms.

[2]Here $\{x\}$ and $\{y\}$ are the singletons existing by virtue of the *reduced* axioms.

by axiom RZFC10, which shows that ZFC2 is indeed implied by the RZFC axioms combined with the schema (CL). □

Theorem 11 above shows that when we restrict ourselves to the standard bivalent truth values, we recover classical (ZFC) set theory.

4.4 Models of Quantum Set Theory

In this section, we describe our intended class of models for quantum set theory, and show that they are indeed models of the RZFC axioms listed in Section 4.3. We note that although we will define candidates for "quantum universes of sets" associated with any orthomodular lattice, not all of these potential "universes" will actually turn out to be models — this is to say that only certain orthomodular lattices L whose properties are similar to those of projection lattices of separable Hilbert spaces will actually satisfy (all of) the RZFC axioms. Additionally, we note that for the ensuing discussion, it is necessary to generalize the notion of *model* defined in Section 2.3 to allow for the underlying *set* in an $\mathcal{L}_{set}$-structure to be extended to a *class*; we also employ the notion of a class function (see Section B in the appendix).

4.4.1 Quantum Universes of Sets

We begin with some preliminary definitions. Let L be an orthomodular lattice, let[1] $\mathfrak{K}$ be a class, and let $f \in L^{\mathfrak{K}}$. The *support of* f (denoted $\sup f$) is given by

$$\sup f := \{k \in \mathfrak{K} : f(k) \neq 0\}. \tag{4.10}$$

[1]Recall that elements of $L^{\mathfrak{K}}$ are maps from $\mathfrak{K}$ to L (see appendix B).

Let $\mathcal{V}$ denote the classical universe of sets. We then define the *L-valued universe (of sets)* (which we denote by $\mathbb{Q}_L$) to be the class[1]

$$\mathbb{Q}_L := \{ f \in L^{\mathcal{V}} \; : \; \sup f \text{ is a (classical) set } \}. \tag{4.11}$$

Further, each $f \in \mathbb{Q}_L$ is called an *L-valued set* or simply *quantum set* if L is clear from the context.[2]

We define the truth function $[\![\in]\!]$ (for L-valued sets f, g) by

$$[\![f \in g]\!] := g(\sup f),$$

and extend this to all $\mathcal{L}_{set}$-wffs in the usual way. The $\mathcal{L}_{set}$-structure $(\mathbb{Q}_L, L, \{[\![\in]\!]\}, \varnothing)$ will be denoted by $\mathcal{Q}_L$. These are the candidates for our intended models of quantum set theory.[3]

4.4.2 Soundness of the Intended Models of Quantum Set Theory

In this section we show that for L a projection lattice of a separable Hilbert space (over $\mathbb{C}$), an $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ as defined above is a model for the RZFC axioms. We note that a number

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[1]Here the "set" builder notation is understood in the sense of classes.

[2]Although the only true models will turn out to be those where L satisfies certain additional properties, we will still call the elements of $\mathbb{Q}_L$ sets since, as will be shown in Section 4.4.2, the L-valued sets for an arbitrary orthomodular lattice L satisfy almost all of the RZFC axioms.

[3]Note that for any $g, h \in \mathbb{Q}_L$ where $\sup g = \sup h$, we have that $g \in f \leftrightarrow h \in f$ (for any $f \in \mathbb{Q}_L$) holds in any $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ defined above. Also note that in $\mathcal{L}_{set}$-structure in which $[\![\in]\!]$ is as defined above, we have that (for any $f \in \mathbb{Q}_L$) $[\![f \in f]\!] = 0$, so that there are no quantum sets (which are even partially) elements of themselves.

of the RZFC axioms actually hold for any complete orthomodular lattice L, while others hold for a larger class of orthomodular lattices than just projection lattices. For each RZFC axiom, the largest class of orthomodular lattices for which we know the axiom holds will be indicated.

We begin with some useful definitions and lemmas. For classical sets $A, B \in \mathcal{V}$, the L-valued set $\delta_A : \mathcal{V} \to L$ is defined by

$$\delta_A(B) := \begin{cases} 1 & \text{if } A = B \\ 0 & \text{if } A \neq B, \end{cases} \tag{4.12}$$

while the L-valued set $\chi_A : \mathcal{V} \to L$ is defined by

$$\chi_A(B) := \begin{cases} 1 & \text{if } B \in A \\ 0 & \text{if } B \notin A. \end{cases} \tag{4.13}$$

Clearly, $\sup \delta_A = \{A\}$ and $\sup \chi_A = A$ for any $A \in \mathcal{V}$. Note that by identifying any classical set A with $\chi_A \in \mathfrak{Q}_L$, we have that the classical sets can be thought of as a subset of the quantum sets associated with $\mathcal{Q}_L$.

Lemma 4.4. Let L be a complete orthomodular lattice, and let $f, g \in \mathfrak{Q}_L$. Then

$$[\![f \doteq g]\!] = \begin{cases} 1 & \text{if } \sup f = \sup g \\ 0 & \text{if } \sup f \neq \sup g \end{cases} \tag{4.14}$$

$$[\![g \in f^*]\!] = \begin{cases} 1 & \text{if } \sup g \in \sup f \\ \\ 0 & \text{if } \sup g \notin \sup f \end{cases} \tag{4.15}$$

$$[\![f^* = g^*]\!] = [\![f \doteq g]\!]. \tag{4.16}$$

Proof. First, using the definition of $f \doteq g$, we have

$$\begin{aligned} [\![f \doteq g]\!] &= [\![(\forall z)(f \in z \leftrightarrow g \in z)]\!] \\ &= \bigwedge_{h \in \mathbb{Q}_L} \big([\![f \in h]\!] \leftrightarrow [\![g \in h]\!]\big) \\ &= \bigwedge_{h \in \mathbb{Q}_L} \big(h(\sup f) \leftrightarrow h(\sup g)\big), \end{aligned}$$

and so we immediately see that if $\sup f = \sup g$, then $h(\sup f) = h(\sup g)$, so that $(h(\sup f) \leftrightarrow h(\sup g)) = 1$ for any $h \in \mathbb{Q}_L$ by Lemma A.12, and so $[\![f \doteq g]\!] = 1$. On the other hand, if $\sup f \neq \sup g$, we have (from equation 4.12 above) that

$$[\![f \doteq g]\!] \leq \big(\delta_{\sup f}(\sup f) \leftrightarrow \delta_{\sup f}(\sup g)\big) = (1 \leftrightarrow 0) = 0,$$

which establishes equation 4.14.

Next, using the definition of $g \in f^*$, we have

$$[\![g \in f^*]\!] = [\![(\exists z)(z \doteq f \wedge g \in z)]\!] = \bigvee_{h \in \mathbb{Q}_L} \big([\![h \doteq f]\!] \wedge [\![g \in h]\!]\big),$$

but from equation 4.14, we have that $[\![h \doteq f]\!]$ is non-zero only when $\sup h = \sup f$, and so only these h's contribute to the join. Hence, we have

$$[\![g \in f^*]\!] = \bigvee_{\substack{h \in \mathbb{Q}_L \\ \sup h = \sup f}} \big(1 \wedge h(\sup g)\big) \geq \chi_{\sup f}(\sup g),$$

so by equation 4.12, we see that if $\sup g \in \sup f$ then $[\![g \in f^*]\!] = 1$. On the other hand, if $\sup g \notin \sup f$, then for any h with $\sup h = \sup f$, we must have $h(\sup g) = 0$, and so $[\![g \in f^*]\!] = 0$, which establishes equation 4.15.

Finally, by definition of '=', and equation 4.15, we have

$$\begin{aligned} [\![f^* = g^*]\!] &= [\![(\forall x)(x \in f^* \leftrightarrow x \in g^*]\!] \\ &= \bigwedge_{h \in \mathbb{Q}_L} [\![h \in f^*]\!] \leftrightarrow [\![h \in g^*]\!]. \end{aligned}$$

Now if $\sup f = \sup g$, by equation 4.15, we have that $([\![h \in f^*]\!] \leftrightarrow [\![h \in g^*]\!]) = 1$ for any $h \in \mathbb{Q}_L$, so that $[\![f^* = g^*]\!] = 1$. If, on the other hand, $\sup f \neq \sup g$, then there exists some classical set A such that $A \in \sup f$ but $A \notin \sup g$ (or vice versa). Either way, for this A, we have (by equation 4.12, equation 4.15, and Lemma A.12) that

$$([\![\chi_A \in f^*]\!] \leftrightarrow [\![\chi_A \in g^*]\!]) = 0,$$

and so $[\![f^* = g^*]\!] = 0$. Hence, by equation 4.14, this establishes equation 4.16. $\square$

Lemma 4.5. Let L be a complete orthomodular lattice, let $f \in \mathbb{Q}_L$, and let $\psi(x)$ be an $\mathcal{L}_{set}$-wff. Then

$$\llbracket \psi(f^*) \rrbracket = \llbracket \psi(\chi_{\sup f}) \rrbracket.$$

Proof. We prove this as a consequence of Lemma 2.7, so we only need show that both $\llbracket f^* \in g \rrbracket = \llbracket \chi_{\sup f} \in g \rrbracket$ and $\llbracket g \in f^* \rrbracket = \llbracket g \in \chi_{\sup f} \rrbracket$ for any two quantum sets f and g. First, for any quantum set g, we have

$$\llbracket f^* \in g \rrbracket = \llbracket f \in g \rrbracket = g(\sup f) = g(\sup \chi_{\sup f}) = \llbracket \chi_{\sup f} \in g \rrbracket$$

by the definition of the expression $f^* \in g$ and $\chi_{\sup f}$. For the other case, we have that $\llbracket g \in f^* \rrbracket = \chi_{\sup f}(\sup g) = \llbracket g \in \chi_{\sup f} \rrbracket$ by Lemma 4.4. □

Theorem 12. Let L be a complete orthomodular lattice. Then $\mathcal{Q}_L$ satisfies RZFC2-RZFC5 as well as RZFC10.

Proof. For RZFC2, we need

$$\mathcal{Q}_L \models (\forall x)\mathbf{T}\big((\exists z)(\forall u)(u \in z \leftrightarrow u^* = x^*\big),$$

i.e. (using Lemma 2.16) it suffices to show that for any quantum set f, there exists some other quantum set g such that

$$\llbracket (\forall u)(u \in g \leftrightarrow u^* = f^*) \rrbracket = 1.$$

Taking $g = \delta_{\sup f}$, we see that

$$[\![(\forall u)(u \in \delta_{\sup f} \leftrightarrow u^* = f^*)]\!] = \bigwedge_{h \in \mathfrak{Q}_L} \left([\![\delta_{\sup f}(\sup h)]\!] \leftrightarrow [\![h^* = f^*]\!]\right) = 1,$$

where the last equality follows from equation 4.12 and equation 4.12, as well as equation 4.16 and equation 4.14 from Lemma 4.4.

To show the RZFC3 holds, we need (for any wff ψ) that

$$\mathcal{Q}_L \vDash (\forall x)(\forall y)(\exists z)\mathbf{T}\big[(\forall u)(u \in z \leftrightarrow u \in x \wedge \psi(u^*, y))\big],$$

and so (again using Lemma 2.16) it suffices to show for any quantum sets f, g that there exists some quantum set h such that

$$[\![(\forall u)(u \in h \leftrightarrow u \in f \wedge \psi(u^*, g))]\!] = 1,$$

i.e. that, for any $j \in \mathfrak{Q}_L$, that $h(\sup j) = f(\sup j) \wedge [\![\psi(j^*, g)]\!]$. But we can simply define, for any $A \in \mathcal{V}$, that

$$h(A) \coloneqq f(A) \wedge [\![\psi(\chi_A, g)]\!].$$

By the (classical) schema of separation, $\sup h$ is indeed a (classical) set (contained in $\sup f$). The result the follows immediately from Lemma 4.5.

Considering RZFC4, we must show

$$\mathcal{Q}_L \vDash (\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow (\exists z)(u \in z \wedge z \in x))\big],$$

and so it is sufficient to find, given an arbitrary quantum sets f, another quantum set g such that

$$\bigwedge_{h \in \mathbb{Q}_L} \Big([\![h \in g]\!] \leftrightarrow \bigvee_{j \in \mathbb{Q}_L} \big([\![h \in j]\!] \wedge [\![j \in f]\!] \big) \Big) = 1.$$

Equivalently, we need to satisfy, for every $h \in \mathbb{Q}_L$, that

$$g(\sup h) = \bigvee_{j \in \mathbb{Q}_L} \big(j(\sup h) \wedge f(\sup j) \big).$$

But clearly to satisfy the previous equation we can simply *define*, for any $A \in \mathcal{V}$,

$$g(A) \coloneqq \bigvee_{j \in \mathbb{Q}_L} \big(j(A) \wedge f(\sup j) \big),$$

and since $g(A) \neq 0$ implies that $A \in \sup j \in \sup f$ for some $j \in \mathbb{Q}_L$, we have that $\sup g$ is indeed a (classical) set, so that RZFC4 is satisfied.

Moving on to RZFC5, we need to show

$$\mathcal{Q}_L \vDash (\forall x)(\exists y)\mathbf{T}\big[(\forall u)(u \in y \leftrightarrow u^* \subseteq x)\big],$$

and so it suffices to demonstrate for any quantum set f, that there is another quantum set g such that

$$[\![(\forall u)(u \in g \leftrightarrow u^* \subseteq f)]\!] = 1.$$

Define g by (for all $A \in \mathcal{V}$)

$$g(A) := [\![\chi_A \subseteq f]\!],$$

and so if $A \in \sup g$, then since

$$[\![\chi_A \subseteq f]\!] = \bigwedge_{h \in \mathbb{Q}_L} \big(\chi_A(\sup h) \to f(\sup h)\big) = \bigwedge_{B \in A} f(B)$$

(using Lemma A.12), we must have $A \subseteq \sup f$, so that $\sup g \subseteq \mathcal{P}(\sup f)$, which shows that $\sup g$ is a (classical) set. Then we have

$$\begin{aligned}[\![(\forall u)(u \in g \leftrightarrow u^* \subseteq f)]\!] &= \bigwedge_{j \in \mathbb{Q}_L} \big([\![j \in g]\!] \leftrightarrow [\![j^* \subseteq f]\!]\big) \\ &= \bigwedge_{j \in \mathbb{Q}_L} \Big(g(\sup j) \leftrightarrow [\![\chi_{\sup j} \subseteq f]\!]\Big) = 1,\end{aligned}$$

where we have used the definition of g and Lemmas 4.5 and A.12 to obtain the final equality.

Considering now RZFC10, we must show that

$$\mathcal{Q}_L \vDash (\forall x)(\forall y)(\exists z)\mathbf{T}[(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)],$$

and so it suffices to show that for quantum sets f, g that there exists some quantum set h such that

$$[\![(\forall u)(u \in h \leftrightarrow u \in f \vee u \in g)]\!] = 1.$$

Define h by (for any $A \in \mathcal{V}$)

$$h(A) := f(A) \vee g(A),$$

and since clearly $\sup h = \sup f \cup \sup g$, we see that $\sup h$ is indeed a (classical) set. Then we have

$$\begin{aligned}[\![(\forall u)(u \in z \leftrightarrow u \in x \vee u \in y)]\!] &= \bigwedge_{j \in \mathcal{Q}_L} \Big([\![j \in h]\!] \leftrightarrow \big([\![j \in f]\!] \vee [\![j \in g]\!] \big) \Big) \\ &= \bigwedge_{j \in \mathcal{Q}_L} \Big(h(\sup j) \leftrightarrow \big(f(\sup j) \vee g(\sup j) \big) \Big) = 1\end{aligned}$$

by the definition of h and Lemma A.12. □

We now consider axioms RZFC1 and RZFC12.

Theorem 13. Let L be a complete irreducible orthomodular lattice satisfying the relative center property.[1] Then $\mathcal{Q}_L$ satisfies RZFC1 and RZFC12.

[1]If unfamiliar, see Definition A.32 in the appendix.

Proof. First we consider RZFC12. Since L has the relative center property, by Lemma A.18, we have, for any evaluated $\mathcal{L}_{set}$-wff ψ, that $\bigwedge_{b\in L}(b \to [\![\psi]\!])$ is in the center of L. But

$$\bigwedge_{b\in L}(b \to [\![\psi]\!]) = \bigwedge_{f,g\in\mathbb{Q}_L}\left([\![f \in g]\!] \to [\![\psi]\!]\right) = [\![\mathbf{T}(\psi)]\!],$$

so that $[\![\mathbf{C}(\mathbf{T}(\psi))]\!] = 1$ by Lemma 2.15 (using that $[\![\varnothing \in \{\varnothing\}]\!] = 1$, for instance). Also, $[\![\mathbf{T}(\psi)]\!] \to [\![\psi]\!] = 1$ by Lemma 2.17. Hence

$$\mathcal{Q}_L \vDash \mathbf{C}(\mathbf{T}(\psi)) \wedge [\mathbf{T}(\psi) \to \psi].$$

Next, considering RZFC1, we must show

$$\mathcal{Q}_L \vDash (\forall x)(\forall y)[\mathbf{T}(x = y) \to x \doteq y],$$

i.e. we must show (by Lemma A.12) for any quantum sets f, g that

$$[\![\mathbf{T}(f = g)]\!] \leq [\![f \doteq g]\!].$$

By the above, $\mathcal{Q}_L \vDash$ RZFC12, so that $[\![\mathbf{T}(f = g)]\!] \in \{0, 1\}$ by Lemma 2.18. The statement trivially holds whenever $[\![f = g]\!] \neq 1$ (since then $[\![\mathbf{T}(f = g)]\!] = 0$ by the aforementioned lemma), so consider the case where $[\![f = g]\!] = 1$. By the definition of '=' this means that $f(A) = g(A)$ for all $A \in \mathcal{V}$, and in particular $\sup f = \sup g$. A simple computation then yields $[\![f \doteq g]\!] = 1$, so that RZFC1 indeed holds in $\mathcal{Q}_L$. This follows trivially from equation 4.16 in Lemma 4.4.

$\square$

Now that we know that axioms RZFC1 and RZFC12, which guarantee the existence of certain quantum sets, hold in $\mathcal{L}_{set}$-structures $\mathcal{Q}_L$ when L satisfies the relative center property, it is safe to make use of the notation defined in Section 4.3 which is utilized in what follows.

Lemma 4.6. Let L be a complete irreducible orthomodular lattice satisfying the relative center property. Then, for $f, g \in \mathbb{Q}_L$, and any $A \in \mathcal{V}$

(i) $\varnothing(A) = 0$

(ii) $\{f\}(A) = \delta_{\sup f}(A)$

(iii) $(f \cap g)(A) = f(A) \wedge g(A)$

(iv) $(f \cup g)(A) = f(A) \vee g(A)$

(v) $f(\sup f) = 0$

(vi) $(f \cup \{f\})(A) = \begin{cases} 1 & \text{if } A = \sup f \\ f(A) & \text{if } A \in \sup f \\ 0 & \text{otherwise.} \end{cases}$

(vii) $\mathcal{P}(f)(A) = \bigwedge_{B \in A} f(B)$

(viii) $\bigcup f(A) = \bigvee_{\substack{B \in \sup f \\ A \in B}} f(B)$

Proof. For number (i) above, the empty set is defined to satisfy $f \in \varnothing \leftrightarrow (f \neq f)$, but

$$[\![f = f]\!] = \bigwedge_{h \in \mathbb{Q}_L} ([\![h \in f]\!] \leftrightarrow [\![h \in f]\!]) = 1,$$

so $[\![f \in \varnothing]\!] = \varnothing(\sup f) = 0$ for all $f \in \mathbb{Q}_L$, hence $\varnothing(A) = 0$ for all $A \in \mathcal{V}$.

Numbers (ii)–(iv) above follow directly from the proof of Theorem 12, extensionality and the fact that $[\![g \in f]\!] := f(\sup g)$.

For number (v) above, we note that by the classical axiom of regularity no set may be a member of itself (Lemma 4.2), and so (v) follows from the fact that $\sup f \notin \sup f$.

For number (vi) above we note that the cases are mutually exclusive by (v) above. We then use (iv) and (ii) above to compute

$$f \cup \{f\}(A) = f(A) \vee \{f\}(A) = f(A) \vee \delta_{\sup f}(A),$$

which gives the above result.

For numbers (vii) and (viii), since the sets which are given by RZFC4 and RZFC5 are unique, from the proof of Lemma 4.2, we know that

$$\mathcal{P}(f)(A) = [\![\chi_A \subseteq f]\!] = \bigwedge_{h \in \mathbb{Q}_L} \big(\chi_A(\sup h) \to f(\sup h)\big) = \bigwedge_{B \in A} f(B),$$

as well as that

$$\bigcup f(A) = \bigvee_{j \in \mathbb{Q}_L} \big(j(A) \wedge f(\sup j)\big) = \bigvee_{B \in \mathcal{V}} \big(\chi_B(A) \wedge f(B)\big) = \bigvee_{\substack{B \in \sup f \\ A \in B}} f(B).$$

$\square$

We are now in a position to show there are quantum sets f and g such that $\bigcup\big(\{f\}\cup\{g\}\big) \neq f\cup g$. From the above we know that for any $A\in\mathcal{V}$ that $(f\cup g)(A) = f(A)\vee g(A)$. However

$$\begin{aligned}\bigcup\big(\{f\}\cup\{g\}\big)(A) &= \bigvee_{\substack{B\in\sup(\{f\}\cup\{g\})\\ A\in B}} \big(\{f\}\cup\{g\}\big)(B)\\ &= \bigvee_{\substack{B\in\{\sup f,\sup g\}\\ A\in B}} \big(\delta_{\sup f}(B)\vee\delta_{\sup g}(B)\big)\\ &= \begin{cases} 1 & \text{if } A\in\sup f\cup\sup g\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

where we have used that

$$\big(\{f\}\cup\{g\}\big)(A) = \delta_{\sup f}(A)\vee\delta_{\sup g}(A),$$

so that $\sup(\{f\}\cup\{g\}) = \{\sup f,\sup g\}$. Hence, we clearly have $\bigcup\big(\{f\}\cup\{g\}\big)\neq f\cup g$ for any quantum sets f,g such that there is some $A\in\mathcal{V}$ with $f(A)\vee g(A)\neq 1$.

Lemma 4.7. Let L be a complete orthomodular lattice, and let ω be the first infinite ordinal. Also, for any $f\in\mathbb{Q}_L$ define (for any $A\in\mathcal{V}$)

$$f^{+}(A) := \begin{cases} 1 & \text{if } A=\sup f\\ f(A) & \text{if } A\in\sup f\\ 0 & \text{otherwise,}\end{cases} \tag{4.17}$$

and define $g_\varnothing \in \mathbb{Q}_L$ by $g_\varnothing(A) := 0$ for every $A \in \mathcal{V}$. Further, for any $f, g \in \mathbb{Q}_L$ assume[1] that $f \cap g \in \mathbb{Q}_L$ which satisfies (iii) in Lemma 4.6 exists.

Then we have

(i) $[\![g_\varnothing \in \chi_\omega]\!] \wedge \bigwedge_{h \in \mathbb{Q}_L} [\![(h \in \chi_\omega \to h^+ \in \chi_\omega)]\!] = 1$

(ii) If $L \neq \{0,1\}$, then

$$\bigwedge_{h \in \mathbb{Q}_L} \Big([\![h \neq g_\varnothing]\!] \to \bigvee_{j \in \mathbb{Q}_L} \big[[\![j \in h]\!] \wedge [\![j \cap h = g_\varnothing]\!] \big] \Big) = 1$$

(iii) If $L \neq \{0,1\}$, then for any $f \in \mathbb{Q}_L$ and any $\mathcal{L}_{set}$-wff $\psi(s,t)$ (with $s, t \in \mathcal{B}_V$)

$$[\![(\forall x)(x \in f \to x \neq g_\varnothing) \to (\exists s)(\forall t)(t \in f \to \psi(s,t))]\!] = 1.$$

Proof. First we consider (i) above. Note that $\varnothing \in \omega$ and for any $\alpha \in \omega$, we have $\alpha + 1 = \alpha \cup \{\alpha\} \in \omega$. Also, for any quantum set h, we have that $\sup h^+ = \sup h \cup \{\sup h\}$. Then, using that $\sup g_\varnothing = \varnothing$, we compute

$$\begin{aligned} [\![g_\varnothing \in \chi_\omega]\!] \wedge \bigwedge_{h \in \mathbb{Q}_L} [\![(h \in \chi_\omega \to h^+ \in \chi_\omega)]\!] &= \chi_\omega(\varnothing) \wedge \bigwedge_{h \in \mathbb{Q}_L} [\chi_\omega(\sup h) \to \chi_\omega(\sup h^+)] \\ &= 1 \wedge \bigwedge_{A \in \mathcal{V}} [\chi_\omega(A) \to \chi_\omega(A \cup \{A\})] = 1, \end{aligned}$$

---

[1] If $f \cap g$ exists as a consequence of the RZFC axioms (such as whenever L satisfies the relative center property by Lemma 4.6), then the assumption is not necessary. Otherwise we can simply take this assumption as the *definition* of $f \cap g$ for purposes of this lemma.

which establishes (i).

Considering (ii) above, for any quantum sets j and h we have (using Lemma A.12)

$$[\![h \neq g_\varnothing]\!] = \neg \bigwedge_{k \in \mathbb{Q}_L} \big(h(\sup k) \leftrightarrow g_\varnothing(\sup k)\big) = \bigvee_{A \in \mathcal{V}} \neg(h(A) \leftrightarrow 0) = \bigvee_{A \in \mathcal{V}} h(A),$$

as well as

$$[\![j \cap h = g_\varnothing]\!] = \bigwedge_{k \in \mathbb{Q}_L} \big([j(\sup k) \wedge h(\sup k)] \leftrightarrow g_\varnothing(\sup k)\big) = \bigwedge_{B \in \mathcal{V}} \big(\neg j(B) \vee \neg h(B)\big).$$

Using these then yields

$$\begin{aligned}\bigwedge_{h \in \mathbb{Q}_L} \Big([\![h \neq g_\varnothing]\!] \rightarrow \bigvee_{j \in \mathbb{Q}_L} \big[[\![j \in h]\!] \wedge [\![j \cap h = g_\varnothing]\!]\big]\Big) \\ = \bigwedge_{h \in \mathbb{Q}_L} \Big(\Big[\bigvee_{A \in \mathcal{V}} h(A)\Big] \rightarrow \bigvee_{j \in \mathbb{Q}_L} \Big[h(\sup j) \wedge \bigwedge_{B \in \mathcal{V}} \big(\neg j(B) \vee \neg h(B)\big)\Big]\Big).\end{aligned}$$

By Lemma A.12, it suffices to show that for any quantum set h, we have

$$\bigvee_{A \in \mathcal{V}} h(A) \leq \bigvee_{j \in \mathbb{Q}_L} \Big[h(\sup j) \wedge \bigwedge_{B \in \mathcal{V}} \big(\neg j(B) \vee \neg h(B)\big)\Big],$$

and hence it is sufficient to show that, for any $A \in \mathcal{V}$, that

$$h(A) \leq \bigvee_{\substack{j \in \mathbb{Q}_L \\ \sup j = A}} \Big[h(\sup j) \wedge \bigwedge_{B \in \sup h \cap \sup j} \big(\neg j(B) \vee \neg h(B) \big) \Big]. \tag{4.18}$$

Now we proceed by cases. First, the case $h(A) = 0$ is trivial, 0 is the bottom element of L (this is always the case if L is trivial). Next, if $A = \varnothing$, then the inequality in equation 4.18 is satisfied since any quantum set j such that $\sup j = \varnothing$ also satisfies $\sup j \cap \sup h = \varnothing$, so that

$$\bigvee_{\substack{j \in \mathbb{Q}_L \\ \sup j = \varnothing}} \Big[h(\sup j) \wedge \bigwedge_{B \in \sup h \cap \sup j} \big(\neg j(B) \vee \neg h(B) \big) \Big] = h(\varnothing).$$

For the third case, assume $A \neq \varnothing$ and also that $h(A) \neq 0$ and $h(A) \neq 1$. Then define a quantum set j_0 by $j_0(B) := \neg h(A)$ for every $B \in \sup h \cap A$, $j_0(B) := 1$ for every $B \in A \backslash \sup h$, and $j_0(B) := 0$ otherwise. Then $\sup j_0 = A$ since $h(A) \neq 1$. Then the inequality in equation 4.18 is satisfied, since we have

$$\bigwedge_{B \in \sup h \cap \sup j_0} \big(\neg j(B) \vee \neg h(B) \big) = \bigwedge_{B \in \sup h \cap A} \big(h(A) \vee \neg h(B) \big) \geq h(A).$$

For the final case, we consider $A \neq \varnothing$, and $h(A) = 1$. Since $L \neq \{0, 1\}$ (and L non-trivial), there exists some $a \in L$ with $a \notin \{0, 1\}$. Define $j_1(B) := a$, and $j_2(B) := \neg a$ for all $B \in \sup h \cap A$, and $j_1(B) := j_2(B) := 1$ for all $B \in A \setminus \sup h$, and $j_1(B) := j_2(B) = 0$ otherwise, so that $\sup j_1 = \sup j_2 = A$. Then the inequality in equation 4.18 is satisfied, since

$$\bigwedge_{B \in \sup h \cap \sup j_1} \big(\neg j_1(B) \vee \neg h(B)\big) = \bigwedge_{B \in \sup h \cap A} \big(\neg a \vee h(B)\big),$$

and

$$\bigwedge_{B \in \sup h \cap \sup j_2} \big(\neg j_2(B) \vee \neg h(B)\big) = \bigwedge_{B \in \sup h \cap A} \big(a \vee h(B)\big),$$

so that

$$\begin{aligned} &\bigvee_{\substack{j \in \mathbb{Q}_L \\ \sup j = A}} \Big[h(\sup j) \wedge \bigwedge_{B \in \sup h \cap \sup j} \big(\neg j(B) \vee \neg h(B)\big)\Big] \\ &\qquad \geq \Big[\bigwedge_{B \in \sup h \cap A} \big(\neg a \vee h(B)\big)\Big] \vee \Big[\bigwedge_{B \in \sup h \cap A} \big(a \vee h(B)\big)\Big] \geq \neg a \vee a = 1 \end{aligned}$$

which establishes (ii).

Finally, we consider (iii) above. First, by Lemma A.12, it suffices to show

$$[\![(\forall x)(x \in f \to x \neq g_{\varnothing})]\!] \leq [\![(\exists s)(\forall t)(t \in f \to \psi(s,t))]\!]. \tag{4.19}$$

Computing the LHS of the inequality in equation 4.19 yields

$$
\begin{aligned}
[\![(\forall x)(x \in f \to x \neq g_{\varnothing})]\!] &= \bigwedge_{h \in \mathbb{Q}_L} \big([\![h \in f]\!] \to [\![h \neq g_{\varnothing}]\!] \big) \\
&= \bigwedge_{h \in \mathbb{Q}_L} \big(f(\sup h) \to \big[\bigvee_{A \in \mathcal{V}} \neg h(A) \big] \big) \\
&= \bigwedge_{B \in \mathcal{V}} \bigwedge_{\substack{h \in \mathbb{Q}_L \\ \sup h = B}} \big(f(B) \to \big[\bigvee_{A \in B} \neg h(A) \big] \big).
\end{aligned}
$$

Then considering the RHS of the inequality in equation 4.19 we have

$$
\begin{aligned}
[\![(\exists s)(\forall t)(t \in f \to \psi(s,t))]\!] &= \bigvee_{h \in \mathbb{Q}_L} \Big[\bigwedge_{j \in \mathbb{Q}_L} \big([\![j \in f]\!] \to [\![\psi(h,j)]\!] \big) \Big] \\
&= \bigvee_{h \in \mathbb{Q}_L} \Big[\bigwedge_{j \in \mathbb{Q}_L} \Big(\neg f(\sup j) \vee \big(f(\sup j) \wedge [\![\psi(h,j)]\!] \big) \Big) \Big] \\
&\geq \bigwedge_{j \in \mathbb{Q}_L} \neg f(\sup j) = \bigwedge_{B \in \mathcal{V}} \neg f(B),
\end{aligned}
$$

and so it suffices to show that, for any $B \in \mathcal{V}$, that

$$
\bigwedge_{\substack{h \in \mathbb{Q}_L \\ \sup h = B}} \big(f(B) \to \big[\bigvee_{A \in B} \neg h(A) \big] \big) \leq \neg f(B). \tag{4.20}
$$

If $f(B) = 0$, then $\neg f(B) = 1$, and so the inequality in equation 4.20 is automatically satisfied. If $f(B) \neq 0$, define $h_0(A) := f(B)$ if $A \in B$, and $h_0(A) = 0$ otherwise, so $\sup h_0 = B$. Then

$$\begin{aligned} \bigwedge_{\substack{h \in \mathfrak{Q}_L \\ \sup h = B}} \Big(f(B) \to \big[\bigvee_{A \in B} \neg h(A)\big]\Big) &\leq \Big(f(B) \to \big[\bigvee_{A \in B} \neg h_0(A)\big]\Big) \\ &= \big(f(B) \to \neg f(B)\big) = \neg f(B) \end{aligned}$$

by Lemma A.12, and so (iii) is established.

$\square$

We now use the above lemma to establish the following theorem.

Theorem 14. Let L be a complete irreducible orthomodular lattice which satisfies the relative center property. Then the $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ satisfies RZFC6, RZFC8, and RZFC9.

Proof. First, we see that $\mathcal{Q}_L \models$ RZFC6 iff

$$[\![(\exists x)\mathbf{T}(\varnothing \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))]\!] = 1,$$

but

$$\begin{aligned} [\![(\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))]\!] &= \bigvee_{j \in \mathfrak{Q}_L} \mathbf{T}\Big([\![g_\varnothing \in j]\!] \wedge \bigwedge_{h \in \mathfrak{Q}_L} [\![(h \in j \to h' \in j)\Big) \\ &\geq [\![g_\varnothing \in \chi_\omega]\!] \wedge \bigwedge_{h \in \mathfrak{Q}_L} [\![(h \in \chi_\omega \to h' \in \chi_\omega)]\!] \\ &= 1 \end{aligned}$$

where in the first equality we have used that $g_\varnothing = \varnothing$ (as quantum sets) as well as that $h' = h \cup \{h\}$ (by Lemma 4.6), and the final equality follows immediately from (i) in Lemma 4.7.

Now, if $L = \{0,1\}$, then our model is just the classical universe, and so RZFC8 and RZFC9 follow by Theorem 11, and so we need only consider the case in which $L \neq \{0,1\}$. Also, the axioms trivially hold if L is trivial, so we assume L non-trivial as well.

To show that $\mathcal{Q}_L \vDash$ RZFC8, we need to show

$$[\![(\forall x)(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing))]\!] = 1,$$

and so we compute (using that $g_\varnothing = \varnothing$)

$$\begin{aligned}&[\![(\forall x)(x \neq \varnothing \to (\exists y)(y \in x \wedge y \cap x = \varnothing))]\!] \\ &\qquad = \bigwedge_{h \in \mathfrak{Q}_L} \Big([\![h \neq g_\varnothing]\!] \to \bigvee_{j \in \mathfrak{Q}_L} \big[[\![j \in h]\!] \wedge [\![j \cap h = g_\varnothing]\!]\big]\Big) = 1\end{aligned}$$

where the last equality follows from (ii) in Lemma 4.7.

Finally, we see that $\mathcal{Q}_L \vDash$RZFC9 iff

$$\begin{aligned}&[\![(\forall z)\big([(\forall x)(\forall y)(x \in z \to x \neq \varnothing) \wedge (x \in z \wedge y \in z \wedge x \neq y \to x \cap y = \varnothing)] \\ &\to (\exists s)\mathbf{T}\big[(\forall t)[t \in z \to (\exists u)(s \cap t = \{u\})]\big]\big)]\!] = 1,\end{aligned}$$

so it suffices to show (by Lemmas A.12 and 2.16, and since $\varnothing = g_\varnothing$), for any quantum set f, that

$$\llbracket (\forall x)(\forall y)(x \in f \to x \neq g_\varnothing) \wedge (x \in f \wedge y \in f \wedge x \neq y \to x \cap y = g_\varnothing) \rrbracket$$
$$\leq \llbracket (\exists s)(\forall t)[t \in f \to (\exists u)(s \cap t = \{u\})] \rrbracket. \tag{4.21}$$

However, trivially we have that

$$\llbracket (\forall x)(\forall y)(x \in f \to x \neq g_\varnothing) \wedge (x \in f \wedge y \in f \wedge x \neq y \to x \cap y = \varnothing) \rrbracket$$
$$\leq \llbracket (\forall x)(\forall y)(x \in f \to x \neq g_\varnothing) \rrbracket. \tag{4.22}$$

Define the $\mathcal{L}_{set}$-wff $\psi(s,t) := (\exists u)(s \cap t = \{u\})$, and then by Lemma 4.7, we have

$$\llbracket (\forall x)(x \in f \to x \neq g_\varnothing) \to (\exists s)(\forall t)(t \in f \to \psi(s,t)) \rrbracket = 1,$$

which by Lemma A.12 is true iff

$$\llbracket (\forall x)(x \in f \to x \neq g_\varnothing) \rrbracket \leq \llbracket (\exists s)(\forall t)(t \in f \to \psi(s,t)) \rrbracket. \tag{4.23}$$

Combining the inequalities in equation 4.22 and equation 4.23 yields the inequality in equation 4.21, which establishes RZFC9. $\square$

We note that it is only in the case in which $L = \{0,1\}$ that we actually need to use the fact that the classical axiom of regularity (respectively, choice) holds in order to establish that the reduced axiom of regularity (respectively, choice) holds — this is to say that when $L \neq \{0,1\}$, the proof given here is independent of whether the classical universe satisfies the regularity (respectively, choice) axiom.

Finally, we consider the remaining two RZFC axioms — namely, RZFC7 and RZFC11, noting that the proofs of these axioms are slightly more subtle.

Lemma 4.8. Let L be a complete orthomodular lattice, and let G be the group of continuous (ortholattice) automorphisms of L. Further let $\psi(x_1,\ldots,x_n)$ be an $\mathcal{L}_{set}$-wff and let $f, f_1, \ldots, f_n \in \mathfrak{Q}_L$. Then for any $\alpha \in G$ we have

(i) $\alpha(0) = 0$ and $\alpha(1) = 1$.

(ii) $\sup \alpha \circ f = \sup f$

(iii) $\mathfrak{Q}_L = \{\alpha \circ g \ : \ g \in \mathfrak{Q}_L\}$

(iv) $\alpha\big([\![\psi(f_1,\ldots,f_n)]\!]\big) = [\![\psi(\alpha \circ f_1,\ldots,\alpha \circ f_n)]\!]$.

Proof. First, since α is an ortholattice automorphism on L, we have that $\alpha(0) = 0$ and $\alpha(1) = 1$ by definition.

Next, since $\alpha(0) = 0$ by (i) above, we know that for any $A \in \mathcal{V}$, $\alpha \circ f(A) = 0$ iff $f(A) = 0$, and hence $\sup \alpha \circ f = \sup f$.

For (iii) above, note that if g is a quantum set, then so is $\alpha \circ g$. Also, this trivially means that $\alpha^{-1} \circ g$ is a quantum set. Then $g = \alpha \circ (\alpha^{-1} \circ g)$, establishing the desired equality.

We then prove (iv) above by induction on the construction of evaluated $\mathcal{L}_{set}$-wffs. For the base case, we see that (for any quantum sets g, h)

$$\alpha\big(\llbracket g \in h \rrbracket\big) = \alpha(h(\sup g)) = \alpha \circ h(\sup \alpha \circ g) = \llbracket \alpha \circ g \in \alpha \circ h \rrbracket,$$

where we have used (ii) above. For the inductive steps, consider evaluated $\mathcal{L}_{set}$-wffs $\psi(g_1, \ldots, g_m)$ and $\xi(g_1, \ldots, g_m)$ with quantum sets $g_1, \ldots, g_m$ for which (iv) holds. Then we have

$$\alpha\big(\llbracket \neg\psi(g_1, \ldots, g_m) \rrbracket\big) = \neg\alpha\big(\llbracket \psi(g_1, \ldots, g_m) \rrbracket\big) = \llbracket \neg\psi(\alpha \circ g_1, \ldots, \alpha \circ g_m) \rrbracket,$$

as well as

$$\begin{aligned} \alpha\big(\llbracket \psi(g_1, \ldots, g_n) \rrbracket \wedge \llbracket \xi(g_1, \ldots, g_m) \big) &= \llbracket \psi(\alpha \circ g_1, \ldots, \alpha \circ g_m) \wedge \xi(\alpha \circ g_1, \ldots, \alpha \circ g_m) \rrbracket \\ &= \llbracket (\psi \wedge \chi)(\alpha \circ g_1, \ldots, \alpha \circ g_m) \rrbracket, \end{aligned}$$

and finally

$$\begin{aligned}\alpha\big(\llbracket(\forall x)\psi(x,g_2,\dots,g_m)\rrbracket\big) &= \bigwedge_{h\in\mathbb{Q}_L}\alpha\big(\llbracket\psi(h,g_2,\dots,g_m)\rrbracket\big)\\ &= \bigwedge_{h\in\mathbb{Q}_L}\llbracket\psi(\alpha\circ h,\alpha\circ g_2,\dots,\alpha\circ g_m)\rrbracket\\ &= \bigwedge_{h\in\mathbb{Q}_L}\llbracket\psi(h,\alpha\circ g_2,\dots,\alpha\circ g_m)\rrbracket\\ &= \llbracket(\forall x)\psi(x,\alpha\circ g_2,\dots,\alpha\circ g_m)\rrbracket,\end{aligned}$$

where the second to last equality follows by (iii) above. □

Lemma 4.9. Let L be a complete orthomodular lattice which is rotatable.[1] Then for any $A, B \in \mathcal{V}$, and any $\mathcal{L}_{set}$-wff $\psi(x,y)$, we have $\llbracket\psi(\chi_A,\chi_B)\rrbracket \in \{0,1\}$.

Proof. Since 0 and 1 are the only fixed points of the group of continuous automorphisms on L (where we denote this group G), it suffices to show that $\llbracket\psi(\chi_A,\chi_B)\rrbracket$ is a fixed point of G. Define $a := \llbracket\psi(\chi_A,\chi_B)\rrbracket$, so by Lemma 4.8 we have

$$\alpha(a) = \alpha\big(\llbracket\psi(\chi_A,\chi_B)\rrbracket\big) = \llbracket\psi(\alpha\circ\chi_A,\alpha\circ\chi_B)\rrbracket = \llbracket\psi(\chi_A,\chi_B)\rrbracket = a,$$

where we have used the following fact: for any $Y \in \mathcal{V}$ and $\alpha \in G$, we have $\alpha\circ\chi_Y = \chi_Y$. To see this, note that for any $Y, Z \in \mathcal{V}$, we have that $\chi_Y(Z) \in \{0,1\}$ by definition, and so $\alpha\circ\chi_Y(Z) = \chi_Y(Z)$ for any $Z \in \mathcal{V}$ by Lemma 4.8. Hence $\alpha\circ\chi_Y = \chi_Y$. □

---

[1]If unfamiliar, see Definition A.33 in the appendix.

Theorem 15. Let L be a complete orthomodular lattice which is rotatable. Then the $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ satisfies RZFC7.

Proof. We need to show

$$\begin{aligned}\mathcal{Q}_L \vDash & \big[(\forall x)(\forall y)(\forall z)(\psi(x,y) \wedge \psi(x,z) \to y = z)\big] \\ & \to (\forall x)(\exists z)\mathbf{T}\big[(\forall u)[u \in z \leftrightarrow (\exists y)(y \in x \wedge \psi(y^*, u^*))\big].\end{aligned}$$

It suffices to assume that

$$[\![(\forall x)(\forall y)(\forall z)(\psi(x,y) \wedge \psi(x,z) \to y = z)]\!] \neq 0,$$

and then prove that, for any quantum set f, there is some other quantum set g such that

$$[\![(\forall u)\big(u \in g \leftrightarrow (\exists y)[y \in f \wedge \psi(y, u^*)]\big)]\!] = 1,$$

i.e. that for any quantum set h

$$g(\sup h) = \bigvee_{j \in \mathbb{Q}_L} \big(f(\sup j) \wedge [\![\psi(j^*, h^*)]\!]\big).$$

But this will be automatically be satisfied (by Lemma 4.5) if we define, for any $A \in \mathcal{V}$,

$$g(A) := \bigvee_{B \in \mathcal{V}} \big(f(B) \wedge [\![\psi(\chi_B, \chi_A)]\!]\big) = \bigvee_{j \in \mathbb{Q}_L} \big(f(\sup j) \wedge [\![\psi(\chi_{\sup j}, \chi_A)]\!]\big),$$

and so we only need show that $\sup g$ is a (classical) set.

Now, we see immediately that, for any $A \in \mathcal{V}$, that $g(A) \neq 0$ iff there is some $B \in \mathcal{V}$ such that $f(B) \wedge [\![\psi(\chi_B, \chi_A)]\!] \neq 0$. But by Lemma 4.9, $[\![\psi(\chi_B, \chi_A)]\!] \in \{0, 1\}$, and so in order for $g(A) \neq 0$ there must exist some $B \in \mathcal{V}$ such that $B \in \sup f$ and $[\![\psi(\chi_B, \chi_A)]\!] = 1$.

We will show that, for any given A, the class of all sets B satisfying both $B \in \sup f$ as well as $[\![\psi(\chi_B, \chi_A)]\!] = 1$ is indeed a (classical) set using the *classical* replacement axiom with regard to the statement $\Psi(B, A)$ which states[1] that "$[\![\psi(\chi_B, \chi_A)]\!] = 1$".

To show that Ψ satisfies the hypothesis of *classical* replacement, we assume that, for generic $X, Y, Z \in \mathcal{V}$, that $\Psi(X, Y)$ and $\Psi(X, Z)$ are true, i.e. that both $[\![\psi(\chi_X, \chi_Y)]\!] = 1$ and $[\![\psi(\chi_X, \chi_Z)]\!] = 1$. Now recall our assumption that

$$[\![(\forall x)(\forall y)(\forall z)(\psi(x, y) \wedge \psi(x, z) \rightarrow y = z)]\!] \neq 0,$$

which implies that,

$$[\![(\psi(\chi_X, \chi_Y)]\!] \wedge [\![\psi(\chi_X, \chi_Z)]\!] \rightarrow [\![Y = Z]\!] \neq 0,$$

but since $\Psi(X, Y)$ and $\Psi(X, Z)$ are true, this means (using Lemma A.12 that

$$\left(1 \rightarrow [\![\chi_Y = \chi_Z]\!]\right) = [\![\chi_Y = \chi_Z]\!] \neq 0,$$

---

[1] We are being rather informal here, since we have not used a formal language when using the classical set theory by which we defined our $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$. However, it would be simple (although tedious) to write every statement in dealing with the construction of $\mathcal{Q}_L$ formally, in which case we would arrive at the statement "$[\![\psi(\chi_B, \chi_A)]\!] = 1$" as a *formal wff* in classical set theory, and it is this formal wff to which $\Psi(B, A)$ refers.

but clearly $[\![\chi_Y = \chi_Z]\!] \in \{0,1\}$, and so we must have $[\![\chi_Y = \chi_Z]\!] = 1$, so that $\chi_Y = \chi_Z$ as quantum sets, which means that $Y = Z$ as classical sets.

Hence $\Psi(X,Y)$ satisfies the hypothesis of classical replacement, and so for any $Z \in \mathcal{V}$, there exists some $S \in \mathcal{V}$ such that for any $T \in \mathcal{V}$, $T \in S$ iff there exists some $U \in \mathcal{V}$ such that $U \in Z$ and $\Psi(U,T)$. Taking $Z = \sup f$, we have that $T \in S$ iff there exists a $U \in \mathcal{V}$ with $U \in \sup f$ and $[\![\psi(\chi_U, \chi_T)]\!] = 1$, which is to say that $T \in S$ iff $T \in \sup g$, so that $\sup g = S$, which is indeed a classical set.

□

Theorem 16. Let L be a complete irreducible atomic orthomodular lattice which satisfies the exchange axiom.[1] Then the $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ satisfies RZFC11.

Proof. If $L = \{0,1\}$, then RZFC11 holds by Theorem 11, so we can assume that $L \neq \{0,1\}$, and in particular the height of L is greater than or equal to 3. We need to show

$$\mathcal{Q} \vDash (\forall x)[(\forall u)\ \mathbf{C}(u \in x) \to x = x^*].$$

By Lemma A.12, it suffices to show, for any quantum set f, that

$$\bigwedge_{h \in \mathbb{Q}_L} [\![\mathbf{C}(h \in f)]\!] \leq [\![f = f^*]\!]. \tag{4.24}$$

---

[1]If unfamiliar, see Definitions A.28 and A.29 in the appendix.

Now, by definition of '**C**', we have

$$\begin{aligned}\bigwedge_{h\in\mathbb{Q}_L} [\![\mathbf{C}(h \in f)]\!] &= \bigwedge_{h\in\mathbb{Q}_L}\bigwedge_{j,k\in\mathbb{Q}_L} \left(\varphi_{[\![j\in k]\!]}([\![h \in f]\!]) \to [\![h \in f]\!]\right) \\ &= \bigwedge_{A\in\mathcal{V}} \Big[\bigwedge_{a\in L} \left(\varphi_a(f(A)) \to f(A)\right)\Big] \\ &= \bigwedge_{A\in\sup f} \Big[\bigwedge_{a\in L} \left(\varphi_a(f(A)) \to f(A)\right)\Big],\end{aligned}$$

where for the last equality we have used that $\varphi_x(0) = 0$ by Lemma A.11, while we also have

$$[\![f = f^*]\!] = \bigwedge_{h\in\mathbb{Q}_L} [\![h \in f \leftrightarrow h \in f^*]\!] = \bigwedge_{A\in\sup f} f(A).$$

by Lemma 4.4. Since for $f = \varnothing$ this gives $[\![f = f^*]\!] = 1$, in order to establish ZFC11 it will suffice to show, for any $b \in L$ with $b \neq 0$, that

$$\Big[\bigwedge_{a\in L} \varphi_a(b) \to b\Big] \leq b.$$

But this follows directly from Lemma A.16, and so RZFC11 is established. □

Finally, we arrive at the following result.

Theorem 17. Let L be a complete, irreducible, atomic, rotatable orthomodular lattice which satisfies the exchange axiom and the relative center property. Then the $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ is a model of RZFC.

Proof. This simply collects the results of Theorems 12, 13, 14, 15, and 16. □

Corollary 4.10. Let L be the projection lattice of a separable complex Hilbert space $\mathcal{H}$. Then the $\mathcal{L}_{set}$-structure $\mathcal{Q}_L$ is a model of RZFC.

Proof. This follows immediately from the previous theorem (i.e. Theorem 17) along with Lemmas 38 and 39, as well as Lemma A.17. □

This corollary shows that the intended class of models $\mathcal{Q}_L$ most strongly motivated by quantum mechanical formalism are models for our quantum set theory.

4.5 Conclusion

In this chapter, we have constructed an axiomatic set theory based on the quantum logic $\mathcal{Q}(\mathcal{L})$, and we showed that the intended class of models for this set theory does indeed satisfy the axioms (axiom schema) of the quantum set theory we constructed. We believe this quantum set theory to be a reasonable first attempt at a foundation for quantum mathematics (in a sense which parallels the foundational role of classical set theory in classical mathematics). This is supported by the fact that the first of two minimal criteria for the quantum set theory is met — namely, we have shown that the quantum set theory is a generalization of classical set theory, and in particular, that those models of our quantum set theory with standard bivalent truth values reduce to models of classical set theory. The second objective which we set forth for our quantum set theory was that it be powerful enough to develop a Peano-like quantum arithmetic. In the following chapter, we use the models $\mathcal{Q}_L$ of quantum set theory in order to construct quantum natural numbers (in these models), as well as develop a Peano-type arithmetic for these quantum natural numbers, along with some consequences of it.

In future work along these lines, we would like to continue to develop (and possibly generalize) the quantum set theory we have constructed — that is, possibly develop a more *fully quantum* version of the set theory, as well as undertake a systematic study of quantum ordinal and cardinal numbers within the quantum set theory — with the ultimate goal that the set theory play a foundational for quantum mathematics in a sense which parallels the foundational role of classical set theory in classical mathematics.

CHAPTER 5

QUANTUM ARITHMETIC

5.1 Introduction

In Chapter 4, we stated certain minimal criteria that any attempt at a quantum set theory should satisfy. One of these is that the set theory should be powerful enough to develop a notion of a 'natural number,' as well as an associated arithmetic. In this chapter, we take on this task.

In ordinary (classical) mathematics, the standard way of constructing the natural numbers is from sets, and properties of those numbers are then proven from known properties of the sets from which they're constructed. In this chapter, we use an analogous approach to construct the quantum natural numbers from the quantum set theory which we've developed. In particular, in each model $\mathcal{Q}_L$ of the quantum set theory,[1] we will identify the quantum natural numbers ω_L which arise from this process, as well as discuss arithmetical properties of these new numbers. Along the way, we compare our results with those of classical arithmetic.

In any model $\mathcal{Q}_L$ with $L = \mathbf{2}$, the above process yields the usual natural numbers $\mathbb{N}$. Moreover, an isomorphic copy of $\mathbb{N}$ sits inside ω_L for each L. However, there are, in general, more quantum natural numbers than ordinary classical numbers; as such, we see that the quantum natural numbers are much richer than their classical counter parts.

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[1]See Section 4.4 for a description of the $\mathcal{L}_{set}$-structures $\mathcal{Q}_L$.

Additionally, we note that in a special class of models which is particularly relevant for quantum theory — namely when $L = \mathscr{P}(\mathcal{H})$, the projection lattice[1] of a Hilbert space $\mathcal{H}$ — it turns out that there exists a 1-1 correspondence between the quantum natural numbers ω_L and bounded observables in quantum theory (i.e. bounded Hermitian operators on $\mathcal{H}$) whose eigenvalues are (ordinary) natural numbers. This 1-1 correspondence is remarkably satisfying, and gives us great confidence in our quantum set theory, as well as its possible future applications in quantum mechanics.

5.1.1 Overview

In what follows, our objective is to describe the quantum natural numbers as they arise from the models $\mathcal{Q}_L$ of our quantum set theory (which were developed in Section 4.4,[2] as well as discuss properties of an arithmetic for these quantum natural numbers.

We begin, in Section 5.2, by briefly reviewing the construction of the natural numbers as they arise from classical set theory. Paralleling this construction, we then define the quantum natural numbers ω_L in the models $\mathcal{Q}_L$ of our quantum set theory. In the process of constructing the sets which correspond to the quantum natural numbers, we define the successor function

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[1]See Section A.4.3 for a discussion of projection lattices.

[2]We note that the orthomodular lattices L for which $\mathcal{Q}_L$ is actually a model of our RZFC axioms is restricted to those orthomodular lattices which are complete, irreducible, atomic, rotatable, and satisfy the relative center property, as well as the exchange axiom. We further note that this class of orthomodular lattices includes projection lattices of separable Hilbert spaces. However, for an arbitrary orthomodular lattice L, we can still construct the structure $\mathcal{Q}_L$ and consider quantum natural numbers associated with it, despite the fact that such a structure isn't truly a model of the axiomatic set theory we've developed (since only axioms RZFC2-5,10 are satisfied in $\mathcal{Q}_L$ for an arbitrary orthomodular lattice L).

on the quantum universe of sets $\mathbb{Q}_L$, and go on in Section 5.3 to prove that the successor, when restricted to ω_L, satisfies a set of axioms which completely characterize the ordinary successor function in the presence of classical logic.

In Section 5.4, we then define an addition $\dot{+}$ and multiplication $\dot{\times}$ on the quantum natural numbers ω_L, after which we demonstrate that all but one of the ordinary arithmetic axioms hold for these arithmetical operations, while the remaining arithmetic axiom holds if and only if L is modular. We go on in Section 5.5 to consider consequences of the full arithmetic (i.e. successor fragment axioms along with the arithmetical axioms for $\dot{+}$ and $\dot{\times}$) for the quantum natural numbers ω_L in the case in which L is modular. Although in such a case we demonstrate that these arithmetical operations behave classically with regard to two-variable identities in ω_L, we show that other consequences of these axioms which involve three or more quantum natural numbers need not hold in general.

Finally, we conclude by considering the quantum natural numbers ω_L for $L = \mathscr{P}(\mathcal{H})$ (the projection lattice of a Hilbert space $\mathcal{H}$). In Section 5.6, we show that in such models, the arithmetical operations $\dot{+}$ and $\dot{\times}$ are the unique operations which satisfy certain desirable criteria.

5.2 The Quantum Natural Numbers

We begin our discussion of the quantum natural numbers and their properties with some preliminary definitions and concepts, as well as a few comments regarding the construction of ordinary natural numbers. For a more detailed discussion of the classical natural numbers, see Enderton (12).

The first thing that is necessary for the construction of the natural numbers (either classical or quantum) is the notion of a *successor* of a set.

We define the *successor* f' of a quantum set f to be given by

$$z \in f' \leftrightarrow [[(\exists g)[(z^\star = g^\star \cup \{g^\star\}) \wedge (g \in f)]] \vee (z^\star = \varnothing)], \tag{5.1}$$

and we note that our axioms for quantum set theory guarantee the existence of such a set in any model $\mathcal{Q}_L$. More precisely, in this class of models, the *successor* f' of a quantum set $f \in \mathbb{Q}_L$ becomes (for $y \in \mathcal{V}$)

$$f'(y) = \begin{cases} 1, & \text{if} \quad y = \varnothing \\ f(x), & \text{if} \quad y = x' \\ 0, & \text{otherwise} \quad . \end{cases} \tag{5.2}$$

See Section 5.3 below for a proof and a more detailed discussion of the successor function and its properties. In ordinary (classical) set theory, the successor S' of a set S is defined by $S' := S \cup \{S\}$.[1] We first note that although the definition of the successor of a quantum set agrees with this definition of the classical successor when the quantum set considered is a

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[1]We note that we can actually keep this same definition for the successor f' of a quantum set $f \in \mathbb{Q}_L$ — that is, we can define $f' := f \cup \{f\}$ — without affecting the definition or construction of the quantum natural numbers. Moreover, all of the proofs of the relevant properties still go through with this standard definition of the successor for quantum sets. However, when we restrict our attention to the quantum natural numbers ω_L as a subset of $\mathbb{Q}_L$, not all of the successor fragment axioms ((SF1)–(SF5) discussed below) hold with this definition of the successor. See Section 5.3 for a more detailed discussion of the axioms and properties of the successor of a quantum set given in equation 5.1 above.

classical natural number,[1] it does not, in general (i.e. for an arbitrary classical set), reduce to the successor function as defined for classical sets. However, since we're only interested in using the successor to construct the quantum natural numbers, the discrepancy is rather innocuous.

We next define and discuss what it means for a set to be *inductive*, as well as what it means for a set to be a *transitive set.*

Definition 5.1. A set $f \in \mathbb{Q}_L$ is said to be *inductive* if $[\![\varnothing \in f]\!] = 1$ and f is closed under the successor function — i.e. $[\![(\forall g)(g \in f \to g' \in f)]\!] = 1$.

We first review how the construction of natural numbers goes in classical set theory. Here, the existence of an inductive set is guaranteed by the infinity axiom. The concept of a natural number can then be defined as a set which belongs to every inductive set. We denote the collection of all such (ordinary) natural numbers by ω_c, which one can show is, itself, an inductive set. Moreover, ω_c is characterized by the property that it is contained in every inductive set — i.e. ω_c is the smallest inductive set. And, since $\varnothing \in \omega_c$, it follows that the set obtained by taking any number of successors of $\varnothing$ is also an element of ω_c. Indeed, all elements of ω_c are of this form. The first few natural numbers are listed below.

- $0 := \varnothing$
- $1 := 0' = \varnothing' = \{\varnothing\} = \{0\}$

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[1] Recall that in any model $\mathcal{Q}_L$, the classical sets can be thought of as a subset of the quantum sets, and hence, in this way, the classical natural numbers can also be thought of as a subset of the quantum sets. Moreover, any standard property of classical sets still holds for the classical sets when considered as a subset of the L-valued universe $\mathbb{Q}_L$.

- $2 := 1' = 0'' = \varnothing'' = \{\varnothing\}' = \{\varnothing, \{\varnothing\}\} = \{0, 1\}$

- $3 := 2' = 1'' = 0''' = \varnothing''' = \{\varnothing, \{\varnothing\}\}' = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} = \{0, 1, 2\}$

Inductively, we define $n := \{0, 1, ...n - 1\}$.

Definition 5.2. A set $f \in \mathbb{Q}_L$ is said to be *a transitive set* if every member of a member of f is itself a member of f — i.e.

$$[\![[(g \in f) \wedge (h \in g)] \to (h \in f)]\!] = 1.$$

We can immediately see that every (classical) natural number is a transitive set, as is ω_c itself. That is, using classical set theory, the transitive properties associated with natural numbers follow from their inductive properties. Indeed, in classical set theory, the property of being in every inductive set and the property of being a finite transitive set such that every element is also a transitive set are equivalent.

However, in our quantum set theory, it turns out that both the property of being in every inductive set and that of being a transitive set are needed to characterize the quantum natural numbers, as these properties of quantum sets no longer have the same relationship that they have classically.[1]

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[1]Note that in our quantum set theory, we have that the existence of an inductive set in any model $\mathcal{Q}_L$ is guaranteed by the infinity axiom (RZFC6). Similarly, the axioms of quantum set theory guarantee the existence of transitive sets in these models.

For notational purposes, let $\mu(f)$ denote the first order statement "f is an element of every inductive set" — i.e.

$$\mu(f) := (\forall h)[[(\varnothing \in h) \wedge (\forall g)(g \in h \rightarrow g' \in h)] \rightarrow f \in h],$$

and let $\lambda(f)$ denote the first order statement "f is a transitive set" — i.e.

$$\lambda(f) := [(g \in f) \wedge (h \in g)] \rightarrow (h \in f).$$

We construct the quantum natural numbers in any model $\mathcal{Q}_L$ to be the elements of the classical set ω_L which is given by

$$\omega_L := \{f \in \mathbb{Q}_L \mid [\![\mu(f)]\!] = 1 \wedge [\![\lambda(f)]\!] = 1\}.$$

That is, we consider the quantum natural numbers to be those quantum sets which are "fully" a member of every inductive set and are "fully" transitive sets, and we discard everything else.

We note that the requirement that $\mu(f)$ evaluates to 1 in the model $\mathcal{Q}_L$ is equivalent to $\sup f$ being an ordinary (classical) natural number. (See Theorem 18 below for details.) That is, $[\![\mu(f)]\!] = 1$ if and only if $\sup f = n$ for some natural number $n \in \omega_c$. On the other hand, the requirement that $\lambda(f)$ evaluates to 1 (i.e. $[\![\lambda(f)]\!] = 1$) gives that (i) $\sup f$ is a transitive set, and also that (ii) for each $n, k \in \omega_c$, we have that $f(k) \leq f(n)$ if $n \in k$ — that is, the $f(k)$'s form a decreasing sequence of elements in L. (See Theorems 19 and 20 below for details.) In what follows, we will sometimes identify $f \in \omega_L$ with its associated decreasing sequence — i.e. we will often represent f by the tuple

$$(f(0), f(1), ... f(\sup f - 1)).$$

Similar to the way in which the classical universe of sets embeds in any L-valued universe of (quantum) sets, the classical natural numbers embed into the set of quantum natural numbers. In particular, $n \in \omega_c$ is represented in ω_L by the quantum set f which has $\sup f = n$ and $f(m) = 1$ for all $m \in n$. (Notice also that when $L = \mathbf{2}$, we have that $\omega_L = \omega_c$).

In the case in which $L = \mathscr{P}(\mathcal{H})$ (i.e. when L is a projection lattice), if $f \in \omega_{\mathscr{P}(\mathcal{H})}$ is such that $f : n \to \mathscr{P}(\mathcal{H})$ where $n \in \omega_c$, then for all $m \in n$ we have that $f(m) = P_m$, where the P_m's are a decreasing sequence of projectors.[1] Such a decreasing sequence can be used to construct a Hermitian operator A_f on $\mathcal{H}$ whose eigenvalues are (classical) natural numbers — in particular,

$$A_f = \sum_{m \in n} P_m. \tag{5.3}$$

It is also clear that any bounded Hermitian operator on $\mathcal{H}$ whose eigenvalues are natural numbers can uniquely be put into the form of equation 5.3, and hence corresponds uniquely to a quantum natural number. This 1-1 correspondence between quantum natural numbers

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[1]For a quantum natural number $f \in \omega_L$ whose support is $n \in \omega_c$, we write $f : n \to \mathscr{P}(\mathcal{H})$, suppressing the portion of the domain $\mathcal{V}$ on which f evaluates to the zero operator. In what follows, we will silently abuse notation in a similar way for other quantum sets.

and bounded observables in quantum theory with natural number eigenvalues is remarkably satisfying, and gives us great confidence in our quantum set theory.[1] As an example, let $dim(\mathcal{H}) = 3$, and let $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ be an orthonormal basis for $\mathcal{H}$. Further, suppose that $f \in \omega_{\mathscr{P}(\mathcal{H})}$ is such that $f : n \to \mathscr{P}(\mathcal{H})$ where $n = 4$, and

$$f(0) := P_0 = I, \quad f(1) := P_1 = P_{Span(|\psi_2\rangle, |\psi_3\rangle)}, \quad f(2) := P_2 = P_{|\psi_3\rangle}, \quad f(3) := P_3 = P_{|\psi_3\rangle}.$$

The corresponding Hermitian operator A_f is given by the sum of these projectors, and in the basis $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$, we have that

$$A_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Additionally, as noted previously, an isomorphic copy of ω_c sits inside ω_L — in particular, in the case in which $L = \mathscr{P}(\mathcal{H})$, for any $n \in \omega_c$, the corresponding Hermitian operator is given by nI, where I is the identity operator on $\mathcal{H}$.

The following theorems (and proofs) fill in some technical detail for results which were already mentioned above, and these details constitute the remainder of this section.

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[1]Takeuti's (22) much more complicated quantum set theory yields the same quantum natural numbers, although the construction is much more cumbersome than ours.

Theorem 18. Consider the class of models $\mathcal{Q}_L$ of quantum set theory, and let $f \in \mathbb{Q}_L$. Then $[\![\mu(f)]\!] = 1$ if and only if $\sup f \in \omega_c$.

Proof. Note that in these models, the first order statement that f is in every inductive set becomes

$$1 = [\![(\forall h)[[(\varnothing \in h) \wedge (\forall g)(g \in h \to g' \in h)] \to f \in h]]\!]$$

$$= \bigwedge_{h \in \mathbb{Q}_L} [[h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')]] \to h(\sup f)].$$

We first assume that f is in every inductive set; as such, the above expression must hold for all $h \in \mathbb{Q}_L$ — in particular, it holds for $z \in \mathbb{Q}_L$, where z is such that $z(x) = 1$ if $x \in \omega_c$ and $z(x) = 0$ if $x \notin \omega_c$ — that is, we have that

$$[[z(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [z(\sup g) \to z(\sup g')]] \to z(\sup f)] = 1.$$

First note that $z(\varnothing) = 1$ since $\varnothing \in \omega_c$ (i.e. $\varnothing$ is the first classical ordinal). Now, if $\sup g \notin \omega_c$, then $z(\sup g) = 0$ by definition of z, so $z(\sup g) \to z(\sup g') = 0 \to a = 1$. On the other hand, if $\sup g \in \omega_c$, then $\sup g' \in \omega_c$ as well, and by definition of z, $z(\sup g) = 1$ and $z(\sup g') = 1$; and so, $z(\sup g) \to z(\sup g') = 1 \to 1 = 1$. (To see that $\sup g \in \omega_c$ implies that $\sup g' \in \omega_c$, note that with the definition of the successor, we have that $x \in \sup g$ implies that $x' \in \sup g'$, so that in the case that $\sup g = n$ for some classical natural number $n \in \omega_c$, we have $i' \in \sup g'$ for every

$i \in n$; and thus, since $\varnothing \in \sup g'$ by construction, it follows that $\sup g' = n+1 \in \omega_c$.) Together these results give that

$$[z(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [z(\sup g) \to z(\sup g')]] = 1.$$

Since we also have that

$$[[z(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [z(\sup g \to z(\sup g')]] \to z(\sup f)] = 1$$

by assumption, it follows that $z(\sup f) = 1$, which gives that $\sup f \in \omega_c$ simply by definition of z. Thus, we see that if f is in every inductive set, then $\sup f \in \omega_c$.

We now want to show that if $\sup f \in \omega_c$, then f is in every inductive set. We again recall that in our models, this property valuates to

$$\bigwedge_{h \in \mathbb{Q}_L} [[h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')]] \to h(\sup f)].$$

Assume that $\sup f \in \omega_c$, and note that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \le h(\sup \hat{g}) \to h(\sup \hat{g}')$$

for any $\hat{g} \in \mathbb{Q}_L$. In particular, it holds if $\sup \hat{g} \in \omega_c$. (And, if $\sup g \in \omega_c$, then $\sup g = \varnothing^{(n)}$, where $\varnothing^0 = \varnothing$, $\varnothing^{(1)} = \{\varnothing\}$, and $\varnothing^{(n+1)} = \varnothing^{(n)} \cup \{\varnothing^{(n)}\}$.) That is, we have that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \leq h(\varnothing^{(n)}) \to h(\varnothing^{(n+1)}).$$

Now, we proceed via proof by induction. We clearly have that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \leq h(\varnothing) = h(\varnothing^{(0)}),$$

which gives the zeroth step. We assume that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \leq h(\varnothing^{(n)}).$$

By the discussion above, we have that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \leq h(\varnothing^{(n)}) \to h(\varnothing^{(n+1)}),$$

and together these results give that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \leq h(\varnothing^{(n)}) \wedge [h(\varnothing^{(n)}) \to h(\varnothing^{(n+1)})].$$

And so, by the orthomodular law (i.e. $a \wedge (a \to b) = a \wedge (\neg a \vee (a \wedge b)) = a \wedge b \leq b$, where the last equality follows from the orthomodular law), we have that

$$h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')] \leq h(\varnothing^{(n+1)}).$$

Since this holds for any $m \in \omega_c$ and any $h \in \mathbb{Q}_L$, and since $\sup f \in \omega_c$ implies that $\sup f = \varnothing^{(k)}$ for some k, we have that

$$\bigwedge_{h \in \mathbb{Q}_L} [[h(\varnothing) \wedge \bigwedge_{g \in \mathbb{Q}_L} [h(\sup g) \to h(\sup g')]] \to h(\sup f)] = 1.$$

(We note that we have used the fact that "$\to$" satisfies Hardegree's "minimal implicative criteria" (16), (17) so that $a \leq b$ iff $a \to b = 1$.) Thus, we have that $\sup f \in \omega_c$ implies that f is in every inductive set.

$\square$

Theorem 19. Consider the class of models $\mathcal{Q}_L$ of quantum set theory, and let $f \in \mathbb{Q}_L$ be such that $[\![\lambda(f)]\!] = 1$. Then $\sup f$ is a (classical) transitive set.

Proof. First note that in a model, we have that first order statement that f is a transitive set becomes

$$[\![\lambda(f)]\!] = f(\sup g) \wedge g(\sup h) \to f(\sup h),$$

and we have that

$$\llbracket \big((g \in f) \wedge (h \in g)\big) \to (h \in f) \rrbracket = f(\sup g) \wedge g(\sup h) \to f(\sup h) = 1 \iff f(\sup g) \wedge g(\sup h) \leq f(\sup h),$$

which is equivalent to

$$f(\sup g) \wedge g(A) \to f(A)$$

since $\sup h$ is (necessarily) some classical set.

Now, we wts that whenever f is a transitive set, $\sup f$ is a transitive set. To see this, suppose that f is a transitive set, but that $\sup f$ is not a transitive set. As such, there exist (classical) sets, B, C such that $B \in C$ and $C \in \sup f$, but $B \notin \sup f$ (i.e. $f(B) = 0$). Letting $C = \sup g^\star$ and $B = \sup h$, we have that

$$f(C) \wedge g^\star(B) = f(C) \wedge 1 = f(C),$$

where $f(C) \neq 0$ since $C \in \sup f$ by assumption, and the fact that $g^\star(B) = 1$ follows from the fact that $B \in \sup g$. However, since $f(B) = 0$ and f is a transitive set, we have that

$$f(C) \wedge g^\star(B) \leq f(B) = 0,$$

which forces $f(C) = 0$, which is a contradiction to the assumption that f is a transitive set — thus, we must have that $\sup f$ is a transitive set. $\square$

Theorem 20. Consider the class of models $\mathcal{Q}_L$ of quantum set theory, and let $f \in \mathbb{Q}_L$. If $[\![\lambda(f)]\!] = 1$, then, for $x, y \in \sup f$, we have that $f(x) \leq f(y)$ whenever $y \in x$ (where "$\leq$" denotes the partial ordering in L).

Proof. First note that in a model, we have that first order statement that f is a transitive set becomes

$$[\![\lambda(f)]\!] f(\sup g) \wedge g(\sup h) \to f(\sup h),$$

and we have that

$$f(\sup g) \wedge g(\sup h) \to f(\sup h) = 1 \Longleftrightarrow f(\sup g) \wedge g(\sup h) \leq f(\sup h),$$

which is equivalent to

$$f(\sup g) \wedge g(A) \to f(A)$$

since $\sup h$ is (necessarily) some classical set.

Now, assume that f is a transitive set, and note that if we have that $\sup f$ is a transitive set, then $f(\sup g) \neq 0$ and $g(A) \neq 0$ imply that $f(A) \neq 0$. To see this, assume that $A \in \sup f$. If $A \neq \varnothing$, then there exists a set B such that $B \in A$ implies that $B \in \sup f$. Take $g(x) = 1$ for all $x \in A$, and $g(x) = 0$ otherwise. Then the transitive set inequality becomes (for all $x \in A$)

$$f(A) \wedge g(x) \leq f(x),$$

from which it follows that $f(A) \leq f(x)$ for all $x \in A$. Thus, we see that whenever a quantum set f is transitive, we have the $f(x)$'s form a decreasing sequence in the truth value algebra L associated with a model. □

5.3 The Successor Fragment of the Quantum Arithmetic

We begin with a proof of equation 5.2.

Lemma 5.1. In the models $\mathcal{Q}_L$, the successor f' of a quantum set $f \in \mathbb{Q}_L$ is given by (for $y \in \mathcal{V}$)

$$f'(y) = \begin{cases} 1, & \text{if} \quad y = \varnothing \\ f(x), & \text{if} \quad y = x' \\ 0, & \text{otherwise} \end{cases} .$$

Proof. As mentioned previously, the first order sentence which defines the successor f' of a quantum set f is given by

$$z \in f' \leftrightarrow [[(\exists g)[(z^\star = g^\star \cup \{g^\star\}) \wedge (g \in f)]] \vee (z^\star = \varnothing)].$$

Recall that in the models $\mathcal{Q}_L$, we have that $[\![z \in f']\!] = f'(\sup z)$, while the RHS of "$\leftrightarrow$" in the expression above valuates to

$$[\bigvee_{g \in \mathbb{Q}_L} [f(\sup g) \wedge \bigwedge_{x \in \mathcal{V}} [z^\star(x) \leftrightarrow (g^\star(x) \vee \{g^\star\}(x))]]] \vee [\bigwedge_{x \in \mathcal{V}} z^\star(x) \leftrightarrow 0]$$

$$= [\bigvee_{g \in \mathfrak{Q}_L} [f(\sup g) \wedge \bigwedge_{\substack{x \in \sup g \\ \text{or} \\ x = \sup g}} [z^\star(x) \wedge \bigwedge_{\substack{x \notin \sup g \\ x \neq \sup g}} \neg z^\star(x)]] \vee [\bigwedge_{x \in \mathcal{V}} z^\star(x) \leftrightarrow 0]].$$

Note that we have used that

$$z^\star(x) \leftrightarrow (g^\star(x) \vee \{g^\star\}(x)) = \begin{cases} z^\star(x) \leftrightarrow 1, & \text{if} \quad x = \sup g \\ z^\star(x) \leftrightarrow g^\star(x), & \text{if} \quad x \in \sup f \\ z^\star(x) \leftrightarrow 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} z^\star(x) \leftrightarrow 1, & \text{if} \quad x = \sup g \quad \text{or} \quad x \in \sup g \\ z^\star(x) \leftrightarrow 0, & \text{otherwise} \end{cases} = \begin{cases} z^\star(x), & \text{if} \quad x = \sup g \quad \text{or} \quad x \in \sup g \\ \neg z^\star(x), & \text{otherwise} \end{cases},$$

so that

$$\bigwedge_{x \in \mathcal{V}} z^\star(x) \leftrightarrow (g^\star(x) \vee \{g^\star\}(x)) = \bigwedge_{\substack{x \in \sup g \\ \text{or} \\ x = \sup g}} z^\star(x) \wedge \bigwedge_{\substack{x \notin \sup g \\ x \neq \sup g}} \neg z^\star(x).$$

Notice that if g is such that

$$\sup z \neq \sup g \cup \{\sup g\},$$

then we have that

$$\bigwedge_{x \in \mathcal{V}} z^\star(x) \leftrightarrow (g^\star(x) \vee \{g^\star\}(x)) = 0.$$

Also note that $\bigwedge_{x \in \mathcal{V}} \neg z^\star(x) = 0$ if $\sup z \neq \varnothing$, while $\bigwedge_{x \in \mathcal{V}} \neg z^\star(x) = 1$ if $\sup z = \varnothing$. As such, for the first term (since we have $\bigvee_{g \in \mathfrak{Q}_L}$), we need only consider the quantum sets $g \in \mathfrak{Q}_L$ such that $\sup z = \sup g \cup \{\sup g\}$, in which case the first term becomes $f(\sup g)$, and the second term is

0 since $\sup z \neq \varnothing$ in this case (i.e. $\sup z = \sup g \cup \{\sup g\}$ implies that $\sup z \neq \varnothing$). And so, we obtain that

$$f'(\sup z) = \begin{cases} 1, & \text{if} \quad \sup z = \varnothing \\ f(\sup g), & \text{if} \quad \sup z = \sup g \cup \{\sup g\} \\ 0, & \text{otherwise} \quad . \end{cases}$$

As such, we see that in the models $\mathcal{Q}_L$, we have that the successor f' of a quantum set $f \in \mathbb{Q}_L$ to be given by (for $y \in \mathcal{V}$)

$$f'(y) = \begin{cases} 1, & \text{if} \quad y = \varnothing \\ f(x), & \text{if} \quad y = x' \\ 0, & \text{otherwise} \quad . \end{cases}$$

$\square$

We note that when the successor function is restricted to the quantum natural numbers, the successor of $f \in \omega_L$ works in such a way as to add 1 (i.e. the top element of L) to the beginning of the decreasing sequence of elements of L associated with f. As such, we see that whenever $f \in \omega_L$, we also have that $f' \in \omega_L$, which shows that ω_L is closed under the successor function. In the case in which $L = \mathscr{P}(\mathcal{H})$, where we have the 1-1 correspondence between the quantum natural numbers and bounded Hermitian operators on $\mathcal{H}$ whose eigenvalues are natural numbers (which was described above), we have that $A_{f'} = A_f + I$. Thus, for the quantum natural numbers which correspond to ordinary (classical) natural numbers, the successor behaves exactly as the classical successor function does — that is, for any $n \in \omega_c$ the successor of nI is given by $(n+1)I$. Hence, the subset of $\omega_{\mathscr{P}(\mathcal{H})}$ which is isomorphic to ω_c is closed under the successor function, as expected. However, we also note that we clearly cannot build up every quantum natural number from 0 via the successor — if this were the case, then the set $\omega_{\mathscr{P}(\mathcal{H})}$ would consist only of the subset isomorphic to ω_c. Thus, we see that, in general, the quantum natural numbers are much richer than their classical counter parts.

Now, we want to consider an M-system $(\mathcal{L}_S, \mathcal{A}_S)$ associated with the successor fragment of Peano arithmetic. We define the language $\langle \mathcal{L}_S, \alpha \rangle$, where $\mathcal{L}_S^P := \{=\}$ and $\mathcal{L}_S^F := \{0,'\}$, with $\alpha(=) := 2$, $\alpha(0) := 0$ and $\alpha(') := 1$. By $\mathcal{A}_S$ we denote the set of axioms (SF1) — (SF5) below.

(SF1) $(\forall x)[x' \neq 0]$

(SF2) $(\forall x)[x \neq x']$, $(\forall x)[x \neq x'']$, ...

(SF3) $(\forall x)(\forall y)[x = y \rightarrow x' = y']$

(SF4) $(\forall x)(\forall y)[x' = y' \rightarrow x = y]$

(SF5) $(\forall x)[(x \neq 0) \rightarrow [(\exists y)(x \neq y')]]$

We note that the axioms listed above comprise an alternative (well-known, but not ubiquitously used) axiomatization of the successor fragment of Peano arithmetic, which, in the presence of classical logic, is equivalent to the standard one (11).[1]

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[1]For a list of the standard successor fragment axioms (axiom schema), see Section 3.2 where they appear as (S1) — (S4). Note that in the axioms above, the induction axiom schema from the standard axiomatization has essentially been replaced by the infinite sequence of axioms in (SF2). Also, recall that in Section 3.2 we have shown that the standard axiomatization is inherently classical; as such, in

We next construct an $\mathcal{L}_S$-structure associated with each $\mathcal{L}_{set}$-structure $\mathcal{Q}_L = (\mathbb{Q}_L, L, \{[\![\in]\!]\}, \varnothing)$ — that is, we consider the $\mathcal{L}_S$-structures

$$\hat{\omega}_L := (\omega_L, L, \{[\![=]\!]\}, \{0, '\}),$$

where $\omega_L \subseteq \mathbb{Q}_L$ are the quantum natural numbers in $\mathcal{Q}_L$, and for $f, g \in \omega_L$,

$$[\![f = g]\!] := \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow g(x)\big).$$

As for the interpretation of the operations in $\mathcal{L}_S^F$, we take $0 \in \mathbb{Q}_L$ to be the zero map (i.e. we have $\sup 0 = \varnothing$ so that 0 is the quantum set which is the analogue of the classical natural number 0), and $'$ is as given in equation 5.2, but restricted to elements of ω_L.

Theorem 21. The successor fragment axioms (SF1) – (SF5) hold in any $\mathcal{L}_S$-structure $\hat{\omega}_L$.

Proof.

(SF1) $(\forall x)[x' \neq 0]$

order to demonstrate the full power and richness of our quantum models, we use *some* alternative (but classically equivalent) axiomitization.

Let $0 \in \mathbb{Q}_L$ be the zero map (i.e. we have $\sup 0 = \varnothing$ so that 0 is the quantum set which is the analogue of the classical natural number 0), and let $f \in \omega_L$ be such that $f : n \to L$ with $n = \sup f \in \omega_c$, and $f(i) \coloneqq P_i$ with $P_{i+1} \leq P_i$. We have that

$$[\![f' = 0]\!] = \bigwedge_{x \in \mathcal{V}} f'(x) \leftrightarrow 0(x) = \bigwedge_{x \in \mathcal{V}} f'(x) \leftrightarrow 0$$

$$= \bigwedge_{x \in \mathcal{V}} \neg f'(x) = \neg f'(\varnothing) \wedge \bigwedge_{x \in \mathcal{V}} \neg f(x) = \neg 1 \wedge \bigwedge_{x \in \mathcal{V}} = 0.$$

Thus, we see that $[\![f' = 0]\!] = 0$, from which it follows that

$$[\![f' \neq 0]\!] = [\![\sim (f' = 0)]\!] = \neg [\![f' = 0]\!] = 1.$$

Thus, we see that for any $f \in \omega_L$, we have that $\hat{\omega}_L vDash f' \neq 0$, so that the axiom $(\forall x)[x' \neq 0]$ holds — i.e. $\hat{\omega}_L \vDash (\forall x)[x' \neq 0]$.

(SF2) $(\forall x)[x \neq x']$, $(\forall x)[x \neq x'']$, ...

In what follows, we consider $(\forall x)[x \neq x']$, noting that the proof is similar for the other axioms in this infinite sequence (e.g. $(\forall x)[x \neq x'']$, $(\forall x)[x \neq x''']$, ...) — as such, we only explicitly consider the first axiom in the infinite sequence.

In what follows, we consider $f \neq f'$, where $f : n \to L$ with $n = \sup f \in \omega_c$, and $f(i) := P_i$ with $P_{i+1} \leq P_i$. We have that

$$[\![f = f']\!] = \bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow f'(x) = (f(\varnothing) \leftrightarrow f'(\varnothing)) \wedge [\bigwedge_{\substack{x \in \mathcal{V} \\ x \neq \varnothing}} f(x) \leftrightarrow f'(x)]$$

$$= (f(\varnothing) \leftrightarrow f'(\varnothing)) \wedge [\bigwedge_{\substack{i \in n \\ i \neq 0}} f(i) \leftrightarrow f'(i)] \wedge (0 \leftrightarrow f'(n-1))$$

$$= P_0 \leftrightarrow I \wedge \neg P_{n-1} \wedge [\bigwedge_{\substack{i \in n \\ i \neq 0}} P_i \leftrightarrow P_{i-1}].$$

We clearly have that $P_0 \leftrightarrow I = P_0$ (by Lemma A.12 (5)), and the first term in $\bigwedge_{\substack{i \in n \\ i \neq 0}} P_i \leftrightarrow P_{i-1}$ is given by $P_1 \leftrightarrow P_0$; additionally, since $P_1 \leq P_0$, Lemma A.12 (2) gives that this simplifies to

$$P_0 \leftrightarrow P_1 = P_0 \to P_1 = \neg P_0 \vee (P_0 \wedge P_1) = \neg P_0 \vee P_1.$$

Together, these two terms give

$$P_0 \wedge (P_0 \leftrightarrow P_1) = P_0 \wedge (\neg P_0 \vee P_1) = \varphi_{P_0}(P_1) = P_1,$$

where the last equality follows from the fact that $\varphi_a(b) = b \Leftrightarrow b \leq a$ (by Lemma A.11 (3)). The next term in $\bigwedge_{\substack{i \in n \\ i \neq 0}} P_i \leftrightarrow P_{i-1}$ is given by $P_2 \leftrightarrow P_1$; and since $P_2 \leq P_1$, Lemma A.12 (2) gives that this simplifies to

$$P_1 \leftrightarrow P_2 = P_1 \to P_2 = \neg P_1 \vee (P_1 \wedge P_2) = \neg P_1 \vee P_2.$$

Combining this with the previous result, we have that

$$P_0 \wedge (P_0 \leftrightarrow P_1) \wedge (P_1 \leftrightarrow P_2) = P_1 \wedge (\neg P_1 \vee P_2) = \varphi_{P_1}(P_2) = P_2,$$

where the last equality again follows from the fact that $\varphi_a(b) = b \Leftrightarrow b \leq a$ (by Lemma A.11 (3)).

And so, for the m^{th} term $\bigwedge_{\substack{i\in n\\ i\neq 0}} P_i \leftrightarrow P_{i-1}$ is given by $P_m \leftrightarrow P_{m-1}$; and since $P_m \leq P_{m-1}$, Lemma A.12 (2) gives that this simplifies to

$$P_{m-1} \leftrightarrow P_m = P_{m-1} \to P_m = \neg P_{m-1} \vee (P_{m-1} \wedge P_m) = \neg P_{m-1} \vee P_m.$$

And so, combining this with the result from all the previous $m-1$ terms, we have that

$$P_0 \wedge [\bigwedge_{\substack{i\in n\\ 0<i<m}} P_i \leftrightarrow P_{i-1}] = P_{m-1} \wedge (\neg P_{m-1} \vee P_m) = \varphi_{P_{m-1}}(P_m) = P_m,$$

where the last equality follows from the fact that $\varphi_a(b) = b \Leftrightarrow b \leq a$ (by Lemma A.11 (3)). Now, this continues until we reach $i = n-1$, which yields

$$P_0 \wedge [\bigwedge_{\substack{i\in n\\ 0<i}} P_i \leftrightarrow P_{i-1}] = P_{n-1}.$$

Thus, we have that

$$[\![f = f']\!] = (P_0 \leftrightarrow I) \wedge [\bigwedge_{\substack{i \in n \\ i \neq 0}} P_i \leftrightarrow P_{i-1}] \wedge \neg P_{n-1} = P_{n-1} \wedge \neg P_{n-1} = 0.$$

As such,

$$[\![f \neq f']\!] = [\![\sim (f = f')]\!] = \neg [\![f = f']\!] = 1,$$

from which we see that for an arbitrary $f \in \omega_L$, we have that $\hat{\omega}_L vDash f \neq f'$, so that the axiom $(\forall x)[x \neq x']$ holds — i.e. $\hat{\omega}_L vDash (\forall x)[x \neq x']$.

(SF3) & (SF4) $(\forall x)(\forall y)[x = y \to x' = y']$ and $(\forall x)(\forall y)[x' = y' \to x = y]$

Let $f, g \in \omega_L$ be such that $f : n \to L$ with $n = \sup f \in \omega_c$, and $f(i) := P_i$ with $P_{i+1} \leq P_i$, and $g : m \to L$ with $m = \sup g \in \omega_c$, and $g(i) := Q_i$ with $Q_{i+1} \leq Q_i$; additionally, we take $m \leq n$ wlog. We have that

$$[\![f = g]\!] = \bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow g(x),$$

and

$$[\![f' = g']\!] = \bigwedge_{x \in \mathcal{V}} f'(x) \leftrightarrow g'(x) = (f'(\varnothing) \leftrightarrow g'(\varnothing)) \wedge [\bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow g(x)] = [\![f = g]\!]$$

$$= (1 \leftrightarrow 1) \wedge [\bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow g(x)] = \bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow g(x).$$

Thus, we have that $[\![f' = g']\!] = [\![f = g]\!]$, which shows that

$$\hat{\omega}_L vDash f = g \to f' = g' \quad \text{and} \quad \hat{\omega}_L vDash f' = g' \to f = g.$$

Since these hold for arbitrary $f, g \in \omega_L$, we have that axioms (SF3) and (SF4) hold.

(SF5) $(\forall x)[(x \neq 0) \to [(\exists y)(x \neq y')]]$

Let $0 \in \mathbb{Q}_L$ be the zero map (i.e. we have $\sup 0 = \varnothing$ so that 0 is the quantum set which is the analogue of the classical natural number 0), and let $f : n \to L$ with $n = \sup f \in \omega_c$, and $f(i) := P_i$ with $P_{i+1} \leq P_i$. Note that in order for this axiom to hold in our models, we must have that $[\![f \neq 0]\!] \leq \bigvee_{h \in \omega_L} [\![f = h']\!]$. First consider $[\![f = 0]\!]$; we have that

$$[\![f = 0]\!] = \bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow 0(x) = \bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow 0 = \bigwedge_{x \in \mathcal{V}} \neg f(x) = \bigwedge_{i \in n} \neg P_i = \neg P_0.$$

Thus, we see that $[\![f' = 0]\!] = \neg P_0$, from which it follows that

$$[\![f' \neq 0]\!] = [\![\sim (f' = 0)]\!] = \neg [\![f' = 0]\!] = P_0.$$

We next consider $[\![(\exists y)(x = y')]\!]$; we have that

$$[\![(\exists y)(x = y')]\!] = \bigvee_{h \in \omega_L} [\![f = h']\!] = \bigvee_{h \in \omega_L} f \leftrightarrow h'.$$

Let $g \in \omega_L$ be such that $f : m \to L$ with $m = \sup g \in \omega_c$, and $g(i) := Q_i$ with $Q_{i+1} \leq Q_i$; we then have that

$$[\![f = g']\!] = \bigwedge_{x \in \mathcal{V}} f(x) \leftrightarrow g'(x) = (f(\varnothing) \leftrightarrow 1) \wedge [\bigwedge_{\substack{x \in \mathcal{V} \\ x \neq \varnothing}} f(x) \leftrightarrow g'(x)]$$

$$= (P_0 \leftrightarrow 1) \wedge [\bigwedge_{\substack{i \in \min(n,m) \\ i \neq 0}} P_i \leftrightarrow Q_{i-1}] = P_0 \wedge [\bigwedge_{\substack{i \in \min(n,m) \\ i \neq 0}} P_i \leftrightarrow Q_{i-1}].$$

However, since we consider

$$[\![(\exists y)(x = y')]\!] = \bigvee_{h \in \omega_L} [\![f = h']\!] = \bigvee_{h \in \omega_L} f \leftrightarrow h',$$

where h runs over all quantum sets $h \in \omega_L$, then the map $g : n - 1 \to L$ defined by $g(i) := P_{i+1}$ for $i \in n - 1$ certainly exists and is included in the join. For this choice of g, we have that

$$[\![f = g']\!] = P_0$$

since

$$\bigwedge_{\substack{i \in n \\ i \neq 0}} P_i \leftrightarrow Q_{i-1} = \bigwedge_{\substack{i \in n \\ i \neq 0}} P_i \leftrightarrow P_i = 1.$$

As such we have that $[\![f \neq 0]\!] \leq \bigvee_{h \in \omega_L} [\![f = h']\!]$ (i.e. $P_0 \leq P_0$), which shows that for any $f \in \omega_L$, we have that $\hat{\omega}_L vDash (f \neq 0) \to [(\exists y)(x = y')]$. It follows from this that the axiom $(\forall x)[(x \neq 0) \to [(\exists y)(x \neq y')]]$ holds — i.e. $\hat{\omega}_L vDash (\forall x)[(x \neq 0) \to [(\exists y)(x \neq y')]]$.

□

We also have the following lemma.

Lemma 5.2. In any $\mathcal{L}_S$-structure $\hat{\omega}_L$, the following properties hold.

(i) $(\forall x)(x = x)$

(ii) $(\forall x)(\forall y)(x = y \to y = x)$

(iii) $(\forall x)(\forall y)(\forall z)(x = y \wedge y = z \to x = z)$

Proof. Recall that for $f, g \in \omega_L$,

$$[\![f = g]\!] := \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow g(x)\big).$$

First note that

$$[\![f = f]\!] = \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow f(x)\big) = 1,$$

which follows from the fact that in any orthomodular lattice L, we have that $a \leftrightarrow a = 1$ for all $a \in L$ (by Lemma A.12 (2)). Since this holds for any $f \in \omega_L$, we have that (i) above holds in $\hat{\omega}_L$ — i.e. $\hat{\omega}_L \vDash (\forall x)(x = x)$. Next note that

$$[\![f = g]\!] = \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow g(x)\big) = \bigwedge_{x \in \mathcal{V}} \big(g(x) \leftrightarrow f(x)\big) = [\![g = f]\!],$$

which follows from the fact that in any orthomodular lattice L, we have that $a \leftrightarrow b = b \leftrightarrow a$ for all $a, b \in L$ (see (1) in Lemma A.12). As such, we have that $[\![f = g \to g = f]\!] = 1$ for all $f, g \in \omega_L$;

and so, it follows that $[\![(\forall x)(\forall y)(x = y \to y = x)]\!] = 1$, or equivalently, that (ii) holds in $\hat{\omega}_L$ — i.e. $\hat{\omega}_L \vDash (\forall x)(\forall y)(x = y \to y = x)$. Finally, we have that for any $f, g, h \in \omega_L$,

$$[\![\big((f = g) \wedge (g = h)\big) \to (f = h)]\!] = \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow g(x) \wedge h(x) \leftrightarrow h(x)\big) = \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow h(x)\big),$$

which follows from (8) in Lemma A.12. However, since this holds for arbitrary $f, g, h \in \omega_L$, we have that $[\![(\forall x)(\forall y)(\forall z)[\big((f = g) \wedge (g = h)\big) \to (f = h)]]\!] = 1$, so that (iii) above holds in $\hat{\omega}_L$ — i.e. $\hat{\omega}_L \vDash (\forall x)(\forall y)(\forall z)[\big((f = g) \wedge (g = h)\big) \to (f = h)]$. □

Note that (i)–(iii) in the above lemma are just the standard equality axioms (E1)–(E3).

And so, Theorem 21 and Lemma 5.2 show that our quantum set theory has given rise to a natural class of (non-standard) models $\hat{\omega}_L$ of the successor fragment of Peano arithmetic.

5.4 An Arithmetic for Quantum Natural Numbers

In this section we discuss an arithmetic for the quantum natural numbers. In what follows, we will define an addition $\dot{+}$ and multiplication $\dot{\times}$ on any ω_L, and then go on to show that the arithmetic axioms due to Peano[1] hold for these operations when L is modular. Further motivation for the following definitions of $\dot{+}$ and $\dot{\times}$ comes, in some sense, from their properties

[1]Here we specifically mean the axioms (A1) — (A4) listed in Section 5.4.1 below. However, we note that *Peano arithmetic* refers specifically to the arithmetic which is associated with the axioms (A1) — (A4) along with the standard successor fragment axioms (axiom schema) (S1) — (S4) listed in Section 3.2. The same arithmetical axioms (A1) — (A4), when combined with the alternative successor fragment axioms (SF1) — (SF5) listed in Section 5.3 above, yield an arithmetic which, in the presence of classical logic, is strictly weaker than Peano arithmetic. It is the quantum analogue of the weaker arithmetic which will occupy our attention below.

in the case in which $L = \mathscr{P}(\mathcal{H})$. In particular, for $f, g \in \omega_{\mathscr{P}(\mathcal{H})}$, the eigenvalues of the Hermitian operator associated with $f \dot{\times} g$ are obtained as products of eigenvalues of A_f and B_g. Similarly, the eigenvalues of the Hermitian operator associated with $f \dot{+} g$ are obtained as sums of eigenvalues of A_f and B_g.[1] Further, when the Hermitian operators A_f and B_g associated with the quantum natural numbers $f, g \in \omega_{\mathscr{P}(\mathcal{H})}$ are such that $[A_f, B_g] = 0$ (i.e. A_f and B_g are commuting linear operators), the operations $\dot{+}$ and $\dot{\times}$ correspond to the ordinary sum and product of linear operators.[2] Moreover, when $\mathscr{P}(\mathcal{H})$ is modular, it can be shown that $\dot{+}$ and $\dot{\times}$ are the *unique* binary operations which satisfy certain desirable criteria.[3]

This being said, we extend the language $\mathcal{L}_S$ to include the additional operations $\dot{+}$ and $\dot{\times}$ — that is, we define the language $\langle \mathcal{L}_A, \alpha \rangle$, where $\mathcal{L}_A^P := \{=\}$ and $\mathcal{L}_A^F := \{0, 1, ', \dot{+}, \dot{\times}\}$, with $\alpha(=) := 2$, $\alpha(0) := 0$, $\alpha(1) := 0$, $\alpha(') := 1$, $\alpha(\dot{+}) := 2$, and $\alpha(\dot{\times}) := 2$. Additionally, we extend the $\mathcal{L}_S$-structures $\hat{\omega}_L$ to $\mathcal{L}_A$-structures $\hat{\omega}(L)$, which we define by

$$\hat{\omega}(L) := (\omega_L, L, \{[\![=]\!]\}, \{0, ', \dot{+}, \dot{\times}\}),$$

where (as before) for $f, g \in \omega_L$,

$$[\![f = g]\!] := \bigwedge_{x \in \mathcal{V}} \big(f(x) \leftrightarrow g(x)\big),$$

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[1]See Theorems 26 and 27 in Section 5.6 below for a proof of these properties.

[2]See Theorems 5.7 and 5.8 below for a proof of these properties.

[3]See Theorems 28 and 29 in Section 5.6 below for a proof of these properties.

and the interpretation of the operations 0 and $'$ in the $\mathcal{L}_A^F$-structures $\hat{\omega}(L)$ remain the same as in $\mathcal{L}_S^F$-structures $\hat{\omega}_L$. As for the interpretation of 1, we take $1 \in \mathbb{Q}_L$ to be the map whose support is the classical natural number $1 \in \omega_c$, and which evaluates to $1 \in L$ on its support — i.e. we have $\sup 1 = \{\varnothing\} = 1$ so that 1 is the quantum set which is the analogue of the classical natural number 1). We now define the interpretation of addition $\dot{+}$ and multiplication $\dot{\times}$[1] in any $\mathcal{L}_A$-structure $\hat{\omega}(L)$. Let $p, q \in \omega_c$, and let $A, B \in \omega_L$ be such that $A : p \to L$, $B : q \to L$, with $A(i) := A_{i+1}$ for $i \in p$ and $A_i \leq A_j$ for $j \in i$, and $B(i) := B_{i+1}$ for $i \in q$ with $B_i \leq B_j$ for $j \in i$. We define

$$(A \dot{+} B)_n := A_n \vee B_n \vee [\bigvee_{k=1}^{n-1} (A_k \wedge B_{n-k})] = A_n \vee B_n \vee [\bigvee_{k+j \geq n} A_k \wedge B_j]$$

and

$$(A \dot{\times} B)_n := \bigvee_{s,t \leq n; s \cdot t \geq n} (P_s \wedge Q_t) = \bigvee_{s=1}^{n} (P_s \wedge Q_{[\frac{n}{s}]}) = \bigvee_{k \cdot j \geq n} A_k \wedge B_j,$$

where $[\frac{n}{s}]$ denotes the smallest integer greater than $\frac{n}{s}$. A proof of the right-most equality in the above expressions for $\dot{+}$ and $\dot{\times}$ is given by Lemmas 5.3 and 5.4 below, respectively.

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[1] Note that from here on, when talking about the arithmetic of the quantum natural numbers, we will denote the quantum natural numbers by $A, B, ...$ instead of $f, g, ...$ that we have been using up to now. This allows us surreptitiously to naturally use the same symbol for a quantum natural number and its associated Hermitian operator when $L = \mathscr{P}(\mathcal{H})$. Also notice that we change the way in which we name the elements of the decreasing sequence associated with a quantum natural number — i.e. we now denote the first element of a sequence by A_1 instead of A_0. This notational shift facilitates future computations and makes certain conditions easier to state.

Lemma 5.3. Let $p, q \in \omega_c$, and let $A, B \in \omega_L$ be such that $A : p \to L$, $B : q \to L$, with $A(i) \coloneqq A_{i+1}$ for $i \in p$ and $A_i \leq A_j$ for $j \in i$, and $B(i) \coloneqq B_{i+1}$ for $i \in q$ with $B_i \leq B_j$ for $j \in i$. Then

$$\bigvee_{k=1}^{n-1} A_k \wedge B_{n-k} = \bigvee_{k+j \geq n} A_k \wedge B_j.$$

Proof. Clearly, we have that

$$\bigvee_{k=1}^{n-1} A_k \wedge B_{n-k} \leq \bigvee_{k+j=n} A_k \wedge B_j.$$

Consider $A_k \wedge B_j$ with $k + j \geq n$. It follows that $j \geq n - k$, and since the B_i's form a decreasing sequence, we have that $B_j \leq B_{n-k}$. As such, we see that anything extra in $\bigvee_{k+j=n} A_k \wedge B_j$ will not contribute to the join because it will be less than some element which is in $\bigvee_{k=1}^{n-1} A_k \wedge B_{n-k}$. Thus, we have that

$$\bigvee_{\substack{k+j=n \\ k \leq n}} A_k \wedge B_j = \bigvee_{k=1}^{n-1} A_k \wedge B_{n-k}.$$

Now, for $n < k$, we have that

$$A_k \wedge B_j \leq A_{n-1} \wedge B_j \leq A_{n-1} \wedge B_1,$$

and we see that these terms won't contribute to the join either. As such, we see that

$$\bigvee_{k+j \geq n} A_k \wedge B_j \leq \bigvee_{k=1}^{n-1} A_k \wedge B_{n-k}.$$

Together, these inequalities give that

$$\bigvee_{k+j\geq n} A_k \wedge B_j = \bigvee_{k=1}^{n-1} A_k \wedge B_{n-k}.$$

□

Lemma 5.4. Let $p, q \in \omega_c$, and let $A, B \in \omega_L$ be such that $A : p \to L$, $B : q \to L$, with $A(i) \coloneqq A_{i+1}$ for $i \in p$ and $A_i \leq A_j$ for $j \in i$, and $B(i) \coloneqq B_{i+1}$ for $i \in q$ with $B_i \leq B_j$ for $j \in i$. Then

$$\bigvee_{s=1}^{n-1} A_s \wedge B_{[\frac{n}{s}]} = \bigvee_{k\cdot j\geq n} A_k \wedge B_j.$$

Proof. Clearly, we have that

$$\bigvee_{s=1}^{n-1} A_s \wedge B_{[\frac{n}{s}]} \leq \bigvee_{k\cdot j\geq n} A_k \wedge B_j$$

since each term in $\bigvee_{s=1}^{n-1} A_s \wedge B_{[\frac{n}{s}]}$ is a term in $\bigvee_{k\cdot j\geq n} A_k \wedge B_j$.

Consider $A_k \wedge B_j$ with $k \cdot j \geq n$. It follows that $j \geq \frac{n}{k}$, so that $[\frac{n}{k}] \leq j$. We then have that $A_k \wedge B_j \leq A_k \wedge B_{[\frac{n}{k}]}$ since the B_i's form a decreasing sequence. As such, we see that anything extra in $\bigvee_{k\cdot j\geq n} A_k \wedge B_j$ will not contribute to the join because it will be less than some element which is in $\bigvee_{s=1}^{n} A_s \wedge B_{[\frac{n}{s}]}$. Thus, we see that

$$\bigvee_{k\cdot j\geq n} A_k \wedge B_j \leq \bigvee_{s=1}^{n} A_s \wedge B_{[\frac{n}{s}]}.$$

Together, these inequalities give that

$$\bigvee_{k\cdot j\geq n} A_k \wedge B_j = \bigvee_{s=1}^{n} A_s \wedge B_{[\frac{n}{s}]}.$$

□

With regard to the arithmetical operations $\dot{\times}$ and $\dot{+}$, we obviously must have that each of $A \dot{\times} B$ and $A \dot{+} B$ is associated with a decreasing sequence of elements of the relevant orthomodular lattice L in order for $\dot{\times}$ and $\dot{+}$ to be well-defined operations in any $\mathcal{L}_A$-structure $\hat{\omega}(L)$. It turns out that this is relatively straight-forward to show, and the details are included below in Lemmas 5.5 and 5.6.

Lemma 5.5. $(A \dot{+} B)_{n+1} \leq (A \dot{+} B)_n$.

Proof. In what follows, we denote $A_n = P_n$ and $B_n = Q_n$ for any $n \in \omega_c$. By definition, we have that

$$(A \dot{+} B)_n = P_n \vee Q_n \vee [\bigvee_{k=1}^{n-1} (P_k \wedge Q_{n-k})]$$

and

$$(A \dot{+} B)_{n+1} = P_{n+1} \vee Q_{n+1} \vee [\bigvee_{k=1}^{n} (P_k \wedge Q_{n+1-k})].$$

However, since both the P_i's and the Q_i's form a decreasing sequence, we have that $P_{n+1} \leq P_n$ and $Q_{n+1} \leq Q_n$. Now, if we consider the terms in $\bigvee_{k=1}^{n-1}(P_k \wedge Q_{n-k})$ and $\bigvee_{k=1}^{n}(P_k \wedge Q_{n_1-k})$, we see that

$$\bigvee_{k=1}^{n-1} (P_k \wedge Q_{n-k}) = (P_1 \wedge Q_{n-1}) \vee (P_2 \wedge Q_{n-2}) \vee ... \vee (P_{n-1} \wedge Q_1)$$

and

$$\bigvee_{k=1}^{n}(P_k \wedge Q_{n+1-k}) = (P_1 \wedge Q_n) \vee (P_2 \wedge Q_{n-1}) \vee \ldots \vee (P_n \wedge Q_1).$$

Since $Q_n \leq Q_{n-1}$, we have that $P_1 \wedge Q_n \leq P_1 \wedge Q_{n-1}$; in general, we have that $P_s \wedge Q_{n+1-s} \leq P_s \wedge Q_{n-s}$. Thus, since every term in $(A \dot{+} B)_{n+1}$ is less than or equal to the corresponding term in $(A \dot{+} B)_n$, we obtain the result

$$(A \dot{+} B)_{n+1} \leq (A \dot{+} B)_n.$$

□

Lemma 5.6. $(A \dot{\times} B)_{n+1} \leq (A \dot{\times} B)_n$.

Proof. In what follows, we denote $A_n = P_n$ and $B_n = Q_n$ for any $n \in \omega_c$. By definition, we have that

$$(A \dot{\times} B)_n = \bigvee_{s=1}^{n}(P_s \wedge Q_{[\frac{n}{s}]})$$

and

$$(A \dot{\times} B)_{n+1} = \bigvee_{s=1}^{n+1}(P_s \wedge Q_{[\frac{n+1}{s}]})$$

Since the Q_i's form a decreasing sequence, we have that $Q_{n+1} \leq Q_n$, from which it follows that $Q_{[\frac{n+1}{s}]} \leq Q_{[\frac{n}{s}]}$ for any classical natural number s. As such,

$$P_s \wedge Q_{[\frac{n+1}{s}]} \leq P_s \wedge Q_{[\frac{n}{s}]},$$

so that every term in $(A \dot{\times} B)_{n+1}$ is less than or equal to the corresponding term in $(A \dot{\times} B)_n$, and we obtain the result

$$(A \dot{\times} B)_{n+1} \leq (A \dot{\times} B)_n.$$

□

One interesting feature of these arithmetical operations in the case in which $L = \mathscr{P}(\mathcal{H})$, is that if two quantum natural numbers are associated with commuting Hermitian operators, then $\dot{\times}$ and $\dot{+}$ correspond to ordinary multiplication and addition, respectively, of these operators. We now prove these properties (Lemmas 5.7 and 5.8 below).

Lemma 5.7. Let $L = \mathscr{P}(\mathcal{H})$, and let A and B be linear operators associated with quantum natural numbers such that $[A, B] = 0$. Then $A \dot{+} B = A + B$, where "+" denotes ordinary addition of linear operators.

Proof. We first note that $[A, B] = 0$ implies that there exists a basis for $\mathcal{H}$ consisting of common eigenvectors of A and B. (In what follows, we denote $A_n = P_n$ and $B_n = Q_n$ for any $n \in \omega_c$.) We also have that the projector R_n associated with $(A + B)_n$ is such that R_n is the projector onto the subspace given by $Span(\{|\psi_i\rangle\}_{i \in I})$, where

$$\{|\psi_i\rangle\}_{i \in I} := \{|\psi_i\rangle \in \mathcal{H} \mid (A + B)|\psi_i\rangle = \lambda_i |\psi_i\rangle \quad \text{with} \quad \lambda_i \geq n\}.$$

Let $A \leftrightarrow P_1, P_2, ...P_k$, where P_n is the projector onto the subspace given by the span of

$$\{|\psi_i\rangle \in \mathcal{H} \mid A|\psi_i\rangle = \gamma_i |\psi_i\rangle \quad \text{with} \quad \gamma_i \geq n\}.$$

And, let $B \leftrightarrow Q_1, Q_2, ...Q_m$, where Q_n is the projector onto the subspace given by the span of

$$\{|\psi_i\rangle \in \mathcal{H} \mid B|\psi_i\rangle = \nu_i|\psi_i\rangle \quad \text{with} \quad \nu_i \geq n\}.$$

Also, recall that by definition, we have that

$$(A \dot{+} B)_n := P_n \vee Q_n \vee [\bigvee_{k=1}^{n-1} (P_k \wedge Q_{n-k})].$$

For $|\psi_i\rangle$ an element of the subspace associated with P_n, we have that $A|\psi_i\rangle = \gamma_i|\psi_i\rangle$ with $\gamma_i \geq n$; also, we have that $(A+B)|\psi_i\rangle = \lambda_i|\psi_i\rangle$, where $\lambda_i = \gamma_i + \mu_i$ (which follows from the fact that A and B are commuting Hermitian operators with positive integer eigenvalues). However, this shows that (since $\lambda_i \geq \gamma_i \geq n$) $|\psi_i\rangle$ is an element of the subspace associated with R_n. Similarly, for $|\psi_i\rangle$ an element of the subspace associated with Q_n, we have that $B|\psi_i\rangle = \mu_i|\psi_i\rangle$ with $\mu_i \geq n$; also, we have that $(A+B)|\psi_i\rangle = \lambda_i|\psi_i\rangle$, where $\lambda_i = \gamma_i + \mu_i$ (which follows from the fact that A and B are commuting Hermitian operators with positive integer eigenvalues). However, this shows that (since $\lambda_i \geq \gamma_i \geq n$) $|\psi_i\rangle$ is an element of the subspace associated with R_n. Since these results give that $P_n \leq R_n$ and $Q_n \leq R_n$, respectively, it follows that $P_n \vee Q_n \leq R_n$.

Next consider $P_k \wedge Q_{n-k}$ for $k \in \{1, 2, ...n-1\}$. Let $|\psi_i\rangle$ be an element of the subspace associated with $P_k \wedge Q_{n-k}$; then we have that $A|\psi_i\rangle = \gamma_i|\psi_i\rangle$ with $\gamma_i \geq k$ and $B|\psi_i\rangle = \mu_i|\psi_i\rangle$ with $\mu_i \geq n-k$. Then $(A+B)|\psi_i\rangle = \lambda_i|\psi_i\rangle$, where $\lambda_i = \gamma_i + \mu_i \geq k + n - k = n$, which shows that $|\psi_i\rangle$ is an element of the subspace associated with R_n; and thus that $P_k \wedge Q_{n-k} \leq R_n$ (for

$k \in \{1, 2, ...n-1\}$). Since each such term is $\leq R_n$, we have that $[\bigvee_{k=1}^{n-1} P_k \wedge Q_{n-k}] \leq R_n$.

The latter result, along with the fact (established above) that $P_n \vee Q_n \leq R_n$ gives that

$$P_n \vee Q_n \vee [\bigvee_{k=1}^{n-1} P_k \wedge Q_{n-k}] \leq R_n,$$

i.e. $(A \dot{+} B)_n \leq R_n = (A + B)_n$.

We next wts that $(A + B)_n = R_n \leq (A \dot{+} B)_n$. Let $|\psi_i\rangle$ be such that it is an element of the subspace associated with R_n; then we have that $(A + B)|\psi_i\rangle = \lambda_i|\psi_i\rangle$, where $\lambda_i =\geq n$. Now, we have that $\lambda_i = \gamma_i + \mu_i \geq n$, where $A|\psi_i\rangle = \gamma_i|\psi_i\rangle$ and $B|\psi_i\rangle = \mu_i|\psi_i\rangle$ since $|\psi_i\rangle$ is a common eigenvector of A and B.

If $\mu_i = 0$, then $\gamma_i \geq n$, and as such, $|\psi_i\rangle$ is an element of the subspace associated with P_n (from which it follows that $|\psi_i\rangle$ is an element of the subspace associated with $(A \dot{+} B)_n$). Similarly, if $\gamma_i = 0$, then $\mu_i \geq n$, and as such, $|\psi_i\rangle$ is an element of the subspace associated with Q_n (from which it follows that $|\psi_i\rangle$ is an element of the subspace associated with $(A \dot{+} B)_n$). Next, suppose that $\lambda_i = n$; if $\mu_i \neq 0$ and $\gamma_i \neq 0$, and $\gamma_i = k$ for some $k \leq n$, then $\mu_i = n - k$ and we have that $|\psi_i\rangle$ is an element of the subspace associated with $P_k \wedge Q_{n-k}$, so that $|\psi_i\rangle$ is an element of the subspace associated with $(A \dot{+} B)_n$. Since any terms $P_m \wedge Q_l$ with $m + l > n$ are such that $P_m \wedge Q_l \leq P_k \wedge Q_{n-k}$ for some $k < n$ (which follows from the fact that both the P_i's and the Q_j's form decreasing sequences), we have that if $|\psi_i\rangle$ is an element of the subspace associated with $P_m \wedge Q_l$, then $|\psi_i\rangle$ is an element of the subspace associated with $(A \dot{+} B)_n$. As such, we see that $R_n \leq (A \dot{+} B)_n$.

Together, the results $R_n \leq (A \dot{+} B)_n$ and $(A \dot{+} B)_n \leq R_n$ give that $R_n = (A \dot{+} B)_n$; and since $R_n = (A + B)_n$, we have that $(A + B)_n = (A \dot{+} B)_n$, from which it follows that $A + B = A \dot{+} B$ whenever $[A, B] = 0$. □

Lemma 5.8. Let $L = \mathscr{P}(\mathcal{H})$, and let A and B be linear operators associated with quantum natural numbers such that $[A, B] = 0$. Then $A \dot{\times} B = A \cdot B$, where "·" denotes ordinary multiplication of linear operators.

Proof. We first note that $[A, B] = 0$ implies that there exists a basis for $\mathcal{H}$ consisting of common eigenvectors of A and B. (In what follows, we denote $A_n = P_n$ and $B_n = Q_n$ for any $n \in \omega_c$.) We also have that the projector R_n associated with $(A \cdot B)_n$ is such that R_n is the projector onto the subspace given by $Span(\{|\psi_i\rangle\}_{i \in I})$, where

$$\{|\psi_i\rangle\}_{i \in I} := \{|\psi_i\rangle \in \mathcal{H} \mid (A + B)|\psi_i\rangle = \lambda_i |\psi_i\rangle \quad \text{with} \quad \lambda_i \geq n\}.$$

Let $A \leftrightarrow P_1, P_2, ...P_k$, where P_n is the projector onto the subspace given by the span of

$$\{|\psi_i\rangle \in \mathcal{H} \mid A|\psi_i\rangle = \gamma_i |\psi_i\rangle \quad \text{with} \quad \gamma_i \geq n\}.$$

And, let $B \leftrightarrow Q_1, Q_2, ...Q_m$, where Q_n is the projector onto the subspace given by the span of

$$\{|\psi_i\rangle \in \mathcal{H} \mid B|\psi_i\rangle = \nu_i|\psi_i\rangle \quad \text{with} \quad \nu_i \geq n\}.$$

Also, recall that by definition, we have that

$$(A \dot{\times} B)_n := \bigvee_{\substack{s,t \leq n \\ s \cdot t \geq n}} (P_s \wedge Q_t) = \bigvee_{s=1}^{n} (P_s \wedge Q_{[\frac{n}{s}]}),$$

where $[\frac{n}{s}]$ denotes the smallest integer greater than $\frac{n}{s}$.

First suppose that $|\psi_i\rangle$ is an element of the subspace associated with $P_s \wedge Q_t$, where $s, t \in \{1, ...n\}$ and $s \cdot t \geq n$. This gives that $A|\psi_i\rangle = \gamma_i|\psi_i\rangle$ with $\gamma_i \geq s$ and $B|\psi_i\rangle = \mu_i|\psi_i\rangle$ with $\mu_i \geq t$ since A and B are commuting linear operators with positive integer eigenvalues. As such, we see that $\gamma_i \cdot \mu_i \geq s \cdot t \geq n$ so that $|\psi_i\rangle$ is an element of the subspace associated with R_n — i.e. $(A \cdot B)|\psi_i\rangle = \lambda_i|\psi_i\rangle$, where $\lambda_i = \gamma_i \cdot \mu_i \geq n$, which shows that $|\psi_i\rangle$ is an element of the subspace associated with R_n. Thus, since this holds for any choice of $s, t \in \{1, ...n\}$ with $s \cdot t \geq n$, we have that $(A \dot{\times} B)_n \leq R_n$.

Now suppose that $|\psi_i\rangle$ is an element of the subspace associated with R_n. This gives that $(A \cdot B)|\psi_i\rangle = \lambda_i|\psi_i\rangle$, where $\lambda_i \geq n$ and $\lambda_i = \gamma_i \cdot \mu_i$ since A and B are commuting linear operators with positive integer eigenvalues. In order to see that $|\psi_i\rangle$ is an element of the subspace associated with $(A \dot{\times} B)_n$, note that since $\gamma_i = k$ for some positive integer k and $\mu_i = m$ for some positive integer m, with $k \cdot m \geq n$, we have that $|\psi_i\rangle$ is an element of the subspace associated with $P_k \wedge Q_m$, which is some element of $\bigvee_{\substack{s,t \leq n \\ s \cdot t \geq n}} (P_s \wedge Q_t) = (A \dot{\times} B)_n$. If, on the other hand, either $\gamma_i \nleq n$ or $\mu_i \nleq n$, we would still have that $|\psi_i\rangle$ is an element of the subspace associated with $P_{\gamma_i} \wedge Q_{\mu_i}$. However, because the P_i's and Q_i's form a decreasing sequence, we have that $P_{\gamma_i} \wedge Q_{\mu_i} \leq P_k \wedge Q_m$ for some positive integers m, k which satisfy the constraint $k \leq n$, $m \leq n$ and $k \cdot m \geq n$. Thus, we have that $R_n \leq (A \dot{\times} B)_n$.

Together, the results $R_n \leq (A \dot{\times} B)_n$ and $(A \dot{\times} B)_n \leq R_n$ give that $R_n = (A \dot{\times} B)_n$; and since $R_n = (A \cdot B)_n$, we have that $(A \cdot B)_n = (A \dot{\times} B)_n$, from which it follows that $A \cdot B = A \dot{\times} B$ whenever $[A, B] = 0$. $\square$

5.4.1 The Arithmetic Axioms

The purely arithmetical axioms we consider (due to Peano) are given by

(A1) $(\forall x)[x \dot{+} 0 = x]$

(A2) $(\forall x)(\forall y)[x \dot{+} y' = (x \dot{+} y)']$

(A3) $(\forall x)[x \dot{\times} 1 = x]$

(A4) $(\forall x)(\forall y)[x \dot{\times} y' = (x \dot{\times} y) \dot{+} x]$

Using the definitions of the arithmetical operations $\dot{\times}$ and $\dot{+}$ given above, we have that the first three arithmetic axioms hold in any $\mathcal{L}_A$-structure $\hat{\omega}(L)$, as Theorem 22 below shows. However, the fourth axiom holds if and only if the L is *modular*; for a proof of this result, see Theorem 23 below.

Theorem 22. The arithmetic axioms (A1) — (A3) hold in any $\mathcal{L}_A$-structure $\hat{\omega}(L)$.

Proof. In what follows, we denote $A_n = P_n$ and $B_n = Q_n$ for any $n \in \omega_c$.

(A1) $(\forall x)[x \dot{+} 0 = x]$

By definition, we have that

$$(A \dot{+} B)_n := P_n \vee Q_n \vee [\bigvee_{k=1}^{n-1} (P_k \wedge Q_{n-k})],$$

and also, $0_n = 0$ for all n. And so,

$$(A \dot{+} 0)_n = P_n \vee 0 \vee [\bigvee_{k=1}^{n-1} (P_k \wedge 0)] = P_n.$$

We clearly have that $A_n = P_n$ for any A. Thus, for any $A \in \omega_L$, we have that $\hat{\omega}(L) vDash A \dot{+} 0 = A$, so that axiom (A1) holds in $\hat{\omega}(L)$ — i.e. $\hat{\omega}(L) \vdash (\forall x)[x \dot{+} 0 = x]$.

(A2) $(\forall x)(\forall y)[x \dot{+} y' = (x \dot{+} y)']$

First note that $(B')_n = B_{n-1}$ for all $n \neq 1$ and $(B')_1 = 1$. Now, by definition, we have that

$$(A \dot{+} B')_n := P_n \vee Q_{n-1} \vee [\bigvee_{k=1}^{n-1} (P_k \wedge Q_{(n-1)-k})]$$

$$= P_n \vee Q_{n-1} \vee [\bigvee_{k=1}^{n-2} P_k \wedge Q_{(n-1)-k}] \vee (P_{n-1} \wedge 1)$$

$$= P_n \vee P_{n-1} \vee Q_{n-1} \vee [\bigvee_{k=1}^{n-2} P_k \wedge Q_{(n-1)-k}]$$

$$= P_{n-1} \vee Q_{n-1} \vee [\bigvee_{k=1}^{n-2} P_k \wedge Q_{(n-1)-k}],$$

where the last equality follows from the fact that $P_{n-1} \vee P_n = P_{n-1}$ since the P_i's form a decreasing sequence. Now consider $[(A \dot{+} B)_n]'$. We have that

$$[(A \dot{+} B)_n]' = (A \dot{+} B)_{n-1} = P_{n-1} \vee Q_{n-1} \vee [\bigvee_{k=1}^{n-2} P_k \wedge Q_{(n-1)-k}].$$

Thus, we see that for arbitrary $A, B \in \omega_L$, we have that $\hat{\omega}(L) vDash [A \dot{+} B]' = A \dot{+} B'$. As such, we have that axiom (A2) holds in $\hat{\omega}(L)$ — i.e.

$$\hat{\omega}(L) \vdash (\forall x)(\forall y)[x \dot{+} y' = (x \dot{+} y)'].$$

(A3) $(\forall x)[x \dot{\times} 1 = x]$

By definition, we have that

$$(A \dot{\times} B)_n := \bigvee_{\substack{s,t \le n \\ s \cdot t \ge n}} (P_s \wedge Q_t) = \bigvee_{s=1}^{n} (P_s \wedge Q_{[\frac{n}{s}]}),$$

where $[\frac{n}{s}]$ denotes the smallest integer greater than $\frac{n}{s}$, and also $1_1 = 1$, while $1_n = 0$ for all $n > 1$. And so,

$$(A \dot{\times} 1)_n = \bigvee_{s=1}^{n} (P_s \wedge 1_{[\frac{n}{s}]}) = P_n \wedge 1 = P_n.$$

We clearly have that $A_n = P_n$ for any A. Thus, for any $A \in \omega_L$, we have that $\hat{\omega}(L) vDash A \dot{\times} 1 = A$, so that axiom (A3) holds in $\hat{\omega}(L)$ — i.e. $\hat{\omega}(L) \vdash (\forall x)[x \dot{\times} 1 = x]$.

□

Theorem 23. The distributive axiom (A4) of arithmetic holds in an $\mathcal{L}_A$-structure $\hat{\omega}(L)$ if and only if L is *modular*.

Proof.

- Let $A = (A_1, A_2, ...A_i...)$ and $B = (B_1, B_2, ...B_q...)$. We have that

$$(A \dot{\times} B')_n = A_n \vee \bigvee_{\substack{k \cdot j \geq n \\ j \neq 1}} A_k \wedge B_{j-1},$$

where we have used Lemma 5.4 along with the fact that $(B')_n = B_{n-1}$ for all $n \neq 1$, and $(B')_1 = 1$. We also have that

$$((A \dot{\times} B) \dot{+} A)_n = A_n \vee \bigvee_{k \cdot j \geq n} (A_k \wedge B_j) \vee \bigvee_{k+j \geq n} [(\bigvee_{s \cdot t \geq k} (A_s \wedge B_t)) \wedge A_j],$$

where we have used Lemma 5.3 along with the fact that by Lemma 5.4, $(A \dot{\times} B)_n = \bigvee_{k \cdot j \geq n} A_k \wedge B_j$. Finally, we recall that the modular identity is given by

$$x \leq b \Longrightarrow x \vee (a \wedge b) = (x \vee a) \wedge b.$$

We first show that $((A \dot{\times} B) \dot{+} A)_n \leq (A \dot{\times} B')_n$. We do this by showing that each piece of the expression for $((A \dot{\times} B) \dot{+} A)_n$ is $\leq$ some piece in $(A \dot{\times} B')_n$; it then follows that the join of all such pieces is $\leq (A \dot{\times} B')_n$. We clearly have that $A_n \leq (A \dot{\times} B')_n$. Now, for the term $\bigvee_{k\cdot j\geq n}(A_k \wedge B_j)$, we consider the following two cases: $j = 1$ and $j \neq 1$. For $j = 1$, we have that $k \cdot j \geq n$ implies that $k \geq n$. Since in such a case (i.e. for $n \leq k$) $A_k \leq A_n$ (since the A_i's form a decreasing sequence), we have that $A_k \wedge B_\alpha \leq A_n$ for any B_α; as such, we have in this case that $A_k \wedge B_j \leq (A \dot{\times} B')_n$. Now suppose that $j \neq 1$ — since the B_i's form a decreasing sequence, we know that $B_j \leq B_{j-1}$, and hence, $A_k \wedge B_j \leq A_k \wedge B_{j-1}$, which shows that $A_k \wedge B_j \leq (A \dot{\times} B')_n$. And so, it follows that

$$\bigvee_{k\cdot j\geq n}(A_k \wedge B_j) \leq (A \dot{\times} B')_n.$$

We next consider the last term in $((A \dot{\times} B) \dot{+} A)_n$ — namely,

$$\bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t) \wedge A_j)].$$

As such, we take $k + j \geq n$, and consider $(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t)) \wedge A_j$ — we have that

$$(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t)) \wedge A_j = (\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}}(A_s \wedge B_t) \vee \bigvee_{\substack{s\cdot t\geq k\\ s<j}}(A_s \wedge B_t)) \wedge A_j.$$

For $j \leq s$, we have that $A_s \leq A_j$ so that $A_s \wedge B_t \leq A_j$; and thus, $[\bigvee_{s\cdot t\geq k, s\geq j}(A_s \wedge B_t)] \leq A_j$. We now use modularity along with the latter inequality (i.e. $[\bigvee_{s\cdot t\geq k, s\geq j}(A_s \wedge B_t)] \leq A_j$) in order to re-associate — i.e. we have that

$$[\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}}(A_s \wedge B_t) \vee \bigvee_{\substack{s\cdot t\geq k\\ s<j}}(A_s \wedge B_t)] \wedge A_j$$

$$= [\bigvee_{s\cdot t\geq k, s\geq j, s,t\leq k} (A_s \wedge B_t)] \vee [A_j \wedge (\bigvee_{s\cdot t\geq k, s<j} (A_s \wedge B_t))].$$

Wlog, we consider $k + j = n$, and as such, we have that $k = n - j$. Then, since $s \geq l$, we have $\frac{l}{s} \leq 1$, so that

$$t \geq \frac{n}{s} - \frac{j}{s} \geq \frac{n}{s} - 1,$$

which implies that $B_t \leq B_{[\frac{n}{s}]-1}$. This in turn, implies that

$$A_s \wedge B_t \leq A_s \wedge B_{[\frac{n}{s}]-1}.$$

(We have that $n \leq s \cdot [\frac{n}{s}]$.) And since

$$\frac{n}{s} = \frac{k+j}{s} = \frac{k}{s} + \frac{j}{s} \geq 1,$$

$A_s \wedge B_{[\frac{n}{s}]-1}$ is some term in $\bigvee_{\substack{k\cdot j\geq n\\ j\neq 1}} A_k \wedge B_{j-1}$; thus, we have that each such term satisfies

$$A_s \wedge B_t \leq \bigvee_{\substack{k\cdot j\geq n\\ j\neq 1}} A_k \wedge B_{j-1} \leq (A \dot{\times} B')_n.$$

It then follows that

$$\bigvee_{\substack{s\cdot t\geq k, s\geq j\\ s,t\leq k}} (A_s \wedge B_t) \leq \bigvee_{\substack{k\cdot j\geq n\\ j\neq 1}} A_k \wedge B_{j-1} \leq (A \dot{\times} B')_n.$$

Finally, we consider $A_j \wedge [\bigvee_{s \cdot t \geq k, s<j}(A_s \wedge B_t)]$. Here we have that (since $\frac{1}{j} \leq \frac{1}{s}$)

$$t \geq \frac{k}{s} = \frac{n-j}{s} \geq \frac{n-j}{j} \geq [\frac{n}{j}] - 1,$$

which implies that $B_t \leq B_{[\frac{n}{j}]-1}$. We also have that $j < n$ (since we take $k + j = n$), and so $2 \leq [\frac{n}{j}]$ (which follows from the fact that $1 < \frac{n}{j}$). And so, we have that $A_j \wedge B_{[\frac{n}{j}]-1}$ is a term in $\bigvee_{\substack{k \cdot j \geq n \\ j \neq 1}}(A_k \wedge B_{j-1})$. Now, since $B_t \leq B_{[\frac{n}{j}]-1}$ and $A_s \wedge B_t \leq B_t$, it follows that $[\bigvee_{\substack{s \cdot t \geq k \\ s<j}}(A_s \wedge B_t)] \leq B_{[\frac{n}{j}]-1}$, and thus,

$$A_j \wedge [\bigvee_{\substack{s \cdot t \geq k \\ s<j}}(A_s \wedge B_t)] \leq A_j \wedge B_{[\frac{n}{j}]-1}.$$

Then, finally, using the above result that $A_j \wedge B_{[\frac{n}{j}]-1}$ is a term in $\bigvee_{\substack{k \cdot j \geq n \\ j \neq 1}}(A_k \wedge B_{j-1})$, we have that

$$A_j \wedge [\bigvee_{\substack{s \cdot t \geq k \\ s<j}}(A_s \wedge B_t)] \leq \bigvee_{\substack{k \cdot j \geq n \\ j \neq 1}}(A_k \wedge B_{j-1}) \leq (A \dot{\times} B')_n.$$

Since we have shown that all terms in $((A \dot{\times} B) \dot{+} A)_n$ is $\leq$ some piece in $(A \dot{\times} B')_n$; it then follows that the join of all such pieces is $\leq (A \dot{\times} B')_n$. From this, we establish that $((A \dot{\times} B) \dot{+} A)_n \leq (A \dot{\times} B')_n$.

We now show that $(A \dot{\times} B')_n \leq ((A \dot{\times} B) \dot{+} A)_n$. We note that this inequality effectively follows from the distributive inequality (which holds in any lattice). Consider $\bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t)) \wedge A_j]$. Using the distributive inequality, we have that

$$\bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t \wedge A_j))] \leq \bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t)) \wedge A_j].$$

We also have that

$$\bigvee_{k+j\geq n}[(\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}}(A_s \wedge B_t \wedge A_j))] \leq \bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t \wedge A_j))].$$

Since $A_s \wedge A_j = A_s$ when $s \geq j$, we have that the above inequality simplifies to

$$\bigvee_{k+j\geq n}[(\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}}(A_s \wedge B_t)] \leq \bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t \wedge A_j)],$$

and thus we see that,

$$\bigvee_{k+j\geq n}[(\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}}(A_s \wedge B_t)] \leq \bigvee_{k+j\geq n}[(\bigvee_{s\cdot t\geq k}(A_s \wedge B_t)) \wedge A_j].$$

Now, $k \geq n - j$ so $s \cdot t \geq k$ becomes $s \cdot t \geq n - j$; also, $j \leq s$ so

$$s(t+1) = s \cdot t + s \geq n - j + s \geq n,$$

where we have used that $s-j \geq 0$ (since $j \leq s$) to establish that $n \leq s(t+1)$. These results show that

$$[\bigvee_{k+j\geq n} (\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}} (A_s \wedge B_t))] \leq \bigvee_{s(t+1)\geq n} (A_s \wedge B_t).$$

We now wts that

$$[\bigvee_{s(t+1)\geq n} (A_s \wedge B_t)] \leq \bigvee_{k+j\geq n} [(\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}} (A_s \wedge B_t))].$$

Assume that $s(t+1) \geq n$; then for $k = s\cdot t$ and $j = s$, we have that $k+j = s(t+1)$. But, $k = s\cdot t$ and $j = s$ show that $s \cdot t \geq k$ and $s \geq j$, so the term $A_s \wedge B_t$ is in $\bigvee_{k+j\geq n}[(\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}}(A_s \wedge B_t))]$, which establishes the above inequality. Together the two inequalities give that

$$[\bigvee_{s(t+1)\geq n} (A_s \wedge B_t)] = \bigvee_{k+j\geq n} [(\bigvee_{\substack{s\cdot t\geq k\\ s\geq j}} (A_s \wedge B_t))].$$

Relabelling $t+1 := m$ (noting that the fact that s and t are positive integers gives that $m \neq 1$), we have that $[\bigvee_{s(t+1)\geq n}(A_s \wedge B_t)]$ becomes $[\bigvee_{\substack{s\cdot m\geq n\\ m\neq 1}}(A_s \wedge B_m)]$, and thus, we see that

$$[\bigvee_{\substack{s\cdot m\geq n\\ m\neq 1}} (A_s \wedge B_m)] \leq \bigvee_{k+j\geq n} [(\bigvee_{s\cdot t\geq k} (A_s \wedge B_t)) \wedge A_j].$$

It then follows that $(A \dot{\times} B')_n \leq ((A \dot{\times} B) \dot{+} A)_n$.

- What follows is a counter-example that shows that the distributive axiom (A4) of arithmetic does not hold generally in our models for addition and multiplication of quantum natural numbers defined by (for $A(i) := P_{i+1}$ for $i \in \sup A$ with $P_i \leq P_j$ for $j \leq i$ and $B(i) := Q_{i+1}$ for $i \in \sup B$ with $Q_i \leq Q_j$ for $j \leq i$)

$$(A \dot{+} B)_n := P_n \vee Q_n \vee [\bigvee_{k=1}^{n-1} (P_k \wedge Q_{n-k})],$$

and

$$(A \dot{\times} B)_n := \bigvee_{\substack{s,t \leq n \\ s \cdot t \geq n}} (P_s \wedge Q_t) = \bigvee_{s=1}^{n} (P_s \wedge Q_{[\frac{n}{s}]}),$$

respectively, where $[\frac{n}{s}]$ denotes the smallest integer greater than $\frac{n}{s}$. We note that this counter-example requires the OML L to be non-modular; as such, it shows that if the OML L is non-modular, then the distributive axiom of Peano arithmetic does not hold — i.e. modularity is necessary for the distributive axiom of Peano arithmetic to hold.

We let

$$A = (P_1, P_2, P_3) \quad \text{and} \quad B = (Q_1, Q_2, Q_3)$$

Also, we recall that any OML which is *not* modular has an M_5 sub-lattice. We take the top element of that M_5 sub-lattice to be $P_1 = Q_1$, and define the bottom element to be x; we take $Q_2 = Q_3$ to be the "nose" of the M_5 sub-lattice and take the elements of the other chain to be P_2 and P_3, where $P_3 \leq P_2$. We compare $(A \dot{\times} B')_5$ and $((A \dot{\times} B) \dot{+} A)_5$.

In order to evaluate $(A \dot{\times} B')_5$, note that $B' = (I, Q_1, Q_2, Q_3)$. We have that

$$(A \dot{\times} B')_5 = \bigvee_{\substack{s,t \le 5 \\ s \cdot t \ge 5}} (A_s \wedge (B')_t) = (A_3 \wedge (B')_2) \vee (A_2 \wedge (B')_3)$$

$$= (P_3 \wedge Q_1) \vee (P_2 \wedge Q_2) = P_3 \vee x = P_3,$$

where we have used the fact that in order to evaluate $\bigvee_{s,t \le p; s \cdot t \ge p} (P_s \wedge Q_t)$, we only need to take (for each fixed P_j) $P_n \wedge Q_m$, where m is the smallest integer with which n can be paired.

Next consider the terms of $A \dot{\times} B$. We have that

$$(A \dot{\times} B)_1 = P_1 \wedge Q_1 = P_1 = Q_1$$

$$(A \dot{\times} B)_2 = (P_1 \wedge Q_2) \vee (P_2 \wedge Q_1) = P_1 = Q_1$$

$$(A \dot{\times} B)_3 = (P_1 \wedge Q_3) \vee (P_2 \wedge Q_2) \vee (P_3 \wedge Q_1) = P_1 = Q_1$$

$$(A \dot{\times} B)_4 = (P_3 \wedge Q_2) \vee (P_2 \wedge Q_2) = (P_2 \wedge Q_2) = x$$

$$(A \dot{\times} B)_5 = (P_3 \wedge Q_2) \vee (P_2 \wedge Q_3) = x.$$

Using these results, we have that

$$((A \dot{\times} B) \dot{+} A)_5$$

$$= P_5 \vee (A \dot{\times} B)_5 \vee (P_1 \wedge (A \dot{\times} B)_4) \vee (P_2 \wedge (A \dot{\times} B)_3) \vee (P_3 \wedge (A \dot{\times} B)_2) \vee (P_4 \wedge (A \dot{\times} B)_1)$$

$$= 0 \vee x \vee (P_1 \wedge x) \vee (P_2 \wedge P_1) \vee (P_3 \wedge P_1) \vee (0 \wedge P_1) = x \vee x \vee P_2 \vee P_3 = P_2.$$

Clearly $P_2 \neq P_3$, so we see that the distributive axiom of Peano arithmetic does not hold.

□

Together, Theorems 22 and 23, along with Lemma 5.2 (with the $\mathcal{L}_S$-wffs that appear in the lemma considered as $\mathcal{L}_A$-wffs) show that the $\mathcal{L}_A$-structures $\hat{\omega}(L)$ are models for M-systems $(\mathcal{L}_A, \mathcal{A}_A)$ if and only if L is modular, where $\mathcal{A}_A$ is taken to consist of the successor fragment axioms (SF1)–(SF5) along with the purely arithmetical axioms (A1)–(A4). Additionally, we note that Theorem 23 is a little surprising — we would not have expected modularity to play this role in our quantum arithmetic. We also note that this result shows that in the context of quantum logic and quantum set theory (on which the arithmetic is built), the algebraic property of modularity has purely arithmetical content! This result also suggests that there should be some natural way of reducing the axiom using modularity — this is to say that we would like some clever way of rewriting (in a form which is equivalent to the usual one in the presence of classical logic) the axiom (A4) so that it holds in any $\mathcal{L}_A$-structure $\hat{\omega}(L)$, and when L is modular, it reduces to the standard distributivity axiom.[1]

5.4.2 Substitution Axioms

The substitution axioms for $\dot{+}$ and $\dot{\times}$ are given by (respectively)

(Sub+) $(\forall x)(\forall y)(\forall z)[x = y \rightarrow x \dot{+} z = y \dot{+} z]$,

(Sub×) $(\forall x)(\forall y)(\forall z)[x = y \rightarrow x \dot{\times} z = y \dot{\times} z]$.

We note that in order for these to hold in our models $\hat{\omega}(L)$, we must have that the following lattice inequalities hold in L (for arbitrary $A, B, C \in \omega_L$):

$$[\![A = B]\!] \leq [\![A \dot{+} C = B \dot{+} C]\!]$$

and

$$[\![A = B]\!] \leq [\![A \dot{\times} C = B \dot{\times} C]\!],$$

respectively.

The following lemmas (i.e. Lemmas 5.9 and 5.10) provide simple examples which show that these inequalities are not satisfied in general.

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[1]This is actually something that, at the time of the writing of this document, we are still working on.

Lemma 5.9. In the models $\hat{\omega}(L)$, the substitution axiom for $\dot{\times}$ does not hold in general.

Proof. In order to see that substitution for $\dot{\times}$ does not hold generally, let $L = \mathscr{P}(\mathcal{H})$ where $dim(\mathcal{H}) = 2$, and let

$$A = (I, I, I, P, P, P), \quad B = (I, I, I, I, P, P), \quad \text{and} \quad C = (I, I, I, I, Q, Q),$$

where $P \neq Q$ and $P, Q \neq I, 0$. Now, note that $[\![A = B]\!] = P$. That this is what $A = B$ valuates to follows from the fact that in $\bigwedge_{i \in I} A_i \leftrightarrow B_i$, we get at least one term that is $P \leftrightarrow 1$, and we get no $P \leftrightarrow 0$ terms; all other terms in $\bigwedge_{i \in I} A_i \leftrightarrow B_i$ are 1. Thus, we have $[\![A = B]\!] = P$.

Now, an explicit calculation gives that

$$A \dot{\times} C = (I, I, ...I, P, P, ...P),$$

where there are 18 I's and 6 P's; also, we obtain $B \dot{\times} C = (I, I, ...I)$, where there are 24 I's.

Finally, we have that $[\![A \dot{\times} C = B \dot{\times} C]\!] = 0$ since $\bigwedge_{i \in I} (A \dot{\times} C)_i \wedge (B \dot{\times} C)_i$ has at least one term that is $P \leftrightarrow Q = 0$. (Note that $P \leftrightarrow Q = 0$ follows from the fact that $P \rightarrow Q = \neg P \vee (P \wedge Q) = \neg P$ and $Q \rightarrow P = \neg Q \vee (P \wedge Q) = \neg Q$, so that $P \leftrightarrow Q = \neg P \wedge \neg Q = 0$ since we consider a two-dimensional example and we take $P \neq Q$ and $P, Q \neq I, 0$.)

And so, since $P \nleq 0$, we see that $[\![A = B]\!] \nleq [\![A \dot{\times} C = B \dot{\times} C]\!]$, which shows that substitution for $\dot{\times}$ does not hold even in the in the two-dimensional case. $\square$

Lemma 5.10. In the models $\hat{\omega}(L)$, the substitution axiom for $\dot{+}$ does not hold in general.

Proof. In order to see that substitution for $\dot{\times}$ does not hold generally, we again let $L = \mathscr{P}(\mathcal{H})$ where $dim(\mathcal{H}) = 2$, and we let

$$A = (I, I, I), \quad B = (Q, Q, Q), \quad \text{and} \quad C = (I, P, P),$$

where $P \neq Q$ and $P, Q \neq I, 0$. Now, note that $[\![A = B]\!] = Q$. That this is what $A = B$ valuates to follows from the fact that in $\bigwedge_{i \in I} A_i \leftrightarrow B_i$, the only contributions come from $I \leftrightarrow Q = Q$ (since all other terms are $0 \leftrightarrow 0 = 1$); thus, we have that $[\![A = B]\!] = Q$. An explicit calculation gives

$$A \dot{+} C = (I, I, I, I, P, P) \quad \text{and} \quad B \dot{+} C = (I, I, I, Q, Q),$$

from which we see that

$$[\![A \dot{\times} C = B \dot{\times} C]\!] = \bigwedge_{i \in I} (A \dot{\times} C)_i \wedge (B \dot{\times} C)_i$$

$$= (I \leftrightarrow I) \wedge (I \leftrightarrow I) \wedge (I \leftrightarrow I) \wedge (I \leftrightarrow Q) \wedge (P \leftrightarrow Q) \wedge (P \leftrightarrow 0) = Q \wedge 0 \wedge P^{\perp} = 0,$$

where we have used the fact that $P \leftrightarrow Q = 0$. As such, we see that $[\![A \dot{+} C = B \dot{+} C]\!] = 0$, and so, since $Q \nleq 0$, we see that $[\![A = B]\!] \nleq [\![A \dot{+} C = B \dot{+} C]\!]$, which shows that substitution for $\dot{+}$ does not hold in the two-dimensional case. $\square$

Given the results of Lemmas 5.9 and 5.10 above, we would like to have a natural way of reducing the substitution axioms for $\dot{+}$ and $\dot{\times}$.

One possibility is given by[1]

(RSub+) $(\forall x)(\forall y)(\forall z)[\hat{\mathbf{T}}(x = y) \to x \dot{+} z = y \dot{+} z]$,

(RSub×) $(\forall x)(\forall y)(\forall z)[\hat{\mathbf{T}}(x = y) \to x \dot{\times} z = y \dot{\times} z]$,

where, for an arbitrary $\mathcal{L}_A$-wff ϕ

$$\hat{\mathbf{T}}(\phi) := (\forall x)(\forall y)[x = y \to \phi].$$

We note that we also need to define

$$\hat{\mathbf{C}}(\phi) := (\forall x)(\forall y)[\varphi_{x=y}(\phi) \to \phi],$$

and add the $\mathcal{L}_A$-wff schema

$$\hat{\mathbf{C}}(\hat{\mathbf{T}}(\phi)) \quad \text{and} \quad \hat{\mathbf{T}}(\phi) \to \phi$$

as axioms in order to insure that $\hat{\mathbf{T}}(\phi)$ behaves the way we want it to in the models $\hat{\omega}(L)$. We note that this method of reducing the substitution axioms is elegant, as well as effective, but may be a little "heavy-handed" for the job we need it to do.

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[1]Recall the discussion in Section 2.5.2 where the "classicality operators" **T** and **C** are defined for an arbitrary predicate.

5.5 Consequences of the Arithmetic for Quantum Natural Numbers

When considering the arithmetic associated with ordinary (classical) natural numbers, one can prove, as simple consequences of the definitions of the addition and multiplication functions, that both are commutative and associative, as well as that "full" distributivity of multiplication over addition holds — i.e. for $m, n, k \in \omega_c$ that

$$n \dot{\times} (m \dot{+} k) = (n \dot{\times} m) \dot{+} (n \dot{\times} k).$$

While the quantum multiplication and addition are both clearly commutative by definition, neither is an associative operation, as the following counter-examples in Lemmas 5.11 and 5.12 show. Moreover, "full" distributivity does not hold in general either for the quantum natural numbers (see Lemma 5.13).

Lemma 5.11. In the models $\hat{\omega}(L)$, the addition $\dot{+}$ is not, in general, associative.

Proof. In order to see that associativity of $\dot{+}$ does not hold in general, let $L = \mathscr{P}(\mathcal{H})$ with $dim(\mathcal{H}) = 2$, and let $A = P$, $B = Q$, and $C = R$ for distinct one-dimensional projectors P, Q, R. We have that

$$A \dot{+} B = (P \vee Q, P \wedge Q) \quad \text{and} \quad B \dot{+} C = (Q \vee R, Q \wedge R),$$

while

$$(A \dot{+} B) \dot{+} C = (P \vee Q \vee R, (P \wedge Q) \vee (R \wedge (P \vee Q)), P \wedge Q \wedge R).$$

Taking $L = MO_n$, we have that

$$(A \dot{+} B) \dot{+} C = (I, R).$$

On the other hand, we have that

$$A \dot{+} (B \dot{+} C) = (P \vee Q \vee R, (Q \wedge R) \vee (P \wedge (Q \vee R)), P \wedge Q \wedge R),$$

so that if we let $L = MO_n$, we have that

$$A \dot{+} (B \dot{+} C) = (I, P).$$

Thus, we see that

$$A \dot{+} (B \dot{+} C) \neq (A \dot{+} B) \dot{+} C,$$

and as such, $\dot{+}$ is not an associative operation. □

Lemma 5.12. In the models $\hat{\omega}(L)$, the multiplication $\dot{\times}$ is not, in general, associative.

Proof. In order to see that associativity of $\dot{\times}$ does not hold in general, let $L = \mathscr{P}(\mathcal{H})$ with $dim(\mathcal{H}) = 2$, and let $A = (I, P)$, $B = (I, Q)$, and $C = R$, where P, Q, R are distinct one-dimensional projectors. Again, letting $L = MO_n$, we have that

$$A \dot{\times} B = (I, P \vee Q, P \wedge Q, P \wedge Q) = (I, I),$$

while

$$B \dot{\times} C = R.$$

Now,

$$(A \dot{\times} B) \dot{\times} C = (R, R),$$

while

$$A \dot{\times} (B \dot{\times} C) = R.$$

Thus, we see that

$$A \dot{\times} (B \dot{\times} C) \neq (A \dot{\times} B) \dot{\times} C,$$

and as such, $\dot{\times}$ is not an associative operation. $\square$

Lemma 5.13. Full distributivity of multiplication over addition does not hold generally in the models $\hat{\omega}(L)$ — i.e. there are models for which

$$A \dot{\times} (B \dot{+} C) \neq (A \dot{\times} B) \dot{+} (A \dot{\times} C)$$

for some $A, B, C \in \omega_L$.

Proof. In order to see that in any model $\hat{\omega}(L)$, full distributivity does not always hold in the arithmetic for the quantum natural numbers, let $L = MO_n$, and let

$$A = (I, I, P, P), \quad B = (I, I, Q, Q, Q, Q), \quad \text{and} \quad C = (I, R, R, R),$$

where $P \neq Q$, $P \neq R$ and $Q \neq R$; also $P, Q, R \neq I, 0$. We have that

$$(B \dot{+} C) = (I, I, I, I, I, I, Q),$$

and an explicit calculation shows that $A \dot{\times} (B \dot{+} C)$ is given by 14 I's and 10 P's. Now, we also have that

$$(A \dot{\times} B) = (I, I, I, I, I, I, I, I, Q, Q, Q, Q),$$

and

$$(A \dot{\times} C) = (I, I, I, I, R, R, R, R).$$

From these, one can then show that $(A \dot{\times} B) \dot{+} (A \dot{\times} C)$ is given by 16 I's. Thus, we see that

$$A \dot{\times} (B \dot{+} C) \neq (A \dot{\times} B) \dot{+} (A \dot{\times} C).$$

$\square$

Although the above counter-examples in Lemmas 5.11, 5.12 and 5.13 show that associativity and full distributivity (of multiplication over addition) do not hold in the models $\hat{\omega}(L)$ in general, we do have that lattice distributivity is sufficient for the associative laws and full (arithmetical) distributivity to hold in the models $\hat{\omega}(L)$. We illustrate this below for distributivity of multiplication over addition, and note that the proofs for associativity are similar.

Theorem 24. Let L be a distributive lattice. Then in the model $\hat{\omega}(L)$, we have that

$$A \dot{\times} (B \dot{+} C) = (A \dot{\times} B) \dot{+} (A \dot{\times} C).$$

holds for all $A, B, C \in \omega_L$.

Proof. First note that

$$[A \dot{\times} (B \dot{+} C)]_n = \bigvee_{\alpha \cdot \beta \geq n} (A_\alpha \wedge (B \dot{+} C)_\beta) = \bigvee_{\alpha \cdot \beta \geq n} (A_\alpha \wedge [B_\beta \vee C_\beta \vee \bigvee_{m+j \geq \beta} (B_m \wedge C_j)])$$

$$= \bigvee_{\substack{\alpha \cdot \beta \geq n \\ m+j \geq \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))],$$

where we have used lattice distributivity to obtain the last equality. We also have that

$$[(A \dot{\times} B) \dot{+} (A \dot{\times} C)]_n = (A \dot{\times} B)_n \vee (A \dot{\times} C)_n \vee \bigvee_{k+l \geq n} [(A \dot{\times} B)_k \wedge (A \dot{\times} C)_l]$$

$$= [\bigvee_{s \cdot t \geq n} (A_s \wedge B_t)] \vee [\bigvee_{u \cdot w \geq n} (A_u \wedge C_w)] \vee \bigvee_{k+l \geq n} [(\bigvee_{p \cdot q \geq k} (A_p \wedge B_q)) \wedge (\bigvee_{y \cdot z \geq l} (A_y \wedge C_z))]$$

$$= [\bigvee_{s \cdot t \geq n} (A_s \wedge B_t)] \vee [\bigvee_{u \cdot w \geq n} (A_u \wedge C_w)] \vee \bigvee_{\substack{k+l \geq n \\ p \cdot q \geq k \\ y \cdot z \geq l}} [(A_p \wedge B_q) \wedge (A_y \wedge C_z)],$$

where we have again used lattice distributivity to obtain the last equality.

We first show that $[A \dot{\times} (B \dot{+} C)]_n \leq [(A \dot{\times} B) \dot{+} (A \dot{\times} C)]_n$. We take an arbitrary term in $[A \dot{\times} (B \dot{+} C)]_n$ — e.g.

$$(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))$$

with $\alpha \cdot \beta \geq n$ and $m + j \geq \beta$. Now, for $\alpha \cdot \beta \geq n$, we have that $A_\alpha \wedge B_\beta$ is some term in $\bigvee_{s \cdot t \geq n}(A_s \wedge B_t)$, which gives that

$$A_\alpha \wedge B_\beta \leq \bigvee_{s \cdot t \geq n} (A_s \wedge B_t).$$

Similarly, for $\alpha \cdot \beta \geq n$, we have that $A_\alpha \wedge C_\beta$ is some term in $\bigvee_{u \cdot w \geq n}(A_u \wedge C_w)$, which gives that

$$A_\alpha \wedge C_\beta \leq \bigvee_{u \cdot w \geq n} (A_u \wedge C_w).$$

Together these results give that

$$(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \leq [\bigvee_{s \cdot t \geq n} (A_s \wedge B_t)] \vee [\bigvee_{u \cdot w \geq n} (A_u \wedge C_w)].$$

We next show that for $\alpha \cdot \beta \geq n$ and $m + j \geq \beta$,

$$A_\alpha \wedge (B_m \wedge C_j) \leq \bigvee_{\substack{k+l \geq n \\ p \cdot q \geq k \\ y \cdot z \geq l}} [(A_p \wedge B_q) \wedge (A_y \wedge C_z)].$$

We have that

$$A_\alpha \wedge (B_m \wedge C_j) = A_\alpha \wedge A_\alpha \wedge (B_m \wedge C_j) = (A_\alpha \wedge B_m) \wedge (A_\alpha \wedge C_j),$$

where $\alpha \cdot \beta \geq n$ and $m + j \geq \beta$. And so, since $\alpha(m + j) = \alpha \cdot m + \alpha \cdot j \geq n$, we see that $(A_\alpha \wedge B_m) \wedge (A_\alpha \wedge C_j)$ is some term in $\bigvee_{\substack{k+l \geq n \\ p \cdot q \geq k \\ y \cdot z \geq l}} [(A_p \wedge B_q) \wedge (A_y \wedge C_z)]$. Thus, it follows that

$$\bigvee_{\substack{\alpha \cdot \beta \geq n \\ m+j \geq \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))]$$

$$\leq [\bigvee_{s \cdot t \geq n} (A_s \wedge B_t)] \vee [\bigvee_{u \cdot w \geq n} (A_u \wedge C_w)] \vee [\bigvee_{\substack{k+l \geq n \\ p \cdot q \geq k \\ y \cdot z \geq l}} [(A_p \wedge B_q) \wedge (A_y \wedge C_z)]],$$

or equivalently,

$$[A \dot{\times} (B \dot{+} C)]_n \leq [(A \dot{\times} B) \dot{+} (A \dot{\times} C)]_n.$$

We now wts that $[(A \dot{\times} B) \dot{+} (A \dot{\times} C)]_n \leq [A \dot{\times} (B \dot{+} C)]_n$. First consider $A_s \wedge B_t$ with $s \cdot t \geq n$; clearly, taking $s = \alpha$ and $t = \beta$ gives that (since $\alpha \cdot \beta \geq n$)

$$A_s \wedge B_t \leq (A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j)) \leq [A \dot{\times} (B \dot{+} C)]_n.$$

And since $A_s \wedge B_t$ is an arbitrary term in $\bigvee_{s \cdot t \geq n} (A_s \wedge B_t)$, it follows that

$$\bigvee_{s \cdot t \geq n} (A_s \wedge B_t) \leq \bigvee_{\substack{\alpha \cdot \beta \geq n \\ m+j \geq \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))] = [A \dot{\times} (B \dot{+} C)]_n.$$

Similarly, we have that

$$\bigvee_{u\cdot w\geq n}(A_u\wedge C_w)\leq\bigvee_{\substack{\alpha\cdot\beta\geq n\\ m+j\geq\beta}}[(A_\alpha\wedge B_\beta)\vee(A_\alpha\wedge C_\beta)\vee(A_\alpha\wedge(B_m\wedge C_j))]=[A\dot{\times}(B\dot{+}C)]_n.$$

Finally, we wts that

$$\bigvee_{\substack{k+l\geq n\\ p\cdot q\geq k\\ y\cdot z\geq l}}[(A_p\wedge B_q)\wedge(A_y\wedge C_z)]$$

$$\leq\bigvee_{\substack{\alpha\cdot\beta\geq n\\ m+j\geq\beta}}[(A_\alpha\wedge B_\beta)\vee(A_\alpha\wedge C_\beta)\vee(A_\alpha\wedge(B_m\wedge C_j))]=[A\dot{\times}(B\dot{+}C)]_n.$$

Consider $(A_p\wedge B_q)\wedge(A_y\wedge C_z)$ with $k+l\geq n$, $p\cdot q\geq k$, and $y\cdot z\geq l$. (Note that these conditions imply that $p\cdot q+y\cdot z\geq n$.) Now, for either $n\leq k\leq p\cdot q$ or $n\leq l\leq y\cdot z$, there exist appropriate choices of α and β which satisfy $\alpha\cdot\beta\geq n$ and are such that

$$A_p\wedge B_q\leq A_\alpha\wedge B_\beta\leq\bigvee_{\substack{\alpha\cdot\beta\geq n\\ m+j\geq\beta}}[(A_\alpha\wedge B_\beta)\vee(A_\alpha\wedge C_\beta)\vee(A_\alpha\wedge(B_m\wedge C_j))],$$

or

$$A_y\wedge C_z\leq A_\alpha\wedge C_\beta\leq\bigvee_{\substack{\alpha\cdot\beta\geq n\\ m+j\geq\beta}}[(A_\alpha\wedge B_\beta)\vee(A_\alpha\wedge C_\beta)\vee(A_\alpha\wedge(B_m\wedge C_j))],$$

respectively. (Note also that the same holds for $k\leq n$ but $p\cdot q\geq n$ and $l\leq n$ but $y\cdot z\geq n$.)

And so, we're left with the cases $k\leq p\cdot q<n$ and $l\leq y\cdot z<n$, where $k+l\geq n$. As above, we consider $(A_p\wedge B_q)\wedge(A_y\wedge C_z)$. If $p\leq y$, then $A_y\leq A_p$, so we have that

$$(A_p\wedge B_q)\wedge(A_y\wedge C_z)=(A_p\wedge A_y)\wedge(B_q\wedge C_z)=A_y\wedge(B_q\wedge C_z),$$

where the second equality follows from the fact that $\wedge$ is associative and the last equality follows from the fact that $A_y \le A_p$. Now, since we assume that $k + l \ge n$, we have that $n \le p \cdot q + y \cdot z$; also, since $p \le y$, $p \cdot q \le y \cdot q$, so it follows that

$$n \le p \cdot q + y \cdot z \le y \cdot q + y \cdot z = y(q + z).$$

Choosing $\alpha = y$, $j = z$, $m = q$ and $\beta := [\frac{n}{\alpha}]$, we get that $\alpha \cdot \beta \ge n$ (since $\alpha \cdot [\frac{n}{\alpha}] \ge n$) as well as $m + j \ge \beta$ (since $n \le y(q + z)$). Thus, we see that $A_y \wedge (B_q \wedge C_z) \le A_\alpha \wedge (B_m \wedge C_j)$. Note also that the argument is exactly the same for the case $y \le p$ (i.e. just interchange y and p in the discussion above); as such, we omit this case in order to avoid excessive repetition.

So, we see that

$$(A_p \wedge B_q) \wedge (A_y \wedge C_z) \le \bigvee_{\substack{\alpha \cdot \beta \ge n \\ m+j \ge \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))],$$

from which it follows that

$$\bigvee_{\substack{k+l \ge n \\ p \cdot q \ge k \\ y \cdot z \ge l}} [(A_p \wedge B_q) \wedge (A_y \wedge C_z)] \le \bigvee_{\substack{\alpha \cdot \beta \ge n \\ m+j \ge \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))].$$

Then, since we also have that

$$\bigvee_{s \cdot t \ge n} (A_s \wedge B_t) \le \bigvee_{\substack{\alpha \cdot \beta \ge n \\ m+j \ge \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))] = [A \dot{\times} (B \dot{+} C)]_n$$

and

$$\bigvee_{u \cdot w \geq n} (A_u \wedge C_w) \leq \bigvee_{\substack{\alpha \cdot \beta \geq n \\ m+j \geq \beta}} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))] = [A \dot{\times} (B \dot{+} C)]_n,$$

it follows that

$$[\bigvee_{s \cdot t \geq n} (A_s \wedge B_t)] \vee [\bigvee_{u \cdot w \geq n} (A_u \wedge C_w)] \vee [\bigvee_{\substack{k+l \geq n \\ p \cdot q \geq k \\ y \cdot z \geq l}} [(A_p \wedge B_q) \wedge (A_y \wedge C_z)]]$$

$$\leq \bigvee_{\alpha \cdot \beta \geq n, m+j \geq \beta} [(A_\alpha \wedge B_\beta) \vee (A_\alpha \wedge C_\beta) \vee (A_\alpha \wedge (B_m \wedge C_j))].$$

That is,

$$[(A \dot{\times} B) \dot{+} (A \dot{\times} C)]_n \leq [A \dot{\times} (B \dot{+} C)]_n.$$

□

5.5.1 Algebraic Identities in Two Variables

In the $\mathcal{L}_A$-structures $\hat{\omega}(L)$ for which L is modular, it turns out that our quantum arithmetic looks classical with regard to two variable expressions — this is to say that if we have two classical polynomials $p(x,y)$ and $q(x,y)$ which are such that $p(x,y)$ can be manipulated into $q(x,y)$ via standard arithmetic, then the corresponding identity for quantum natural numbers

$\hat{p}(A,B) = \hat{q}(A,B)$ will also hold[1] — that is $[\![\hat{p}(A,B) = \hat{q}(A,B)]\!] = 1$ when L is modular. This remarkable property follows easily from Theorem 25 below.

Theorem 25. Let L be a complete modular ortholattice, and let $A, B \in \omega_L$. Then, for any $n \in \omega_c$

$$\hat{p}(A,B)_n = \bigvee_{p(j,k) \geq n} (A_j \wedge B_k), \tag{5.4}$$

where p is any two-variable polynomial expression.

Proof. We proceed via proof by induction on the construction of (two-variable) polynomials p. There are there are two base cases to consider — namely, we consider case $p(x,y) = 0$, as well as the case $p(x,y) = x$. For the case $p(x,y) = 0$, we recall that for any model $\hat{\omega}(L)$, the quantum natural number described by the map which sends every $n \in \omega_c$ to 0 in L is the analogue of $\varnothing$ (or the ordinary classical number 0). As such, by definition $0_0 = 1$ and $0_n = 0$ for all $n \geq 1$. (Recall that $X_0 := 1$ for any $X \in \omega_L$, but that this is not part of the sequence which defines X.) We see by inspection that this agrees with

$$0_n = \bigvee_{0 \geq n} (A_n \wedge B_k)$$

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[1]We use $\hat{p}(A,B)$ to denote the quantum natural number constructed in the same way that the ordinary classical polynomial expression $p(x,y)$ is constructed from x and y using classical addition and multiplication, but instead using the arithmetical operations $\dot{+}$ and $\dot{\times}$ and quantum natural numbers A and B.

from the fact that $A_0 = B_0 = 1$ and the join over an empty set is the bottom element of L. The other base case for a term is $p(x,y) = x$, and we easily see that for this p,

$$\bigvee_{p(j,k)\geq n} (A_j \wedge B_k) = \bigvee_{j\geq n} (A_j \wedge B_k) = \bigvee_{j\geq n} (A_j) = A_n = \hat{p}(A,B).$$

We now consider the inductive steps. The first operation to consider is the successor. Assume that $p(x,y)$ satisfies equation 5.4. We need to show that $[p(x,y)]'$ also satisfies this equation. It is easy to see that for any quantum natural number C, we have that $(C')_n = C_{n-1}$ for all $n \geq 1$ (and, of course, $(C')_0 = 1$). From this, we compute (for $n \geq 1$ — the $n = 0$ case is trivial)

$$[p(A,B)']_n = [p(A,B)]_{n-1} = \bigvee_{p(j,k)\geq n-1} (A_j \wedge B_k) = \bigvee_{p(j,k)'\geq n} (A_j \wedge B_k)$$

for all $n \geq 1$ (since the classical successor function effectively just gives that $k' = k + 1$ for any $k \in \omega_c$).

For the next inductive step, we consider the sum and product simultaneously — to this end, let $* \in \{\dot{+}, \dot{\times}\}$ and assume that equation 5.4 holds for the two terms $p(x,y)$ and $q(x,y)$. First, using Theorem 32, we see that the set of all A_i's and B_j's generate a distributive sub-lattice of L. Hence, we can use this (lattice) distributivity with abandon.[1]

$$\begin{aligned}
[p(A,B) \dot{*} q(A,B)]_n &= \bigvee_{k*l\geq n} \Big(\big[\bigvee_{p(a,b)\geq k} (A_a \wedge B_b) \big] \wedge \big[\bigvee_{q(c,d)\geq l} (A_c \wedge B_d) \big] \Big) \\
&= \bigvee_{k+l\geq n} \Big[\bigvee_{\substack{p(a,b)\geq k \\ q(c,d)\geq l}} \big(A_a \wedge B_b \wedge A_c \wedge B_d \big) \Big] \\
&= \bigvee_{k+l\geq n} \Big[\bigvee_{\substack{p(a,b)\geq k \\ q(c,d)\geq l}} \big(A_{\max(a,c)} \wedge B_{\max(b,d)} \big) \Big],
\end{aligned}$$

where we have used that the A_i's and B_j's form a decreasing sequence. Now, first note that we trivially have

$$\bigvee_{k+l\geq n} \Big[\bigvee_{\substack{p(a,b)\geq k \\ q(c,d)\geq l}} \big(A_{\max(a,c)} \wedge B_{\max(b,d)} \big) \Big] \geq \bigvee_{p(r,s)*q(r,s)\geq n} (A_r \wedge B_s)$$

as we can see that (for any r and s satisfying $p(r,s) * q(r,s) \geq n$) the term $A_r \wedge B_s$ is in the join on the LHS of the above expression by taking $a = c = r$ and $b = d = s$ along with $k = p(r,s)$ and $l = q(r,s)$. To get the other inequality, note that for any $k, l, a, b, c, d \in \omega_c$ such that $k + l \geq n$, $p(a,b) \geq k$ and $q(c,d) \geq l$, we can take $r = \max(a,c)$ and $s = \max(b,d)$. Then $p(r,s) \geq p(a,b)$ and $q(r,s) \geq q(c,d)$ (due to the monotonicity of polynomials in standard arithmetic on natural

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[1] Although we appear to have infinite joins, since both A and B are quantum natural numbers, in fact the joins only run over a finite number of terms.

numbers), and this means that $p(r,s) + q(r,s) \geq k + l \geq n$, so that any term in the LHS of the above expression is also in the right, yielding that in fact we have

$$[p(A,B) \dot{*} q(A,B)]_n = \bigvee_{k+l \geq n} \Big[\bigvee_{\substack{p(a,b) \geq k \\ q(c,d) \geq l}} \big(A_{\max(a,c)} \wedge B_{\max(b,d)}\big)\Big] = \bigvee_{p(r,s) * q(r,s) \geq n} (A_r \wedge B_s)$$

which completes the induction. □

5.6 Properties of a Class of Models Relevant for Quantum Theory

In what follows, we consider properties of the class of models $\hat{\omega}(\mathscr{P}(\mathcal{H}))$. We begin by noting that the eigenvalues of $A \dot{\times} B$ are actually a subset of the products of eigenvalues of A and eigenvalues of B; similarly, the eigenvalues of $A \dot{+} B$ are a subset of the sums of eigenvalues of A and eigenvalues of B. Theorems 26 and 27 below establish these remarkable results, which point to a possible interpretation of $\dot{+}$ and $\dot{\times}$ in terms of measurement.

Theorem 26. The eigenvalues of $A \dot{\times} B$ are a subset of the products of eigenvalues of A and eigenvalues of B.

Proof. Consider a model $\hat{\omega}(\mathscr{P}(\mathcal{H}))$, and let A be an arbitrary quantum natural number thought of as a Hermitian operator on $\mathcal{H}$ (whose eigenvalues are ordinary natural numbers) — that is, let $A = \sum_{i \in k} P_i$, where $k = \sup A$. Further, for any $\lambda \in \mathbb{R}$, let S_A^λ denote the projector onto the subspace defined by

$$\{|\psi\rangle \in \mathcal{H} \mid A|\psi\rangle = a|\psi\rangle \quad \text{with} \quad a \geq \lambda\}.$$

Then we have that $P_{[\lambda]} = S_A^\lambda$, where $[\lambda]$ denotes the smallest integer greater than λ. To see that this is so, first note that by definition,

$$S_A^\lambda|\psi\rangle = \begin{cases} |\psi\rangle, & \text{if} \quad \lambda \leq a \\ 0, & \text{if} \quad \lambda > a. \end{cases}$$

On the other hand, if we have $|\psi\rangle \in \mathcal{H}$ such that $A|\psi\rangle = \alpha|\psi\rangle$, then $P_\alpha|\psi\rangle = |\psi\rangle$, while $P_{\alpha+1}|\psi\rangle = 0$. Since the P_i's form a decreasing sequence, it follows immediately that $P_n|\psi\rangle = |\psi\rangle$ for all $n \leq \alpha$, as well as that $P_m|\psi\rangle = 0$ for all $m \geq \alpha + 1$. As such, we see that $P_{[\lambda]} = S_A^\lambda$.

Given this relationship, we re-write the formula for $A \dot{\times} B$ in terms of the S^λ's. Let A and B be a quantum real numbers thought of as Hermitian operators on $\mathcal{H}$, and let $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ denote the eigenvalues of A and B, respectively. We have that

$$S^\lambda_{A\dot{\times}B} = \bigvee_{\alpha\cdot\beta\geq\lambda}(S_A^\alpha \wedge S_B^\beta) = \bigvee_{0\nleq\alpha\leq a_1, \alpha\cdot\beta\geq\lambda}(S_A^{a_1} \wedge S_B^\beta) \vee ... \vee \bigvee_{a_{n-1}\nleq\alpha\leq a_n, \alpha\cdot\beta\geq\lambda}(S_A^{a_n} \wedge S_B^\beta).$$

Now, since the S_B^χ's form a decreasing sequence, we can, wlog, choose $\beta = \frac{\lambda}{\alpha}$ to be the smallest value in the range (which corresponds to taking α to be the largest value in each range). We then obtain that

$$S^\lambda_{A\dot{\times}B} = \bigvee_{0\nleq\alpha\leq a_1}(S_A^{a_1} \wedge S_B^{\frac{\lambda}{\alpha}}) \vee ... \vee \bigvee_{a_{n-1}\nleq\alpha\leq a_n}(S_A^{a_n} \wedge S_B^{\frac{\lambda}{\alpha}}) = \bigvee_{i=1}^{n}(S_A^{a_i} \wedge S_B^{\frac{\lambda}{a_i}}).$$

Now, we're interested in where $S^{\lambda}_{A \dot{\times} B}$ jumps as a function of λ. However, the only λ dependence is in $S_B^{\frac{\lambda}{a_i}}$ — and since S_B^{χ} jumps at exactly the eigenvalues of B, we must have that the places where $S^{\lambda}_{A \dot{\times} B}$ jumps are such that $\frac{\lambda}{a_i} = b_j$ for some eigenvalue b_j of B. That is, $\lambda = a_i \cdot b_j$, which shows that the eigenvalues of $A \dot{\times} B$ are products of eigenvalues of A with eigenvalues of B. □

Theorem 27. The eigenvalues of $A \dot{+} B$ are a subset of the sums of eigenvalues of A and eigenvalues of B.

Proof. Consider a model $\hat{\omega}(\mathscr{P}(\mathcal{H}))$, and let A be an arbitrary quantum natural number thought of as a Hermitian operator on $\mathcal{H}$ (whose eigenvalues are ordinary natural numbers) — that is, let $A = \sum_{i \in k} P_i$, where $k = \sup A$. Further, for any $\lambda \in \mathbb{R}$, let S^{λ}_A denote the projector onto the subspace defined by

$$\{|\psi\rangle \in \mathcal{H} \mid A|\psi\rangle = a|\psi\rangle \quad \text{with} \quad a \geq \lambda\}.$$

Then we have that $P_{[\lambda]} = S^{\lambda}_A$, where $[\lambda]$ denotes the smallest integer greater than λ. To see that this is so, first note that by definition,

$$S^{\lambda}_A|\psi\rangle = \begin{cases} |\psi\rangle, & \text{if} \quad \lambda \leq a \\ \\ 0, & \text{if} \quad \lambda > a. \end{cases}$$

On the other hand, if we have $|\psi\rangle \in \mathcal{H}$ such that $A|\psi\rangle = \alpha|\psi\rangle$, then $P_{\alpha}|\psi\rangle = |\psi\rangle$, while $P_{\alpha+1}|\psi\rangle = 0$. Since the P_i's form a decreasing sequence, it follows immediately that $P_n|\psi\rangle = |\psi\rangle$ for all $n \leq \alpha$, as well as that $P_m|\psi\rangle = 0$ for all $m \geq \alpha + 1$. As such, we see that $P_{[\lambda]} = S_A^{\lambda}$.

Given this relationship, we re-write the formula for $A \dot{+} B$ in terms of the S^{λ}'s. Let A and B be a quantum real numbers thought of as Hermitian operators on $\mathcal{H}$, and let $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ denote the eigenvalues of A and B, respectively. We have that

$$S^{\lambda}_{A\dot{+}B} = \bigvee_{\alpha+\beta\geq\lambda}(S_A^{\alpha} \wedge S_B^{\beta}) = \bigvee_{0\nleq\alpha\leq a_1,\alpha+\beta\geq\lambda}(S_A^{a_1} \wedge S_B^{\beta}) \vee ... \vee \bigvee_{a_{n-1}\nleq\alpha\leq a_n,\alpha+\beta\geq\lambda}(S_A^{a_n} \wedge S_B^{\beta}).$$

Now, since the S_B^{χ}'s form a decreasing sequence, we can, wlog, choose $\beta = \lambda - \alpha$ to be the smallest value in the range (which corresponds to taking α to be the largest value in each range). We then obtain that

$$S^{\lambda}_{A\dot{\times}B} = \bigvee_{0\nleq\alpha\leq a_1}(S_A^{a_1} \wedge S_B^{\lambda-\alpha}) \vee ... \vee \bigvee_{a_{n-1}\nleq\alpha\leq a_n}(S_A^{a_n} \wedge S_B^{\lambda-\alpha}) = \bigvee_{i=1}^{n}(S_A^{a_i} \wedge S_B^{\lambda-a_i}).$$

Now, we're interested in where $S^{\lambda}_{A\dot{\times}B}$ jumps as a function of λ. However, the only λ dependence is in $S_B^{\lambda-a_i}$ — and since S_B^{χ} jumps at exactly the eigenvalues of B, we must have that the places where $S^{\lambda}_{A\dot{\times}B}$ jumps are such that $\lambda - a_i = b_j$ for some eigenvalue b_j of B. That is, $\lambda = a_i + b_j$, which shows that the eigenvalues of $A \dot{+} B$ are sums of eigenvalues of A with eigenvalues of B. $\square$

We also have that whenever $\mathscr{P}(\mathcal{H})$ is modular (i.e. $dim(\mathcal{H}) < \infty$), we can show that $\dot{\times}$ and $\dot{+}$ (as defined previously) are the unique operations which satisfy a certain set of criteria (Theorems 28 and 29 below). We begin by establishing a result (Lemma 5.14) which is useful in the proofs of Theorems 28 and 29.

Let $A = (P_1, P_2, ...P_n)$ be a quantum natural number (thought of as a linear operator on $\mathcal{H}$), and let $|\psi\rangle \in \mathcal{H}$. Also, let $P_0 := I$, but here note that P_0 is *not* part of the sequence that defines A.

We define a (classical) natural number $\#_A^{|\psi\rangle}$ associated with any pair $|\psi\rangle, A$ by

$$P_{\#_A^{|\psi\rangle}}|\psi\rangle = |\psi\rangle \quad \text{and} \quad P_{\#_A^{|\psi\rangle}+1}|\psi\rangle \neq |\psi\rangle.$$

The number $\#_A^{|\psi\rangle}$ always exists for any $|\psi\rangle \in \mathcal{H}$ and quantum natural number A. Note that if $A = nI$ for $n \in \omega_c$, then $\#_A^{|\psi\rangle} = n$ for any $|\psi\rangle \in \mathcal{H}$.

Lemma 5.14. $|\psi\rangle$ is an eigenvector of A if and only if $P_{\#_A^{|\psi\rangle}+1}|\psi\rangle = 0$. In this case, the associated eigenvalue is exactly $\#_A^{|\psi\rangle}$.

Proof. First assume that $|\psi\rangle$ is an eigenvector of A. Then we have that $A|\psi\rangle = \alpha|\psi\rangle$, where $A = \sum_i P_i$. By definition we have that $P_{\#_A^{|\psi\rangle}}|\psi\rangle = |\psi\rangle$. Since the P_i's form a decreasing sequence, $P_{\#_A^{|\psi\rangle}} \leq P_i$ for all $i \leq \#_A^{|\psi\rangle}$, so that for all such i, $P_i|\psi\rangle = |\psi\rangle$. As such, we have that $\#_A^{|\psi\rangle} \leq \alpha$. However, we also have (by definition) that $P_{\#_A^{|\psi\rangle}+1}|\psi\rangle \neq |\psi\rangle$. Note that if $P_{\#_A^{|\psi\rangle}+1}|\psi\rangle \neq 0$, then we have a contradiction to the fact that $|\psi\rangle$ is an eigenvector of A — this follows from the fact that if

$$P_{\#_A^{|\psi\rangle}+1}|\psi\rangle = |\phi\rangle \neq |\psi\rangle,$$

then $A|\psi\rangle = \lambda|\psi\rangle + \mu|\phi\rangle$; however, since the P_i's form a decreasing sequence, $P_{\#_A^{|\psi\rangle}+1} \leq P_i$ for all i such that $i \leq \#_A^{|\psi\rangle} + 1$, which implies that $P_i|\psi\rangle = |\phi\rangle$, which is a contradiction. And so, we see that $P_{\#_A^{|\psi\rangle}+1}|\psi\rangle = 0$, and also that it follows that $\#_A^{|\psi\rangle} = \alpha$.

Now assume that $P_{\#_A^{|\psi\rangle}+1}|\psi\rangle = 0$. By definition, $P_{\#_A^{|\psi\rangle}}|\psi\rangle = |\psi\rangle$, and since the P_i's form a decreasing sequence, $P_{\#_A^{|\psi\rangle}} \leq P_i$ for all i's such that $i \leq \#_A^{|\psi\rangle}$, so that $P_i|\psi\rangle = |\psi\rangle$ for all such i's. Thus, we see that

$$A|\psi\rangle = \sum_i P_i|\psi\rangle = \#_A^{|\psi\rangle}|\psi\rangle,$$

which shows that $|\psi\rangle$ is an eigenvector of A with eigenvalue $\#_A^{|\psi\rangle}$. $\square$

We now have the uniqueness theorems for $\dot{\times}$ and $\dot{+}$.

Theorem 28. *Uniqueness of* $\dot{\times}$: Let $L = \mathscr{P}(\mathcal{H})$ be modular, and let A, B, C be quantum natural numbers given by

$$A = (P_1, P_2, ...), \quad B = (Q_1, Q_2, ...), \quad \text{and} \quad C = (\widehat{R}_1, \widehat{R}_2, ...).$$

Further, assume that C satisfies:

1. Each $\widehat{R}_n$ is a lattice polynomial (i.e. a polynomial in $\wedge$ and $\vee$, but not $\neg$) in the P_i's and Q_j's. That is, we assume that the polynomial in the P_i's and Q_j's is the same no matter what A and B are inputs.

2. For $|\psi\rangle \in \mathcal{H}$,

$$P_n|\psi\rangle = Q_m|\psi\rangle = |\psi\rangle \Longrightarrow \widehat{R}_{n\cdot m}|\psi\rangle = |\psi\rangle.$$

3. If $|\psi\rangle$ is a simultaneous eigenstate of A and B with eigenvalues α and β, respectively, then $|\psi\rangle$ is an eigenstate of C with eigenvalue $\alpha \cdot \beta$.

Then $C = A \dot{\times} B$.

Proof. We wts that $\widehat{R}_n = R_n$ for all n. Recall that

$$[A \dot{\times} B]_n := R_n = \bigvee_{\substack{s\cdot t\geq n\\ s,t=1}}^{\infty} (P_s \wedge Q_t).$$

(i) We first show that $\widehat{R}_i \geq R_i$ for all i (where $\geq$ is the ordering in the lattice). From assumption/criterion (2) above, we have that

$$P_k|\psi\rangle = Q_m|\psi\rangle = |\psi\rangle \Longrightarrow \widehat{R}_{k\cdot m}|\psi\rangle = |\psi\rangle.$$

We also have that

$$P_k|\psi\rangle = Q_m|\psi\rangle = |\psi\rangle \Longrightarrow (P_k \wedge Q_m)|\psi\rangle = |\psi\rangle.$$

As such, we know that $\widehat{R}_{k \cdot m} \geq P_k \wedge Q_m$. However, we also know that $P_k \wedge Q_m \geq P_s \wedge Q_t$ for $s \geq k$ and $t \geq m$; and as such, it follows that $P_k \wedge Q_m \geq \bigvee_{\substack{s \geq k \\ t \geq m}} (P_s \wedge Q_t)$. And so, we have that

$$\widehat{R}_\lambda \geq \widehat{R}_{\mu \cdot \gamma} \geq P_\mu \wedge Q_\gamma,$$

where μ and γ are such that $\mu \cdot \gamma \geq \lambda$. Thus,

$$\widehat{R}_\lambda \geq \widehat{R}_{\mu \cdot \gamma} \geq \bigvee_{\mu \cdot \gamma \geq \lambda} (P_\mu \wedge Q_\gamma) = R_\lambda.$$

i.e. we have that $\widehat{R}_\lambda \geq R_\lambda$.

(ii) We have that

$$\widehat{R}_n = (P_{\alpha_1} \wedge Q_{\beta_1}) \vee (P_{\alpha_2} \wedge Q_{\beta_2}) \vee \ldots$$

That this is true follows from the fact that for any element y in a distributive lattice L, $y = a_1 \vee a_2 \vee \ldots$, where $a_i = g_1 \wedge g_2 \wedge \ldots \wedge g_q$ for $g_w \in G \subseteq L$ (where G is a generating set for L) — i.e. any element in a distributive lattice can be expressed in disjunctive normal form, which is the join of finite meets of generating elements. In our case, the P_i's and Q_j's are the generating set, and we know by Jonsson's Theorem (Theorem 32) that the sub-lattice of a modular lattice generated by two chains is a distributive sub-lattice. As such, we can always put any element R_n in the form above. Note that the fact that this is so follows from the fact that since the P_i's and Q_j's always each form a decreasing sequence, the meet of any subset of P_i's and Q_j's always comes down to just $P_p \wedge Q_q$ for a single projector from each chain.

(iii) Since by (i) above we have that $\widehat{R}_i \geq R_i$, and by (ii) above we have that

$$\widehat{R}_n = (P_{\alpha_1} \wedge Q_{\beta_1}) \vee (P_{\alpha_2} \wedge Q_{\beta_2}) \vee ...,$$

it follows that

$$\widehat{R}_n = R_n \vee (P_{\alpha_1} \wedge Q_{\beta_1}) \vee (P_{\alpha_2} \wedge Q_{\beta_2}) \vee ...$$

Now, if each $P_{\alpha_j} \wedge Q_{\beta_j}$ is such that $P_{\alpha_j} \wedge Q_{\beta_j} \leq R_n$, then $\widehat{R}_n = R_n$ for all n, and we're done with the proof. So, suppose that $\widehat{R}_n > R_n$. Then we have that there exists some $P_\alpha \wedge Q_\beta$ which is such that $P_\alpha \wedge Q_\beta \nleq R_n$ — i.e. wlog, take $\alpha \cdot \beta < n$. In particular, suppose that $P_\alpha = Q_\beta$, $P_{\alpha+1} = Q_{\beta+1} \neq P_\alpha$, and consider $|\psi\rangle \in \mathcal{H}$ such that $P_\alpha|\psi\rangle = |\psi\rangle$ (and as such $Q_\beta|\psi\rangle = |\psi\rangle$) and $P_{\alpha+1}|\psi\rangle = 0$ (and as such $Q_{\beta+1}|\psi\rangle = 0$). And since we assume that $\alpha \cdot \beta \nleq n$, we have that $\widehat{R}_n|\psi\rangle = 0$. Now, by Lemma 5.14 above, we have that

$$A|\psi\rangle = \alpha|\psi\rangle \quad \text{and} \quad B|\psi\rangle = \beta|\psi\rangle,$$

which by assumption/criterion (3) above implies that $|\psi\rangle$ is an eigenvector of C with eigenvalue $\alpha \cdot \beta$ — i.e. $C|\psi\rangle = \alpha \cdot \beta|\psi\rangle$ — so that by Lemma 5.14, $\widehat{R}_{\alpha\cdot\beta}|\psi\rangle = |\psi\rangle$ and $\widehat{R}_{\alpha\cdot\beta+1}|\psi\rangle = 0$. (Actually, we have that $\widehat{R}_n|\psi\rangle = 0$ for all $n > \alpha \cdot \beta$.) However, since

$(P_\alpha \wedge Q_\beta)|\psi\rangle = |\psi\rangle$, we see that $\widehat{R}_n|\psi\rangle \neq 0$, which is a contradiction. As such, we must have that $\widehat{R}_n = R_n$.

□

Theorem 29. *Uniqueness of* $\dot{+}$: Let $L = \mathscr{P}(\mathcal{H})$ be modular, and let A, B, C be quantum natural numbers given by

$$A = (P_1, P_2, \ldots), \quad B = (Q_1, Q_2, \ldots), \quad \text{and} \quad C = (\widehat{S}_1, \widehat{S}_2, \ldots).$$

Further, assume that C satisfies:

1. Each $\widehat{S}_n$ is a lattice polynomial (i.e. a polynomial in $\wedge$ and $\vee$, but not $\neg$) in the P_i's and Q_j's. That is, we assume that the polynomial in the P_i's and Q_j's is the same no matter what A and B are inputs.
2. For $|\psi\rangle \in \mathcal{H}$,
$$P_n|\psi\rangle = Q_m|\psi\rangle = |\psi\rangle \Longrightarrow \widehat{S}_{n+m}|\psi\rangle = |\psi\rangle.$$
3. If $|\psi\rangle$ is a simultaneous eigenstate of A and B with eigenvalues α and β, respectively, then $|\psi\rangle$ is an eigenstate of C with eigenvalue $\alpha + \beta$.

Then $C = A \dot{+} B$.

Proof. We wts that $\widehat{S}_n = S_n$ for all n. Recall that

$$[A \dot{+} B]_n := S_n = P_n \vee Q_n \vee \bigvee_{\substack{k+j \geq n \\ k,j=1}}^{\infty} (P_k \wedge Q_j) = \bigvee_{\substack{k+j \geq n \\ k,j=0}}^{\infty} (P_k \wedge Q_j),$$

where the last equality follows from the fact that we can include a "fictitious" $P_0 = Q_0 = I$ in the join.

(i) We first show that $\widehat{S}_i \geq S_i$ for all i (where $\geq$ is the ordering in the lattice). From assumption/criterion (2) above, we have that

$$P_i|\psi\rangle = Q_m|\psi\rangle = |\psi\rangle \Longrightarrow \widehat{S}_{i+m}|\psi\rangle = |\psi\rangle.$$

We also have that

$$P_i|\psi\rangle = Q_m|\psi\rangle = |\psi\rangle \Longrightarrow (P_i \wedge Q_m)|\psi\rangle = |\psi\rangle.$$

As such, we know that $\widehat{S}_{i+m} \geq P_i \wedge Q_m$. However, we also know that $P_i \wedge Q_m \geq P_k \wedge Q_j$ for $k \geq i$ and $j \geq m$; and as such, it follows that $P_i \wedge Q_m \geq \bigvee_{\substack{k \geq i \\ j \geq m}} (P_k \wedge Q_j)$. And so, we have that

$$\widehat{S}_\lambda \geq \widehat{S}_{\mu+\gamma} \geq P_\mu \wedge Q_\gamma,$$

where μ and γ are such that $\mu + \gamma \geq \lambda$. Thus,

$$\widehat{S}_\lambda \geq \widehat{S}_{\mu+\gamma} \geq \bigvee_{\mu+\gamma \geq \lambda} (P_\mu \wedge Q_\gamma) = S_\lambda.$$

i.e. we have that $\widehat{S}_\lambda \geq S_\lambda$.

(ii) We have that

$$\widehat{S}_n = (P_{\alpha_1} \wedge Q_{\beta_1}) \vee (P_{\alpha_2} \wedge Q_{\beta_2}) \vee \ldots$$

That this is true follows from the fact that for any element y in a distributive lattice L, $y = a_1 \vee a_2 \vee \ldots$, where $a_i = g_1 \wedge g_2 \wedge \ldots \wedge g_q$ for $g_w \in G \subseteq L$ (where G is a generating set for L) — i.e. any element in a distributive lattice can be expressed in disjunctive normal form, which is the join of finite meets of generating elements. In our case, the P_i's and Q_j's are the generating set, and we know by Jonsson's Theorem (Theorem 32) that the sub-lattice of a modular lattice generated by two chains is a distributive sub-lattice. As such, we can always put any element R_n in the form above. Note that the fact that this is so follows from the fact that since the P_i's and Q_j's always each form a decreasing sequence, the meet of any subset of P_i's and Q_j's always comes down to just $P_p \wedge Q_q$ for a single projector from each chain.

(iii) Since by (i) above we have that $\widehat{S}_i \geq S_i$, and by (ii) above we have that

$$\widehat{S}_n = (P_{\alpha_1} \wedge Q_{\beta_1}) \vee (P_{\alpha_2} \wedge Q_{\beta_2}) \vee \ldots,$$

it follows that

$$\widehat{S}_n = S_n \vee (P_{\alpha_1} \wedge Q_{\beta_1}) \vee (P_{\alpha_2} \wedge Q_{\beta_2}) \vee \ldots$$

Now, if each $P_{\alpha_j} \wedge Q_{\beta_j}$ is such that $P_{\alpha_j} \wedge Q_{\beta_j} \leq S_n$, then $\widehat{S}_n = S_n$ for all n, and we're done with the proof. So, suppose that $\widehat{S}_n > S_n$. Then we have that there exists some $P_\alpha \wedge Q_\beta$ which is such that $P_\alpha \wedge Q_\beta \nleq S_n$ — i.e. wlog, take $\alpha + \beta \nleq n$. In particular, suppose that $P_\alpha = Q_\beta$, $P_{\alpha+1} = Q_{\beta+1} \neq P_\alpha$, and consider $|\psi\rangle \in \mathcal{H}$ such that $P_\alpha|\psi\rangle = |\psi\rangle$ (and as such $Q_\beta|\psi\rangle = |\psi\rangle$) and $P_{\alpha+1}|\psi\rangle = 0$ (and as such $Q_{\beta+1}|\psi\rangle = 0$). And since we assume that $\alpha + \beta < n$, we have that $\widehat{S}_n|\psi\rangle = 0$. Now, by Lemma 5.14 above, we have that

$$A|\psi\rangle = \alpha|\psi\rangle \quad \text{and} \quad B|\psi\rangle = \beta|\psi\rangle,$$

which by assumption/criterion (3) above implies that $|\psi\rangle$ is an eigenvector of C with eigenvalue $\alpha + \beta$ — i.e. $C|\psi\rangle = \alpha + \beta|\psi\rangle$ — so that by Lemma 5.14, $\widehat{S}_{\alpha+\beta}|\psi\rangle = |\psi\rangle$ and $\widehat{S}_{\alpha+\beta+1}|\psi\rangle = 0$. (Actually, we have that $\widehat{S}_n|\psi\rangle = 0$ for all $n > \alpha + \beta$.) However, since $(P_\alpha \wedge Q_\beta)|\psi\rangle = |\psi\rangle$, we see that $\widehat{S}_n|\psi\rangle \neq 0$, which is a contradiction. As such, we must have that $\widehat{S}_n = S_n$.

$\square$

5.7 Conclusion

In this chapter we have, using the models $\mathcal{Q}_L$ of quantum set theory, constructed the quantum natural numbers ω_L in these models, as well as developed an arithmetic for these new numbers. In future work, we would like to further explore consequences of the arithmetical axioms in the presence of the quantum logic $\mathcal{Q}(\mathcal{L})$. We would also like to consider additional applications of the new arithmetic, particularly with regard to the models $\hat{\omega}(\mathscr{P}(\mathcal{H}))$, as well as find a physical interpretation of the arithmetical operations in these models, as such an interpretation may lend insight into foundational questions in quantum theory (particularly with regard to measurement). Additionally, we would like to extend our construction of the quantum natural numbers to obtain the quantum analogue of the real numbers, as well as extend our arithmetic to the quantum real numbers.

CHAPTER 6

SUMMARY & CONCLUSION

In this work, we began by defining, for any first order language $\mathcal{L}$, the quantum logic $\mathcal{Q}(\mathcal{L})$, which consists of the axioms (Q1) – (Q6) and inference rules (R1) – (R5). We then defined notions of formal deduction and derivability in $\mathcal{Q}(\mathcal{L})$. We also noted that an axiomatization of classical logic can be obtained from $\mathcal{Q}(\mathcal{L})$ by simply adding one additional axiom, which illustrated the fact that this quantum logic is sub-classical — i.e. every theorem of $\mathcal{Q}(\mathcal{L})$ is also a theorem of classical logic (but not vice versa).

We then went on to define a semantics for $\mathcal{Q}(\mathcal{L})$. Recalling that an M-system (or mathematical system) is a language $\mathcal{L}$ along with a set of $\mathcal{L}$-wffs $\mathcal{A}$ (which is effectively the set of mathematical axioms), we constructed $\mathcal{L}$-structures $\hat{A}$ for M-systems to consist of (i) an underlying set A in which the variables are interpreted, (ii) a truth value algebra L, where L is a complete orthomodular lattice, (iii) a map $[\![P]\!]$ which assigns truth values (in L) to the atomic sentences for each predicate $P \in \mathcal{L}^P$, and (iv) for every $f \in \mathcal{L}^F$, an interpretation of f in $\hat{A}$. Then, an $\mathcal{L}$-structure $\hat{A}$ is a model for $(\mathcal{L}, \mathcal{A})$ if all of the axioms $\mathcal{A} \cup \mathcal{Q}_{\mathcal{A}}(\mathcal{L})$ (or $\mathcal{A} \cup \mathcal{Q}_{\mathcal{A}}(\mathcal{L}) \cup \mathcal{E}(\mathcal{L})$ if $\approx \in \mathcal{L}^P$) hold in $\hat{A}$. Additionally, we demonstrated soundness and completeness for our formal deductive system relative to this (model theoretic) semantics.

We then applied this formalism to specific mathematical structures. That is, we considered several mathematical systems using the first order quantum logic as the underlying logic, and illustrated some interesting features of these quantum mathematical systems. In particular, we have shown that axiomatizations of M-systems which are equivalent in the presence of classical logic (in the sense that they have exactly the same theorems) are not necessarily equivalent when $\mathcal{Q}(\mathcal{L})$ is used for the underlying logic, which suggests a richness in the structure of mathematics which is classically inaccessible. Additionally, interesting examples of M-systems which admit no non-standard models have been given, as have examples of conservative and non-conservative models. Also, we have encountered classes of models which are extremely natural from the point of view of quantum theory, as well as examined relationships between some of them. Moreover, we have demonstrated that certain classical properties — namely, substitution for (some) operations with respect to equality, and strong transitivity of equality — no longer hold in these natural models. Given the naturalness of these models, we have interpreted these results as a manifestation of the true behavior[1] of "equality" in quantum mathematics, and have begun to consider what this is suggesting about quantum theory.

We next constructed an axiomatic set theory based on the quantum logic $\mathcal{Q}(\mathcal{L})$, with a particular intended class of models. We believe this quantum set theory to be a reasonable first attempt at a foundation for quantum mathematics (in a sense which parallels the foundational role of classical set theory in classical mathematics). For instance, we show that this quantum set theory is robust enough to provide a foundation on which to construct *quantum natural numbers*. Moreover, we have seen that in a special class of models, there exists a 1-1

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[1]As noted previously, although it is not impossible to find (non-standard) models in which substitution and/or strong transitivity of equality hold, we expect that such models will be few and far between — i.e. "most" non-standard models will not satisfy these properties. Moreover, these properties do not, in general, seem to hold in the most natural (from the point of view of quantum theory) classes of models.

correspondence between the quantum natural numbers and bounded observables in quantum theory whose eigenvalues are (ordinary) natural numbers. This 1-1 correspondence is remarkably satisfying, and not only gives us great confidence in our quantum set theory, but indicates the naturalness of such models for quantum theory. Additionally, we went on to consider an elegant arithmetic for the new numbers in these natural models, as well as some consequences of this arithmetic.

From all of this, we see that our initial investigations into quantum mathematics have yielded results which provide strong evidence that it has a richness and complexity worthy of further attention.

In future work, we would like to further explore the foundational role of the quantum logic $\mathcal{Q}(\mathcal{L})$, one aspect of which involves continuing our examination of the different properties and features of a variety of M-systems in the presence of $\mathcal{Q}(\mathcal{L})$. In particular, we will continue to develop the quantum set theory we have constructed, with an eye toward developing a more *fully quantum* version of set theory. We would also like to undertake a systematic study of quantum ordinal and cardinal numbers within quantum set theory.

With regard to the quantum arithmetic motivated by our quantum set theory, we would like to further explore consequences of the arithmetical axioms we have presented. We would also like to consider additional applications of this arithmetic, particularly with regard to the models $\hat{\omega}(\mathscr{P}(\mathcal{H}))$, as well as find a quantum mechanical interpretation of the arithmetical operations in these models. Such an interpretation may potentially lend insight into foundational questions in quantum theory (particularly with respect to measurement). Additionally, we would like to extend our construction of the quantum natural numbers to obtain the quantum analog of the real numbers, as well as extend our quantum arithmetic to include these as well.

Finally, we also intend to explore the possibility of an axiomatization of quantum mechanics built on quantum mathematics, extending the initial results described here. The hope is that allowing quantum mechanics to "speak" through its own native mathematics will naturally address (or at least suggest how to resolve) some troublesome questions and paradoxes that have been plaguing the interpretation of quantum mechanics for decades.

APPENDICES

Appendix A

LATTICE THEORY

A.1 Basic Concepts and Definitions

Definition A.1. Let X be a set, and let $\circledast$ be a binary relation on X. Then $\circledast$ is said to be

- *reflexive* if $a \circledast a$ for all $a \in X$;
- *anti-symmetric* if $a \circledast b$ and $b \circledast a$ imply that $a = b$;
- *symmetric* if $a \circledast b$ implies that $b \circledast a$ for all $a \in X$;
- *transitive* if $a \circledast b$ and $b \circledast c$ imply that $a \circledast c$.

Definition A.2. A binary relation $\leq$ on a set X is a *partial order* if it is reflexive, anti-symmetric and transitive. A set X equipped with a partial order $\leq$ is called a *partially ordered set* (or *poset*), and is denoted $\langle X, \leq \rangle$.

Definition A.3. Let $\langle X, \leq \rangle$ be a poset and let $S \subseteq X$. An element $u \in X$ is called an *upper bound* of S if $s \leq u$ $\forall s \in S$. We say that u is a *least upper bound* (abbreviated LUB) (or *join*) of S if it is an upper bound of S and we have that $u \leq v$ for any upper bound v of S. *Lower bound* and *greatest lower bound* (abbreviated GLB) (or *meet*) are defined analogously.

Let $\langle X, \leq \rangle$ be a poset, with $A \subseteq X$. Then the *meet of A* (or *GLB of A*), denoted $\bigwedge A$, is defined to be the element $z \in X$ (if it exists) such that

(i) $z \leq a$ for all $a \in A$

Appendix A (Continued)

(ii) For any $b \in X$ such that $b \leq a$ for all $a \in A$, we have $b \leq z$.

We then also denote $a \wedge b := \bigwedge\{a, b\}$. Similarly, the *join of* A (or *LUB of* A), denoted $\bigvee A$, is defined to be the element $y \in X$ (if it exists) such that

(i) $a \leq y$ for all $a \in A$

(ii) For any $b \in X$ such that $a \leq b$ for all $a \in A$, we have $y \leq b$.

We then also denote $a \vee b := \bigvee\{a, b\}$.

Definition A.4. Let $\langle X, \leq \rangle$ be a poset, with $a, b \in X$. Then b is said to *cover* a if for any $c \in X$ with $a \leq c \leq b$, we have either $a = c$ or $b = c$.

Any finite poset $\langle X, \leq \rangle$ can be represented graphically by a *Hasse diagram* — this representation for a poset can be obtained by drawing a dot for each element of X, and for every $a, b \in X$ such that b covers a, we draw a line from a up to b. One of the simplest examples of a non-trivial Hasse diagram is shown below.

Example A.1. Let $B_2 = \{0, 1\}$ with the usual partial ordering. Then the Hasse diagram for B_2 is given in Figure 1. Note that we will also refer to B_2 as $\mathbf{2}$.

Definition A.5. Let $\langle X, \leq \rangle$ be a poset. X is called a *chain* if for every $a, b \in X$, either $a \leq b$ or $b \leq a$. If X is finite, the *length of* X (denoted $l(X)$) is the number of elements of X.

Note that another common convention is to define the length of X as the number of elements of X *minus 1*.

Appendix A (Continued)

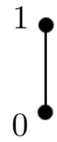

Figure 1. The Hasse diagram for $B_2 = \{0, 1\}$ from example A.1.

Definition A.6. Let $\langle X, \leq \rangle$ be a poset. The *height of* X (denoted Ht X is defined to be the supremum of the set

$$\{l(C) \ : \ C \subseteq X \text{ and } C \text{ is a finite chain under } \leq\}$$

if the supremum exists; otherwise Ht X is defined to be '∞'. If the height of X is finite, then X is said to be of *finite height*.

Definition A.7. A poset $\langle X, \leq \rangle$ is called a *lattice* if every pair of elements of X has both a greatest lower bound (GLB) and a least upper bound (LUB). If X is further equipped with an involution, then we will call X an *involutive lattice*. A lattice which has a top element (denoted by 1) and bottom element (denoted by 0) is said to be *bounded*.

Theorem 30. Let L be a set and let '$\vee$' and '$\wedge$' be binary operations on L satisfying (for all $a, b, c \in L$)

Commutativity: $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$.

Associativity: $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Appendix A (Continued)

Absorption: $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$.

Further define the relation '$\leq$' on L by $a \leq b$ iff $a \wedge b = a$. Then $\leq$ is a partial order and L is a lattice under $\leq$. Conversely, if L is a lattice under $\leq$, then the meet and join (of pairs) satisfy the three properties above.

Definition A.8. A bounded lattice L is said to be $\wedge$-*complete* if for any non-empty subset of L the meet exists. L is said to be $\vee$-*complete* if for any non-empty subset the join exists. L is said to be *complete* if it is both $\wedge$-complete and $\vee$-complete. L is said to be σ-*complete* if any countable subset has a meet and a join.

Lemma A.2. Let L be a lattice, with $a, b, c \in L$. Then

1. If $a \leq b$ and $a \leq c$, then $a \leq b \wedge c$.
2. If L is complete, with $B \subseteq L$, and $a \leq b$ for every $b \in B$, then $a \leq \bigwedge_{b \in B} b = \bigwedge B$.
3. $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$.

Definition A.9. A subset S of a lattice L is called a *sub-lattice* of L if for every $a, b \in S$ we have that $a \vee b \in L$ and $a \wedge b \in L$. If S is a sub-lattice such that $S \subsetneq L$, we say S is a *proper sub-lattice* of L.

Definition A.10. A bounded lattice L is said to be *complemented* if for any $a \in L$ there exists some element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$. We refer to a as the *complement* of b, or vice versa.

Definition A.11. Let L be a lattice with $a \in L$. An *involution* is a map $f : L \to L$ which satisfies (for all $a, b \in L$)

1. $f(f(a)) = a$

2. if $a \leq b$, then $f(b) \leq f(a)$

The involution of an element a will often be denoted by $\neg a$.

Definition A.12. Let L be an involutive lattice. *De Morgan's law* is given by (for $a, b \in L$)

$$\neg(\neg a \vee \neg b) = a \wedge b.$$

Note that the dual version of this law is given by $\neg(\neg a \wedge \neg b) = a \vee b$, as well as that de Morgan's law holds in any involutive lattice.

An equivalent statement of De Morgan's law is $f(a_1 \vee a_2) = f(a_1) \wedge f(a_2)$.

DeMorgan's Law relates the three algebraic operations $(\wedge \vee \neg)$ in a fundamental way in lattices with involution; these algebraic operations are not independent, but rather, given either meet or join DeMorgan's Law automatically gives the other.

Definition A.13. A bounded lattice in which the involution is a complementation is called an *orthocomplemented lattice*, or simply an *ortholattice* — that is, an ortholattice is a bounded involutive lattice L which satisfies both (for all $a, b \in L$)

$$a \wedge \neg a = 0 \quad \text{and} \quad b \vee \neg b = 1.$$

Lemma A.3. Let L be a non-trivial ortholattice. Then

1. Ht $L \geq 2$

2. Ht $L = 2$ iff $L = \{0, 1\}$.

Proof. Since L is non-trivial, we have that $0 \neq 1$, and since $0 < 1$ we have that Ht $L \geq 2$, establishing (1) above.

If $L \neq \{0, 1\}$, then there is some $a \in L$ with $a \notin \{0, 1\}$, and hence $0 < a$ and $a < 1$, so that $\{0, a, 1\}$ is a chain with three elements so if $L \neq \{0, 1\}$, then Ht $L \geq 3$. For the other implication, if Ht $L \geq 3$, then there is some chain $C \subseteq L$ with at least 3 elements, so that $L \neq \{0, 1\}$. □

A.1.1 Maps

Definition A.14. Let $\langle X_1, \leq_1 \rangle$ and $\langle X_2, \leq_2 \rangle$ be posets, and let $a, b \in X_1$. Then a map $f : X_1 \to X_2$ is called *isotone* if whenever $a \leq_1 b$, we have that $f(a) \leq_2 f(b)$. f is called *antitone* if whenever $a \leq_1 b$, we have that $f(b) \leq_2 f(a)$.

This is to say that isotone maps are order preserving, while antitone maps are order reversing.

Definition A.15. An *order isomorphism* between posets $\langle X_1, \leq_1 \rangle$ and $\langle X_2, \leq_2 \rangle$ is a bijective map $f : X_1 \to X_2$ for which both f and f^{-1} are isotone.

Note that order isomorphisms constitute an equivalence relation on the set of all posets. This is to say that two posets are equivalent if and only if they are order isomorphic to one another.

Definition A.16. Let L_1, L_2 be lattices. A *lattice homomorphism* from L_1 to L_2 is a map $f : L_1 \to L_2$ which satisfies, for all $a, b \in L_1$

1. $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$

2. $f(a \vee_1 b) = f(a) \vee_2 f(b)$

If the map, $f : L_1 \to L_2$ satisfies only property 1, we say that it is a $\wedge$-*homomorphism*; if the map satisfies only property 2, we say that it is a $\vee$-*homomorphism*. If L_1 and L_2 are involutive trellises, and f satisfies $f(\neg_1 a) = \neg_2 f(a)$ for all $a \in L_1$, then f is called a $\neg$-homomorphism.

Definition A.17. Let L_1 and L_2 be lattices, with $h : L_1 \to L_2$ a lattice homomorphism. Then h is said to be *continuous* if, for every $A \subseteq L_1$, we have that

$$h(\bigvee A) = \bigvee h(A) \quad \text{and} \quad h(\bigwedge A) = \bigwedge h(A).$$

Definition A.18. Let L_1 and L_2 be lattices. A bijective lattice homomorphism $f : L_1 \to L_2$ is called a *lattice isomorphism* (or *L-isomorphism*) from L_1 to L_2. If there exists a lattice isomorphism between two lattices L_1 and L_2, we say that L_1 is *isomorphic* to L_2, and denote this by $L_1 \simeq L_2$. If L_1 and L_2 are involutive, then a bijective involutive lattice homomorphism is called an *IL-isomorphism* from L_1 to L_2.

A.1.2 Orthogonality and Compatibility

Definition A.19. Let L be an ortholattice, and let $a, b \in L$. If $a \leq \neg b$, then a is said to be *orthogonal* to b (denoted by $a \perp b$).

Note that $a \perp b \Rightarrow b \perp a$ for $a, b \in L$ — i.e. $\perp$ is a symmetric binary relation on L. This follows trivially from the fact that

$$a \leq \neg b \Rightarrow \neg(\neg b) \leq \neg a \Rightarrow b \leq \neg a := b \perp a.$$

Also, $a \not\perp a$ for any $a \in L$ — i.e. $\perp$ is an irreflexive relation on L. Note also that $a \perp b$ does not imply $\neg a \perp \neg b$. Additionally, we have the following properties.

Lemma A.4. Let $a, b, c \in L$ an ortholattice. The following properties hold concerning the relation $\perp$ of Definition A.19 above.

1. $a \perp b$ and $a \perp c \Rightarrow a \perp b \vee c$;
2. $a \perp b$ or $a \perp c \Rightarrow a \perp b \wedge c$;
3. $a \perp a \Rightarrow a = 0$;
4. $a \perp b \Rightarrow a \wedge b = 0$.

Proof.

1. $a \leq \neg b$ and $a \leq \neg c$ implies that $a \leq \neg b \wedge \neg c = \neg(b \vee c)$, and hence $a \perp b \vee c$.
2. $a \leq \neg b$ or $a \leq \neg c$ implies that $a \leq \neg b \vee \neg c = \neg(b \wedge c)$, and hence $a \perp b \wedge c$.
3. $a \leq \neg a$ implies that $a \wedge \neg a = a$, but $a \wedge \neg a = 0$ since L is an ortholattice, and hence $a = 0$.
4. Joining both sides of $a \leq \neg b$ with $\neg a$ gives $a \vee \neg a = 1 \leq \neg b \vee \neg a = \neg(a \wedge b)$, but this means that $1 = \neg(a \wedge b)$ and hence $0 = a \wedge b$.

□

Definition A.20. Let L be an ortholattice. Then for $a, b \in L$ we say that a is *compatible with* b (denoted $a\ \mathcal{C}\ b$) if

$$a = (a \wedge b) \vee (a \wedge \neg b).$$

Alternatively, we will sometimes say that a *commutes with* b when we have that $a\ \mathcal{C}\ b$.

Lemma A.5. Let L be an ortholattice, and let $a, b \in L$. The following properties hold concerning the relation $\mathcal{C}$ of Definition A.20 above.

1. $a\ \mathcal{C}\ b \iff a\ \mathcal{C}\ \neg b$;
2. $a\ \mathcal{C}\ 1$, $a\ \mathcal{C}\ 0$, $0\ \mathcal{C}\ b$, $1\ \mathcal{C}\ b \quad \forall a, b \in L$;
3. $a \perp b \Longrightarrow a\ \mathcal{C}\ b$ and $b\ \mathcal{C}\ a$;
4. $a \leq b \Longrightarrow a\ \mathcal{C}\ b$ (and therefore $a\ \mathcal{C}\ a$).

Proof.

1. $(a \wedge \neg b) \vee (a \wedge b) = a$ because $a\ \mathcal{C}\ b$, and hence $a\ \mathcal{C}\ \neg b$.
2. (a) $(a \wedge 1) \vee (a \wedge \neg 1) = a \vee 0 = a$, so $a\ \mathcal{C}\ 1$.
   (b) $(a \wedge 0) \vee (a \wedge \neg 0) = 0 \vee a = a$, and so $a\ \mathcal{C}\ 0$.
   (c) $(0 \wedge b) \vee (0 \wedge \neg b) = 0 \vee 0 = 0$, and so $0\ \mathcal{C}\ b$.
   (d) $(1 \wedge b) \vee (1 \wedge \neg b) = b \vee \neg b = 1$ because L is an ortholattice, so $1\ \mathcal{C}\ b$.
3. $a \leq \neg b$ implies that $(a \wedge b) \vee (a \wedge \neg b) = (a \wedge b) \vee a = a$, and so $a\ \mathcal{C}\ b$. Since $a \perp b$ implies $b \perp a$, $b\ \mathcal{C}\ a$ as well.

4. $a \le b$ implies that $(a \wedge b) \vee (a \wedge \neg b) = a \vee (a \wedge \neg b) = a$, and so $a\ \mathcal{C}\ b$.

□

Note that in general $a\ \mathcal{C}\ b$ does not imply $b\ \mathcal{C}\ a$ — that is, compatibility is not symmetric in general — nor do the combination of $a\ \mathcal{C}\ b$ and $b\ \mathcal{C}\ c$ imply $a\ \mathcal{C}\ c$ — that is, compatibility is not transitive.

A.2 Distributive Lattices

Lemma A.6. In a lattice L, the following inequalities hold for all $a, b \in L$.

$$a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$$

For proof see (3), p.9.

Definition A.21. A lattice L is called *distributive* if the following equalities hold for all $a, b, c \in L$.

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Example A.7. A simple distributive lattice is given by $D = \{0, x, y, 1\}$ with $0 \le x \le 1$ and $0 \le y \le 1$ and no other orderings. Then the Hasse diagram for D is given in Figure 2.

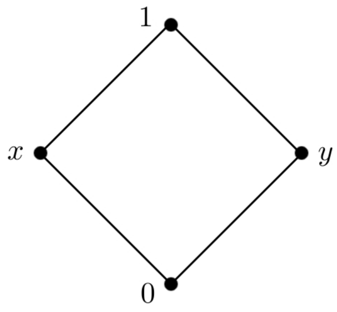

Figure 2. The Hasse diagram for $D = \{0, x, y, 1\}$ from example A.7.

Definition A.22. An ortholattice which satisfies distributivity is called a *Boolean Algebra* (or *Boolean Lattice*).

In the example above, if $y = \neg x$, then D becomes the *Boolean diamond*.

A.3 Modular Lattices

Definition A.23. A lattice L is called *modular* if for $a, b, c \in L$ with $c \leq a$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee c.$$

Note that every distributive lattice is modular — that is, modularity is a weakening of distributivity.

A.3.0.0.1 M_3 and N_5:

These are the two smallest non-distributive lattices.

Appendix A (Continued)

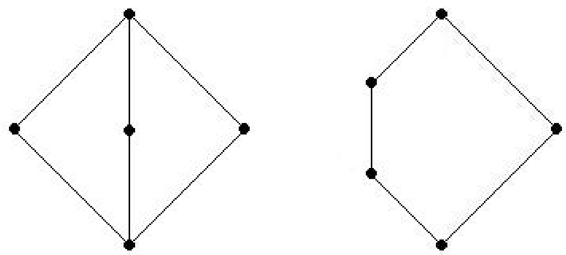

Figure 3. M_3 and N_5

Theorem 31. *The M_3-N_5 Theorem*:

1. A lattice is modular if and only if it has no sub-lattices isomorphic to N_5.

2. A lattice is distributive if and only if it has no sub-lattices isomorphic to M_3 or N_5.

Theorem 32. *Jonsson's Theorem*: Let L be a modular lattice, let $p \in \mathbb{Z}^+$, and let X_1, X_2, $\ldots X_p$ be non-empty chains of L. In order that the sub-lattice of L generated by

$$X_1 \cup X_2 \cup \ldots \cup X_p$$

be distributive, it is necessary and sufficient that, for any $x_1 \in X_1$, $x_2 \in X_2$, $\ldots x_p \in X_p$, the sub-lattice of L generated by the set $\{x_1, x_2, \ldots x_p\}$ be distributive.

A.4 Orthomodular Lattices

A standard reference for orthomodular lattice theory is Kalmbach (21).

Definition A.24. Let L be an ortholattice. If all $a, b \in L$ satisfy

$$a \wedge b = a \wedge (\neg a \vee (a \wedge b)),$$

then L is said to be an *orthomodular lattice* (or *OML*).

Note that when L is an ortholattice, modularity implies orthomodularity (see (21)). Also, note that orthomodularity is a weakening of the distributivity law satisfied by Boolean algebras, and as such, every Boolean algebra is an OML. Additionally, there is a simple way to characterize when an OML is a Boolean algebra.

Proposition A.8. Let L be an orthomodular lattice. Then L is a Boolean algebra if and only if for every $a, b \in L$ we have $a\ \mathcal{C}\ b$.

Proof. This is a trivial corollary of Theorem 33. $\square$

Also, note that in an OML, we have that (for every $a, b \in L$)

$$a\ \mathcal{C}\ b \quad \text{if and only if} \quad b\ \mathcal{C}\ a.$$

That is, '$\mathcal{C}$' is a symmetric relation in an OML.

Definition A.25. Let L be an OML. Then the *center of* L is defined to be the set

$$\{c \in L\ :\ c\ C\ a \text{ for all } a \in L\},$$

and is denoted by $Z(L)$.

The following theorems are both proven in Kalmbach (21), p. 24.

Theorem 33. Let L be an OML. Then $Z(L)$ is a sub-algebra of L which is a Boolean Algebra. If L is also complete, then so is $Z(L)$.

Theorem 34. Let L be an OML with $a \in L$. Then the set of elements of L which commute with a form a sub-algebra which is closed under infinite meets and joins (whenever they exist).

Definition A.26. Let L be an OML. We define, for any $a, b \in L$, the *commutator of a and b* (denoted $c(a,b)$) to be

$$c(a,b) := [(a \wedge b) \vee (a \wedge \neg b)] \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

The following theorem is also proven in Kalmbach (21), p. 26.

Proposition A.9. Let L be an OML, with $a, b \in L$. Then $c(a,b) = 1$ if and only if $a\ C\ b$.

We now consider a particular class of orthomodular lattices that will be extremely useful for constructing models — namely, the modular ortholattices MO_k, for some finite cardinal $k \in \{1, 2, \ldots\}$.[1] More precisely,

$$\mathrm{MO}_n := \{v_i |\ i < n+1\} \cup \{\neg v_i |\ i < n+1\},$$

[1]There is an obvious generalization to the infinite cardinals, so that the lattice MO_α is defined for any infinite cardinal α.

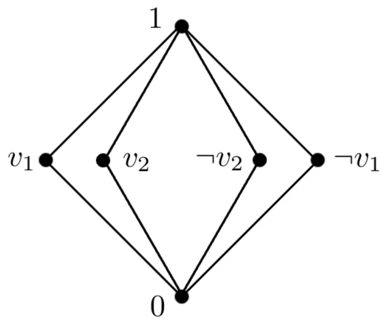

Figure 4. The Hasse diagram for $\mathrm{MO}_2 = \{0, 1, v_1, v_2, \neg v_1, \neg v_2\}$ of example A.10.

where $v_0 = 1$ (hence $0 = \neg v_0$), and where for every distinct $v, w \in \mathrm{MO}_n \backslash \{0, 1\}$ we have $v \wedge w = 0$ and $v \vee w = 1$. Note that MO_1 is, in fact, a reducible Boolean algebra (isomorphic to $\mathbf{2} \times \mathbf{2}$), and also that MO_n is irreducible for every $n \geq 2$.

Example A.10. We define MO_2 to be the set $\{0, 1, v_1, v_2, \neg v_1, \neg v_2\}$ (with the obvious involution), and with partial order given as in the Hasse diagram of Figure 4. Then MO_2 is a modular ortholattice (and hence an orthomodular lattice) which is not a Boolean algebra.

A.4.1 The Sasaki Projection and the Sasaki Hook

Definition A.27. Let L be an orthomodular lattice, and let $a, b \in L$. We define the *Sasaki Hook* (denoted '$\cdot \to \cdot$') to be the binary operation on L given by

$$a \to b := \neg a \vee (a \wedge b) \tag{A.1}$$

and the *Sasaki projection* (denoted '$\varphi.(\cdot)$') to be the binary operation defined by

$$\varphi_a(b) := a \vee (\neg a \wedge b). \tag{A.2}$$

Finally, define

$$a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

Note that for any $a, b \in L$, the Sasaki hook and the Sasaki projection are related by $a \to b = \neg\varphi_a(\neg b)$.

Lemma A.11. Let L be an orthomodular lattice, and let φ be the Sasaki projection. Then for any $a, b \in L$,

1. $a \wedge b \leq \varphi_b(a) \leq b$
2. $a \; C \; b \quad \text{iff} \quad \varphi_a(b) \leq b$
3. $\varphi_a(0) = 0$
4. $\varphi_b(a) = a \quad \text{iff} \quad a \leq b$

Proof.

1. We have that $\varphi_b(a) = b \wedge (\neg b \vee a)$, from which we clearly see that $\varphi_b \leq b$. Also, because $a \leq \neg b \vee a$, we clearly have that $a \wedge b \leq \varphi_b(a)$.
2. This is 12.(iii) in Kalmbach (21), p. 156.
3. $\varphi_a(0) = a \wedge (\neg a \vee 0) = a \wedge \neg a = 0$.

4. First assume that $a \leq b$; we have that $a \leq b$ is equivalent to $\neg b \leq \neg a$, and so by orthomodularity we have $\neg a = \neg b \vee (b \wedge \neg a)$. Taking the negation of both sides of this expression gives $a = b \wedge (\neg b \vee a) \equiv \varphi_b(a)$. Now consider $\varphi_b(a) = a$. We have (by (1) above) that $\varphi_b(a) \leq b$, but by assumption $\varphi_b(a) = a$, and hence $a \leq b$.

□

Lemma A.12. Let L be an orthomodular lattice, let '$\rightarrow$' be the Sasaki hook (of Definition A.27 above), and let $a, b, c \in L$.

1. $a \leftrightarrow b = b \leftrightarrow a$
2. $a \rightarrow b = 1$ iff $a \leq b$
3. $1 \rightarrow a = a$
4. $a \leftrightarrow b = 1$ iff $a = b$
5. $a \leftrightarrow 1 = a$
6. $a \leftrightarrow 0 = a \rightarrow 0 = \neg a$
7. $a \rightarrow \neg a = \neg a$
8. $a \leftrightarrow b \wedge b \leftrightarrow c \leq a \leftrightarrow c$
9. $a \leftrightarrow b = (a \wedge b) \vee (\neg a \wedge \neg b) = \neg a \leftrightarrow \neg b$.

Proof.

1. Follows trivially from the definition.

2. If $a \to b = \neg a \vee (a \wedge b) = 1$ then meeting both sides with a yields (using orthomodularity) that

$$a = a \wedge (\neg a \vee (a \wedge b)) = a \wedge b,$$

i.e. that $a \leq b$. Conversely, if $a \leq b$, then $a \wedge b = a$ so $a \to b = \neg a \vee a = 1$.

3. $1 \to a = \neg 1 \vee (1 \wedge a) = 0 \vee a = a$

4. $a \leftrightarrow b = 1$ iff $a \to b = b \to a = 1$, by (2) this is true iff $a \leq b$ and $b \leq a$, i.e. iff $a = b$.

5. Trivially by (2) and (3) above.

6. By (2) above $0 \to a = 1$, so $a \leftrightarrow 0 = a \to 0 = \neg a \vee (a \wedge 0) = \neg a$.

7. $a \to \neg a = \neg a \vee (a \wedge \neg a) = \neg a \vee 0 = \neg a$.

8. That this holds can be found in (8). However, we can easily use the Foulis-Holland Theorem to compute this. Since we have that $a \wedge b \mathcal{C} \neg a$ and $a \wedge b \mathcal{C} \neg b$, we simply 'undistribute' to obtain

$$a \leftrightarrow b = [\neg a \vee (a \wedge b)] \wedge [\neg b \vee (b \wedge a)] = (\neg a \wedge \neg b) \vee (b \wedge a).$$

9. This can be found in (8) (where $a \Delta b \coloneqq \neg[a \leftrightarrow b]$). See p. 437.

□

A.4.2 The Exchange Property & Some Others

Definition A.28. Let L be a lattice with bottom element 0, and let $a \in L$. Then a is said to be an *atom* of L if a covers 0. L is said to be *atomic* if for every $x \in L$ with $x \neq 0$, there exists an atom $p \in L$ such that $p \leq x$.

Appendix A (Continued)

Note that we will often denote the set of all atoms of a lattice L by $\Omega(L)$.

The following two propositions are proven in Kalmbach (21), p. 140.

Lemma A.13. Let L be an atomic OML. Then L is atomistic.

Theorem 35. The following three conditions are equivalent in any orthomodular lattice L.

1. If (for any $a, b \in L$) a covers $a \wedge b$, then $a \vee b$ covers b.

2. If (for any $a \in L$ and atom $p \in L$) $p \nleq a$, then $p \vee a$ covers a.

3. If (for any $a \in L$ and atoms $p, q \in L$) the conditions $p \leq q \vee a$ and $p \wedge a = 0$ imply $q \leq p \vee a$.

Definition A.29. Let L be an atomic OML. If L satisfies any of the three equivalent conditions of proposition 35, then L is said to satisfy the *exchange axiom* (EA).

Definition A.30. Let L be an atomic OML. If, for every pair of atoms $p, q \in L$ such that $p \perp q$, we have that there exists an atom $r \in L$ such that $r \leq p \vee q$ and also $r \not\subset p$ and $r \not\subset q$, then L is said to satisfy the *atomic bisection property* (or *ABP*).

Definition A.31. Let L be an atomic OML. If L satisfies ABP and also, for every triple of distinct atoms $p, q, r \in L$ we have that $p \leq q \vee r$ imply both $q \leq p \vee r$ and $r \leq p \vee q$, then L is said to satisfy the *superposition principle* (or *SP*).

Lemma A.14. Let L be an atomic OML satisfying the exchange axiom. Then L is irreducible if and only if L satisfies SP.

Lemma A.15. L atomic OML, with $a, s \in L$, s an atom, and s and a commute. Then $s \leq a$ or $s \perp a$.

Proof. To see this, note that we have

$$s\ C\ a \leftrightarrow s = (s \wedge a) \vee (s \wedge \neg a),$$

and also $s \in \Omega(L)$ implies that $s \wedge x = s$ or $s \wedge x = 0$, and if $s \wedge a = 0$, then $s \wedge \neg a = s$ (or vice versa) — thus, $s \leq a$ or $s \perp a$. □

Lemma A.16. Let $L \neq \{0, 1\}$ be a complete, atomic, irreducible OML satisfying EA. Then for any $y \in L$ with $y \neq 0, 1$, we have

$$\bigwedge_{x \in L} \big(\varphi_x(y) \to y\big) = 0.$$

Proof. First, since $L \neq \{0, 1\}$ then L is height greater than 2 (Lemma A.3). Then since L is an atomic, irreducible, OML such that EA holds and the height of L is greater than 2, both superposition (SP) and the atomic bisection property (ABP) hold (Lemma A.14). Let $\Omega(L)$ denote the set of all atoms of L. Since L is an atomic OML, L is atomistic (Lemma A.13), so for any $y \in L$, we have that both

$$y = \bigvee_{\substack{p \leq y \\ p \in \Omega(L)}} p \qquad \text{and} \qquad \neg y = \bigvee_{\substack{q \leq \neg y \\ q \in \Omega(L)}} q,$$

and so

$$1 = y \vee \neg y = \Big[\bigvee_{\substack{p \leq y \\ p \in \Omega(L)}} p\Big] \vee \Big[\bigvee_{\substack{q \leq \neg y \\ q \in \Omega(L)}} q\Big] = \bigvee_{\substack{p \leq y, q \leq \neg y \\ p, q \in \Omega(L)}} (p \vee q).$$

Appendix A (Continued)

Now, for each $p, q \in \Omega(L)$ with $p \leq y$, $q \leq \neg y$, we have $p \perp q$ (since $p \leq y \leq \neg q$), and so since L satisfies ABP, we choose some $r_{pq} \in \Omega(L)$ which is such that $r_{pq} \leq p \vee q$ and r_{pq} does not commute with either p or q, and furthermore, by SP, we have that $p \vee q = q \vee r_{pq} = p \vee r_{pq}$. We further claim that r_{pq} does not commute with y. By Lemma A.15, r_{pq} commutes with y iff either $r_{pq} \leq y$ or $r_{pq} \leq \neg y$. But if $r_{pq} \leq y$, then $r_{pq} \vee p = p \vee q \leq y$, so that $q \leq y \wedge \neg y = 0$ which is a contradiction since $q \in \Omega(L)$. A similar contradiction results if $r_{pq} \leq \neg y$, and so r_{pq} cannot commute with y.

Now, $r_{pq} \in \Omega(L)$ implies that $\neg r_{pq}$ is a coatom, and $r_{pq} \leq p \vee q$ implies that $p \vee q \nleq \neg r_{pq}$ (by transitivity of $\leq$ since $r_{pq} \neq 0$). Then, $\neg r_{pq} \wedge (p \vee q)$ is an atom by EA.

But then, by remark 9 in Kalmbach p.143 (using that $p \nleq r_{pq}$ since both $p, r_{pq} \in \Omega(L)$, and also $r_{pq} \perp \neg r_{pq}$) we see that $z_{pq} := \neg r_{pq} \wedge (p \vee r_{pq})$ is an atom. Since $z_{pq} \leq \neg r_{pq}$, clearly $z_{pq} \neq r_{pq}$, and then by SP we have

$$z_{pq} \vee r_{pq} = p \vee z_{pq} = p \vee q.$$

Additionally, we must have that z_{pq} does not commute with y, by the same argument that r_{pq} does not commute with y. Hence, we have that

$$1 = \bigvee_{\substack{p \leq y, q \leq \neg y \\ p,q \in \Omega_L}} (p \vee q) = \bigvee_{\substack{p \leq y, q \leq \neg y \\ p,q \in \Omega_L}} (r_{pq} \vee z_{pq}) \leq \bigvee_{\substack{r \in \Omega(L) \\ r \not\subset y}} r,$$

and taking the negation yields,

$$0 = \bigwedge_{\substack{r \in \Omega(L) \\ r \not\subset y}} \neg r. \tag{A.3}$$

Appendix A (Continued)

But for r any atom that doesn't commute with y, we have

$$\varphi_r(y) = r \wedge (\neg r \vee y) = r \wedge 1 = r,$$

since $y \nleq \neg r$ and $\neg r$ is a coatom, so that

$$\varphi_r(y) \to y = \neg r \vee (r \wedge y) = \neg r \vee 0 = \neg r.$$

Plugging this back into equation A.3 yields

$$0 = \bigwedge_{\substack{r \in \Omega(L) \\ r \not\mathrel{\mathscr{C}} y}} \neg r = \bigwedge_{\substack{r \in \Omega(L) \\ r \not\mathrel{\mathscr{C}} y}} \big(\varphi_r(y) \to y\big) \geq \bigwedge_{\substack{x \in L \\ x \not\mathrel{\mathscr{C}} y}} \big(\varphi_x(y) \to y\big).$$

□

A.4.3 Projection/Subspace Lattices

We begin by defining some notation. For any Hilbert space $\mathcal{H}$ we will use the Dirac bra-ket notation for the inner product, '$|0\rangle$' to represent the zero vector, and for any $S \subseteq \mathcal{H}$, we define

$$S^{\perp} := \{\ |\psi\rangle \in \mathcal{H} \mid \langle\psi|\phi\rangle = 0 \text{ for all } |\phi\rangle \in S\ \},$$

(which is always a closed subspace). Also, for any set $S \subseteq \mathcal{H}$, we denote the linear span of S by $\mathrm{span}(S)$, and the closure of the span of S by $\overline{\mathrm{span}}(S)$. Also recall that an *orthogonal projection* *operator on* $\mathcal{H}$ is a Hermitian operator P such that $P^2 = P$. Finally, for any linear operator A on $\mathcal{H}$ we define $\ker(A) := \{|\psi\rangle \in \mathcal{H} : A|\psi\rangle = |0\rangle\}$.

Theorem 36. Let $\mathcal{H}$ be a separable Hilbert space. Then the set of closed linear subspaces (ordered under inclusion) form an orthomodular lattice with $S \mapsto S^{\perp}$ as the involution. This OML is called the *subspace lattice of* $\mathcal{H}$.

Note that for two closed subspaces $V, W \subseteq \mathcal{H}$, we have that $V \wedge W = V \cap W$ (i.e. the meet is the intersection), and $V \vee W = \overline{\text{span}}(V \cup W)$ (i.e. the join is the closure of the span of the union). Also, the closed linear subspaces of a separable Hilbert space are in 1-1 correspondence with the orthogonal projection operators[1] on $\mathcal{H}$, and so the projection operators of $\mathcal{H}$ form an OML (called the *projection lattice*) which is naturally isomorphic to the subspace lattice of $\mathcal{H}$. Given this correspondence, we will frequently go back and forth between the two. See (19).

For a given quantum system described by some Hilbert space $\mathcal{H}$, the orthogonal projections on $\mathcal{H}$ have a natural interpretation as (equivalence classes of) propositions concerning that quantum system — this follows from the fact that the measurement of such an operator always yields an outcome of '1' or '0'. One may think of the relevant proposition for a projection operator P as being the declarative statement 'the state of the system is in the subspace $\ker(P)^{\perp}$'.

As noted by Birkhoff and von Neumann (2), when $\mathcal{H}$ is a finite dimensional Hibert space, its corresponding projection lattice has the following nice property.

---

[1]An orthogonal projection operator P corresponds to its image (i.e. with $\ker(P)^{\perp}$) under this 1-1 correspondence.

Theorem 37. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Then the subspace lattice of $\mathcal{H}$ is a modular lattice.

A.4.4 Additional Properties

Definition A.32. Let L be an orthomodular lattice. L is said to satisfy the *relative center property* if for any $a \in L$, the center of any interval $[0, a]$ is exactly the set $\{a \wedge b \ : b \text{ is in the center of } L\}$.

Lemma A.17. The following OMLs satisfy the relative center property.

1. Any complete modular ortholattice;
2. Projection lattices of any Hilbert space;
3. Projection lattices of any von Neumann algebra.

Proof. For (1) and (3) see Theorem 14 in Kalmbach (21), pages 110-111. (2) is a special case of (3). □

Lemma A.18. Let L be a complete orthomodular lattice which satisfies the relative center property, and let $a \in L$. Then $\bigwedge_{b \in L} b \to a$ is in the center of L.

Proof. Note that an element is in the center of L iff its negation is, and

$$\neg \bigwedge_{b \in L} b \to a = \bigvee_{b \in L} \neg(\neg b \vee (b \wedge a)) = \bigvee_{b \in L} \varphi_b(\neg a),$$

(with $\varphi_b(a)$ the Sasaki projection), and so it suffices to show that $\bigvee_{b \in L} \varphi_b(a)$ is in the center of L if L satisfies the relative center property. But this then follows directly from Theorem 14 in Kalmbach (21) (page 110) and Theorem 7 (page 108) along with proposition 10 in Chevalier (6) (specifically the equivalence of (b) and (d)). □

Definition A.33. Let L be a complete OML, and let G be the group of continuous ortholattice automorphisms on L. L is said to be *rotatable* if the only fixed points of L under the action of G are 0 and 1.

Theorem 38. The projection lattice of a separable complex Hilbert space is rotatable.

Proof. For any unitary transformation U, the map which takes any projection operator P to $U^{\dagger}PU$ induces an automorphism on the projection lattice. However, the only projectors fixed by all such automorphisms are I and 0. □

Theorem 39. The projection lattice of a separable complex Hilbert space is an atomic, irreducible OML satisfying the exchange axiom.

Appendix B

ELEMENTARY SET THEORY

In this appendix we attempt to provide some basic classical set theory; here we follow the approach of naive set theory, the interested reader is referred to (15) for additional details, or to (12) to see this material developed axiomatically.

B.1 Basic Concepts

In naive set theory, there are three primitive notions — namely, that of *object*, *set*, and *membership*. Everything in the universe of discourse is an object, but only certain objects are sets. Membership (often used synonymously with 'being an element of') is a binary relation between an object and a set, and for a given object a and set A we write $a \in A$ to mean that a is an element of A, and $a \notin A$ to mean that a is not an element of A. Moreover, sets are determined entirely by their members, so that two sets are said to be *identically equal* if and only if they have the same members. This is known as the property of *extensionality*. One set which is relevant in many contexts is the *power set* of a set — for a set A, the *power set of* A is the set whose elements are exactly the subsets of A (including $\varnothing$ and A itself), and is denoted by $\mathcal{P}(A)$.

We also note that for any set A and any property P, we can form the set whose elements are exactly those elements $a \in A$ such that a satisfies P, which we denote by

$$\{a \in A : a \text{ satisfies } P\},$$

and often refer to this as *set builder notation.* We note that when we use set builder notation, we must begin with some set A (rather than just taking the collection of all objects satisfying some property P) in order to avoid Russel's paradox.[1] However, we will occasionally use set-builder notation without this restriction, but doing so will (in general) only result in a class[2], not necessarily a set.

B.2 Relations and Maps

In order to define what we mean by a relation on a set, we first need a notion of ordered pair. The basic idea is that, for two objects a and b, we want a new object (a, b) such that $(a, b) = (c, d)$ if and only if $a = b$ and $c = d$. To this end, we define (for objects a and b)

$$(a, b) := \{\{a\}, \{a, b\}\},$$

and then any such (a, b) is called an *ordered pair.* Then, for any objects $a_1, \ldots, a_n$ define inductively

$$(a_1, \ldots, a_n) := ((a_1, \ldots, a_{n-1}), a_n),$$

and call $(a_1, \ldots, a_n)$ an *ordered n-tuple.* We then have the following lemma, a proof of which can be found in Enderton (12).

---

[1]We take the property P to be the property of not being an element of oneself. Then if we consider the set S of all sets which are not members of themselves (i.e. $S := \{x \ : \ \psi(x)\}$, where $\psi(x) := \neg\ x \in x$). If $S \notin S$, then $S \in S$ by definition of $\psi(x)$, but if $S \in S$, then by the definition of $\psi(x)$, we have$S \notin S$. Either possibility leads to a contradiction, and so S cannot be a set.

[2]See Section B.2.3

Appendix B (Continued)

Lemma B.1. Let a, b, c, d be objects. Then $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Using the notion of an ordered pair, we can define, for sets A and B, the *product of A and B* (which we denote by $A \times B$) by

$$A \times B := \{(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B)) : a \in A \text{ and } b \in B\}.$$

Let $A_1, \ldots, A_n$ be a finite list of sets. Then define the *product of the* A_i's (denote $A_1 \times \cdots \times A_n$, or $\prod_{i=1}^{n} A_i$) inductively by

$$\prod_{i=1}^{n} A_i := (A_1 \times \cdots \times A_{n-1}) \times A_n.$$

If $A_1 = \cdots = A_n$ we define $A^n := A_1 \times \cdots \times A_n$, and define $A^0 := \{\varnothing\}$.

B.2.1 Relations

We are now in a position to define relations on sets.

Definition B.1. Let $n \in \{1, 2, 3, \ldots\}$, and let $A_1, \ldots A_n$ be sets. Then an *n-ary relation* between $A_1, \ldots,$ and A_n is a subset $R \subseteq A_1 \times \cdots \times A_n$. If $A_1 = \cdots = A_n = A$, then we say that R is an *n-ary relation on A.* If $n = 1$ we call R *unary*, and if $n = 2$ we call the relation *binary.* For R a binary relation we define $aRb := (a, b) \in R$.

Note: Unary relations are essentially just predicates, for example if we define $P \subseteq \mathbb{Z}$ by $P := \{n \in \mathbb{Z} : z > 0\}$ then P is just the predicate of positivity.

Some of the most important relations are *equivalence relations.*

Appendix B (Continued)

Definition B.2. Let A be a set, and $\eqsim$ be a binary relation on A. Then $\eqsim$ is called an *equivalence relation* on A if $\eqsim$ satisfies the following three properties.

Reflexivity: $a \eqsim a$ for all $a \in A$

Symmetry: If $a \eqsim b$, then $b \eqsim a$ for all $a, b \in A$

Transitivity: If $a \eqsim b$ and $b \eqsim c$, then $a \eqsim c$ for all $a, b, c \in A$

If $\eqsim$ is an equivalence relation, then for any $a \in A$, the set $\{b \in A \ : \ b \sim a\}$ is called the *equivalence class of a modulo* $\eqsim$, and we denote this equivalence class $[a]_{\eqsim}$.

There is a natural relationship between equivalence relations and partitions.

Definition B.3. Let A be a set. Then a set $S \subseteq \mathcal{P}(A)$ is called a *partition of* A if both

1. $X \cap Y = \varnothing$ for all $X, Y \in S$,
2. $\bigcup S = A$.

The natural relationship between equivalence relations on a set and partitions of that set is made explicit in the lemma below. See Enderton (12) for a proof.

Lemma B.2. Let A be a set, let S be a partition of A, and let $\equiv$ be an equivalence relation on A. Define the binary relation $\eqsim$ on A by (for all $a, b \in A$)

$$a \eqsim b \quad \text{iff there exists some } X \in S \text{ with both } a, b \in X,$$

and for every $a \in A$ let X_a be the equivalence class of a modulo $\equiv$, and define

$$T := \{X \in \mathcal{P}(A) \ : \ X = X_a \text{ for some } a \in A.\}.$$

Then $\eqsim$ is an equivalence relation, T is a partition of A, and $\eqsim$ & $\equiv$ are the same equivalence relation if and only if $S = T$.

B.2.2 Maps

We ordinarily think of a map f from some set A to another set B, as a rule such that for every $a \in A$, we assign some object $f(a)$ which must be in B. Using relations, this can be formalized. This is to say that for sets A and B, a binary relation f between A and B is said to be a *map* (or equivalently a *function*) *from A to B* (denoted by $f : A \to B$) if for every $a \in A$, there is a unique $b \in B$ such that $(a, b) \in f$. For a given a, we denote this unique b by $f(a)$, and if f is a function from A to B, we say that A is the *domain of f* (denoted dom f), and define the *range of f* (denote ran f) to be the set $\{b \in B \ : \ b = f(a) \text{ for some } a \in A\}$.

Ex: Consider the function on the integers which assigns to any number its square, i.e. $n \mapsto n^2$. The relation f which corresponds to this function is $\{(n, n^2) \in \mathbb{Z} \times \mathbb{Z} \ : \ n \in \mathbb{Z}\}$.

The following sets associated with functions are frequently used:

Let A, B, and C, be sets, and $f : A \to B$, and $g : B \to C$. Further let $A_0 \subseteq A$ and $B_0 \subseteq B$.

1. The *image of A_0 under f* (denoted $f(A_0)$) is defined by

$$f(A_0) := \{b \in B \ : \ b = f(a) \text{ for some } a \in A_0\}.$$

Appendix B (Continued)

2. The *pre-image of* B_0 *under* f (denoted $f^{-1}(B_0)$) is defined by

$$f^{-1}(B_0) := \{a \in A \ : \ f(a) \in B_0\}.$$

3. The *composition of* f *with* g (denoted $g \circ f$) is the set

$$g \circ f := \{(a,c) \in A \times C \ : \ (a,b) \in f \text{ and } (b,c) \in g \text{ for some } b \in B\}.$$

4. The *restriction of* f *to* A_0 (denoted $f|_{A_0}$) is given by

$$f|_{A_0} := \{(a,b) \in f \ : \ a \in A_0\}.$$

Note that the image of A under f is just the range of f, i.e. $f(A) = \text{ran } f$. Also, the composition $g \circ f$ is clearly a function from A to C, and the restriction $f|_{A_0}$ is a map from A_0 to B.

Additionally, for sets A and B, we denote the set of all maps from A to B by

$$B^A := \{f \in A \times B \ : \ f \text{ is a map from } A \text{ to } B\}. \tag{B.1}$$

We next consider several useful properties of maps between sets. Consider the sets A and B, and a map $f : A \to B$ between them. If for any $a, b \in A$, we have that $f(a) = f(b)$ implies that $a = b$, we say that f is *1-1* (or *injective*), and if, for any $b \in B$ there exists some $a \in A$ such that $f(a) = b$, then we say f is *onto* (or *surjective*). If f is both injective and surjective, we say that f is *bijective*; in such a case we will sometimes say that A and B are *isomorphic as sets*.

B.2.3 Classes and Class Functions

In passing, we noted that the collection of all sets satisfying some given property may not be a set. However, for some property P, it is often convenient to be able to refer to the collection of all sets satisfying this property as *something*, and this motivates the definition of a *class*. That is, we take a *class* to be any collection of objects. If a class is not a set, then we will often refer to it as a *proper class*. For example, we have that the collection of all sets is a proper class, while the collection of all sets contained in a given set A is a class, but is not a proper class — this class is just $\mathcal{P}(A)$.

For a given property P, we will use the following notation (similar to set-builder) to designate classes. Namely,

$$\{a \ : \ a \text{ satsifies property } P\} \tag{B.2}$$

is defined to be the class of all objects satisfying property P. We then abuse the '$\in$' notation, and so for a given class $\mathfrak{K}$ and object a, we write '$a \in \mathfrak{K}$' to mean that 'the object a is in the class $\mathfrak{K}$'. We employ similar abuses of notation for $\cup$, $\cap$, $\subseteq$, etc.

Further, we define the notion of a class function.

Definition B.4. Let $\mathfrak{F}$ be a class such that every element of $\mathfrak{F}$ is an ordered pair, and for any objects a, b and c such that $(a, b) \in \mathfrak{F}$ and $(a, c) \in \mathfrak{F}$, we have $b = c$. Then $\mathfrak{F}$ is said to be a *class* *function*, and we define $\mathfrak{F}(a)$ to be the unique object b such that $(a,b) \in \mathfrak{F}$. We also say that, for classes $\mathfrak{L}$ and $\mathfrak{K}$, a class function $\mathfrak{F} \in \mathfrak{L}^{\mathfrak{K}}$ if for all $(a,b) \in \mathfrak{F}$, we have $a \in \mathfrak{F}$ and $b \in \mathfrak{L}$.

We note that if a class function is a set, then it is an ordinary function (as defined above), and as such, functions can be seen as a special case of class functions. We now define some notation for specifying class functions. First, when the intended domain of a class function is clearly seen to be some class $\mathfrak{A}$, we will specify a class function $\mathfrak{F}$ by stating 'for all $a \in \mathfrak{A}$, $a \mapsto b$' to mean that $(a,b) \in \mathfrak{F}$. Also, rather than using the usual '$\mathfrak{F}(a)$' for the image of an object under a given class function $\mathfrak{F}$, we may instead use some expression such as a^*. In such a case, we will use the symbol ' $\cdot$ ' to indicate where the symbol representing the object(s) should go. For example, for $a \mapsto a^*$ we would use ' $\cdot^*$ to represent the associated class function, and for $(a,b) \mapsto \varphi_a(b)$ we would use $\varphi_\cdot(\cdot)$ to represent the asssociated class function.

Additionally, we define the domain and range of a class function $\mathfrak{F}$ in such a way that these concepts agree with those already defined when $\mathfrak{F}$ is a set. That is, for a class function $\mathfrak{F}$, we define the *domain of* $\mathfrak{F}$ (denoted dom $\mathfrak{F}$) by

$$\operatorname{dom} \mathfrak{F} := \{a \ : \ \text{for some object } b,\ (a,b) \in \mathfrak{F}\},$$

and define the *range of* $\mathfrak{F}$ (denoted ran $\mathfrak{F}$) by

$$\operatorname{ran} \mathfrak{F} := \{b \ : \ \text{for some object } a,\ (a,b) \in \mathfrak{F}\},$$

and for any subclass $\mathfrak{K} \subseteq \text{dom } \mathfrak{F}$, we define

$$\mathfrak{F}(\mathfrak{K}) := \{b \ : \ a \in \mathfrak{K} \text{ and } (a,b) \in \mathfrak{F}\}.$$

Also, we will write $\mathfrak{F} : \mathfrak{A} \to \mathfrak{B}$ to mean that $\text{dom } \mathfrak{F} = \mathfrak{A}$ and $\text{ran } \mathfrak{F} \subseteq \mathfrak{B}$.

Finally, we will need the following notion in discussions of the models of our axiomatic set theory.

Definition B.5. Let $\mathfrak{K}$ be a class, and $n \in \{1, 2, \ldots\}$. Define

$$\mathfrak{K}^n := \{(k_1, \ldots, k_n) \ : \ k_1, \ldots, k_n \in \mathfrak{K}\},$$

and $\mathfrak{K}^0 := \{\varnothing\}$. Then an *n-ary operation on* $\mathfrak{K}$ is a class function $\mathfrak{F}$ such that for any pair $(a,b) \in \mathfrak{F}$, we have that both $a \in \mathfrak{K}^n$ and $b \in \mathfrak{K}$.

B.3 Ordinal Numbers and Transfinite Induction

In this section we define the ordinal numbers. In order to do so, we need to discuss proofs by transfinite induction as well as definitions by transfinite recursion — these concepts are used in the construction of the classical universe of sets, as well as our models of quantum set theory. We note that this discussion follows Chapter 7 in Enderton (12) closely, and the reader is referred there for details and proofs.

B.3.1 Transfinite Induction and Recursion

We begin with the basic notions of a strict linear ordering[1] a well-ordering, and a segment. That is, if A is a set, and $<$ be a binary relation on A, we say that $<$ is a *strict linear ordering* on A if $<$ satisfies

1. Transitivity: For every $a, b, c \in A$ with $a < b$ and $b < c$ we have $a < c$.
2. Trichotomy: For every $a, b \in A$, exactly one of the following

$$a < b, \qquad b < a, \qquad a = b$$

holds.

Definition B.6. Let A be a set and $<$ a strict linear ordering on A. We say that $<$ is a *well-ordering* if every non-empty subset $X \subseteq A$ has a least element under $<$. The pair $(A, <)$ is then called a *well-ordered structure.*

Ex: The quintessential example of a well-ordered set is the natural numbers $\mathbb{N}$ with the usual strict ordering.

Definition B.7. Let A be a set with $a \in A$, and $<$ a strict linear ordering. Define the *segment of A before a* (denoted seg a) by

$$\text{seg } a := \{b \in A \ : \ b < a\}.$$

---

[1] We could develop the theory in this section using chains rather than strict linear orderings, but we follow Enderton (12) for ease of reference.

Appendix B (Continued)

We now consider the principle of transfinite induction.

Theorem 40. Let A be a set and $<$ a well-ordering on A. Further let $B \subseteq A$, and assume that seg $a \subseteq B$ implies that $a \in B$ for every $a \in A$. Then $B = A$.

Theorem 40 extends the ordinary principle of induction, enabling us to do proofs by transfinite induction on any well-ordered set — e.g. the above theorem can be used to prove that (for some well-ordered set A) every $a \in A$ satisfies some property P. This is to say that if we can demonstrate that whenever every element of seg a satisfies property P then a does as well, the above theorem then gives that all of A satisfies P (simply by taking B to be the set of elements satisfying P.)

We next discuss definition by transfinite recursion, which is necessary for defining the ordinal numbers. We note that the following theorem requires the axiom of replacement (see Section 4.2) for its proof.

Theorem 41. Let $(A, <)$ be a well-ordered structure, and let $\mathfrak{F}$ be a class function such that for every set x there exists an object y with $(x, y) \in \mathfrak{F}$. Then there exists a unique function f with domain A such that

$$(f|_{\text{seg } a}, f(a)) \in \mathfrak{F}$$

for all $a \in A$.

B.3.2 Ordinal Numbers

We can now use definition by transfinite recursion to define the ordinal numbers.

Appendix B (Continued)

Definition B.8. Let $(A, <)$ be a well-ordered structure, and take $\mathfrak{K} := \{(x, \text{ran } x) \; : \; x \text{ is a set}\}$. Then let f be the unique function with domain A given by Theorem **??** such that

$$f(a) = \text{ran } f_{\text{seg a}} = f(\text{seg } a) = \{f(x) \; : \; x < a\}.$$

Then ran f is called the $\in$-image of $(A, <)$. If, for a given set α, there exists a well-ordered structure $(A, <)$ such that α is the $\in$-image of $(A, <)$, then α is called the *ordinal number of* $(A, <)$, or just an *ordinal number*. We denote the class of all ordinals by **Ord**.

Ex: Take A to be r, s, t with $r < s < t$. Then we compute the function f to be

$$f(r) = \{f(x) \; : \; x < r\} = \varnothing$$

$$f(s) = \{f(x) \; : \; x < s\} = \{f(r)\} = \{\varnothing\}$$

$$f(t) = \{f(x) \; : \; x < t\} = \{f(r), f(s)\} = \{\varnothing, \{\varnothing\}\}.$$

We note that the class of all ordinals **Ord** is a proper class, and this statement is known as the Burali-Forti Theorem.

From the above example we can see the beginning of a pattern. In fact this allows us to consider the natural numbers as a special case of the ordinal numbers. To see this, let $n \in \mathbb{N} := \{0, 1, \ldots\}$, with $<$ the standard ordering on $\mathbb{N}$. Then define *the* n^{th} *ordinal* (denoted $\mathbf{n}$) to be the ordinal number of $(\{m \leq n\}, <)$. We define the first infinite ordinal ω to be the ordinal number of $(\mathbb{N}, <)$. As such, we have that $\mathbf{0} = \varnothing$, $\mathbf{1} = \{\varnothing\}$, $\mathbf{2} = \{\varnothing, \{\varnothing\}\}$, etc.

Now, the following definition and theorem are needed for the construction of the classical universe of sets.

Definition B.9. Let α be an ordinal number. Define $\alpha + 1 := \alpha \cup \{\alpha\}$. If there is some ordinal number β such that $\alpha = \beta + 1$ then α is said to be a *successor ordinal.* If there is no such β, and $\alpha \neq \varnothing$, then α is said to be a *limit ordinal.*

Theorem 42. Let α be an ordinal number, and $\mathcal{A}$ a set of ordinal numbers. Define $\in_\alpha :=$ $\{(A, B) \in \alpha \times \alpha \ : \ A \in B\}$. Then

1. $(\alpha, \in_\alpha)$ is a well-ordered structure.
2. $\varnothing = \alpha$ or $\varnothing \in \alpha$.
3. $\alpha + 1$ is an ordinal number.
4. $\bigcup \mathcal{A}$ is an ordinal number.

B.4 The Axiom of Choice and Zorn's Lemma

We conclude with a brief discussion of the Axiom of Choice and Zorn's Lemma. The content of the *axiom of choice*[1] is essentially that

> For any set $\mathcal{A}$ consisting of disjoint, non-empty sets, there is a set B such that every $b \in B$ is a member of exactly one set $A \in \mathcal{A}$.

This axiom of set theory is certainly intuitive for finite collections — in fact, it can be proven from the other (ZF) axioms of set theory if we restrict $\mathcal{A}$ above to be a finite collection

[1] See Section 4.2 for a formal treatment.

(by simply making a choice of an element from each $A \in \mathcal{A}$ and forming B from those chosen elements). However, the axiom of choice enables us to extend this procedure to arbitrary infinite collections.

We note that although the statement of this axiom seems relatively intuitive, there are certain other equivalent[1] statements which are seem much less intuitive — e.g. it has been shown (4) that the statement 'every vector space has a basis' is equivalent to the axiom of choice. Zorn's Lemma is another (equivalent) way of stating the axiom of choice.

Lemma B.3. *Zorn's Lemma:* Let A be a set partially ordered under $\leq$, such that for every chain $C \subseteq A$, there exists some $a \in A$ such that a is an upper bound for C. Then A has a maximal element under $\leq$.

Zorn's Lemma makes frequent appearences in classical mathematics, where it is used to prove that every vector space has a basis, that every ring has a maximal ideal, etc. (e.g. see Aluffi (1)). This lemma is used in the proof of Theorem 5.

[1] At least in the presence of the other axioms of set theory under classical logic.

Appendix C

BASICS OF UNIVERSAL ALGEBRA

Here we define some basic concepts from universal algebra that are needed in the construction of the model theory for our first order quantum logic $\mathcal{Q}(\mathcal{L})$, as well as in the proof of the soundness and completeness theorems for the deductive system relative to the semantics for our first order logic developed in Chapter 2. See (5) for a comprehensive introduction to universal algebra.

C.1 Algebras and Homomorphisms

Definition C.1. Let A and F be nonempty sets such that F, as well as every $f \in F$ is a map from A^n to A for some $n \in \mathbb{N} \cup \{\mathbb{N}\}$. Then the pair (A, F) (as well as just the set A itself) is said to be an *algebra with operations* F. If $f : A^n \to A$, then f is said to be an *n-ary operation on* A, and f is said to be of *arity* n. The *type of* (A, F) is the map $\alpha : F \to \mathbb{N} \cup \{\mathbb{N}\}$ assigning to each $f \in F$ its airity. If ran $\alpha \subseteq \mathbb{N}$, then (A, F) is said to be of *finite type*. If F is finite, (A, F) is said to have a finite number of operations. If F has m elements, with $m \in \mathbb{N}$, then (A, F) is said to be a $(n_1, n_2, \ldots, n_m)$-algebra, where $n_1, \ldots, n_m$ are the airities of the elements of F in non-increasing order.

As an example, consider a group G with idenity 'e', multiplication '$\cdot$', and inverse operation $\cdot^{-1}$. Using the above definition, we can construct several different algebras for this group —

namely, $(G, \{\cdot, \cdot^{-1}, e\})$ is a $(2,1,0)$-algebra (with $\alpha(e) = 0$, $\alpha(\cdot) = 2$, and $\alpha(\cdot^{-1}) = 1$), while $(G, \{\cdot\})$ as a (2)-algebra, and so on.

Now, for (A_1, F_1) and (A_2, F_2) algebras of type α and β, respectively, if there exists a bijection $\nu : F_1 \to F_2$ such that $\beta \circ \nu = \alpha$, then A_1 and A_2 are said to be of the *same type*, and the bijection ν is called a *type identification.* These notions enable the definition of a structure-preserving map between algebras of the same type.

Definition C.2. Let (A_1, F_1) be an algebra of type α, and let (A_2, F_2) be another algebra of the same type with type identifier ν. Then a F_1*-homomorphism from* A_1 *to* A_2 *wrt* ν is a map $h : A_1 \to A_2$ such that

$$\nu(f)(h(a_1), \ldots, h(a_{\alpha(f)}) = f(a_1, \ldots, a_{\alpha(f)}).$$

for every $f \in F_1$ and $a_1, \ldots, a_{\alpha(f)} \in A_1$.

We note that, in practice, the type identifier is usually implicit, as the algebras under consideration will use the same symbols to denote identified operations; in such a case, the map h is simply called a homomorphism. Further, if (A_1, F_1) and (A_2, F_2) are algebras of the same type, and $h : A_1 \to A_2$ is a homomorphism which is also a bijection, an *algebraic isomorphism*, or simply an *isomorphism.* If there exists an algebraic isomorphism from (A_1, F_1) to (A_2, F_2), then (A_1, F_1) and (A_2, F_2) are said to be *isomorphic.*

Appendix C (Continued)

C.2 Products and Irreducibility

In what follows, we consider products of two algebras; although the concept of a product of algebras can be defined more generally, the following discussion suffices for our purposes.

Definition C.3. Let (A_1, F_1) and (A_2, F_2) be algebras of the same type with type identifier ν. For each $f \in F_1$, define the $\alpha(f)$-ary operation $\hat{f}$ on $A_1 \times A_2$ by (for every $a_1 \in A_1$ and $a_2 \in A_2$)

$$\hat{f}(a_1, a_2) \coloneqq (f(a), \nu(f)(a)),$$

and let $\hat{F} \coloneqq \{\hat{f} \ : \ f \in F_1\}$. Then *the product of* (A_1, F_1) *and* (A_2, F_2) is the algebra given by $(A_1 \times A_2, \hat{F})$.

We note that if (A, F) is the product of (A_1, F_1) and (A_2, F_2), then (A, F) is of the same type as (A_1, F_1) and (A_2, F_2). Finally, if (A, F) is an algebra and there exist two other algebras (A_1, F_1) and (A_2, F_2) which are of the same type as (A, F), and further, if (A, F) is isomorphic to the product of (A_1, F_1) and (A_2, F_2), then (A, F) is said to be *reducible.* Otherwise, (A, F) is said to be *irreducible.*

C.3 Free Algebras, Congruences and Quotients

For purposes of our discussion we only need to consider free algebras of finite type. A proof that such algebras are well-defined and always exist can be found in Burris (5).

Definition C.4. Let F be a set with a map $\alpha : F \to \mathbb{N}$, and let A be a nonempty set. Then the *free algebra with operations F on A* (denoted by $\mathcal{F}(A)$) is the unique algebra (up to isomorphism) with operations F of type α such that for any algebra (B, F') (which is also of type

α) and every set theoretic map $f : A \to B$, there exists a unique homomorphism $\hat{f} : (\mathcal{F}(A) \to B$ such that $\hat{f}(a) = f(a)$ for every $a \in A$. Also, for any $a \in A$, a is called a *basic element of* $\mathcal{F}(A)$.

Just as a set can be partitioned into equivalence classes, the set of which forms a new set, it is possible to make a similar construction for algebras — that is, for a given algebra (A, F), it is possible to take a partition of the underlying set A (with the right properties), and from this form an algebra on the equivalence classes of A which is of the same type as (A, F). To this end, we consider the following definition of a congruence (which is effectively an equivalence relation which respects the algebraic properties of the algebra).

Definition C.5. Let (A, F) be an algebra of type α, let $f \in F$, and let '$\eqsim$' be an equivalence relation on A. Then '$\eqsim$' is called a *congruence wrt* f if, for any $a_1, \ldots, a_{\alpha(f)}, b_1, \ldots, b_{\alpha(f)} \in A$ such that $a_i \eqsim b_i$ for $i \in \{1, \ldots, \alpha(f)\}$,

$$f(a_1, \ldots, a_n) \eqsim f(b_1, \ldots, b_n).$$

If '$\eqsim$' is a congruence wrt every $f \in F$, then '$\eqsim$' is simply called a *congruence on* A.

Using this definition, we can now define a quotient algebra.

Theorem 43. Let (A, F) be an algebra of type α, and let '$\eqsim$' be a congruence on A. Then for each $f \in F$, define[1] $\hat{f} : (A/\eqsim)^{\alpha(f)} \to (A/\eqsim)$ by $\hat{f}([a]_{\eqsim}) := [f(a)]_{\eqsim}$. Let $\hat{F} := \{\hat{f} \;\; f \in F\}$ and define $\beta : \hat{F} \to \mathbb{N}$ by $\beta(\hat{f}) := \alpha(f)$. Then

---

[1]That the map $\hat{f}$ is well-defined requires proof. The details can be found in Burris (5).

Appendix C (Continued)

1. $(A/\eqsim, \hat{F})$ is an algebra of type β.

2. (A, F) and $(A/\eqsim, \hat{F})$ are of the same type, with type identifier $f \mapsto \hat{f}$.

3. The map given by $a \mapsto [a]_{\eqsim}$ (for every $a \in A$) is a surjective algebra homomorphism.

Now, if (A, F) is an algebra of type α and '$\eqsim$' is a congruence on A, then the algebra $(A/\eqsim, \hat{F})$ from Theorem 43 is called the *quotient algebra of A by '$\eqsim$'*.

Finally, we conclude this section by noting that while the above theorem shows that quotients give rise to homomorphisms, the theorem below shows that homomorphisms give rise to quotients.

Theorem 44. Let (A_1, F_1) and (A_2, F_2) be algebras of the same type, and let $h : A_1 \to A_2$ a homomorphism. The binary relation '$\eqsim$' on A_1 given by

$$a \eqsim b \quad \text{iff} \quad h(a) = h(b)$$

is a congruence on A.

CITED LITERATURE

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Kimberly Frey
Curriculum Vitae

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Education

2013 University of Illinois at Chicago — Chicago, IL
PhD in Physics
Dissertation Advisor: Professor Tom Imbo

2002 Wayne State University — Detroit, MI
Master of Science in Physics
Thesis Advisors: Professors Talbert Stein & Walter Kauppila

1998 University of Michigan — Ann Arbor, MI
Bachelor of Arts in Literature
Secondary Teaching Certificate – English and Physics

Professional Experience

2005-2013 Graduate Teaching Assistant
University of Illinois at Chicago — Chicago, IL

2008 Instructor of Physics (Adjunct)
St. Xavier University — Chicago, IL

2003-2005 Instructor of Physics
Armstrong Atlantic State University — Savannah, GA

2002-2003 High School Physics Teacher
Calvary Day School — Savannah, GA

1999-2002 Graduate Teaching Assistant & Research Assistant
Wayne State University — Detroit, MI

Research Experience

Experience with using algebraic approaches to formal languages and logic to study the logic of physical theories in particular, the logic which underlies quantum theory.

Experience with the phenomenology of chiral symmetries of vector-like gauge theories in arbitrary dimensions, as well as the relevant mathematics and representation theory.

Experience with examining simple quantum systems in the presence of closed time-like curves via a model which utilizes methods of quantum computation.

Experience with ab initio (first principles) investigations of the size-dependence of quasiparticle gaps (electron affinities and ionization potentials), optical gaps, exciton binding energies and dielectric screening in silicon nanoshells; familiarity with use of the real-space higher-order finite-difference pseudopotenital method.

Experience with and preliminary work on the phase diagram of QCD through numerical methods based on the NJL model.

Experience with synthesis & characterization of ZnS nanoparticles by a variety of techniques including Ultraviolet-Visible & Fourier Transform Infrared Spectroscopy, Nuclear Magnetic Resonance, and Scanning Tunneling Microscopy.

Preliminary work with developing organic light-emitting diodes (using MEH-PPV and Ru(bpy)32+) on indium tin oxide (ITO) substrates for determination of I-V characteristics (turn-on voltage & voltage that corresponds to the maximum intensity of emitted light).

Experience with positron–alkali metal atom scattering experiments — specifically the measurements of the positronium formation cross-section for positrons scattered by cesium atoms. Familiarity with vacuum technology, positron beam technology and the associated electronics, as well as with LabVIEW.

Publications

R. DeJonghe, K. Frey and T. Imbo, "Quantum Set Theory," in preparation.

R. DeJonghe, K. Frey and T. Imbo, "Quantum Arithmetic," in preparation.

R. DeJonghe, K. Frey and T. Imbo, "Mathematics with Quantum Logic," in preparation.

R. DeJonghe, K. Frey and T. Imbo, "Bott Periodicity and Realizations of Chiral Symmetry in Arbitrary Dimensions," Phys. Lett. B, 718 603-609 (2012).

R. DeJonghe, K. Frey and T. Imbo, "Discontinuous Quantum Evolutions in the Presence of Closed Timelike Curves," Phys. Rev. D, 81 087501 (2010).

K. Frey, J. C. Idrobo, M.L. Tiago, F. Reboredo and S. Ogut, "Quasiparticle Gaps and Exciton Coulomb Energies in Si Nanoshells: First-principles Calculations," Phys. Rev. B, 80, 153411 (2009).

E. Surdutovich, W. E. Kauppila, C. K. Kwan, E. G. Miller, S. P. Parikh, K. A. Price(Frey), and T. S. Stein, "Measurements of Total and Inelastic Cross-sections in Positron-Cs and Electron-Cs Collisions," Physical Review A, 75, 32720 (2007).

E. Surdutovich, W.E. Kauppila, C.K. Kwan, E.G. Miller, S.P. Parikh, K.A. Price(Frey), T.S. Stein, "Measurements of Cross-sections for Positrons and Electrons Scattered by Cs Atoms," Nuclear Instruments and Methods in Physics Research B, 221 97-99 (2004).

Presentations

K. Frey, R. DeJonghe, T. Imbo, "Quantum Mathematics," presented at University of Illinois at Chicago, July 10, 2013.

K. Frey, R. DeJonghe, T. Imbo, "Bott Periodicity and Realizations of Chiral Symmetry in Arbitrary Dimensions," presented at University of Illinois at Chicago, August 31, 2011.

K. Frey, R. DeJonghe, T. Imbo, "The Nature of Quantum Truth," presented at University of Illinois at Chicago, March 2, 2009.

K. Frey, W.E. Lynch, D.A. Nivens, "Effects of Particle Size and Capping Entities on Nanocluster Properties," presented at the Annual American Physical Society March Meeting, Los Angeles, CA, March 21-25, 2005.

E. Surdutovich, W.E. Kauppila, E.G. Miller, K.A. Price(Frey), T.S. Stein, "Measurements of Cross-sections for Positrons Scattered by Cs Atoms," 35th Annual Division of Atomic, Molecular and Optical Physics Meeting, Tucson, AZ, May 25-29, 2004.

E. Surdutovich, W.E. Kauppila, K.A. Price(Frey), T.S. Stein, "Preliminary Measurements of Positronium Formation Cross-sections for Positrons Scattered by Cs Atoms," 33rd Annual Division of Atomic, Molecular and Optical Physics Meeting, Williamsburg, VA, May 29-June 2, 2002.

Honors, Awards & Recognition

2010 Deans Fellowship UIC

2010 James Kouvel Fellowship UIC Department of Physics

2006 Outstanding Teaching Assistant AwardUIC Department of Physics

2002 AAPT Outstanding Teaching Assistant Award

Service Activities

20082011 Co-Founder & Organizer of the Student Colloquium Series for Physics (UIC)

2008-2009 Physics Department Representative on the Graduate Student Council (UIC)

2003-2005 Faculty Sponsor of the AASU Physics Club

Professional Memberships

American Physical Society

American Association of Physics Teachers

Courses Taught

UIC

PHYS 106	Introductory Physics I Discussion
PHYS 108	Introductory Physics II Discussion
PHYS 108	Introductory Physics II Lab
PHYS 121	Physical Universe (T.A. & grader)
PHYS 141	General Physics I Lab
PHYS 511/512	Graduate Quantum Mechanics (T.A. & grader)
PHYS 513/514	Graduate Quantum Field Theory (T.A. & grader)

SXU

PHYS 201	Introductory Physics I
PHYS 201L	Introductory Physics I Lab

AASU

PHYS 1111	Introductory Physics I
PHYS 1112	Introductory Physics II
PHYS 1211	Physical Environment
PHYS 4990	Advanced Undergraduate Research
PHYS 1111L/2211L	Introductory Physics I Lab
PHYS 1112L	Introductory Physics II Lab

WSU

PHYS 2130	Quiz Section, Introductory Physics I
PHYS 2140	Quiz Section, Introductory Physics II
PHYS 2170	Quiz Section, General Physics I
PHYS 2131/2171	Intro/General Physics I Lab
PHYS 2141/2181	Intro/General Physics II Lab