Algorithm XXX: Fortran 90 subroutines for computing the expansion coefficients of Mathieu functions using Blanch's algorithm

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A translation to Fortran 90 of Gertrude Blanch's algorithm to compute the expansion coefficients of the series that represent Mathieu functions is presented. Its advantages are portability, higher precision, practicality of use and extended documentation. In addition, numerical validations and comparisons with other existing methods are presented.

Categories and Subject Descriptors: G.4 [**Mathematical Software**]: documentation; J.2 [**Physical Sciences and Engineering**]: engineering, mathematics, physics

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1. INTRODUCTION

Some problems of mathematical physics find their natural formulation in the elliptic cylinder coordinate system and, therefore, require use of Mathieu functions [Mathieu 1868]. These functions have been studied by many authors, including Stratton [1941], Meixner and Schäfke [1954], and McLachlan [1964]. The computation of Mathieu functions is not a trivial problem and software packages that provide support to compute them have been developed by, among others, Clemm [1969; 1970], Hodge [1972], Frisch [1972], Baker [1992], Shirts [1993a], [1993b], IMSL [1994], Zhang and Jin [1996], Alhargan [2001], and Mathematica [2003].

This article presents the translation to Fortran 90 of the algorithm developed by Gertrude Blanch [1966] to compute the expansion coefficients of the series that define Mathieu functions. In addition, this article presents a subroutine that performs validations of the translation and a sample driver program for its use.

There are multiple motivations for this work. Blanch's algorithm is associated with a very detailed numerical analysis that justifies its convergence; therefore, in this regard, it should be preferred over other algorithms because of the documentation available for it. In addition, these subroutines are written in Fortran 90, which is used by many scientists, and the subroutines are written to allow for the change

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of precision in order to improve portability among different platforms. These subroutines compute Mathieu functions according to three different normalizations: 1) the one introduced by Stratton, Morse and Chu [Stratton 1941], which is used in many other works such as in Bowman *et al.* [1987]; 2) the Goldstein-Ince normalization [Goldstein 1927], [Ince 1932], used for example in [Abramovitz and Stegun 1970]; and 3) the neutral normalization.

2. MATHIEU FUNCTIONS

Many different notations have been introduced to identify Mathieu functions. The notation adopted in this work is the one of Blanch and Rhodes [1955] (which is also available in [National Bureau of Standards 1951] and [Staff of the computation Laboratory 1967]) and Stratton [1941], which is justified by its usefulness in many applications. Mathieu functions come from the solution of Mathieu's differential equation

$$\frac{d^2y}{dx^2} + (b - s\cos^2 x)y = 0. \tag{1}$$

Here only positive real values of s will be considered. When $s = 0$ the solutions of Mathieu's equation are simple and of the form $y(x) = y(0)\exp(\pm i\sqrt{b}x)$. When $s > 0$, Mathieu's equation contains a periodic coefficient. In many physical applications only periodic solutions are of interest and for a given s there exist two countable sets of values of b for which equation (1) admits periodic solutions. These values of b are called characteristic values and, depending upon the set, the period of the solution is either π or 2π. There are four kinds of periodic solutions of (1) associated with the characteristic values b:

$$\mathrm{Se}_{2r}(s,x) = \sum_{k=0}^{\infty} \mathrm{De}_{2k}^{(2r)} \cos 2kx \qquad \text{(of period } \pi) \tag{2}$$

$$\mathrm{Se}_{2r+1}(s,x) = \sum_{k=0}^{\infty} \mathrm{De}_{2k+1}^{(2r+1)} \cos(2k+1)x \qquad \text{(of period } 2\pi) \tag{3}$$

$$\mathrm{So}_{2r}(s,x) = \sum_{k=1}^{\infty} \mathrm{Do}_{2k}^{(2r)} \sin 2kx \qquad \text{(of period } \pi) \tag{4}$$

$$\mathrm{So}_{2r+1}(s,x) = \sum_{k=0}^{\infty} \mathrm{Do}_{2k+1}^{(2r+1)} \sin(2k+1)x \qquad \text{(of period } 2\pi) \tag{5}$$

Because of their periodicity and their meaning in physical applications, these functions are also called Mathieu angular functions and they are computed by the function `MathieuAngular`. The Mathieu angular functions are indicated by Stratton with the symbols $\mathrm{Se}_n(s,\cos x)$ and $\mathrm{So}_n(s,\cos x)$. Unfortunately, this is a misleading notation because it would suggest, for example, that $\mathrm{So}_n(s,\cos x)$ would be an even function of x, which is clearly wrong given the definitions (4-5). However, the notation of Blanch adopted in this work avoids any misinterpretation.

If x is replaced by ix in (1) one obtains

$$\frac{d^2y}{dx^2} - (b - s\cosh^2 x)y = 0, \tag{6}$$

which is known as Mathieu's modified equation. The functions $\mathrm{Se}_n(s, ix)$ and $\mathrm{So}_n(s, ix)$ clearly satisfy (6) for the same characteristic values b, but (2)-(5) converge slowly. Therefore the solutions of (6) are written in terms of rapidly converging series of products of Bessel functions associated with the same coefficients $\mathrm{De}_m, \mathrm{Do}_m$ of the angular functions. These new solutions, proportional to $\mathrm{Se}_n(s, ix)$ and $\mathrm{So}_n(s, ix)$, are referred to as Mathieu modified functions of the first kind. Their meaning in many physical applications suggests the additional name of Mathieu radial functions of the first kind. Their expressions are:

$$\mathrm{Re}_{2r}^{(1)}(s,x) = \frac{(-1)^r}{\mathrm{De}_0^{(2r)}}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}(-1)^k \mathrm{De}_{2k}^{(2r)} J_k(u) J_k(v), \tag{7}$$

$$\mathrm{Re}_{2r+1}^{(1)}(s,x) = \frac{(-1)^r}{\mathrm{De}_1^{(2r+1)}}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}(-1)^k \mathrm{De}_{2k+1}^{(2r+1)} \left[J_{k+1}(u)J_k(v) + J_k(u)J_{k+1}(v)\right], \tag{8}$$

$$\mathrm{Ro}_{2r}^{(1)}(s,x) = \frac{(-1)^r}{\mathrm{Do}_2^{(2r)}}\sqrt{\frac{\pi}{2}}\sum_{k=1}^{\infty}(-1)^k \mathrm{Do}_{2k}^{(2r)} \left[J_{k+1}(u)J_{k-1}(v) - J_{k+1}(v)J_{k-1}(u)\right], \tag{9}$$

$$\mathrm{Ro}_{2r+1}^{(1)}(s,x) = \frac{(-1)^r}{\mathrm{Do}_1^{(2r+1)}}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}(-1)^k \mathrm{Do}_{2k+1}^{(2r+1)} \left[J_{k+1}(u)J_k(v) - J_k(u)J_{k+1}(v)\right], \tag{10}$$

where

$$u = \frac{\sqrt{s}}{2}e^{x}, \quad v = \frac{\sqrt{s}}{2}e^{-x}. \tag{11}$$

The radial functions of the first kind have parity either even, $\mathrm{Re}_n^{(1)}$, or odd, $\mathrm{Ro}_n^{(1)}$. A second set of solutions for the modified Mathieu's equation is obtained by replacing the Bessel functions $J_m(u)$ in the previous equations with the Bessel functions $Y_m(u)$. This substitution yields the modified functions of the second kind. In many physical applications they are referred to as radial functions of the second kind. They have parity either even, $\mathrm{Re}_n^{(2)}$, or odd, $\mathrm{Ro}_n^{(2)}$:

$$\mathrm{Re}_{2r}^{(2)}(s,x) = \frac{(-1)^r}{\mathrm{De}_0^{(2r)}}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}(-1)^k \mathrm{De}_{2k}^{(2r)} Y_k(u) J_k(v), \tag{12}$$

$$\mathrm{Re}_{2r+1}^{(2)}(s,x) = \frac{(-1)^r}{\mathrm{De}_1^{(2r+1)}}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}(-1)^k \mathrm{De}_{2k+1}^{(2r+1)} \left[Y_{k+1}(u)J_k(v) + Y_k(u)J_{k+1}(v)\right], \tag{13}$$

$$\mathrm{Ro}_{2r}^{(2)}(s,x) = \frac{(-1)^r}{\mathrm{Do}_2^{(2r)}}\sqrt{\frac{\pi}{2}}\sum_{k=1}^{\infty}(-1)^k \mathrm{Do}_{2k}^{(2r)} \left[Y_{k+1}(u)J_{k-1}(v) - Y_{k-1}(u)J_{k+1}(v)\right], \tag{14}$$

$$\mathrm{Ro}_{2r+1}^{(2)}(s,x) = \frac{(-1)^r}{\mathrm{Do}_1^{(2r+1)}}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{\infty}(-1)^k \mathrm{Do}_{2k+1}^{(2r+1)} \left[Y_{k+1}(u)J_k(v) - Y_k(u)J_{k+1}(v)\right]. \tag{15}$$

Similar to Hankel functions, one defines modified functions of the third and fourth kinds. They are also referred to as radial functions and they have even parity

$$\mathrm{Re}_n^{(3)} = \mathrm{Re}_n^{(1)} + i\mathrm{Re}_n^{(2)}, \tag{16}$$
$$\mathrm{Re}_n^{(4)} = \mathrm{Re}_n^{(1)} - i\mathrm{Re}_n^{(2)}, \tag{17}$$

and odd parity

$$\mathrm{Ro}_n^{(3)} = \mathrm{Ro}_n^{(1)} + i\mathrm{Ro}_n^{(2)}, \tag{18}$$
$$\mathrm{Ro}_n^{(4)} = \mathrm{Ro}_n^{(1)} - i\mathrm{Ro}_n^{(2)}. \tag{19}$$

All Mathieu radial functions are computed by the function `MathieuRadial`.

3. THE COMPUTATION OF MATHIEU FUNCTIONS

The computation of both angular and radial Mathieu functions occurs in three steps 1) for a given order n and a parameter s the Mathieu eigenvalue b is generated; 2) the expansion coefficients De_m or Do_m are computed using n, s, and b; and, 3) the series expansions (2)-(5), (7)-(10), (12)-(19) are evaluated.

The main contribution of the present work is the subroutine `Blanch_Coefficients`, which produces the expansion coefficients according to Blanch's algorithm [1966]. Her algorithm exploits recurrence relations that are obtained for the coefficients $\mathrm{De}_m, \mathrm{Do}_m$ when expressions (2)-(5) are introduced into Mathieu's equation (1). The four periodic solutions (2)-(5) produce four different recurrence relations among the coefficients. As an example, one particular recurrence relation is obtained when (2) is introduced into (1) and $\cos^2 x$ is replaced by $(1 + \cos 2x)/2$ yielding

$$\mathrm{De}_2 - V_0\mathrm{De}_0 = 0 \tag{20}$$
$$\mathrm{De}_4 - V_2\mathrm{De}_2 + 2\mathrm{De}_0 = 0 \tag{21}$$
$$\mathrm{De}_{m+2} + \mathrm{De}_{m-2} - V_m\mathrm{De}_m = 0, \quad V_m = \frac{4b - 2s - 4m^2}{s}, \quad m \geq 3. \tag{22}$$

By introducing the definitions

$$G_m = \mathrm{De}_m/\mathrm{De}_{m-2} \tag{23}$$
$$H_m = 1/G_m \tag{24}$$

the previous recurrence relations define a forward rule to generate the coefficients G_m. Introducing the notation $G_{m,1}$ to refer to the coefficients G_m produced using the forward rule, the recurrence relations (20)-(22) are written as

$$G_{2,1} = V_0 \tag{25}$$
$$G_{m,1} = V_{m-2,1} - c_{m-4}H_{m-2,1}, \qquad \text{with } c_0 = 2 \text{ and } c_m = 1, m \geq 2. \tag{26}$$

The coefficients G_m may also be generated using a backward rule that is expressed using a continuous fraction

$$G_{m,2} = \frac{c_{m-2}}{V_m - \cdots}\frac{1}{V_{m+2} - \cdots} \tag{27}$$

Since each G_m may be generated in two independent ways, in particular, the value $G_{2,2}$ obtained from (27) must equal V_0 from (25) so that

$$G_{2,1} = V_0 = \frac{2}{V_2 - G_{4,2}} = G_{2,2} \tag{28}$$

The previous relation is a sufficient and necessary condition that must hold when b is the eigenvalue of Mathieu's equation.

Blanch's algorithm consists in generating two sequences $G_{m,1}(b)$ and $G_{m,2}(b)$. The two sequences are joined at $m = m_1$, which is determined so that the numerical precision is maximized. This is possible because Blanch proved that b is an eigenvalue of Mathieu's equation if and only if the difference

$$G_{m,1}(b) - G_{m,2}(b) = 0 \tag{29}$$

for some m for which $G_{m,1}(b)$ is finite. Hence, the two sequences may be joined not only at $m = 2$, as in (28), but also at other convenient values. Blanch's method determines the value m_1 so that $|G_{m_1,1}(b)|$ and $|G_{m_1,2}(b)|$ are of the same order of magnitude. Further details on Blanch's algorithm are found in [Blanch 1966].

`Blanch_Coefficients` normalizes the expansion coefficients according to three different methods.

1. Stratton, Morse, Chu normalization. The expansion coefficients satisfy

$$\sum_{k=0}^{\infty} \mathrm{De}_{2k+p} = 1, \qquad \sum_{k=0}^{\infty} (2k+p)\mathrm{Do}_{2k+p} = 1, \qquad p = 0, 1 \tag{30}$$

2. Goldstein-Ince normalization The expansion coefficients satisfy

$$\mathrm{Se}_{2r} : 2\left(\mathrm{De}_0^{(2r)}\right)^2 + \sum_{k=1}^{\infty}\left(\mathrm{De}_{2k}^{(2r)}\right)^2 = 1, \tag{31}$$

$$\mathrm{Se}_{2r+1} : \qquad \sum_{k=0}^{\infty}\left(\mathrm{De}_{2k+1}^{(2r+1)}\right)^2 = 1 \tag{32}$$

$$\mathrm{So}_{2r} : \qquad \sum_{k=1}^{\infty}\left(\mathrm{Do}_{2k}^{(2r)}\right)^2 = 1 \tag{33}$$

$$\mathrm{So}_{2r+1} : \qquad \sum_{k=0}^{\infty}\left(\mathrm{Do}_{2k+1}^{(2r+1)}\right)^2 = 1 \tag{34}$$

3. Neutral normalization. Once a set of expansion coefficients has been obtained, all the coefficients are divided by the largest one so that the numerically largest coefficient becomes one.

4. SOFTWARE ASSOCIATED WITH THIS TRANSLATION

The software associated with this translation is organized in three modules and one driver program. The main contribution of this work is the subroutine `Blanch_Coefficients`, which is contained in the module `Blanch`.

In addition to the computation of the expansion coefficients, one must generate the Mathieu eigenvalue and also evaluate the series expansions. The subroutines that perform the eigenvalue computation are taken, with modifications, from Zhang and Jin [1996] and are located in the module `Mathieu_Zhang_Jin`. The following modifications were introduced to translate these subroutines to Fortran 90. All `implicit none` statements were removed and all variables have been explicitly declared. The `kind` function was introduced to allow for a change of the precision of all variables. Most of the `GOTO` statements were removed. The subroutines that evaluate the series expansions are contained in the module `Blanch`.

The driver program, `driver`, contains examples that show how to use the various subroutines. The module `constants` contains definitions used by the other two modules and the driver program.

5. VALIDATION AND COMPARISONS WITH OTHER EXISTING ALGORITHMS

In this section, we validate the expansion coefficients computed by `Blanch_Coefficients` by making comparisons with three other methods. The comparisons are based on the wronskian property

$$W_1 = \mathrm{Re}_n^{(1)}(s,x)\frac{d}{dx}\mathrm{Re}_n^{(2)}(s,x) - \mathrm{Re}_n^{(2)}(s,x)\frac{d}{dx}\mathrm{Re}_n^{(1)}(s,x) = 1 \tag{35}$$

$$W_2 = \mathrm{Ro}_n^{(1)}(s,x)\frac{d}{dx}\mathrm{Ro}_n^{(2)}(s,x) - \mathrm{Ro}_n^{(2)}(s,x)\frac{d}{dx}\mathrm{Ro}_n^{(1)}(s,x) = 1 \tag{36}$$

Since the wronskian quantities W_1 and W_2 must equal one, the accuracy in the evaluation of the radial functions is assessed by computing the differences $W_1 - 1$ and $W_2 - 1$ for various cases. Three existing algorithms are considered: 1) IMSL subroutines; 2) Shirts' method ; and 3) Zhang and Jin's method. All methods have in common that they need to generate the Mathieu eigenvalue b, for a given pair of values of the parameter s and the order m, and produce the expansion coefficients. Some details of the methods considered for these comparisons are given below.

5.1 IMSL

The IMSL subroutines [Visual Numerics, Inc. 1994] do not compute radial functions, but only angular functions, which are normalized according to Goldstein and Ince and are based on an algorithm developed by Hodge [1972]. Hodge casts the eigenvalue problem associated with Mathieu equation in matrix form. The resulting matrix is symmetric, tridiagonal, and the eigenvalues are computed using the bisection method. The expansion coefficients are then obtained by introducing the eigenvalue into the recurrence relationships for the expansion coefficients.

Since the angular functions (2)-(5) and the radial functions (7)-(10) share the expansion coefficients, it is possible to generate the radial functions once the sequence of expansion coefficients is available. As an example, one may call the IMSL subroutines `DM2TCE` and `DM2TSE`, which compute even and odd angular functions, respectively. With `DM2TCE` and `DM2TSE` one can provide explicitly the workspace using [Visual Numerics, Inc. 1994] `M2TCE (X, Q, N, CE, NORDER, NEEDEV, EVAL0, EVAL1, COEF, WORK, BSJ)` and `M2TSE (X, Q, N, CE, NORDER, NEEDEV, EVAL0, EVAL1, COEF, WORK, BSJ)` where, in particular, the array `COEF` contains the expansion coefficients. To obtain the expansion coefficients normalized according to Stratton, it is necessary to execute either `DM2TCE` or `DM2TSE` for the order n, with argument x equal to zero, and the parameter $q = s/4$. DM2TCE computes

$$\mathrm{ce}_{2r+p}(q,x) = \sum_{k=0}^{\infty} A_{2k+p}^{(2r+p)}(q)\cos\left[(2k+p)x\right], \qquad p = 0,1 \tag{37}$$

where the coefficients $A_{2k+p}^{(2r+p)}(q)$ are normalized according to Goldstein-Ince, see (31)-(32), and the relationship between the coefficients $\mathrm{De}_{2k+p}^{(2r+p)}(s)$ and $A_{2k+p}^{(2r+p)}(q)$ is

$$\mathrm{De}_{2k+p}^{(2r+p)}(s) = \frac{A_{2k+p}^{(2r+p)}(s/4)}{\mathrm{ce}_{2r+p}(s/4,0)} \qquad p = 0,1 \tag{38}$$

DM2TSE computes

$$\mathrm{se}_{2r+p}(q,x) = \sum_{k=0}^{\infty} B_{2k+p}^{(2r+p)}(q)\sin\left[(2k+p)x\right], \qquad p = 0,1 \tag{39}$$

where the coefficients $B_{2k+p}^{(2r+p)}(q)$ are normalized according to Goldstein-Ince, see (33)-(34), and the relationship between the coefficients $\mathrm{Do}_{2k+p}^{(2r+p)}(s)$ and $B_{2k+p}^{(2r+p)}(q)$ is

$$\mathrm{Do}_{2k+p}^{(2r+p)}(s) = \frac{B_{2k+p}^{(2r+p)}(s/4)}{\sum_{\ell=0}^{\infty}(2\ell+p)B_{2\ell+p}^{(2r+p)}(s/4)} \qquad p = 0,1 \qquad k \geq 1 \tag{40}$$

In this author's experience, the coefficients extracted in this way are correct only when the order n is odd. So, when n is odd one can obtain the expansion coefficients from IMSL subroutines using the described method and applying the relationships (38) and (40).

5.2 Shirts

Shirts' subroutines are described in [1993a] and [1993b]. His subroutines do not compute radial functions but provide angular Mathieu functions for real values of the order. For the purpose of this comparison, only integer values of the order are of interest. Shirts subroutines assume that the solution to Mathieu's equation is written as

$$y(x) = e^{i\nu x}\sum_k c_{2k}e^{i2kx} \tag{41}$$

where ν is the real order. The expansion coefficients are obtained calling the subroutine `mtieu1(anu, q, a, ia0b1, ivec, amurd, ic0, n0, mdim, maxdim, diag, subd, vec, ncof, ier)` that returns, in particular, the matrix `vec` and the indices `ic0` and `n0`. The expansion coefficients c_{2k} corresponding to the order `anu=`ν, parameter $q = s/4$, and parity even or odd depending on `ia0b1`=0 or 1, respectively, are located in column `n0` of `vec` and the coefficient c_0 is found along this column at row `ic0`. As an alternative to `mtieu1`, which is more general, `mtieu2` is used when the order $v < 10.5$. Note that `mtieu2` uses a variant of Blanch's algorithm [1966] when the order is integer. The relationship between the coefficients c_{2k} and the coefficients De_m, Do_m is:

$$\mathrm{De}_p^{(2r+p)}(s) = \frac{(1+p)c_0}{\mathrm{ce}_{2r+p}(s/4,0)} \qquad p = 0,1 \tag{42}$$

$$\mathrm{De}_{2k+p}^{(2r+p)}(s) = \frac{2c_{2k}}{\mathrm{ce}_{2r+p}(s/4,0)} \qquad p = 0,1 \qquad k \geq 1 \tag{43}$$

$$\mathrm{Do}_{2k+2-p}^{(2r+p)}(s) = \frac{2c_{2(k+1-p)}}{\sum_{\ell=1}^{\infty} 2\left((2\ell-1)+p\right)c_{2\ell}} \qquad p = 0,1 \qquad k \geq 0 \tag{44}$$

where the quantity $\mathrm{ce}_{2r+p}(s/4,0)$ is the same defined in (37) and corresponds to the value `fr` returned by subroutine `mtfctn(x, fr, fi, vec, ncof, ic0, maxdim, anu,phase, ier)`, which evaluates the Mathieu functions. Hence, the coefficients De_n and Do_n are created using the three previous relations.

5.3 Zhang and Jin

Zhang and Jin [1996] developed subroutines to compute the Mathieu functions (37) and (39) and the modified Mathieu functions that correspond to (7)-(10) provided that the former are multiplied by the factor $\sqrt{\pi/2}$. Hence, with a simple change of normalization it is possible to compare Zhang and Jin's results with those presented in this paper. It is worth noting that Zhang and Jin compute the expansion coefficients using a mixed forward and backward recurrence relation to ensure stability; however, they do not apply Blanch's algorithm.

5.4 Discussion of the results

The computation of the wronskians (35), (36) requires the evaluation of the radial Mathieu functions.

The radial Mathieu functions are computed in the following way. Values of the order n, the parameter s, and the variable x are assigned. Using these three values, the Mathieu eigenvalue is calculated. In particular, the methods of Blanch, and Zhang and Jin use the same Mathieu eigenvalues because the eigenvalue is computed with the same subroutines. Shirts' method and the IMSL method generate their own Mathieu eigenvalue. Then, each radial Mathieu function is evaluated using the definitions (7)-(15) that involve the computation of expansion coefficients and Bessel functions. The Bessel functions are always the same for all methods, while the expansion coefficients are evaluated using Blanch's algorithm and the other three methods.

In order to compare the performance of the three methods, various tests were performed for various values of n, s, and x. This article only reports the results shown in Tables I-V. In these tables, the columns k_m represent the number of expansion coefficients that were computed for the evaluation of the Mathieu radial functions involved for each order n.

In particular, for Blanch's method the number k_m of expansion coefficients is determined according to the following criterion, which is explained in the appendix of Blanch's article [1966]. Let us assume that the natural normalization is considered and that a set of expansion coefficients A_{2k} is required to have a truncation error less than a preassigned value ν_2, i.e.

$$|A_{2s+2m}| < \nu_2, \tag{45}$$

Table I. Verification of the accuracy using the wronskian property (35). Data computed using $s = 30$ and $x = 5$.

n	$W_1 - 1$(Blanch)	k_m	$W_1 - 1$(Zhang-Jin)	k_m	$W_1 - 1$(Shirts)	k_m	$W_1 - 1$(IMSL)	k_m
1	$-0.155431E-14$	23	$-0.155431E-14$	25	$-0.199840E-14$	10	$-0.199840E-14$	27
9	$-0.155431E-14$	23	$-0.177636E-14$	29	$-0.188738E-14$	18	$-0.188738E-14$	27
17	$-0.177636E-14$	26	$-0.222045E-14$	33	$-0.222045E-14$	15	$-0.199840E-14$	29
25	$-0.222045E-14$	29	$-0.188738E-14$	37	$-0.732747E-14$	19	$-0.199840E-14$	37
33	$-0.233147E-14$	32	$-0.255351E-14$	41	$-0.105471E-13$	23	$-0.277556E-14$	45
41	$-0.888178E-15$	36	$-0.177636E-14$	45	$-0.877076E-14$	27	$-0.222045E-14$	53
49	$-0.210942E-14$	39	$-0.122125E-14$	49	$-0.766054E-14$	31	$-0.188738E-14$	61
57	$-0.788258E-14$	43	$-0.766054E-14$	53	$-0.832667E-14$	35	$-0.766054E-14$	69
65	$-0.111022E-15$	46	$0.310862E-14$	57	$0.976996E-14$	39	$-0.399680E-14$	77
73	$0.488498E-14$	50	$0.401901E-13$	61	$0.426326E-13$	43	$0.688338E-14$	85
81	$0.803801E-13$	54	$-0.757172E-13$	65	$0.172307E-12$	47	$0.127010E-12$	93
89	$-0.400679E-12$	58	$0.220912E-11$	69	$0.197375E-11$	50	$-0.293099E-13$	101
97	$-0.104339E-11$	62	$-0.299882E-11$	73	$-0.216422E-10$	54	$-0.866818E-11$	109

Table II. Verification of the accuracy using the wronskian property (36). Data computed using $s = 30$ and $x = 5$.

n	$W_2 - 1$(Blanch)	k_m	$W_2 - 1$(Zhang-Jin)	k_m	$W_2 - 1$(Shirts)	k_m	$W_2 - 1$(IMSL)	k_m
1	$-0.222045E-14$	24	$-0.199840E-14$	25	$-0.222045E-14$	9	$-0.244249E-14$	27
9	$-0.177636E-14$	23	$-0.222045E-14$	29	$-0.188738E-14$	17	$-0.188738E-14$	27
17	$-0.244249E-14$	26	$-0.244249E-14$	33	$-0.266454E-14$	15	$-0.222045E-14$	29
25	$-0.199840E-14$	29	$-0.177636E-14$	37	$-0.444089E-14$	19	$-0.244249E-14$	37
33	$-0.155431E-14$	32	$-0.188738E-14$	41	$-0.943690E-14$	23	$-0.222045E-14$	45
41	$-0.133227E-14$	36	$-0.188738E-14$	45	$-0.104361E-13$	27	$-0.122125E-14$	53
49	$-0.166533E-14$	39	$-0.244249E-14$	49	$-0.109912E-13$	31	$-0.388578E-14$	61
57	$-0.999201E-14$	43	$-0.510703E-14$	53	$-0.202061E-13$	35	$-0.144329E-13$	69
65	$0.111022E-14$	46	$0.133227E-14$	57	$-0.464073E-13$	39	$-0.410783E-14$	77
73	$0.421885E-14$	50	$0.111910E-12$	61	$-0.113243E-13$	42	$0.226485E-13$	85
81	$0.187406E-12$	54	$0.139000E-12$	65	$0.507372E-12$	46	$-0.167866E-12$	93
89	$-0.690004E-12$	58	$0.106404E-11$	69	$-0.444456E-11$	50	$0.410338E-12$	101
97	$0.382383E-11$	62	$-0.161027E-11$	73	$-0.161449E-11$	54	$-0.215761E-11$	109

where $2s$ is the coefficient such that $|G_{n,2}| < |G_{2s,2}|$ for all $n > 2s$. Then, one can prove that $|A_{2s+2m}| < |A_{2s}||G_{2s,2}|^m$ so that if one defines $\nu_2 = |A_{2s}||G_{2s,2}|^m$, it gives

$$m = -\frac{\log |A_{2s}|/\nu_2}{\log G_{2s,2}} \tag{46}$$

Hence, the maximum number of coefficients is given by either $k_m = s$ or, when $m > 0$, by $k_m = s + m + 1$.

In all tables, the data related to the columns labeled Shirts were obtained using Shirts' subroutine `mtieu2`, when the order $n < 10.5$, and Shirts' subroutine `mtieu1` for all other cases.

Tables I and II examine the case $s = 30$, $x = 5$ for values of the order n that range between 1 and 97. All methods successfully pass the Wronksian tests.

Tables III and IV examine the case $s = 8$ and $x = 3$. These combination of values of s and x is more challenging because all methods provide equivalent results in the sense that when the order $n \leq 43$ they all pass the wronskian tests. However, for larger values of n, the wronskian tests are not passed and it is interesting to observe that increasing the number of expansion coefficients does not seem to provide any

Table III. Verification of the accuracy using the wronskian property (35). Data computed using $s = 8$ and $x = 3$.

n	$W_1 - 1$(Blanch)	k_m	$W_1 - 1$(Zhang-Jin)	k_m	$W_1 - 1$(Shirts)	k_m	$W_1 - 1$(IMSL)	k_m
1	$-0.310862E-14$	18	$-0.333067E-14$	21	$-0.355271E-14$	8	$-0.377476E-14$	27
9	$-0.277556E-14$	19	$-0.333067E-14$	25	$-0.288658E-14$	16	$-0.310862E-14$	27
17	$-0.488498E-14$	22	$0.355271E-14$	29	$-0.878186E-13$	13	$-0.455191E-14$	29
25	$0.167866E-12$	26	$-0.266454E-14$	33	$0.196132E-11$	17	$-0.455525E-12$	37
33	$0.135973E-08$	29	$0.356233E-09$	37	$0.218332E-09$	21	$-0.123166E-08$	45
41	$0.170580E-04$	33	$-0.125333E-03$	41	$0.429813E-03$	25	$0.543152E-03$	53
43	$\mathbf{-0.111009E-01}$	**34**	$\mathbf{0.862081E-02}$	**42**	$\mathbf{0.183499E-01}$	**26**	$\mathbf{0.379809E-02}$	**55**
45	$0.126223E+00$	35	$0.924758E+00$	43	$-0.830670E+00$	27	$0.415111E+00$	57
47	$-0.371876E+02$	36	$0.678331E+01$	44	$0.104278E+03$	28	$-0.256366E+02$	59
49	$-0.121586E+04$	37	$-0.902926E+03$	45	$-0.710050E+04$	29	$0.148814E+04$	61
57	$0.166533E+12$	40	$0.315763E+12$	49	$-0.243745E+12$	33	$0.389361E+11$	69
65	$0.935600E+20$	44	$0.174713E+21$	53	$-0.717971E+20$	36	$0.853870E+20$	77
73	$-0.311705E+29$	48	$0.104033E+30$	57	$-0.962324E+28$	40	$0.968566E+29$	85
81	$-0.849862E+38$	52	$-0.362686E+37$	61	$-0.117225E+39$	44	$0.887920E+38$	93
89	$0.579300E+47$	56	$0.930446E+47$	65	$-0.404699E+48$	48	$-0.155601E+48$	101
97	$0.555711E+55$	60	$-0.588792E+57$	69	$0.163762E+58$	52	$0.107233E+58$	109

Table IV. Verification of the accuracy using the wronskian property (36). Data computed using $s = 8$ and $x = 3$.

n	$W_2 - 1$(Blanch)	k_m	$W_2 - 1$(Zhang-Jin)	k_m	$W_2 - 1$(Shirts)	k_m	$W_2 - 1$(IMSL)	k_m
1	$-0.310862E-14$	18	$-0.333067E-14$	21	$-0.366374E-14$	7	$-0.366374E-14$	27
9	$-0.333067E-14$	19	$-0.333067E-14$	25	$-0.321965E-14$	15	$-0.355271E-14$	27
17	$-0.466294E-14$	22	$0.355271E-14$	29	$-0.768274E-13$	13	$-0.666134E-15$	29
25	$-0.492828E-12$	26	$0.369482E-12$	33	$0.322409E-11$	17	$-0.832667E-14$	37
33	$-0.318299E-10$	29	$0.183282E-09$	37	$0.402022E-09$	21	$-0.124873E-08$	45
41	$0.770921E-04$	33	$-0.217444E-03$	41	$-0.900555E-03$	25	$0.163031E-03$	53
43	$\mathbf{-0.644405E-02}$	**34**	$\mathbf{0.197056E-01}$	**42**	$\mathbf{0.132903E-01}$	**26**	$0.119965E-01$	55
45	$-0.584493E+00$	35	$0.288946E+00$	43	$-0.150908E+01$	27	$\mathbf{0.669110E-01}$	**57**
47	$-0.472322E+02$	36	$0.182705E+02$	44	$0.797088E+02$	27	$0.215676E+02$	59
49	$0.504007E+03$	37	$0.668637E+03$	45	$-0.500060E+03$	28	$-0.658586E+03$	61
57	$0.131858E+12$	40	$0.225046E+12$	49	$0.289661E+12$	32	$0.206643E+12$	69
65	$0.815463E+20$	44	$0.303869E+20$	53	$-0.986744E+20$	36	$0.122297E+21$	77
73	$-0.817995E+29$	48	$0.566317E+29$	57	$-0.450960E+29$	40	$0.121246E+29$	85
81	$-0.119869E+39$	52	$-0.106416E+39$	61	$-0.134335E+39$	44	$-0.617735E+38$	93
89	$0.187295E+47$	56	$0.251657E+48$	65	$-0.332797E+48$	48	$-0.130842E+48$	101
97	$-0.132210E+58$	60	$-0.181295E+58$	69	$0.158417E+58$	52	$0.130904E+58$	109

improvement.

For example, when $n = 49$, Shirts' method uses 29 expansion coefficients, while the IMSL method uses 61 expansion coefficients and, even though the IMSL method employs more the double of the expansion coefficients, the results of the two methods are of the same order of magnitude. It is likely that the failure for larger values of the order n depends on the form of the series and in not having enough significant digits to carry out all the necessary operations.

To verify the previous statement, the test for $s = 8$ and $x = 3$ is repeated using this translation of Blanch's method when the computations are executed using quadruple precision variables. The corresponding results are reported in Table V, where only Blanch's method results are reported because the subroutines for the other cases cannot be executed at this level of precision. The wronskian test is passed satisfactorily up to $n = 63$ for the even radial functions and $n = 61$ for

Table V. Results obtained for the wronskian property (35) and (36) using quadruple precision numbers for the case $s = 8$ and $x = 3$.

n	$W_1 - 1$(Blanch)	k_m	$W_2 - 1$(Blanch)	k_m
...	...	...		
33	$0.38188119E-008$	83	$0.38188119E-008$	83
41	$0.38188119E-008$	103	$0.38188119E-008$	103
49	$0.38188105E-008$	123	$0.38188114E-008$	123
51	$0.38187709E-008$	128	$0.38187517E-008$	128
57	$-0.17247124E-006$	143	$-0.54942269E-007$	143
59	$-0.38699220E-005$	148	$-0.87023435E-005$	148
61	$0.16601005E-002$	153	$\mathbf{0.56675987E-003}$	**153**
63	$\mathbf{0.14941168E-001}$	158	$-0.27859398E+000$	158
65	$0.87101088E+001$	163	$0.61224252E+002$	163
67	$-0.35399794E+004$	168	$-0.41447462E+004$	168
69	$0.82162356E+006$	173	$0.10020799E+007$	173
71	$0.12000783E+009$	178	$-0.52041024E+008$	178
73	$-0.47577352E+011$	183	$0.39455250E+010$	183
81	$-0.16691999E+021$	203	$-0.24579136E+020$	203
89	$-0.23712579E+030$	223	$0.54556887E+029$	223
97	$0.30174552E+039$	243	$0.16582758E+040$	243

the odd radial functions. For larger values of n even the quadruple precision computation finds this case challenging. Nevertheless, for this example, the quadruple precision variables allows for the computation of an additional twenty orders, which may be important for some applications. The results shown in table V were computed introducing a modification to Blanch's algorithm. Specifically, the number of expansion coefficients that are actually computed is forced to be at least 2.5 times the order.

Finally, applications where quadruple precision computations were applied, are reported in [Erricolo 2003], [Erricolo and Uslenghi 2004], [Erricolo et al. 2005a], [Erricolo et al. 2005b], [Erricolo and Uslenghi 2005], [Erricolo et al. 2005].

5.5 Comments on other software to compute Mathieu functions

To the best of this author's knowledge, Blanch's algorithm was first implemented in Fortran by Clemm [1969], [1970]. Clemm's code is not easily available and uses many obsolete functions, so that the present translation should be more beneficial. Clemm's code was also converted to the C programming language by Baker [1992]. Alhargan [2001] developed C language subroutines to compute Mathieu functions of integer order and claimed that Blanch's algorithm should be used when his fails, so that Blanch's algorithm should be considered more accurate.

6. CONCLUSIONS

The subroutine `Blanch_Coefficients` presented in this paper implements in Fortran 90 the algorithm developed by Gertrude Blanch [1966] to compute the expansion coefficients of the series (2-5),(7-10) and (12-19) that define Mathieu functions. The advantage of Blanch's algorithm is that it is associated with a very detailed numerical analysis.

In order to verify the accuracy of the Mathieu functions computed using the expansion coefficients evaluated by the subroutine `Blanch_Coefficients`, some tests that involved comparisons with three other independent methods were made. All tests showed that this translation provides results that are at least as good as those provided by the other methods. In addition, when the subroutine `Blanch_Coefficients` is run at higher precision, its performance exceeds the other methods.

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