# Isorefractivity: teaching and research perspectives

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***Abstract*—The isorefractive condition augments the set of geometries for which electromagnetic boundary value problems are solvable with the mode matching method. From a teaching viewpoint, understanding the isorefractive condition enables to appreciate the underpinnings of the special functions used in certain coordinate systems. From a research viewpoint, the isorefractive condition extends the set of problems for which an exact solution is known, thus leading to new canonical problems that can be used as challenging benchmarks to validate numerical approaches.**

## I. Introduction

Uslenghi coined the term *isorefractive* [1] to refer to a special condition between two media. This condition enables the application of the mode-matching technique to solve boundary value problems. Other authors have addressed this condition only in the case of a dielectric wedge referring to it as diaphanous wedge [2]. Consider two materials characterized by dielectric permittivity $\varepsilon_1$ and magnetic permeability $\mu_1$ and another with dielectric permittivity $\varepsilon_2$ and magnetic permeability $\mu_2$, respectively. These materials are isorefractive when

$$\varepsilon_1\mu_1 = \varepsilon_2\mu_2, \tag{1}$$

so that the wavevector does not change change going from one medium to the other, however, the intrinsic impedance of the material can still change since $Z_1 = \sqrt{\mu_1/\varepsilon_1} \neq \sqrt{\mu_2/\varepsilon_2} = Z_2$.

## II. Teaching perspective

Let us consider a simple two-dimensional scattering problem where the mode matching method cannot be applied, unless the isorefractive condition is satisfied. Referring to Fig. 1, a plane wave with the electric field polarized parallel to the $z$ axis, with wavevector $\beta$, and propagating along a direction making the angle $\varphi_0$ with the negative x axis is incident upon an infinite elliptic dielectric cylinder whose axis of symmetry is the $z$ axis. In any plane orthogonal to $z$, using elliptic coordinates $x = d/2\cosh u \cos v$, $y = d/2 \sinh u \sin v$, where $d$ is the interfocal distance, the cross section of the cylinder is an ellipse with $u = u_1$ and oriented so that the $x$ axis overlaps the major axis of the ellipse. Inside the elliptic cylinder, the material has dielectric permittivity $\varepsilon_2$ and magnetic permeability $\mu_2$ and outside the material has permittivity $\varepsilon_1$ and permeability $\mu_1$. For this problem, using the notation of Stratton [3], the incident electric field may be written as

$$\begin{aligned} E_z^i &= E_0 e^{j\beta(x\cos\varphi_0 + y\sin\varphi_0)} \\ &= \sqrt{8\pi}E_0 \sum_{n=0}^{\infty} j^n \left[ \frac{1}{N_{1n}^{(e)}} \mathrm{Re}_n^{(1)}(c_1,u)\mathrm{Se}_n(c_1,\varphi_0)\mathrm{Se}_n(c_1,v) \right. \\ &\left. + \frac{1}{N_{1n}^{(o)}} \mathrm{Ro}_n^{(1)}(c_1,u)\mathrm{So}_n(c_1,\varphi_0)\mathrm{So}_n(c_1,v) \right] \end{aligned} \tag{2}$$

and the scattered electric field is

$$\begin{aligned} E_z^s = \sqrt{8\pi}E_0 \sum_{n=0}^{\infty} j^n & \left[ \frac{a_n}{N_{1n}^{(e)}} \mathrm{Re}_n^{(4)}(c_1,u)\mathrm{Se}_n(c_1,\varphi) \right. \\ & \left. + \frac{b_n}{N_{1n}^{(o)}} \mathrm{Ro}_n^{(4)}(c_1,u)\mathrm{So}_n(c_1,\varphi) \right] \end{aligned} \tag{3}$$

where the radial Mathieu functions of the fourth kind $\mathrm{Re}_n^{(4)}$ and $\mathrm{Ro}_n^{(4)}$ guarantee the satisfaction of the boundary condition at infinity [3], [4]. The field inside the dielectric region is written

$$\begin{aligned} E_{2z} = \sqrt{8\pi}E_0 \sum_{n=0}^{\infty} j^n & \left[ \frac{c_n}{N_{2n}^{(e)}} \mathrm{Re}_n^{(1)}(c_2,u)\mathrm{Se}_n(c_2,v) \right. \\ & \left. + \frac{d_n}{N_{2n}^{(o)}} \mathrm{Ro}_n^{(1)}(c_2,u)\mathrm{So}_n(c_2,v) \right] \end{aligned} \tag{4}$$

where only the radial Mathieu functions of the first kind $\mathrm{Re}_n^{(1)}$ and $\mathrm{Ro}_n^{(1)}$ are present to avoid any singularity. The

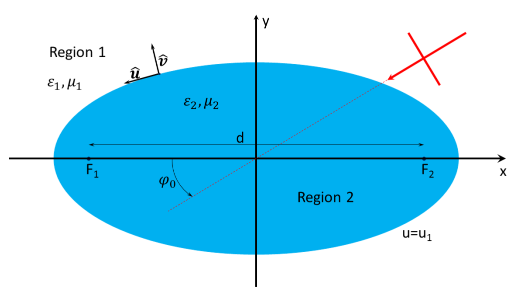


Fig. 1. Cross section of the problem geometry.

parameter $c = \beta d/2$ depends on the material properties and so do the normalization coefficients, therefore they are indicated as $N_{1n}^{(e)}$, $N_{1n}^{(o)}$ in region 1 and as $N_{2n}^{(o)}$, $N_{2n}^{(o)}$ in region 2. The boundary condition on the continuity of the tangential component of the total electric field, i.e.

$$\sum_{n=0}^{\infty} j^n \left\{ \frac{1}{N_{1n}^{(e)}} \left[ \mathrm{Re}_n^{(1)}(c_1,u_1)\mathrm{Se}_n(c_1,\varphi_0)\mathrm{Se}_n(c_1,v) \right.\right.$$
$$\left. + \ a_n \mathrm{Re}_n^{(4)}(c_1,u_1) \right] \mathrm{Se}_n(c_1,v)$$
$$+ \ \frac{1}{N_{1n}^{(o)}} \left[ \mathrm{Ro}_n^{(1)}(c_1,u_1)\mathrm{So}_n(c_1,\varphi_0)\mathrm{So}_n(c_1,v) \right.$$
$$\left.\left. + b_n \mathrm{Ro}_n^{(4)}(c_1,u_1) \right] \mathrm{So}_n(c_1,v) \right\}$$
$$= \sum_{n=0}^{\infty} j^n \left[ \frac{c_n}{N_{2n}^{(e)}} \mathrm{Re}_n^{(1)}(c_2,u_1)\mathrm{Se}_n(c_2,v) \right.$$
$$\left. + \ \frac{d_n}{N_{2n}^{(o)}} \mathrm{Ro}_n^{(1)}(c_2,u_1)\mathrm{So}_n(c_2,v) \right], \qquad (5)$$

requires that the sum of the series at the LHS equals the sum of the series at the RHS. The terms of the series are similar to the case of the circular cylinder in that there are radial factors ($\mathrm{Re}_n^{(1),(4)}$,$\mathrm{Ro}_n^{(1),(4)}$) and angular factors ($\mathrm{Se}_n$, $\mathrm{So}_n$), with the additional feature that even and odd parity are required for both radial and angular Mathieu functions. Along the boundary at $u = u_1$, the angular factors at the LHS depend on $\mathrm{Se}_n(c_1, v)$ or $\mathrm{So}_n(c_1, v)$ while at the RHS they depend on $\mathrm{Se}_n(c_2, v)$ or $\mathrm{So}_n(c_2, v)$. Hence, for both even and odd parts, the behavior at the LHS depends on the material outside the elliptic cylinder, which is associated with the parameter $c_1$ and this is different from the behavior at the RHS that depends on the material inside the elliptic cylinder, which is associated with the parameter $c_2$. Therefore, in general, it is not possible to apply the mode-matching method thus precluding an exact analytical solution leading to closed form expressions of the expansion coefficients. However, when the isorefractive condition (1) is satisfied,

$$c_1 = \frac{d}{2}\omega\sqrt{\varepsilon_1\mu_1} = \frac{d}{2}\omega\sqrt{\varepsilon_2\mu_2} = c_2, \qquad (6)$$

the angular functions at the LHS and RHS of (5) have the same behavior and the mode matching method can be applied. Once the boundary condition on the continuity of the total tangential component of the magnetic field is applied, the expressions of the expansion coefficients are obtained as

$$a_n = \frac{(Z_1 - Z_2)\mathrm{Re}_n^{(1)}(c,u_1)\mathrm{Re}_n^{(1)\prime}(c,u_1)\mathrm{Se}_n(c,\varphi_0)}{Z_2\mathrm{Re}_n^{(1)}(c,u_1)\mathrm{Re}_n^{(4)\prime}(c,u_1) - Z_1\mathrm{Re}_n^{(1)\prime}(c,u_1)\mathrm{Re}_n^{(4)}(c,u_1)} \qquad (7)$$

$$b_n = \frac{(Z_1 - Z_2)\mathrm{Ro}_n^{(1)}(c,u_1)\mathrm{Ro}_n^{(1)\prime}(c,u_1)\mathrm{So}_n(c,\varphi_0)}{Z_2\mathrm{Ro}_n^{(1)}(c,u_1)\mathrm{Ro}_n^{(4)\prime}(c,u_1) - Z_1\mathrm{Ro}_n^{(1)\prime}(c,u_1)\mathrm{Ro}_n^{(4)}(c,u_1)} \qquad (8)$$

$$c_n = \mathrm{Se}_n(c,\varphi_0) + \frac{\mathrm{Re}_n^{(4)}}{\mathrm{Re}_n^{(1)}} a_n, d_n = \mathrm{So}_n(c,\varphi_0) + \frac{\mathrm{Ro}_n^{(4)}}{\mathrm{Ro}_n^{(1)}} a_n \qquad (9)$$

Therefore an exact analytical expression for the expansion coefficients is obtained and a new canonical problem is solved for the isorefractive elliptic cylinder.

## III. Research perspective

The isorefractive condition led to many new exact analytical solutions for canonical geometries in many coordinate systems, including: the circular cylinder [5], the elliptic cylinder [6], [7], the oblate spheroidal [8], the prolate spheroidal [9], and the paraboloidal [10]. These new solutions provide additional benchmarks for the validation of computational electromagnetic software. For example, computational software is frequently tested in the 2D case by making comparisons with the exact solution for the circular cylinder. However, a comparison with the isorefractive geometry of Fig. 2, investigated in [11], would provide a more challenging test due to the presence of a cavity, sharp edges, an aperture, and different materials.

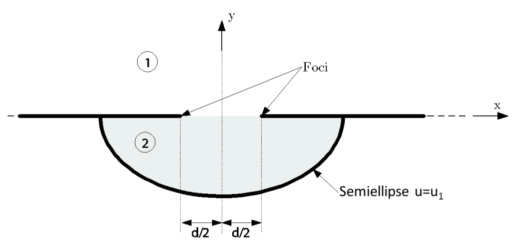


Fig. 2. Sample 2D geometry for a more challenging benchmark.

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