Deformations and Products of Polish Groups

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THESIS

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SUMMARY

Polish groups are studied in the general context of large-scale geometry. In this context many well-known Polish groups can be equipped with a canonically defined quasi-isometry type which allows for the application of techniques from geometric group theory.

Deformation retracts of Polish groups receive special attention. The main result concerning deformation retracts is that the existence of a sufficiently tame deformation of a Polish group implies that the group has a well-defined quasi-isometry type.

Polish groups which are knit products also receive their own study. In a number of examples it was noticed that a Polish group's quasi-isometry type was given by its knit product structure. The main result on knit products provides a way to understand the geometry of all these examples at once.

As an application of these ideas we study the large-scale geometry of Polish groups whose elements are absolutely continuous homeomorphisms.

CHAPTER 1

INTRODUCTION

1.1 Context

For a group G generated by a subset S there is the associated left-invariant *word metric* ρ_S defined by

$$\rho_S(x, y) = \min\left\{k \geqslant 0 \mid x^{-1}y \in \left(S \cup S^{-1}\right)^k\right\}$$

for all $x, y \in G$. A foundational observation in the subject of geometric group theory is that on any finitely generated group the word metrics obtained from finite generating sets are mutually quasi-isometric, and so any such group has a well-defined quasi-isometry type; namely, the quasi-isometry equivalence class of these word metrics. This allows for the study of large-scale geometry of finitely generated groups, which links algebraic properties of finitely generated groups with metric properties of their quasi-isometry types. Celebrated instances of this include Gromov's theorem on polynomial growth (1), Stallings' theorem on ends (2), and Sela's work on word-hyperbolic groups (3). By now the study of large-scale geometry of finitely generated groups has been covered in far greater detail than is possible here. See, for example, (4).

There are several classes of topological groups where a similar observation allows for the application of large-scale geometry. For example, on any compactly generated locally compact group the word metrics obtained from compact generating sets are mutually quasi-isometric, and so there is a large-scale geometry inherent to these topological groups. We refer to (5) for a recent survey of the geometry of locally compact groups.

In this project we work within the general context of large-scale geometry of Polish groups, so recall that a Polish space is a topological space which is separable and completely metrizable and a Polish group is a topological group whose topology is Polish. The general theory of large-scale geometry of Polish groups is due to C. Rosendal and is the subject of the manuscript (6). The following concepts are key: A subset of a Polish group G is *coarsely bounded in* G if it is bounded in every compatible, left-invariant metric on G, and a Polish group is *coarsely bounded* if it is a coarsely bounded subset of itself. For now, we only mention that any Polish group which is generated by a coarsely bounded subset can be equipped with a quasi-isometry type in a way that is analogous to the previous cases discussed. This allows us to speak of quasi-isometric properties of certain Polish groups. For example, whenever we say that a function $\mathsf{G} \to \mathsf{H}$ is a quasi-isometry of Polish groups we mean that both G and H are generated by coarsely bounded subsets and that the function is a quasi-isometry of metric spaces when appropriate metrics are chosen on the groups. A more general and detailed background on coarse geometry of Polish groups is provided in the next chapter, but this will suffice to state our results.

1.2 Summary

Throughout we use I to denote the compact interval $[0, 1]$. This document collects some results related to the large-scale geometry of a Polish group G when we are in either one of the two situations:

1. There is a deformation retract of G, i.e. a continuous function $\mathrm{I} \times \mathsf{G} \to \mathsf{G}$, which gives information about its geometry.

2. The group G is a knit product, i.e. contains subgroups H and K so that $\mathsf{G} = \mathsf{HK}$ and $\mathsf{H} \cap \mathsf{K} = \{1\}$, and the geometry of G is reflected in this product structure.

All the relevant background material is collected in separate sections in Chapter 2, and then each of the above points receives its own chapter.

Chapter 3 contains a study of deformation retracts as in point 1. We define a number of properties on a deformation retract of a Polish group and study their effects on the geometry of the group. Two such properties we have named being *left-restrained* and *locally left-restrained*. See Definition 3.1. We offer Proposition 1.1 here as a representative of the kinds of statements we are looking to deduce.

Proposition 1.1. *Suppose* $\mathcal{H} : \mathrm{I} \times \mathsf{G} \to \mathsf{G}$ *is a deformation retract of a Polish group* G *onto the trivial subgroup. Then the following implications hold.*

1. *If* $\mathcal{H}$ *is left-restrained then* G *is coarsely bounded.*

2. *If* $\mathcal{H}$ *is locally left-restrained then* G *is generated by a coarsely bounded subset.*

We give examples of deformations of the Polish groups $\mathrm{Homeo}_+(\mathrm{I})$ and $\mathrm{Diff}^1_+(\mathrm{I})$ which fit into the above framework and, by Proposition 1.1, whose existence implies $\mathrm{Homeo}_+(\mathrm{I})$ is coarsely bounded and $\mathrm{Diff}^1_+(\mathrm{I})$ is generated by a coarsely bounded subset. We also define and discuss a property which is expressed by saying that a deformation of a Polish group has *conjugate moments*.

Chapter 4 contains the geometric results on knit products. The main result is the following theorem.

Theorem 1.2. *Suppose* G *is a Polish group which is a knit product of closed subgroups* H *and* K*. Also suppose* H *is generated by a subset* $\mathsf{S} \subset \mathsf{H}$ *which is coarsely bounded in* H*,* K *is a coarsely bounded group when equipped with the subspace topology, and* $\mathsf{SK} = \mathsf{KS}$*. Then the inclusion* $\mathsf{H} \to \mathsf{G}$ *is a quasi-isometry of Polish groups.*

In the abstract the two situations studied here, deformations and products, are unrelated; however, there are a number of nontrivial examples where a suitably chosen deformation retract of one factor subgroup within a knit product implies that the factor is geometrically trivial, and from this a quasi-isometric equivalence between the product group and the other factor subgroup can be deduced. As we will see, a concrete example of this is the Polish group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms of $\mathbb{R}$ that commute with integer translations. An interest in this and related examples is the underlying motivation for this work.

For a compact connected $\mathsf{1}$–manifold M^1 (so the interval I or circle $\mathbb{S}^1$) we let $\mathrm{AC}_+\left(\mathsf{M}^1\right)$ denote the Polish group of orientation-preserving homeomorphisms of M^1 which are absolutely continuous and whose inverses are absolutely continuous. We introduce the Polish group $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ which is algebraically the subgroup of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of homeomorphisms which are locally absolutely continuous and whose inverses are locally absolutely continuous. We apply our investigation of deformations and products to these groups and obtain the following result.

Theorem 1.3. *The Polish groups* $\mathrm{AC}_+(\mathrm{I})$ *and* $\mathrm{AC}_+\left(\mathbb{S}^1\right)$ *are coarsely bounded and the Polish group* $\mathrm{AC}_{\mathbb{Z}}^{\mathrm{loc}}(\mathbb{R})$ *is generated by a coarsely bounded subset. Further, the quasi-isometry type of* $\mathrm{AC}_{\mathbb{Z}}^{\mathrm{loc}}(\mathbb{R})$ *is that of the infinite cyclic group.*

Section 2.3 contains background material on groups of homeomorphisms of 1–manifolds and includes more detailed definitions of the Polish groups mentioned in Theorem 1.3. The proof of Theorem 1.3 is postponed until Chapter 5.

CHAPTER 2

DEFORMATIONS AND BOUNDEDNESS

2.1 Uniform structures

We review the notion of a uniform space which is due to A. Weil (7). For a set X we write $\Delta(X)$, or just Δ, to denote the diagonal subset $\{(x,x)|x \in X\}$ of the Cartesian square $X \times X$. For subsets $E, F \subset X \times X$ we let

$$E \circ F = \{(x,z) \mid \exists y \in X \ (x,y) \in E, (y,z) \in F\}$$

and $E^{-1} = \{(y,x)|(x,y) \in E\}$. For a subset $E \subset X \times X$ and a positive integer $n > 0$ we define E^n inductively by $E^n = E$ if $n = 1$ and $E^n = E \circ E^{n-1}$ for $n > 1$.

A *uniform space* is a set X along with a family $\mathcal{U}$ of subsets of $X \times X$ which satisfy the following conditions.

1. For all $E \in \mathcal{U}$, $\Delta \subset E$.

2. If $E \in \mathcal{U}$ and $E \subset F \subset X \times X$, then $F \in \mathcal{U}$.

3. If $E, F \in \mathcal{U}$, then $E \cap F \in \mathcal{U}$.

4. If $E \in \mathcal{U}$, then there is $F \in \mathcal{U}$ such that $F \circ F \subset E$.

5. If $E \in \mathcal{U}$, then $E^{-1} \in \mathcal{U}$.

Elements of $\mathcal{U}$ are called *entourages* and $\mathcal{U}$ is called a *uniform structure* or *uniformity* on X.

A *fundamental system* for a uniform space $(X, \mathcal{U})$ is a subcollection $\mathcal{B}$ of $\mathcal{U}$ such that for all $E \in \mathcal{U}$ there exists $F \in \mathcal{B}$ such that $F \subset E$. By the second condition above, a fundamental system for a uniform space is enough to determine that uniform space.

For a uniform space $(X, \mathcal{U})$, an entourage $E \in \mathcal{U}$, and a point $x \in X$ we write $E[x] = \{y|(x, y) \in E\}$. For a subset $D \subset X$ we write $E[D] = \{y| \ \exists x \in D \ (x, y) \in E\}$.

We get a topology on the uniform space X by declaring a subset $O \subset X$ to be open if for every point $x \in O$ there is an entourage $E \in \mathcal{E}$ such that $E[x] \subset O$. This is the *topology induced by the uniformity.*

Whenever (X, d) is a metric space we can always induce a uniform structure $\mathcal{U}_d$ on X by setting

$$E_\alpha = \{(x, y) \mid d(x, y) < \alpha\}$$

for each real number $\alpha > 0$ and taking

$$\mathcal{U}_d = \{E \subset X \times X \mid \exists \alpha > 0 \ E_\alpha \subset E\}.$$

When $(X, \mathcal{U})$ is a uniform space and d is a metric on X with $\mathcal{U}_d = \mathcal{U}$ then we say that d is a *compatible* metric on $(X, \mathcal{U})$. The topology induced by the uniformity $\mathcal{U}_d$ is, of course, the same topology induced directly by d.

Similarly to how the notion of continuity extends from metric spaces to topological spaces, the notion of uniform continuity extends from metric spaces to uniform spaces. A function $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ between uniform spaces is *uniformly continuous* if for every $F \in \mathcal{V}$ there is

$E \in \mathcal{U}$ such that for every $(x_1, x_2) \in E$ we have $(f(x_1), f(x_2)) \in F$. We also mention here that a family of functions

$$(f_\alpha : (X, \mathcal{U}) \to (Y, \mathcal{V}))_\alpha$$

is *equi-uniformly continuous* if for every $F \in \mathcal{V}$ there is a $E \in \mathcal{U}$ such that for all indices α and all $(x_1, x_2) \in E$ we have $(f_\alpha(x_1), f_\alpha(x_2)) \in F$.

Every uniform space comes with a notion of boundedness as given by the following definition.

Definition 2.1. A subset S of a uniform space $(X, \mathcal{E})$ is *bounded* if for each entourage E there is a finite set $D \subset X$ and an integer $n \geqslant 1$ such that $S \subset E^n[D]$.

2.1.1 Uniformities on a topological group

There are a number of ways to define a uniform structure on an abstract topological group G. The left, right, two-sided, and Roelcke uniformities have fundamental systems given by, respectively,

1. $\{(x, y) \in G \times G \mid x^{-1}y \in W)\}$
2. $\{(x, y) \in G \times G \mid xy^{-1} \in W)\}$
3. $\{(x, y) \in G \times G \mid x^{-1}y \in W \text{ and } xy^{-1} \in W\}$
4. $\{(x, y) \in G \times G \mid y \in WxW\}$

where W varies over symmetric neighborhoods of the identity in G. In general each of these uniformities is distinct. In any case, they all induce the original topology on the underlying topological group.

As stated in the introduction, a subset of a Polish group is coarsely bounded if it is bounded in any metric which is compatible with the topology and left-invariant, so it is important to mention that on any Polish group there are always metrics with these properties. This is due to the Birkhoff—Kakutani Theorem (8; 9), which states the equivalence of the following three items for an arbitrary topological group G.

1. G is Hausdorff and first countable.
2. There is a metric on G which is compatible with the topology.
3. There is a metric on G which is compatible with the topology and which is left-invariant.

If G is a topological group and d is a metric on G as in part 3 of the Birkhoff-Kakutani Theorem, then d is compatible with the left uniformity on G. Additionally, compatible metrics for the other uniformities on G may be written in terms of d. Let

$$d_r(x, y) = d(x^{-1}, y^{-1})$$

for all $x, y \in G$, then d_r is a compatible metric for the right uniformity, and it follows that $d + d_r$ is a compatible metric for the two-sided uniformity. Let

$$d_R(x, y) = \inf_{z \in G} d(x, z) + d_r(z, y)$$

for all $x, y \in G$, then d_R is a compatible metric for the Roelcke uniformity. In the special case that G is Polish then the metric $d + d_r$ is complete.

2.1.2 Boundedness in a uniformity on a topological group

For each of the left, right, two-sided, and Roelcke uniformities on a topological group we state a reformulation of when a subset of the uniformity is bounded.

Definition 2.2. Let G be a topological group and let S be a subset.

1. S is *bounded in the left uniformity* if for any open $V \ni 1$ there is a finite set F and an integer $k \geqslant 1$ such that $S \subset FV^k$.
2. S is *bounded in the right uniformity* if for any open $V \ni 1$ there is a finite set F and an integer $k \geqslant 1$ such that $S \subset V^k F$.
3. S is *bounded in the two-sided uniformity* if for any open $V \ni 1$ there is a finite set $F \subset G$ and integer $k \geqslant 1$ so that for any $s \in S$ there are $x_0 \in F, x_1, \ldots, x_{k-1} \in G$ and $x_k = s$ so that $x_{i+1} \in x_i V$ and $x_{i+1}^{-1} \in x_i^{-1} V$ for all i.
4. S is *bounded in the Roelcke uniformity* if for any open $V \ni 1$ there is a finite set $F \subset G$ and an integer $k \geqslant 1$ such that $S \subset V^k F V^k$.

See (10, Section 1.5) for a systematic study of the above concepts when the subset S is the whole group G. It is straightforward to check that each of the above restatements of boundedness coincides with the notion which comes from Definition 2.1.

2.2 Coarse structures

Coarse spaces are the large-scale analog to uniform spaces. The notion is due to J. Roe (11). For a set X and subsets $E, F \subset X \times X$, the subsets $\Delta, E \circ F$, and E^{-1} of $X \times X$ are as in the previous section on uniform spaces.

A *coarse space* is a set X along with a family $\mathcal{E}$ of subsets of $X \times X$ which satisfies the following conditions.

1. $\Delta \in \mathcal{E}$.

2. If $E \in \mathcal{E}$ and $F \subset E$, then $F \in \mathcal{E}$.

3. If $E, F \in \mathcal{E}$, then $E \cup F \in \mathcal{E}$.

4. If $E, F \in \mathcal{E}$, then $E \circ F \in \mathcal{E}$.

5. If $E \in \mathcal{E}$, then $E^{-1} \in \mathcal{E}$.

Elements of $\mathcal{E}$ are, as with uniform spaces, called *entourages* and $\mathcal{E}$ is called a *coarse structure* on X.

Also paralleling the situation from uniform spaces, whenever (X, d) is a metric space there is an induced coarse structure $\mathcal{E}_d$ on X defined as follows. Again, for all $\alpha > 0$, E_α is the subset of $X \times X$ of all pairs (x, y) with $d(x, y) < \alpha$ and now take

$$\mathcal{E}_d = \{E \subset X \times X \mid \exists \alpha > 0 \ E \subset E_\alpha\}.$$

The coarse structure $\mathcal{E}_d$ is called the *bounded coarse structure*.

Suppose $\alpha, \beta : Z \to X$ are functions from a set Z to a coarse space $(X, \mathcal{E})$. We say α and β are *close* if there is some $E \in \mathcal{E}$ so that $(\alpha(z), \beta(z)) \in E$ for all $z \in Z$. Now suppose $f : (X, \mathcal{E}) \to (Y, \mathcal{F})$ is a function of coarse spaces. We say that f is *bornologous* if $(f \times f)[\mathcal{E}] \subset \mathcal{F}$, meaning $(x_1, x_2) \mapsto (f(x_1), f(x_2))$ takes entourages in X to entourages in Y. A map $f : X \to Y$

of coarse spaces is a *coarse equivalence* if it is bornologous and there is a bornologous map $g : Y \to X$ such that $f \circ g$ is close to id_X and $g \circ f$ is close to id_Y.

Recall that a map $f : (X, d_X) \to (Y, d_Y)$ of metric spaces is a *quasi-isometry* when there are constants $A \geqslant 1, B \geqslant 0$, and $C \geqslant 0$ so that the following two conditions hold:

1. For every $x_1, x_2 \in X$,

$$\frac{1}{A}\ d_X(x_1, x_2) - B \leqslant d_Y(f(x_1), f(x_2)) \leqslant A\ d_X(x_1, x_2) + B.$$

2. For every $y \in Y$ there is $x \in X$ so that $d_Y(y, f(x)) \leqslant C$.

Every quasi-isometry is a coarse equivalence when the metric spaces involved are equipped with their respective bounded coarse structures. Standard arguments show that coarse equivalence of coarse spaces and quasi-isometric equivalence of metric spaces are equivalence relations.

2.2.1 Coarse structures on topological groups

If G is a group we say that a subset $E \subset G \times G$ is *left-invariant* if for all $x, y, z \in G$, whenever $(x, y) \in E$ we have $(zx, zy) \in E$. A coarse structure on a group is *left-invariant* if every entourage is contained in a left-invariant entourage.

For a group G, a left-invariant subset $E \subset G \times G$ may be recovered from the set

$$A_E = \{x \in G \mid (1, x) \in E\}$$

by observing that $E = \{(x, y) \in G \times G \mid x^{-1}y \in A_E\}$, and a subset $A \subset G$ may be recovered from the left-invariant set

$$E_A = \{(x, y) \in G \times G \mid x^{-1}y \in A\}$$

by noting that $A = \{x \in G \mid (1, x) \in E_A\}$.

Given a set X, an *ideal* on X is a nonempty subset of the powerset of X which is closed under taking subsets and unions. The following proposition is (6, Proposition 2.12).

Proposition 2.3. *Let* G *be a group. Then*

$$\mathcal{E} \mapsto \mathcal{A}_{\mathcal{E}} = \{A \mid A \subset A_E \textit{ for some } E \in \mathcal{E}\}$$

with inverse

$$\mathcal{A} \mapsto \mathcal{E}_{\mathcal{A}} = \{E \mid E \subset E_A \textit{ for some } A \in \mathcal{A}\}$$

defines a bijection between the collection of left-invariant coarse structures $\mathcal{E}$ *on* G *and the collection of ideals* $\mathcal{A}$ *on* G *which contain* $\{1\}$*, are closed under inversion* $A \mapsto A^{-1}$*, and closed under products* $(A, B) \mapsto AB$*.*

For a topological group G the *group-compact coarse structure* is the left-invariant coarse structure on G which is identified with the ideal of precompact subsets. There are other coarse structures which may be defined on an arbitrary topological group and are very much worth considering.

For an arbitrary topological group G the following ideals define left-invariant coarse structures.

$$\mathcal{V} = \{\mathsf{A} \subset \mathsf{G} \mid \forall \mathsf{V} \ni 1 \text{ open } \exists \mathsf{k} \geqslant 1 \ \mathsf{A} \subset \mathsf{V}^{\mathsf{k}}\}$$

$$\mathcal{F} = \{\mathsf{A} \subset \mathsf{G} \mid \forall \mathsf{V} \ni 1 \text{ open } \exists \mathsf{k} \geqslant 1 \ \exists \mathsf{F} \subset \mathsf{G} \text{ finite } \mathsf{A} \subset (\mathsf{F}\mathsf{V})^{\mathsf{k}}\}$$

Elements of $\mathcal{V}$ are sometimes called *topologically bounded.* In a connected topological group G any nonempty open subset V must algebraically generate G, and so for any finite subset $\mathsf{F} \subset \mathsf{G}$ there is an integer $\mathsf{k} \geqslant 1$ such that $\mathsf{F} \subset \mathsf{V}^{\mathsf{k}}$. It follows that in every connected topological group we have $\mathcal{V} = \mathcal{F}$.

A central observation in the general theory of coarse geometry of Polish groups is that in a Polish group G the elements of $\mathcal{F}$ are exactly those subsets of G which are coarsely bounded, i.e. they are bounded in any metric on G which is compatible and left-invariant.

A Polish group is *locally bounded* if it has a coarsely bounded identity neighborhood. Whenever we say a function $\mathsf{G} \to \mathsf{H}$ of Polish groups is a *coarse equivalence* we mean it is a coarse equivalence when G and H are equipped with the coarse structure coming from the coarsely bounded subsets.

By a theorem of R. Struble (12), every locally compact, second countable group has a compatible, left-invariant metric which is also *proper*, i.e. the bounded subsets are precompact. A corresponding result holds in the setting of Polish groups. To be precise, every Polish group which is algebraically generated by a coarsely bounded subset has a compatible, left-invariant metric which is also *coarsely proper*, i.e. the bounded subsets are coarsely bounded. Now, suppose G is such a Polish group. By taking the closure of any coarsely bounded set we obtain a closed, coarsely bounded set, and so G has a closed, coarsely bounded generating set. By an application of the Baire Category theorem, any of the word metrics obtained from coarsely bounded, closed generating sets are mutually quasi-isometric. Thus, we say the quasi-isometry type of G is the quasi-isometry equivalence class of these metric spaces. The coarse structure obtained from the quasi-isometry type is the same coarse structure obtained directly from considering the coarse structure associated to the coarsely bounded subsets.

The choice to define coarse boundedness in terms of left-invariant metrics as opposed to right-invariant metrics is nearly arbitrary. For a left-invariant metric d on a group, the formula $d_r(x, y) = d(x^{-1}, y^{-1})$ defines a right-invariant metric with the same bounded subsets. When working within the general theory we are in the habit of working with left-invariant metrics. In many examples, including the groups of homeomorphisms we intend to study here, the natural metrics to work with are right-invariant.

Let us mention a few notable examples of Polish groups which fit into this theory. In a discrete countable group a subset is coarsely bounded if and only if it is finite, and so the quasi-isometry type of a finitely generated group coincides with its quasi-isometry type as a Polish group. In a locally compact Polish group a subset is coarsely bounded if and only if it is precompact, so the coarse structures inherent to the group when viewed as a locally compact group and as a Polish group are one and the same. Examples which are not locally compact include the underlying abelian group of any separable Banach space, wherein a subset is coarsely bounded if and only if it is norm bounded. We also mention the Polish groups $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms of a compact manifold M. The fragmentation metric on $\mathrm{Homeo}_0(M)$ introduced by R. D. Edwards and R.C. Kirby (13) gives the same quasi-isometry type of $\mathrm{Homeo}_0(M)$ when it is viewed in Rosendal's theory.

We will also consider the notion of a topological group being ultralocally bounded. Let U be a subset of a group G. The set of U–*admissible* products is the smallest set of products such that

1. if $x \in U$ then the single factor product x is U–admissible,
2. if $x_1 \cdots x_n$ and $y_1 \cdots y_m$ are U–admissible and $x_1 \cdots x_n \cdot y_1 \cdots y_m \in U$, then also $x_1 \cdots x_n \cdot y_1 \cdots y_m \in U$ is U-admissible.

Definition 2.4. A topological group G is *ultralocally bounded* if every identity neighborhood U contains a further identity neighborhood V so that, for every identity neighborhood W there is a finite set F and $k \geqslant 1$ for which every element $v \in V$ can be written as a U-admissible product $v = x_1 \cdots x_k$ with terms $x_i \in F \cup W$.

2.3 Groups of homeomorphisms

The purpose of this section is to collect notation and background material on the transformation groups of $I, \mathbb{S}^1$ and $\mathbb{R}$ that will be needed.

2.3.1 Homeomorphisms of compact 1-manifolds

We identify I with the unit interval $[0, 1]$ and $\mathbb{S}^1$ with the group of complex numbers with unit norm. Let $\pi : \mathbb{R} \to \mathbb{S}^1$ be the covering map $x \mapsto e^{2\pi i x}$. We use $d_{\mathbb{S}^1}$ to denote the metric on $\mathbb{S}^1$ which is defined by taking the minimum distance between $\pi^{-1}(x)$ and $\pi^{-1}(y)$ in $\mathbb{R}$ for all $x, y \in \mathbb{S}^1$. We use d_∞ to denote both the metric defined by

$$d_\infty(f, g) = \sup_{x \in I} |f(x) - g(x)|$$

on Homeo$_+$(I) and the metric defined by

$$d_\infty(f, g) = \sup_{x \in \mathbb{S}^1} d_{\mathbb{S}^1}(f(x), g(x))$$

on Homeo$_+$($\mathbb{S}^1$). This allows us to treat the Polish groups Homeo$_+$(I) and Homeo$_+$($\mathbb{S}^1$) with some ambiguity because on either group d_∞ denotes a right-invariant metric which induces the standard Polish group topology.

For a compact connected 1–manifold M^1 and for each integer $k \geqslant 1$ we use Diff$^k_+$(M^1) to denote the Polish group of orientation preserving C^k-diffeomorphisms of M^1. For each k we use d_{C^k} to denote a compatible metric on Diff$^k_+$(M^1), namely the one defined by

$$d_{C^k}(f, g) = d_\infty(f, g) + \sum_{i=1}^{k} \sup_{x \in M^1} \left| f^{(i)}(x) - g^{(i)}(x) \right|$$

for all $f, g \in$ Diff$^k_+$(M^1).

A function $f : J \to \mathbb{R}$ whose domain is an interval is *absolutely continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every finite sequence $(a_1, b_1), \ldots, (a_n, b_n)$ of disjoint subintervals of J, if $\sum_{i=1}^{n}(b_i - a_i) < \delta$ then $\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$. A function $f : J \to \mathbb{R}$ whose domain is an interval is *locally absolutely continuous* if the restriction of f to every compact subinterval of J is absolutely continuous. For every homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ there is a unique homeomorphism $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with $\tilde{f}(0) \in [0, 1)$ and $\pi \circ \tilde{f} = f \circ \pi$. A homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ is *absolutely continuous* if $\tilde{f}$ is locally absolutely continuous. For $M = \mathrm{I}$ and $M = \mathbb{S}^1$ we let $\mathrm{AC}_+(M)$ denote the group of orientation-preserving homeomorphisms $f : M \to M$ such that both f and f^{-1} are absolutely continuous.

Suppose J is a compact interval and $f : \mathrm{J} \to \mathbb{R}$ is continuous and nondecreasing. The fundamental theorem of Lebesgue integration states that f is absolutely continuous if and only if f has a derivative f' almost everywhere on J with respect to Lebesgue measure, $f' \in L^1(\mathrm{J})$, and for all $x \in \mathrm{J}$

$$f(x) = f(a) + \int_a^x f'(t)\ dt.$$

See, for instance, (14, Theorem 7.18).

We set

$$d_{\mathrm{AC}}(f, g) = \int_0^1 \left| f'(t) - g'(t) \right|\ dt$$

for all $f, g \in \mathrm{AC}_+(\mathrm{I})$. In the proof of (15, Lemma 2.4) S. Solecki shows d_{AC} defines a right-invariant metric on $\mathrm{AC}_+(\mathrm{I})$ which induces a Polish group topology.

Lemma 2.5. *On* $\mathrm{AC}_+(\mathrm{I})$ *the metrics* d_∞ *and* d_{AC} *satisfy* $d_\infty \leqslant d_{\mathrm{AC}}$. *Consequently, the inclusion* $\mathrm{AC}_+(\mathrm{I}) \to \mathrm{Homeo}_+(\mathrm{I})$ *is continuous.*

Proof. Let $f, g \in AC_+(I)$ and let $x \in I$. Then

$$f(x) - g(x) = \int_0^x f'(t) - g'(t)\ dt \leqslant \int_0^x |f'(t) - g'(t)|\ dt \leqslant d_{AC}(f, g)$$

and likewise $g(x) - f(x) \leqslant d_{AC}(f, g)$. So $d_\infty \leqslant d_{AC}$ on $AC_+(I)$. This says the inclusion $(AC_+(I), d_{AC}) \to (\mathrm{Homeo}_+(I), d_\infty)$ is a contraction mapping and so it is continuous. □

Lemma 2.5 implies the metrics d_{AC} and $d_\infty + d_{AC}$ induce the same topology on $AC_+(I)$.

2.3.2 Commuting with integer translations

For each $r \in \mathbb{R}$ we let $\tau_r : \mathbb{R} \to \mathbb{R}$ be translation $x \mapsto x + r$. A homeomorphism $f : \mathbb{R} \to \mathbb{R}$ *commutes with integer translations* if

$$f \circ \tau_n = \tau_n \circ f$$

for all $n \in \mathbb{Z}$. The group of all homeomorphisms of $\mathbb{R}$ that commute with integer translations is denoted $\mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$. We fix a compatible metric d_∞ defined by

$$d_\infty(f, g) = \sup_{x \in I} |f(x) - g(x)|$$

for all $f, g \in \mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$. Equivalently, because $x \mapsto f(x) - g(x)$ defines a period 1 function $\mathbb{R} \to \mathbb{R}$, the supremum in the definition of d_∞ may be taken over all $x \in \mathbb{R}$.

The group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is the group of lifts of elements of $\mathrm{Homeo}_+(\mathbb{S}^1)$ to homeomorphisms of $\mathbb{R}$, so there is a short exact sequence

$$1 \to \mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathbb{S}^1) \to 1$$

of Polish groups. There is also a short exact sequence

$$1 \to \mathbb{R} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathrm{I}) \to 1$$

which one arrives at by considering the inclusion $\mathbb{R} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}), r \mapsto \tau_r$ and noting that the isotropy subgroup of 0 in $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is isomorphic to $\mathrm{Homeo}_+(\mathrm{I})$ via the restriction map $f \mapsto f|_{\mathrm{I}}$. Although, we must come clean and confess that this is not a short exact sequence of groups, but of the underlying Polish spaces. To be explicit, the map $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}_+(\mathrm{I})$ is defined by

$$f \mapsto f \circ \tau_{f^{-1}(0)}|_{\mathrm{I}},$$

which is not a group homomorphism. Both $\mathrm{Homeo}_+(\mathbb{S}^1)$ and $\mathrm{Homeo}_+(\mathrm{I})$ are coarsely bounded groups and so are coarsely trivial, and in both short exact sequences the inclusion is a coarse equivalence. Thus, it seems that either one of these sequences may be seen as being responsible for the quasi-isometry type of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

The Polish group $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is an example of one which is not locally compact and which has the quasi-isometry type of $\mathbb{Z}$. We introduce another. Let $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ denote the group of homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ such that f commutes with integer translations and both f and f^{-1} are locally absolutely continuous. We set

$$d_{AC}(f, g) = \int_0^1 \left|f'(t) - g'(t)\right| \, dt$$

for all $f, g \in AC^{loc}_{\mathbb{Z}}(I)$. In other words, on $AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ the quantity d_{AC} is given by the same formula which defines the metric d_{AC} on $AC_+(I)$. It is straightforward to check that d_{AC} defines a pseudometric on $AC^{loc}_{\mathbb{Z}}(\mathbb{R})$. For $f, g \in AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ we have $d_{AC}(f, g) = 0$ if and only if $f - g$ is a constant function, so d_{AC} does not define a metric on $AC^{loc}_{\mathbb{Z}}(\mathbb{R})$.

Proposition 2.6. *On* $AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ *the pseudometric* d_{AC} *is right-invariant and satisfies*

$$d_{AC}(\tau_r \circ f, \tau_s \circ g) = d_{AC}(f, g)$$

for all $r, s \in \mathbb{R}$ *and all* $f, g \in AC^{loc}_{\mathbb{Z}}(\mathbb{R})$.

Proof. For each $f \in AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ we have

$$f' = (\tau_1 \circ f)' = (f \circ \tau_1)' = f' \circ \tau_1$$

so f' is periodic with period 1. From this it follows that the integrand $|f'(t)-g'(t)|$ that appears in the definition of $d_{AC}(f, g)$ is periodic in t with period 1 and so

$$d_{AC}(f, g) = \int_a^b |f'(t) - g'(t)| \, dt$$

for all $f, g \in AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ and all $a, b \in \mathbb{R}$ with $b - a = 1$. Now for any $f, g, u \in AC^{loc}_{\mathbb{Z}}(\mathbb{R})$

$$\begin{aligned} d_{AC}(f \circ u, g \circ u) &= \int_0^1 \left|(f \circ u)'(t) - (g \circ u)'(t)\right| \, dt \\ &= \int_{u(0)}^{u(1)} \left|f'(t) - g'(t)\right| dt \\ &= d_{AC}(f, g) \end{aligned}$$

by integration by substitution and because $u(1) - u(0) = 1$, and so d_{AC} is right-invariant.

For all $r \in \mathbb{R}$ and $f \in AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ we have $(\tau_r \circ f)' = f'$ which implies the equality in the proposition. □

The sum of a right-invariant metric and right-invariant pseudometric is always a right-invariant metric, so $d_\infty + d_{AC}$ defines a right-invariant metric on $AC^{loc}_{\mathbb{Z}}(\mathbb{R})$. We also set

$$d_{AC}(f, g) = \int_0^1 \left|\tilde{f}'(t) - \tilde{g}'(t)\right| \, dt$$

for all $f, g \in AC_+\left(\mathbb{S}^1\right)$ where $\tilde{f}$ and $\tilde{g}$ are the unique homeomorphisms $\mathbb{R} \to \mathbb{R}$ defined in Section 2.3.1. Following similar reasoning, d_{AC} defines a right-invariant pseudometric and $d_\infty + d_{AC}$ defines a right-invariant metric on $AC_+\left(\mathbb{S}^1\right)$.

Later we prove that for both $G = AC_+\left(\mathbb{S}^1\right)$ and $G = AC^{loc}_{\mathbb{Z}}(\mathbb{R})$ the metric $d_\infty + d_{AC}$ is compatible with a Polish group topology on G. In both cases the proof uses the fact that G is a knit product. See Propositions 5.3 and 5.4.

2.4 Deformation retracts

The precise definition of a deformation retract of a topological space has slight variations from source to source. We fix our definitions here.

For a topological space X and a subspace A, a *deformation retract of* X *onto* A is a continuous map $\mathcal{H} : I \times X \to X$ such that

1. $\mathcal{H}(0, x) = x$,
2. $\mathcal{H}(1, x) \in A$, and
3. $\mathcal{H}(1, a) = a$

for every $x \in X$ and every $a \in A$. If also $\mathcal{H}(t, a) = a$ for every $t \in I$ and all $a \in A$ then $\mathcal{H}$ is a *strong deformation retract of* X *onto* A. A space X is *contractible* if there is a deformation retract $I \times X \to X$ onto a point in X.

If $\mathcal{H} : I \times X \to X$ is a deformation retract of a space X onto a subspace A we write $\mathcal{H}_t(x)$ for $\mathcal{H}(t, x)$ so that $\mathcal{H}_t$ defines a continuous map $X \to X$ for each $t \in I$. For each $x \in X$ we write $\phi_{x,\mathcal{H}}(t)$ for $\mathcal{H}_t(x)$ so that $\phi_{x,\mathcal{H}}$ defines a path $I \to X$ from x to $\mathcal{H}_1(x)$ for each $x \in X$. For a subset $J \subset I$ and a subset $Y \subset X$ we use both $\mathcal{H}[J \times Y]$ and $\mathcal{H}_J[Y]$ to denote the image of $J \times Y$ under $\mathcal{H}$.

The author is unaware if the topological notions expressed by Definitions 2.7 and 2.8 exist elsewhere. They are relevant to our study of large-scale geometry, so we make them here.

Definition 2.7. A deformation retract $\mathcal{H} : I \times X \to X$ of a space X is *gradual at time* t for $t \in I$ if for every open $U \supseteq \mathcal{H}_t[X]$ there is $\delta > 0$ such that $U \supseteq \mathcal{H}_s[X]$ for all $s \in I$ with $|s - t| < \delta$. A deformation retract is *gradual* if it is gradual for all times $t \in I$.

To explain the vocabulary in Definition 2.7, consider a nontrivial normed vector space X over $\mathbb{R}$ and let $\mathcal{R} : I \times X \to X$ be the straight line deformation retract of X onto 0_X defined by $\mathcal{R}(t, x) = (1 - t) \cdot x$. For all $t < 1$ we have $\mathcal{R}(t, X) = X$, so we imagine the deformation abruptly collapsing X to a point at time 1, and indeed, $\mathcal{R}$ is gradual for all times t except $t = 1$.

Definition 2.8. A deformation retract $\mathcal{H} : I \times X \to X$ of a space X onto a subspace A *has local restrictions* if for every open $U \supseteq A$ there is open $V \supseteq A$ such that $\mathcal{H}_I(V) \subset U$.

In other words, a deformation of X onto A has local restrictions if for every open $U \supseteq A$ there is an open $V \supseteq A$ so that paths starting in V stay inside U.

2.4.1 Deformations of $\mathrm{Homeo}_+(I)$

We fix notation for three deformations of the group $\mathrm{Homeo}_+(I)$.

For all $t \in I$ and $f \in \mathrm{Homeo}_+(I)$ let $\mathcal{V}, \mathcal{T}$, and $\mathcal{A}$ be the homeomorphisms of I defined by

$$\mathcal{V}(t,f)(x) = (1-t)\cdot f(x) + t\cdot x$$

$$\mathcal{T}(t,f)(x) = \begin{cases} x+1-t & \text{if } x+1-t \leqslant f(x) \\ x-1+t & \text{if } f(x) \leqslant x-1+t \\ f(x) & \text{if } |f(x)-x| \leqslant 1-t \end{cases}$$

$$\mathcal{A}(t,f)(x) = \begin{cases} x & \text{if } t=1 \text{ or } 1-t \leqslant x \leqslant 1 \\ (1-t)\cdot f\left(\frac{x}{1-t}\right) & \text{if } t \neq 1 \text{ and } 0 \leqslant x \leqslant 1-t \end{cases}$$

for all $x \in I$.

We think of $\mathcal{V}$ as deforming the graph of a homeomorphism of I along vertical lines towards the diagonal, the deformation $\mathcal{T}$ acts on graphs by truncating them along lines which are parallel to the diagonal, and $\mathcal{A}$ is the Alexander homotopy, which scales the graph of a function by progressively smaller values. Figure Figure 1 is an attempt to illustrate all this.

2.5 Knit products

A group G is a *knit product* (16), *Zappa-Szép product* (17; **?**), or *general product* (18) of subgroups H and K if $H \cap K = \{1\}$ and $G = HK$. This is equivalent to requiring that the group operation $G \times G \to G$ restricts to a bijection $H \times K \to G$. A semidirect product of groups is then a knit product with the added requirement that one of the two factor subgroups is normal.

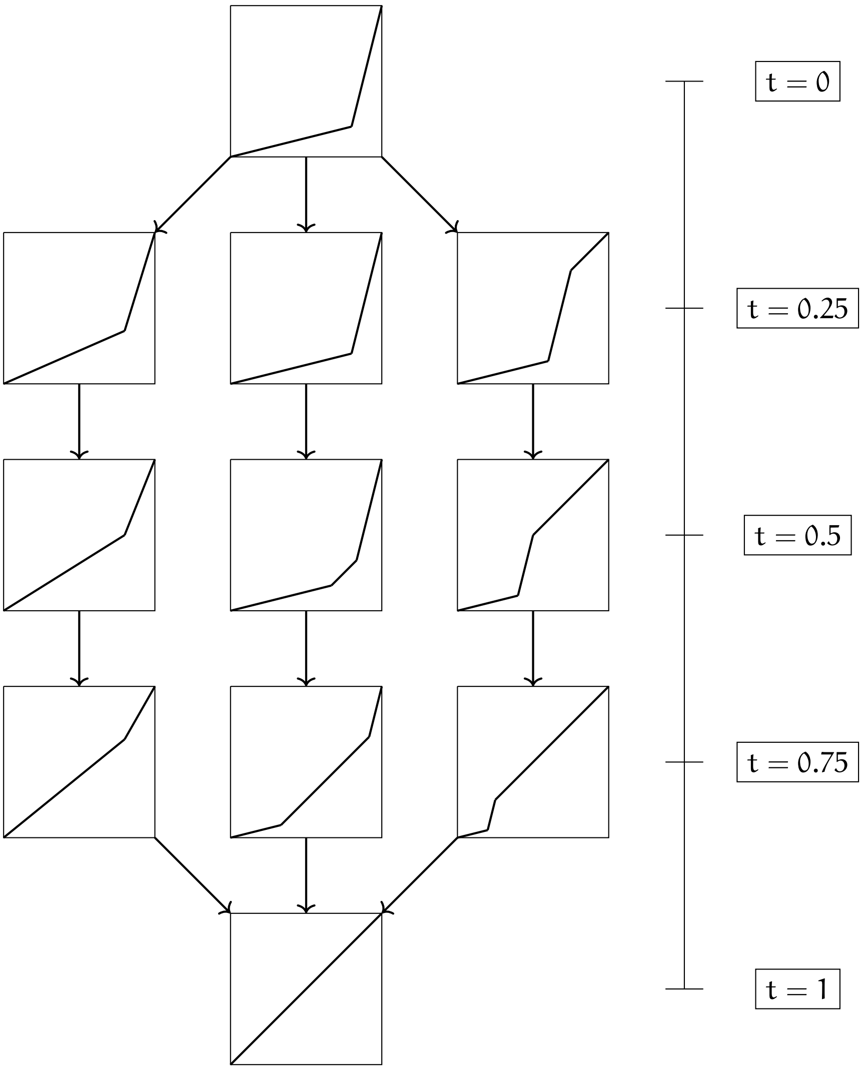

Figure 1: Three ways to deform a homeomorphism of the interval to the identity homeomorphism. From left to right, along a path corresponding to the deformation $\mathcal{V}, \mathcal{T}$, and $\mathcal{A}$.

2.5.1 External definition

As with the semidirect product there is both an internal and external definition for the knit product.

One way to define an external knit product is as follows: Suppose H and K are any groups and $\alpha : K \times H \to H$ and $\beta : K \times H \to K$ are functions. On $H \times K$ define a binary operation by

$$(h_1, k_1)(h_2, k_2) = (h_1\alpha(k_1, h_2), \beta(k_1, h_2)k_2)$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K$. If (and only if) this operation makes $H \times K$ a group and also makes the injections

$$H \to H \times K, h \mapsto (h, 1_K)$$

and

$$K \to H \times K, k \mapsto (1_H, k)$$

group homomorphisms, then the *external knit product* of H and K with respect to α and β is $H \times K$ equipped with this operation. The identity element in the external product is $(1_H, 1_K)$ and the inverse of (h, k) is

$$(\alpha(k^{-1}, h^{-1}), \beta(k^{-1}, h^{-1}))$$

for all $h \in H$ and all $k \in K$. As is hopefully clear, the external product of H and K is an internal product of the subgroups $H \times \{1_K\}$ and $\{1_H\} \times K$, for otherwise this definition would not be very meaningful.

Given an internal knit product G of subgroups H and K there are functions $\alpha : K \times H \to H$ and $\beta : K \times H \to K$ which are uniquely determined by the equation

$$kh = \alpha(k, h)\beta(k, h)$$

for all $h \in H$ and $k \in K$. The binary operation defined above on $H \times K$ makes it the external knit product with respect to α and β and makes the bijection $H \times K \to G, (h, k) \mapsto hk$ a group isomorphism. It follows that there is a natural correspondence between internal and external knit products which is analogous to the correspondence which holds for semidirect products.

For a group G with subgroups H and K the subset HK of G is a subgroup if and only if $HK = KH$, so in the definition of the knit product the two factor subgroups play a symmetric role.

2.5.2 As transformation groups

For any set X we let Bij(X) denote the group of all bijections of X with composition as the group operation. For any $S, T \subset$ Bij(X) and $Y \subset X$ we let

$$S \circ T = \{f \circ g \mid f \in S, g \in T\}$$

and

$$S[Y] = \{f(x) \mid f \in S, x \in Y\}.$$

So if $X = H$ is a group then Bij(H) denotes the group of all bijections of the underlying set H.

For a group H and an element $\mathsf{h} \in \mathsf{H}$ we let $\lambda_{\mathsf{h}} : \mathsf{H} \to \mathsf{H}$ be left translation $\mathsf{h} \mapsto \mathsf{h}x$, and we let

$$\Lambda_{\mathsf{H}} = \{\lambda_{\mathsf{h}} \mid \mathsf{h} \in \mathsf{H}\}$$

denote the subgroup of $\mathrm{Bij}(\mathsf{H})$ consisting of left translations.

Notation 2.9. Suppose H is a group and $\mathsf{G} \leqslant \mathrm{Bij}(\mathsf{H})$ is a subgroup such that $\Lambda_{\mathsf{H}} \leqslant \mathsf{G}$. We let

$$\mathsf{K}_{\mathsf{G}} = \{\mathsf{k} \in \mathsf{G} \mid \mathsf{k}\,(1_{\mathsf{H}}) = 1_{\mathsf{H}}\}$$

denote the isotropy subgroup of 1_{H} in G. We let

$$\Omega : \mathsf{H} \times \mathsf{K}_{\mathsf{G}} \to \mathsf{G}$$

be the function $(\mathsf{h}, \mathsf{k}) \mapsto \lambda_{\mathsf{h}} \circ \mathsf{k}$.

The assumption $\Lambda_{\mathsf{H}} \leqslant \mathsf{G}$ in Notation 2.9 ensures that G is a knit product of Λ_{H} and K_{G}. It is clear that $\Lambda_{\mathsf{H}} \cap \mathsf{K}_{\mathsf{G}}$ is the trivial subgroup of G. To see $\mathsf{G} = \Lambda_{\mathsf{H}} \circ \mathsf{K}_{\mathsf{G}}$ note that any $\mathsf{g} \in \mathsf{G}$ may be decomposed

$$\mathsf{g} = \lambda_{\mathsf{g}(1_{\mathsf{H}})} \circ \left(\lambda^{-1}_{\mathsf{g}(1_{\mathsf{H}})} \circ \mathsf{g}\right)$$

and the composition in parentheses is an element of K_{G}. Similarly the assumption $\Lambda_{\mathsf{H}} \leqslant \mathsf{G}$ ensures that Ω is a bijection.

2.5.3 Topological groups which are knit products

Suppose G is a topological group which is a knit product of subgroups H and K, then the group operation restricts to a continuous bijection $\mathsf{H} \times \mathsf{K} \to \mathsf{G}$ on the product space $\mathsf{H} \times \mathsf{K}$. The following is (6, Theorem A.3).

Theorem 2.10 (Rosendal)**.** *Suppose G is a Polish group which is a knit product of closed subgroups H and K. Then the group operation is a homeomorphism $\mathsf{H} \times \mathsf{K} \to \mathsf{G}$.*

Corollary 2.11. *Suppose G is a Polish group which is a knit product of closed subgroups H and K. Then $\overline{\mathsf{S}}\,\overline{\mathsf{T}} = \overline{\mathsf{S}\mathsf{T}}$ for any subsets $\mathsf{S} \subset \mathsf{H}$ and $\mathsf{T} \subset \mathsf{K}$.*

Proof. Let $\mathsf{S} \subset \mathsf{H}$ and $\mathsf{T} \subset \mathsf{K}$. Continuity of the group operation $\mathsf{H} \times \mathsf{K} \to \mathsf{G}$ implies $\overline{\mathsf{S}}\,\overline{\mathsf{T}} \subset \overline{\mathsf{S}\mathsf{T}}$, and by Theorem 2.10 $\overline{\mathsf{S}}\,\overline{\mathsf{T}}$ is a closed subset of G which contains $\mathsf{S}\mathsf{T}$, so $\overline{\mathsf{S}\mathsf{T}} \subset \overline{\mathsf{S}}\,\overline{\mathsf{T}}$. □

Corollary 2.12. *Suppose H is a Polish group and $\mathsf{G} \leqslant \mathrm{Homeo}(\mathsf{H})$ is a subgroup such that $\Lambda_{\mathsf{H}} \leqslant \mathsf{G}$. Let $\Omega : \mathsf{H} \times \mathsf{K}_{\mathsf{G}} \to \mathsf{G}$ be the bijection from Notation 2.9. Suppose G is equipped with a Polish group topology such that*

1. *$\mathsf{H} \to \mathsf{G}, \mathsf{h} \mapsto \lambda_{\mathsf{h}}$ is a topological embedding and*
2. *K_{G} is closed.*

Then Ω is a homeomorphism.

Proof. Let $\Phi : \mathsf{H} \times \mathsf{K}_{\mathsf{G}} \to \Lambda_{\mathsf{H}} \times \mathsf{K}_{\mathsf{G}}$ be $(\mathsf{h}, \mathsf{k}) \mapsto (\lambda_{\mathsf{h}}, \mathsf{k})$ and let $\Psi : \Lambda_{\mathsf{H}} \times \mathsf{K}_{\mathsf{G}} \to \mathsf{G}$ be composition of functions.

By assumption $\mathsf{H} \to \Lambda_{\mathsf{H}}, \mathsf{h} \mapsto \lambda_{\mathsf{h}}$ is a homeomorphism and so Φ is a homeomorphism as well. By virtue of being Polish it follows that Λ_{H} is a closed subgroup of G and so applying Theorem 2.10 we get that Ψ is also a homeomorphism. As $\Omega = \Psi \circ \Phi$ this implies Ω is a homeomorphism. □

Without the topological assumptions in Corollary 2.12 it is possible to have a knit product $\mathsf{G} \leqslant \mathrm{Homeo}(\mathsf{H})$ and a Polish group topology on G for which Ω is not a homeomorphism. If H is a group that supports multiple Polish group topologies then considering one topology on H and another topology on $\mathsf{G} = \Lambda_{\mathsf{H}}$ provides a counterexample.

Proposition 2.13 is admittedly a bit wordy. What it does is describe a way to construct a group topology on a knit product. Say we have a topological group H and a knit product $\mathsf{G} \leqslant \mathrm{Homeo}(\mathsf{H})$ for which the isotropy subgroup K_{G} already has some known group topology. This implies the existence of a product topology on G, and Proposition 2.13 reformulates when this topology makes G a topological group. The proposition also happens to be a convenient place to state a compatible metric on G when metrics are known on H and K_{G}.

Proposition 2.13. *Suppose* H *is a topological group and* $\mathsf{G} \leqslant \mathrm{Homeo}(\mathsf{H})$ *is a subgroup such that* $\Lambda_{\mathsf{H}} \leqslant \mathsf{G}$. *Let* $\Omega : \mathsf{H} \times \mathsf{K}_{\mathsf{G}} \to \mathsf{G}$ *be the bijection from Notation 2.9. Also suppose there is a topology on* K_{G} *which makes it a topological group. Then there is a unique topology on* G *which makes* Ω *a homeomorphism. This topology makes* G *a topological group if and only if evaluation*

$$\mathsf{K}_{\mathsf{G}} \times \mathsf{H} \to \mathsf{H}, (\mathsf{k}, \mathsf{h}) \mapsto \mathsf{k}(\mathsf{h})$$

and the function

$$K_G \times H \to K_G, (k, h) \mapsto \lambda_{k(h)}^{-1} \circ k \circ \lambda_h$$

are continuous. If d_H *and* d_K *are compatible metrics on* H *and* K_G*, respectively, then* d *defined by*

$$d(f, g) = d_H(f(1_H), g(1_H)) + d_K\left(\lambda_{f(1_H)}^{-1} \circ f, \lambda_{g(1_H)}^{-1} \circ g\right)$$

is a compatible metric on G.

Proof. The unique topology on G which makes Ω a homeomorphism is clearly the one obtained by declaring $U \subset G$ open if and only if $\Omega^{-1}(U) \subset H \times K_G$ is open.

For all $h \in H$ and all $k \in K_G$ let $\phi(k, h) = k(h)$ and let $\psi(k, h) = \lambda_{k(h)}^{-1} \circ k \circ \lambda_h$. Let $\otimes$ be the binary operation on $H \times K_G$ defined by

$$(h_1, k_1) \otimes (h_2, k_2) = (h_1\phi(k_1, h_2), \psi(k_1, h_2) \circ k_2)$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K_G$. Then

$$\begin{aligned}
\Omega(h_1, k_1) \circ \Omega(h_2, k_2) &= \lambda_{h_1} \circ k_1 \circ \lambda_{h_2} \circ k_2 \\
&= \lambda_{h_1 k_1(h_2)} \circ \left(\lambda_{k_1(h_2)}^{-1} \circ k_1 \circ \lambda_{h_2} \circ k_2\right) \\
&= \Omega\left(h_1\phi(k_1, h_2), \psi(k_1, h_2) \circ k_2\right) \\
&= \Omega\left((h_1, k_1) \otimes (h_2, k_2)\right)
\end{aligned}$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K_G$. This says $\Omega : (H \times K_G, \otimes) \to (G, \circ)$ is an operation-preserving bijection and thus a group isomorphism. Indeed, $(H \times K_G, \otimes)$ is the external knit product of H and K_G with respect to ϕ and ψ.

The proposition states the equivalence between (1) and (3) among the following three equivalent conditions.

1. G is a topological group with the topology from Ω.
2. $(H \times K_G, \otimes)$ is a topological group with the product topology.
3. The functions ϕ and ψ are continuous.

The equivalence between (1) and (2) is immediate because Ω is a group isomorphism and a homeomorphism.

Suppose for a moment that H and K are some arbitrary topological groups and $G = H \times K$ is an external knit product with respect to functions $\alpha : K \times H \to H$ and $\beta : K \times H \to K$. The claim is that the product topology $H \times K$ makes G a topological group if and only if α and β are continuous. For one direction, if α and β are continuous then the group operation and inversion in G have continuous coordinate functions and so are continuous themselves. For the reverse direction, if the group operation in G is continuous with respect to the product topology $H \times K$ then

$$(k, h) \mapsto (1_H, k)(h, 1_K) = (\alpha(k, h), \beta(k, h))$$

defines a continuous function $K \times H \to H \times K$ and so α and β are continuous. Now returning to the setting of the current proposition, $(H \times K_G, \otimes)$ is the external knit product with respect to ϕ and ψ so (2) and (3) are equivalent.

If d_H and d_K are compatible metrics on H and K_G, respectively, then D defined

$$D((h_1, k_1), (h_2, k_2)) = d_H(h_1, h_2) + d_K(k_1, k_2)$$

for all $h_1, h_2 \in H$ and all $k_1, k_2 \in K_G$ is a metric on $H \times K$ which is compatible with the product topology. With d as in the proposition we have

$$D((h_1, k_1), (h_2, k_2)) = d(\Omega(h_1, k_1), \Omega(h_2, k_2))$$

so d is a metric on G and $\Omega : (H \times K_G, D) \to (G, d)$ is an isometry. Hence d is compatible with the topology on G. $\square$

The product of two Polish spaces is a Polish space, and so in our applications of Proposition 2.13 once we know G is a topological group it is then obvious it is a Polish group.

CHAPTER 3

DEFORMATIONS AND BOUNDEDNESS

Throughout this chapter $\mathcal{H} : \mathrm{I} \times \mathsf{G} \to \mathsf{G}$ is an arbitrary deformation retract of a Polish group G onto a subspace A, so $\mathsf{A} = \mathcal{H}_1[\mathsf{G}]$. The idea at play here is that if $\mathcal{H}$ is suitably well-behaved then boundedness of A may be inherited upwards by supersets $\mathsf{B} \supset \mathsf{A}$.

3.1 Restrained deformations

We recall that a family of functions $\mathsf{f}_\alpha : \mathsf{X} \to \mathsf{Y}$ of uniform spaces is equi-uniformly continuous if for every entourage $\mathsf{F} \subset \mathsf{Y} \times \mathsf{Y}$ there is an entourage $\mathsf{E} \subset \mathsf{X} \times \mathsf{X}$ such that for all indices α and all $(x_1, x_2) \in \mathsf{E}$ we have $(\mathsf{f}_\alpha(x_1), \mathsf{f}_\alpha(x_2)) \in \mathsf{F}$. The uniformity on I in the following definition is, of course, the one induced by the standard Euclidean metric.

Definition 3.1. For a subset $\mathsf{B} \subset \mathsf{G}$, the deformation retract $\mathcal{H} : \mathrm{I} \times \mathsf{G} \to \mathsf{G}$ is *left-restrained on* B if the family

$$(\phi_{b,\mathcal{H}})_{b \in \mathsf{B}}$$

of paths which start in B is equi-uniformly continuous with respect to the left uniformity on G.

A deformation retract of a topological group is *left-restrained* if it is left-restrained on the whole group and *locally left-restrained* if there is an identity neighborhood on which it is left-restrained.

In more workable terms, $\mathcal{H}$ is left-restrained on B if for every identity neighborhood $V \subset G$ there is a $\delta > 0$ so that

$$\mathcal{H}_t(b) \in \mathcal{H}_s(b) \cdot V$$

for all $b \in B$ and all $s, t \in I$ with $|s - t| < \delta$.

Proposition 3.2. *Suppose the deformation retract $\mathcal{H} : I \times G \to G$ of G onto A is left-restrained on a subset $B \subset G$. Then for every identity neighborhood $V \subset G$ there exists an integer $k \geqslant 1$ such that $B \subset AV^k$. Consequently, if A is bounded in any one of the following ways, then so is B.*

1. *bounded in the left uniformity*
2. *bounded in the Roelcke uniformity*
3. *coarsely bounded*

Proof. Let $V \subset G$ be an identity neighborhood, let $\delta > 0$ be given by the assumption that $\mathcal{H}$ is left-restrained on B, so

$$\mathcal{H}(s, b)^{-1}\mathcal{H}(t, b) \in V$$

for all $s, t \in I$ with $|s - t| < \delta$ and all $b \in B$. Let k be an integer with $k > \delta^{-1}$. For every $b \in B$,

$$\mathcal{H}(1, b)^{-1}b = \prod_{i=1}^{k} \left[\mathcal{H}\left(\frac{i}{k}, b\right)^{-1} \mathcal{H}\left(\frac{i-1}{k}, b\right) \right] \in V^k$$

and so

$$b \in \mathcal{H}(1, b)V^k \subset AV^k$$

which implies $B \subset AV^k$.

1. If A is bounded in the left uniformity then for every identity neighborhood V there is a finite subset $F \subset G$ and an integer $j \geqslant 1$ with $A \subset FV^j$. By the above there is an integer $k \geqslant 1$ with
$$B \subset FV^jV^k = FV^{j+k},$$
so B is also bounded in the left uniformity.

2. If A is bounded in the Roelcke uniformity then for every identity neighborhood V there is a finite subset $F \subset G$ and an integer $j \geqslant 1$ with $A \subset V^jFV^j$. By the above there is an integer $k \geqslant 1$ with
$$B \subset V^jFV^jV^k \subset V^{j+k}FV^{j+k},$$
so B is bounded in the Roelcke uniformity.

3. If A is coarsely bounded then for every identity neighborhood V there is a finite subset $F \subset G$ which contains 1_G and an integer $j \geqslant 1$ with $A \subset (FV)^j$. By the above there is an integer $k \geqslant 1$ with
$$B \subset (FV)^jV^k \subset (FV)^{j+k}$$
so B is coarsely bounded.

This completes the proof. □

Proposition 1.1, which was stated in the introduction, is a straightforward application of Proposition 3.2.(3) by taking $A = \{1\}$ and then taking B to be either the whole group or taking it to be an identity neighborhood.

We let

$$\mathcal{H}^* : I \times G \to G$$

be the deformation retract of G defined by

$$\mathcal{H}^*(t, g) = \mathcal{H}(t, g^{-1})^{-1}$$

for all $t \in I$ and all $g \in G$.

Suppose $B \subset G$ is a subset. We make the obvious right-handed version of Definition 3.1, $\mathcal{H}$ is *right-restrained on* B if the family of paths which start in B is equi-uniformly continuous with respect to the right uniformity on G. We only comment that $\mathcal{H}$ is left-restrained on B if and only if $\mathcal{H}^*$ is right-restrained on B. So for anything we say about a deformation being left-restrained there is a symmetric statement for some right-restrained deformation. The definitions for being globally *right-restrained* and *locally right-restrained* are the same as in Definition 3.1, except with “right” replacing “left.”

The ideas in this section are motivated from (10), where Rosendal uses a deformation of the Polish group $\mathrm{Homeo}_+(I)$ to show that the group is bounded in the two-sided uniformity. The following definition isolates a property of the deformation that is used in that proof.

Definition 3.3. The deformation retract $\mathcal{H} : I \times G \to G$ of G is *two-sided-restrained* if the family of paths which start in G is equi-uniformly continuous with respect to the two-sided uniformity on G.

This means that for every identity neighborhood $V \subset G$ there is some $\delta > 0$ so that

$$\mathcal{H}_t(g) \in \mathcal{H}_s(g) \cdot V$$

and

$$\mathcal{H}_t(g)^{-1} \in \mathcal{H}_s(g)^{-1} \cdot V$$

for all $g \in G$ and all $s, t \in I$ with $|s - t| < \delta$.

Proposition 3.4. *Suppose the deformation retract $\mathcal{H} : I \times G \to G$ of G onto A is two-sided-restrained and A is finite. Then G is bounded in the two-sided uniformity.*

Proof. Let $V \subset G$ be an identity neighborhood, let $\delta > 0$ be given by the assumption that $\mathcal{H}$ is two-sided-restrained and let k be an integer with $k > \delta^{-1}$. Set

$$x_i = \mathcal{H}\left(\frac{i}{k}, g\right)$$

for all $i = 0, \ldots, k$. For every $g \in G$ we have $x_k \in A$ and

$$\mathcal{H}\left(\frac{i}{k}, g\right) \in \mathcal{H}\left(\frac{i+1}{k}, g\right) \cdot V$$

and

$$\mathcal{H}\left(\frac{i}{k}, g\right)^{-1} \in \mathcal{H}\left(\frac{i+1}{k}, g\right)^{-1} \cdot V$$

so Definition 2.2.(3) is satisfied with $G = S, A = F$, and with the order of the indices of the x_i reversed, so $i \mapsto k - i$. □

Proposition 3.5. *Suppose the deformation retract $\mathcal{H} : I \times G \to G$ of G onto A is locally left-restrained, has local restrictions, and A is finite. Then G is ultralocally bounded.*

Proof. Let U be an open set such that $\mathcal{H}$ is left-restrained on U and let V be an open set with $U \supseteq V \supseteq F$ and $\mathcal{H}_I[V] \subset U$. Let $U' \ni 1_G$ be an arbitrary identity neighborhood and set $V' = U' \cap V$. Then $\mathcal{H}$ is left-restrained on V' because it is a subset of U. Let $W \ni 1$ be open, let $\delta > 0$ witness that $\mathcal{H}$ is left-restrained on V', and let $k > \delta^{-1}$ be an integer. For any $v \in V'$,

$$v = \mathcal{H}(1, v) \prod_{i=1}^{k} \left[\mathcal{H}\left(\frac{i}{k}, v\right)^{-1} \mathcal{H}\left(\frac{i-1}{k}, v\right)\right]$$

and so v can be written as a U'–admissible product with $k+1$ terms in $F \cup W$. □

3.1.1 Examples

See Section 2.4.1 for the definitions of the deformations $\mathcal{V}, \mathcal{T}$, and $\mathcal{A}$ of Homeo$_+$(I).

Proposition 3.6. *The deformation $\mathcal{V}$ is right-restrained on* Homeo$_+$(I) *and restricts to a right-restrained deformation of* AC$_+$(I).

Proof. For any $f \in \text{Homeo}_+(I)$ and $s, t \in I$ we have

$$d_\infty(\mathcal{V}_t(f), \mathcal{V}_s(f)) = |t - s| \sup_{x \in I} |f(x) - x| \leqslant |t - s|$$

and because the upper bound is independent of f this implies that the family of paths defined by $\mathcal{V}$ is equi-uniformly continuous with respect to the metric d_∞ on $\text{Homeo}_+(I)$. This metric is right-invariant and so compatible with the right uniformity on $\text{Homeo}_+(I)$ and so $\mathcal{V}$ is right-restrained.

For any $f, g \in AC_+(I)$ we have

$$\begin{aligned} d_{AC}(f, g) &\leqslant \int_0^1 |f'(t)|\ dt + \int_0^1 |g'(t)|\ dt \\ &= \int_0^1 f'(t)\ dt + \int_0^1 g'(t)\ dt \\ &= 2 \end{aligned}$$

by the triangle inequality, the fact that f and g are increasing, and the fundamental theorem. For any $f \in AC_+(I)$ and any $s, t \in I$

$$\begin{aligned} d_{AC}(\mathcal{V}_t(f), \mathcal{V}_s(f)) &= |t - s| \cdot \int_0^1 \left| f'(x) - 1 \right| \\ &= |t - s| \cdot d_{AC}(f, \text{id}) \\ &\leqslant |t - s| \cdot 2 \end{aligned}$$

so the restriction of $\mathcal{V}$ to $\mathrm{AC}_+(\mathrm{I})$ defines a deformation of $\mathrm{AC}_+(\mathrm{I})$ which is right-restrained with respect to the metric d_{AC}. This metric is right-invariant on $\mathrm{AC}_+(\mathrm{I})$ so the deformation we get by restricting $\mathcal{V}$ to $\mathrm{AC}_+(\mathrm{I})$ is right-restrained.

□

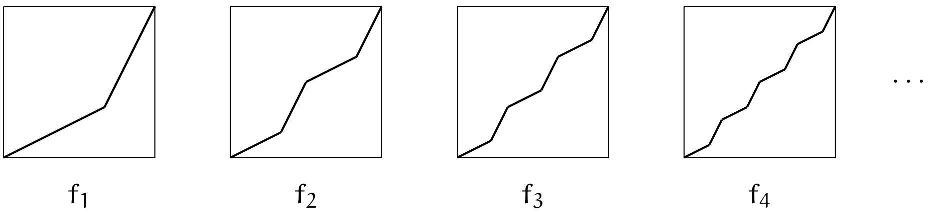

Figure 2: The family of homeomorphisms that appear in the proof of Proposition 3.7.

Proposition 3.7. *The deformation $\mathcal{T}$ is two-sided-restrained on* $\mathrm{Homeo}_+(\mathrm{I})$ *but the restriction to a deformation of* $\mathrm{AC}_+(\mathrm{I})$ *is neither left-restrained nor right-restrained.*

Proof. For any $f \in \mathrm{Homeo}_+(\mathrm{I})$ and any $s, t \in \mathrm{I}$ we have

$$d_\infty(\mathcal{T}_s(f), \mathcal{T}_t(f)) \leqslant |t - s|.$$

To see this, recall that $\mathcal{T}$ acts on graphs by truncating them along the lines $y = x + s$ and $y = x - s$ for $s \in \mathrm{I}$, and so the graphs of $\mathcal{T}_s(f)$ and $\mathcal{T}_t(f)$ are equal except possibly between the lines $y = x + s$ and $y = x + t$ or between the lines $y = x - s$ and $y = x - t$, and thus the greatest vertical and horizontal distances between these graphs, i.e. $d_\infty(\mathcal{T}_s(f), \mathcal{T}_t(f))$ and $d_\infty(\mathcal{T}_s(f)^{-1}, \mathcal{T}_t(f)^{-1})$, are bounded by $|t-s|$. These bounds, together with the fact that d_∞ is right-invariant, imply that $\mathcal{T}$ is two-sided restrained.

We find a family of homeomorphisms $(f_n : I \to I)_{n\in\mathbb{N}}$ which are elements of $AC_+(I)$ and so that $d_\infty(f_n, \mathrm{id}) \to 0$ as $n \to \infty$ and $d_{AC}(f_n, \mathrm{id}) \geqslant 1/2$ for all n. We have $\mathcal{T}_t(f) = f$ for all $t \in I$ and $f \in \mathrm{Homeo}_+(I)$ with $d_\infty(f, \mathrm{id}) \leqslant t$, so the existence of such a family means that for any time t which is arbitrarily close to 1 there are always some f_n which, by time t, have not yet been moved by the deformation $\mathcal{T}$, and so these f_n are still at least d_{AC}–distance $1/2$ from the identity. As d_{AC} is right-invariant, this says that the restriction of $\mathcal{T}$ to a deformation of $AC_+(I)$ is not right-restrained. By symmetry over the diagonal, it is easy to modify the argument to show the deformation is also not left-restrained.

We use the Alexander homotopy $\mathcal{A}$ of $\mathrm{Homeo}_+(I)$ to define the family (f_n), although the reader may prefer to just look at Figure Figure 2. Let

$$f_1(x) = \begin{cases} \frac{x}{2} & \text{if } x \leqslant \frac{2}{3} \\ 2x-1 & \text{if } \frac{2}{3} \leqslant x \end{cases}$$

and for all $n \geqslant 2$ let f_n be the function defined by

$$f_n(x) = \mathcal{A}\left(1 - \frac{1}{n}, f_1\right)\left(x - \frac{i-1}{n}\right) + \frac{i-1}{n}$$

for x in the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ for all integers i with $0 < i \leqslant n$. So f_n is the function that repeats a scaled copy of f_1 along the diagonal n times. Then $d_\infty(f_n, \mathrm{id}) \leqslant 1/n$ and $f_n'(x) \geqslant 1/2$ for almost every $x \in I$ so $d_{AC}(f_n, \mathrm{id}) \geqslant 1/2$ as required.

$\square$

Taken together, Propositions 3.6 and 3.7 indicate a somewhat complicated situation. Offhandedly, they say that $\mathcal{V}$ is half as good on both $\mathrm{Homeo}_+(I)$ and $\mathrm{AC}_+(I)$ as $\mathcal{T}$ is on $\mathrm{Homeo}_+(I)$, but $\mathcal{T}$ is not good at all on $\mathrm{AC}_+(I)$.

Question 3.8. Is $\mathrm{AC}_+(I)$ bounded in the two-sided uniformity? If so, can this be witnessed by some two-sided restrained deformation?

By a result of M. Cohen (19), for any compact connected 1–manifold M^1 and any integer $k \geqslant 1$, the Polish group $\mathrm{Diff}^k_+(M^1)$ has a nontrivial quasi-isometry type. This means that the restriction of $\mathcal{V}$ to a deformation of $\mathrm{Diff}^k_+(I)$ cannot possibly be left-restrained or right-restrained, as by Proposition 1.1 this would imply the group is quasi-isometrically trivial. We prove that in the case $k = 1$ the deformation is locally right-restrained. Things are more complicated here because the natural metric to use on $\mathrm{Diff}^1_+(I)$ is not right-invariant, as was the case with the previous Polish groups $\mathrm{Homeo}_+(I)$ and $\mathrm{AC}_+(I)$.

Lemma 3.9. *The inequalities*

$$d_{C^1}(f \circ h, g \circ h) \leqslant d_{C^1}(f, g) \cdot \|h'\|_\infty$$

and

$$d_{C^1}(f \circ g^{-1}, \mathrm{id}) \leqslant d_{C^1}(f, g) \cdot \left\| \frac{1}{g'} \right\|_\infty$$

hold for all $f, g, h \in \mathrm{Diff}^1_+(I)$.

Proof. The first inequality is by the chain rule and right-invariance of d_∞. The second inequality is by making the substitution $h \mapsto g^{-1}$ in the first inequality, and then applying the formula

$$\left(g^{-1}\right)' = \frac{1}{g'} \circ g^{-1}$$

and right-invariance once again. □

Proposition 3.10. *The deformation* $\mathcal{V}$ *restricts to a locally right-restrained deformation of* $\mathrm{Diff}^1_+(I)$.

Proof. Let B be any subset of $\mathrm{Diff}^1_+(I)$ such that

$$\sup_{f \in B} \sup_{x \in I} |\log f'(x)| < \infty.$$

We claim that $\mathcal{V}$ restricts to a deformation of $\mathrm{Diff}^1_+(I)$ which is right-restrained on B.

Note that

$$
\begin{aligned}
d_{C^1}\left(\mathcal{V}_s(f), \mathcal{V}_t(f)\right) &= \| \mathcal{V}_s(f)' - \mathcal{V}_t(f)' \|_\infty \\
&= \sup_{x \in I} |(1-s)f'(x) + s - (1-t)f'(x) - t| \\
&= \sup_{x \in I} |(t-s)f'(x) - t + s| \\
&\leqslant |t-s| \cdot \sup_{x \in I} \left| f'(x) \right| + |t-s| \\
&= |t-s| \cdot \left(\|f'\|_\infty + 1 \right)
\end{aligned}
$$

and

$$
\left\| \frac{1}{\mathcal{V}_t(f)'} \right\|_\infty = \left\| \frac{1}{(1-t)\cdot f' + t} \right\|_\infty
$$

for all $s, t \in I$ and $f \in \mathrm{Diff}^1_+(I)$.

Combining the above with the second inequality of Lemma 3.9 we have

$$
\begin{aligned}
d_{C^1}(\mathcal{V}_s(f) \circ \mathcal{V}_t(f)^{-1}, \mathrm{id}) &\leqslant d_{C^1}\left(\mathcal{V}_s(f), \mathcal{V}_t(f)\right) \cdot \left\| \frac{1}{\mathcal{V}_t(f)'} \right\|_\infty \\
&\leqslant |t-s| \cdot \left(\|f'\|_\infty + 1 \right) \cdot \left\| \frac{1}{(1-t)\cdot f' + t} \right\|_\infty
\end{aligned}
$$

for all $s, t \in I$ and all $f \in \mathrm{Diff}^1_+(I)$.

By assumption

$$
\sup_{f \in B} \sup_{x \in I} |\log f'(x)| < \infty
$$

for all $f \in B$. It follows there is a real number $j > 0$ so that

$$\sup_{x \in I} |f'(x)| < j$$

and

$$\sup_{x \in I} |\left(f^{-1}\right)'(x)| < j$$

for all $f \in B$. So we have

$$\begin{aligned} d_{C^1}\left(\mathcal{V}_s(f) \circ \mathcal{V}_t(f)^{-1}, \mathrm{id}\right) &\leqslant (|t-s| \cdot j + |t-s|) \cdot j \cdot (1+j) \\ &= |t-s| \cdot (j + 2j^2 + j^3) \end{aligned}$$

for all $f \in B$. The upper bound does not depend on f, and so it follows that $\mathcal{V}$ is right-restrained on B.

It suffices to find such a B that is an identity neighborhood. For instance,

$$B = \left\{ f \;\middle|\; \sup_x |\log f'(x)| < 1 \right\}$$

works fine. $\square$

By Proposition 1.1 these examples show that $\mathrm{Homeo}_+(I)$ and $\mathrm{AC}_+(I)$ are coarsely bounded groups and $\mathrm{Diff}^1_+(I)$ is generated by a coarsely bounded subset. The fact that $\mathrm{AC}_+(I)$ is coarsely bounded is new and is the first part of Theorem 1.3. We also point out that on each of these groups $\mathcal{V}$ defines a deformation which has local restrictions (see Definition 2.8), so by a right-handed version of Proposition 3.5 each of these groups is ultralocally bounded.

3.2 Conjugate moments

By $\mathbb{D}^n$ we mean the compact ball of topological dimension n. In the proof that the Polish group $\mathrm{Homeo}_+(\mathbb{D}^n)$ is coarsely bounded that appears in (20), Rosendal and K. Mann make use of the Alexander homotopy on the group. We extract from that proof the following definition.

Definition 3.11. The deformation retract $\mathcal{H} : I \times G \to G$ *has conjugate moments* if there is a function

$$\varphi : (0,1)^2 \to G$$

with

$$\lim_{x \to (0,0)} \varphi(x) = 1_G$$

and

$$\mathcal{H}_s(G)^{\varphi(s,t)} \subset \mathcal{H}_t(G)$$

for all $s, t \in (0,1)$.

Proposition 3.12. *Suppose the deformation retract $\mathcal{H} : I \times G \to G$ contracts G onto 1_G, is gradual at 1, and has conjugate moments. Then for every symmetric open $V \ni 1_G$ there exists a finite subset $F \subset G$ such that $(FV)^4 = G$. In particular, G is coarsely bounded.*

Proof. Fix a left-invariant metric d on G and let B_ϵ denote the open d–ball of radius ϵ about 1_G for all $\epsilon > 0$.

Let V be any symmetric neighborhood of 1_G. Let $\epsilon_0 \in (0,1)$ satisfy $B_{\epsilon_0} \subset V$. As $\mathcal{H}$ is gradual at 1 we choose $\epsilon_1 \in (0,1)$ with $\mathcal{H}_{[1-\epsilon_1,1]}(G) \subset V$. Let $\varphi : (0,1)^2 \to G$ witness that $\mathcal{H}$ has conjugate moments and let $\epsilon_2 \in (0,1)$ satisfy $(0,\epsilon_2)^2 \subset \varphi^{-1}(V)$. Set $\epsilon = \min\{\epsilon_0, \epsilon_1, \epsilon_2\}$ and set

$$F = \left\{1_G, \varphi(\epsilon, 1-\epsilon), \varphi(\epsilon, 1-\epsilon)^{-1}\right\}.$$

We claim $(FV)^4 = G$. Let $g \in G$ be arbitary. By continuity of $\mathcal{H}$ we choose $t \in (0,\epsilon)$ with

$$g \in \mathcal{H}_t(g) \cdot B_\epsilon.$$

We have

$$\left(\mathcal{H}_t(G)^{\varphi(t,\epsilon)}\right)^{\varphi(\epsilon,1-\epsilon)} \subset \mathcal{H}_{1-\epsilon}(G) \subset V$$

so

$$\varphi(\epsilon, 1-\epsilon) \cdot \varphi(t,\epsilon) \cdot \mathcal{H}_t(g) \cdot \varphi(t,\epsilon)^{-1} \cdot \varphi(\epsilon, 1-\epsilon)^{-1} \in V$$

and then

$$\mathcal{H}_t(g) \in V \cdot \varphi(\epsilon, 1-\epsilon)^{-1} \cdot V \cdot \varphi(\epsilon, 1-\epsilon) \cdot V.$$

Combining this gives

$$g \in \mathcal{H}_t(g) \cdot B_\epsilon \subset V \cdot \varphi(\epsilon, 1-\epsilon)^{-1} \cdot V \cdot \varphi(\epsilon, 1-\epsilon) \cdot V^2$$

and so $g \in (FV)^4$ which proves the claim. The claim immediately implies G is coarsely bounded.

$\square$

Take $G = \mathrm{Homeo}_+(I)$. We record the details that Proposition 3.12 applies with $\mathcal{H} = \mathcal{A}$. It is clear that

$$\mathcal{A}_t(G) = \{f \in G \mid f \text{ is supported on } [0, 1-t]\}$$

and so

$$d_\infty\left(\mathcal{A}_t(f), \mathcal{A}_t(g)\right) \leqslant (1-t) \cdot d_\infty(f, g)$$

for all $t \in I$ and all $f, g \in G$. This implies $\mathcal{A}$ is gradual at 1. For all $s, t \in (0,1)$ let $\varphi_{s,t} : I \to I$ be the element of G which is defined

$$\varphi_{s,t}(x) = \begin{cases} \frac{1-t}{1-s} \cdot x & \text{if } x \in [0, 1-s] \\ \frac{t}{s} \cdot x + \frac{s-t}{s} & \text{if } x \in [1-s, 1] \end{cases}$$

for all $x \in I$. We have

$$d_\infty(\varphi_{s,t}, \mathrm{id}) = |s-t|$$

so

$$\lim_{x \to (0,0)} \varphi(x) = \mathrm{id}$$

and $\varphi_{s,t}$ satisfies

$$\mathcal{A}_s(g)^{\varphi_{s,t}} = \mathcal{A}_t(g)$$

for all $g \in G$. This implies $\mathcal{A}$ has conjugate moments.

Also

$$d_{\mathrm{AC}}(\mathcal{A}_t(f), \mathcal{A}_t(g)) = (1-t) \cdot d_{\mathrm{AC}}(f, g)$$

and

$$d_{\mathrm{AC}}(\varphi_{s,t}, \mathrm{id}) = 2 \cdot |s-t|$$

for all $s, t \in (0,1)$ so $\mathcal{A}$ restricts to a deformation of $\mathrm{AC}_+(I)$ which is gradual at 1 and has conjugate moments.

CHAPTER 4

GEOMETRY OF KNIT PRODUCTS

The theory in this chapter is organized around Lemma 4.1. The lemma says that certain word metrics on the factor subgroups of a knit product imply the existence of a word metric on the product itself, and further, the lemma gives a condition which implies that the factor subgroups are isometrically embedded. Theorem 1.2 then brings this observation to the context of an abstract Polish group which is a knit product of closed subgroups. Even with the assumption that one of the two factor subgroups is coarsely trivial we are still able to use this result to "compute" the quasi-isometry type of a number of interesting Polish groups. Many of these examples arise as transformation groups as in Section 2.5.3, and so we also prove a corollary to Theorem 1.2 which is suited to this setting.

4.1 General results

For a group G generated by a subset S we recall the left-invariant *word metric* ρ_S defined by

$$\rho_S(x, y) = \min\left\{k \geqslant 0 \mid x^{-1}y \in \left(S \cup S^{-1}\right)^k\right\}$$

for all $x, y \in G$.

Lemma 4.1. *Suppose* G *is a knit product of subgroups* H *and* K*. Also suppose* H *is generated by a symmetric subset* $S \subset H$ *and* K *is generated by a symmetric subset* $T \subset K$ *with* $1 \in S \cap T$ *and* $ST = TS$*. Then the word metric* ρ_{ST} *is defined on* G *and*

$$\rho_{ST}(hk, 1) = \max\{\rho_S(h, 1), \rho_T(k, 1)\}$$

for all $h \in H$ *and* $k \in K$*. Consequently, the inclusions of the two factor subgroups* $(H, \rho_S) \to (G, \rho_{ST})$ *and* $(K, \rho_T) \to (G, \rho_{ST})$ *are isometric embeddings.*

Proof. For any integer $n \geqslant 0$ we have $S^n T^n = (ST)^n$ by repeatedly applying the assumption $ST = TS$.

Let $h \in H$ and $k \in K$ and set $M = \max\{\rho_S(h, 1), \rho_T(k, 1)\}$. Because $1 \in S \cap T$ we have $hk \in S^M T^M = (ST)^M$ so ρ_{ST} is defined on G and $\rho_{ST}(hk, 1) \leqslant M$. If $n \geqslant 0$ is an integer with $\rho_{ST}(hk, 1) \leqslant n$ then $hk \in (ST)^n = S^n T^n$ so there exists $s_1, \ldots, s_n \in S$ and $t_1, \ldots, t_n \in T$ with

$$hk = s_1 \cdots s_n t_1 \cdots t_n$$

and because the group opertion $H \times K \to G$ is injective this implies $h = s_1 \cdots s_n$ and $k = t_1 \cdots t_n$, so $\rho_S(h, 1) \leqslant n$ and $\rho_T(k, 1) \leqslant n$. This holds for any $n \geqslant 0$ so $\rho_{ST}(hk, 1) \geqslant \rho_S(h, 1)$ and $\rho_{ST}(hk, 1) \geqslant \rho_T(k, 1)$, and hence $\rho_{ST}(hk, 1) \geqslant M$. This proves the equality in the lemma.

Now

$$\rho_{ST}(h_1^{-1} h_2, 1) = \max\left\{\rho_S(h_1^{-1} h_2, 1), \rho_T(1, 1)\right\} = \rho_S(h_1^{-1} h_2, 1)$$

for all $h_1, h_2 \in H$. By left invariance it follows that the inclusion $(H, \rho_S) \to (G, \rho_{ST})$ is an isometric embedding. Similarly $(K, \rho_T) \to (G, \rho_{ST})$ is an isometric embedding. $\square$

We restate and prove Theorem 1.2 from the introduction.

Theorem 4.2. *Suppose G is a Polish group which is a knit product of closed subgroups H and K. Also suppose H is generated by a subset $S \subset H$ which is coarsely bounded in H, K is a coarsely bounded group when equipped with the subspace topology, and $SK = KS$. Then the inclusion $H \to G$ is a quasi-isometry of Polish groups.*

Proof. Set $\mathcal{S} = S \cup \{1\} \cup S^{-1}$. As $K = K^{-1}$ and $SK = KS$ we have

$$S^{-1}K = (KS)^{-1} = (SK)^{-1} = KS^{-1}$$

and

$$\mathcal{S}K = SK \cup K \cup S^{-1}K = KS \cup K \cup KS^{-1} = K\mathcal{S}$$

so by Corollary 2.11

$$\overline{\mathcal{S}}K = \overline{\mathcal{S}K} = \overline{K\mathcal{S}} = K\overline{\mathcal{S}}$$

because $\overline{\mathcal{S}}$ and K are closed. Now by Lemma 4.1 the inclusion $(H, \rho_{\overline{\mathcal{S}}}) \to (G, \rho_{\overline{\mathcal{S}}K})$ is an isometric embedding.

For every $g \in G$ there is $h \in H$ and $k \in K$ with $g = hk$, and by left invariance

$$\rho_{\overline{\mathcal{S}}K}(g, h) = \rho_{\overline{\mathcal{S}}K}(k, 1) \leqslant 1$$

so the inclusion $(\mathsf{H}, \rho_{\overline{\mathsf{S}}}) \to (\mathsf{G}, \rho_{\overline{\mathsf{S}}\mathsf{K}})$ is a quasi-isometry of metric spaces.

As $\overline{\mathsf{S}}$ is a symmetric generating set for H which is closed and coarsely bounded in H the quasi-isometry type of H is that of $(\mathsf{H}, \rho_{\overline{\mathsf{S}}})$. It remains to show that the quasi-isometry type of G is that of $(\mathsf{G}, \rho_{\overline{\mathsf{S}}\mathsf{K}})$. We know $\overline{\mathsf{S}}\mathsf{K} = \overline{\mathsf{S}\mathsf{K}}$ is a generating set for G which is closed, so we must show $\overline{\mathsf{S}}\mathsf{K}$ is coarsely bounded in G. Let d be a compatible, left-invariant metric on G. Then d restricts to a compatible, left-invariant metric on both H and K. The subsets $\overline{\mathsf{S}}$ and K are coarsely bounded in H and K, respectively, so these subsets are bounded in d. By left invariance and the triangle inequality it follows that $\overline{\mathsf{S}}\mathsf{K}$ is bounded in d, and thus $\overline{\mathsf{S}}\mathsf{K}$ is coarsely bounded in G. This means the quasi-isometry type of G is that of $(\mathsf{G}, \rho_{\overline{\mathsf{S}}\mathsf{K}})$ as required to make the inclusion $\mathsf{H} \to \mathsf{G}$ a quasi-isometry of Polish groups. $\square$

Corollary 4.3. *Suppose* H *is a Polish group and* $\mathsf{G} \leqslant \mathrm{Homeo}(\mathsf{H})$ *is a subgroup such that* $\Lambda_{\mathsf{H}} \leqslant \mathsf{G}$*. Let* $\mathsf{K}_{\mathsf{G}} \leqslant \mathsf{G}$ *be the isotropy subgroup from Notation 2.9. Suppose* G *is equipped with a Polish group topology which satisfies the assumptions of Corollary 2.12. Also suppose* H *is generated by a subset* $\mathsf{S} \subset \mathsf{H}$ *which is coarsely bounded in* H*,* K_{G} *is a coarsely bounded group when equipped with the subspace topology,* $\mathsf{K}_{\mathsf{G}}[\mathsf{S}] \subset \mathsf{S}$*, and* $\mathsf{K}_{\mathsf{G}}\left[\mathsf{S}^{-1}\right] \subset \mathsf{S}^{-1}$*. Then* $\mathsf{H} \to \mathsf{G}, \mathsf{h} \mapsto \lambda_{\mathsf{h}}$ *is a quasi-isometry of Polish groups.*

Proof. First note that for all $\mathsf{h} \in \mathsf{H}$ and $\mathsf{k} \in \mathsf{K}_{\mathsf{G}}$

$$\mathsf{k} \circ \lambda_{\mathsf{h}} = \lambda_{\mathsf{k}(\mathsf{h})} \circ \left(\lambda_{\mathsf{k}(\mathsf{h})}^{-1} \circ \mathsf{k} \circ \lambda_{\mathsf{h}}\right)$$

and

$$\lambda_h \circ k = \left(\lambda_h \circ k \circ \lambda_{k^{-1}(h^{-1})}\right) \circ \lambda^{-1}_{k^{-1}(h^{-1})}$$

and in both equations the composition in parentheses is an element of K_G.

Set $\Lambda_S = \{\lambda_s | s \in S\}$. By assumption $k(s) \in S$ and $k^{-1}(s^{-1}) \in S^{-1}$ for every $s \in S$ and $k \in K_G$, so the first equation above implies $K_G \circ \Lambda_S \subset \Lambda_S \circ K_G$ and the second equation implies $\Lambda_S \circ K_G \subset K_G \circ \Lambda_S$, and thus $\Lambda_S \circ K_G = K_G \circ \Lambda_S$.

By Theorem 4.2 with Λ_S in place of S it follows that the inclusion $\Lambda_H \to G$ is a quasi-isometry of Polish groups. As $H \to \Lambda_H, h \mapsto \lambda_h$ is an isomorphism of Polish groups it is a quasi-isometry of Polish groups, and so Corollary 4.3 follows by composing the quasi-isometry $H \to \Lambda_H$ with the quasi-isometry $\Lambda_H \to G$. $\square$

We should mention what is known outside of the case that one of the two factor subgroups is generated by a coarsely bounded subset. In the time since the above results were established, Rosendal has shown the following as (6, Theorem A.12). Suppose G is a Polish group which is a knit product of closed subgroups H and K. Let $\pi_H : G \to H$ and $\pi_K : G \to K$ be the functions determined by the equation $g = \pi_H(g)\pi_K(g)$ for all $g \in G$.

Theorem 4.4 (Rosendal)**.** *Suppose* H *is locally bounded. Then the bijection* $H \times K \to G$ *is a coarse equivalence if and only if* $\pi_H[X^K]$ *is coarsely bounded in* H *and* $\pi_K[X^K]$ *is coarsely bounded in* K *for every coarsely bounded subset* $X \subset H$.

Given a knit product G of H and K we have bijections $H \times K \to G$ and $K \times H \to G$ defined by the group operation. The asymmetry between the two factor subgroups in Theorem 4.4 is a reflection of the observation that requiring $H \times K \to G$ to be a coarse equivalence is not the same as requiring that of $K \times H \to G$.

4.2 Examples

By definition a semidirect product of groups is a knit product where at least one of the two factor subgroups is normal. If G is a semidirect product of subgroups H and K with K normal then $SK = KS$ for every subset $S \subset H$ and so the condition relating S and K in Theorem 1.2 is satisfied with no extra verification. In this case the theorem simplifies to the following.

Example 4.5. Suppose G is a Polish group which is a semidirect product of closed subgroups H and K with K normal. If K is coarsely bounded and H is generated by a coarsely bounded subset then the inclusion $H \to G$ is a quasi-isometry.

Isometry groups of left-invariant metrics provide another class of examples where Theorem 1.2 applies.

Example 4.6. Let H be a locally compact Polish group that admits a compatible, complete, proper, left-invariant metric d whose closed unit ball B generates H and let $G = \mathrm{Isom}(H, d)$ be the isometry group with the topology of pointwise convergence. Because H is locally compact the isotropy subgroup K_G is compact. Applying Corollary 4.3 with $S = B$ we conclude that $H \to G, h \mapsto \lambda_h$ is a quasi-isometry. The condition relating S and K_G in the corollary is satisfied by left-invariance: For every $h \in H$ and $k \in K_G$

$$d(k(h), 1_H) = d(k(h), k(1_H)) = d(h, 1_H)$$

which implies $\mathsf{K}_\mathsf{G}[\mathsf{S}] \subset \mathsf{S}$, and since $\mathsf{S} = \mathsf{S}^{-1}$ this also says $\mathsf{K}_\mathsf{G}[\mathsf{S}^{-1}] \subset \mathsf{S}^{-1}$.

So for instance we can take H to be a finitely generated group and $\mathsf{G} = \mathrm{Isom}(\mathsf{H}, \rho_\mathsf{S})$ for any finite generating set $\mathsf{S} \subset \mathsf{H}$.

The idea for Theorem 1.2 was primarily motivated by studying the large-scale geometry of $\mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$, so we record this group as a special example.

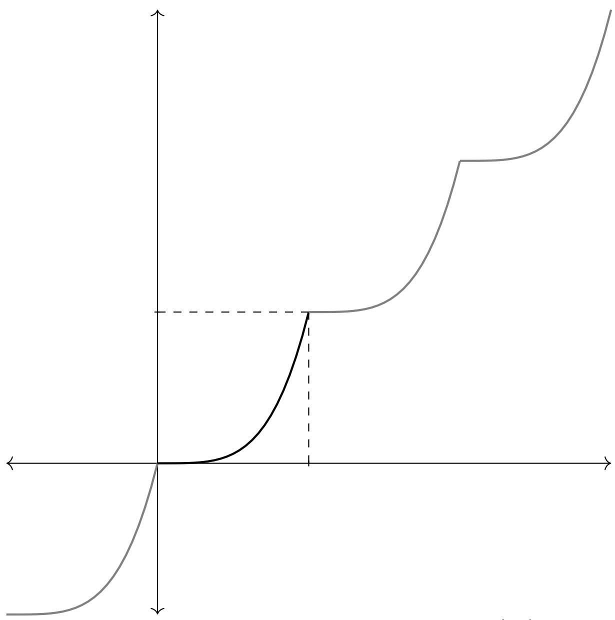

Figure 3: The graph of an element of $\mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$ which fixes 0.

Example 4.7. To apply Corollary 4.3 with $\mathsf{G} = \mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$ we need a compact generating set S for $\mathsf{H} = \mathbb{R}$ so that $\mathsf{K}_\mathsf{G}[\mathsf{S}] \subset \mathsf{S}$. Note that any element of $\mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$ which fixes 0 must also fix every integer (as in Figure Figure 3), and so taking $\mathsf{S} = \mathrm{I}$ is sufficient. We remind the reader that the isotropy subgroup K_G is isomorphic to $\mathrm{Homeo}_+(I)$ and so it is a coarsely bounded group. It follows that $\mathbb{R} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}), r \mapsto \tau_r$ is a quasi-isometry.

So $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ is a knit product of the group $\mathbb{R}$, which is locally compact but geometrically nontrivial, and the group $\mathrm{Homeo}_+(I)$ which is geometrically trivial but not locally compact. Thus, we think of the geometry and topology of the two factor subgroups as in a balancing act: One factor subgroup has an interesting geometry while the other factor subgroup has an interesting topology.

By Cohen's result, for $k = 1$ the Polish group $\mathrm{Diff}^k_+(\mathbb{S}^1)$ is quasi-isometric to the Banach space $\mathcal{C}[I]$ of continuous functions $I \to \mathbb{R}$. Although not the method used in (19), one way to see this is by noting that $G = \mathrm{Diff}^k_+(\mathbb{S}^1)$ is a knit product of $\mathbb{S}^1$ and the isotropy subgroup K_G. In this case K_G is isomorphic to the closed subgroup of $\mathrm{Diff}^k_+(I)$ of diffeomorphisms $f : I \to I$ such that $f'(0) = f'(1)$. It is possible to first show K_G is quasi-isometric to $\mathcal{C}[I]$ and then use Theorem 1.2 to conclude that the inclusion $K_G \to G$ is a quasi-isometry. Contrasting with Example 4.7, in this case G is a knit product of one group which is locally compact and geometrically trivial and another group which is neither locally compact nor geometrically trivial.

Example 4.8. Let $\mathbf{T}$ be the countably infinite regular tree, so $\mathbf{T}$ is isomorphic to the Cayley graph of the free group $\mathbb{F}_\infty$ on a countably infinite set of generators. We can view the group $\mathrm{Aut}(\mathbf{T})$ as a subgroup of the homeomorphism group of the countable discrete group $\mathbb{F}_\infty$. It follows that $\mathrm{Aut}(\mathbf{T})$ is a knit product of $\mathbb{F}_\infty$ and the isotropy subgroup of $\mathrm{Aut}(\mathbf{T})$. The latter group is coarsely bounded as a consequence of (21, Theorem 6.31(ii)) and so by the main result of this chapter $\mathrm{Aut}(\mathbf{T})$ is quasi-isometric to $\mathbf{T}$.

CHAPTER 5

EXAMPLE: ABSOLUTE CONTINUITY

This chapter contains the details needed to finish the proof of Theorem 1.3, so recall the theorem states that $\mathrm{AC}_+(\mathrm{I})$ and $\mathrm{AC}_+(\mathbb{S}^1)$ are coarsely bounded groups and $\mathrm{AC}_{\mathbb{Z}}^{\mathrm{loc}}(\mathbb{Z})$ is quasi-isometric to the infinite cyclic group. We have already shown $\mathrm{AC}_+(\mathrm{I})$ is coarsely bounded by exhibiting a right-restrained deformation retract of $\mathrm{AC}_+(\mathrm{I})$ onto its trivial subgroup. As for $\mathrm{AC}_+(\mathbb{S}^1)$ and $\mathrm{AC}_{\mathbb{Z}}^{\mathrm{loc}}(\mathbb{Z})$, we have yet to even verify that the topologies defined on these groups in Section 2.3.2 are group topologies.

In both the case $\mathsf{G} = \mathrm{AC}_+(\mathbb{S}^1)$ and $\mathsf{G} = \mathrm{AC}_{\mathbb{Z}}^{\mathrm{loc}}(\mathbb{Z})$ the group G is a knit product of two groups which are already known to have group topologies. In both cases we apply Proposition 2.13 to show that the topology induced on G by the knit product structure is a Polish group topology, and further, that the right-invariant metric $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$ defined in Section 2.3.2 on either group induces the same topology.

We set

$$\mathsf{K}_* = \left\{\mathsf{k} \in \mathrm{AC}_{\mathbb{Z}}^{\mathrm{loc}}(\mathbb{R}) \mid \mathsf{k}(0) = 0\right\}$$

and note that $\mathsf{k} \mapsto \mathsf{k}|_{\mathrm{I}}$ defines an isomorphism of groups $\mathsf{K}_* \to \mathrm{AC}_+(\mathrm{I})$ which preserves d_{AC}. As d_{AC} is a right-invariant metric on $\mathrm{AC}_+(\mathrm{I})$ which induces a Polish group topology it follows that d_{AC} is also a right-invariant metric on K_* which induces a Polish group topology, and the Polish groups K_* and $\mathrm{AC}_+(\mathrm{I})$ are isomorphic.

Lemma 5.1. *The evaluation*

$$\mathsf{K}_* \times \mathbb{R} \to \mathbb{R}, (\mathsf{k}, \mathsf{r}) \mapsto \mathsf{k}(\mathsf{r})$$

is continuous.

Proof. Let

$$\mathsf{K}_\infty = \{\mathsf{k} \in \mathrm{Homeo}_\mathbb{Z}(\mathbb{R}) \mid \mathsf{k}(0) = 0\},$$

let

$$\Phi : \mathsf{K}_* \times \mathbb{R} \to \mathsf{K}_\infty \times \mathbb{R}$$

be the inclusion, and let

$$\Psi : \mathsf{K}_\infty \times \mathbb{R} \to \mathbb{R}$$

be evaluation. By Lemma 2.5 inclusion $\mathrm{AC}_+(\mathrm{I}) \to \mathrm{Homeo}_+(\mathrm{I})$ is continuous so also Φ is continuous, and by Proposition 2.13 (with $\mathsf{H} = \mathbb{R}$ and $\mathsf{G} = \mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$) Ψ is continuous. Evaluation $\mathsf{K}_* \times \mathsf{H} \to \mathsf{H}$ is the composition $\Psi \circ \Phi$, and so it is continuous. □

Lemma 5.2. *The function*

$$\mathsf{K}_* \times \mathbb{R} \to \mathsf{K}_*, (\mathsf{k}, \mathsf{r}) \mapsto \tau_{\mathsf{k}(\mathsf{r})}^{-1} \circ \mathsf{k} \circ \tau_\mathsf{r}$$

is continuous.

Proof. For all $r \in \mathbb{R}$ and $k \in K_*$ let $\psi(k,r) = \tau_{k(r)}^{-1} \circ k \circ \tau_r$. Now fix $(k,r) \in K_* \times \mathbb{R}$ and let $\epsilon > 0$ be given. For any compact interval J the collection of continuous functions $\mathcal{C}(J)$ is a dense subset of $L^1(J)$, so there exists some continuous function $\gamma : [-\epsilon, 1+\epsilon] \to \mathbb{R}$ with

$$\int_{-\epsilon}^{1+\epsilon} |k'(t) - \gamma(t)| \; dt < \frac{\epsilon}{4}$$

and by uniform continuity of γ there exists some $\delta > 0$ so that

$$|\gamma(x) - \gamma(y)| < \frac{\epsilon}{4}$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$. Using the properties of d_{AC} from Proposition 2.6, for all $s \in \mathbb{R}$ and $l \in K_*$

$$d_{\mathrm{AC}}\left(\psi(k,r), \psi(l,s)\right) = d_{\mathrm{AC}}(k, l \circ \tau_{s-r})$$

and if $d_{AC}(k, l) + |r - s| < \min\{\delta, \epsilon/4\}$ then

$$\begin{aligned}
d_{AC}(k, l \circ \tau_{s-r}) &\leqslant \int_0^1 |k'(t) - \gamma(t)| \, dt \\
&\quad + \int_0^1 |\gamma(t) - \gamma(t + s - r)| \, dt \\
&\quad + \int_0^1 |\gamma(t + s - r) - k'(t + s - r)| \, dt \\
&\quad + \int_0^1 |k'(t + s - r) - l'(t + s - r)| \, dt \\
&< \epsilon
\end{aligned}$$

so ψ is continuous at (k, r). Because the argument given works for an arbitrary point $(k, r) \in K_* \times \mathbb{R}$ it follows that ψ is continuous on $K_* \times \mathbb{R}$. □

Proposition 5.3. *Set* $H = \mathbb{R}$ *and* $G = AC_{\mathbb{Z}}^{loc}(\mathbb{R})$ *and let* $\Omega : H \times K_G \to G$ *be the bijection from Notation 2.9. Then the unique topology on* G *which makes* Ω *a homeomorphism also makes* G *a Polish group, and the metric* $d_\infty + d_{AC}$ *is compatible with this topology.*

Proof. In the present notation $K_* = K_G$. Evaluation $K_G \times H \to H$ and the function

$$K_G \times H \to K_G, (k, h) \mapsto \lambda_{k(h)}^{-1} \circ k \circ \lambda_h$$

are continuous by Lemmas 5.1 and 5.2. Now Proposition 2.13 applies and so the topology that makes Ω a homeomorphism also makes G a Polish group. A compatible metric d on G is given by

$$d(f, g) = |f(0) - g(0)| + d_{AC}\left(\tau_{f(0)}^{-1} \circ f, \tau_{g(0)}^{-1} \circ g\right) = |f(0) - g(0)| + d_*(f, g)$$

for all $f, g \in G$.

For all $f, g \in G$

$$d(f,g) = |f(0) - g(0)| + d_{AC}(f,g) \leqslant d_\infty(f,g) + d_{AC}(f,g)$$

and

$$\begin{aligned} d_\infty(f,g) \leqslant & |f(0) - g(0)| + d_\infty\left(\tau_{f(0)}^{-1} \circ f, \tau_{g(0)}^{-1} \circ g\right) \\ \leqslant & |f(0) - g(0)| + d_{AC}\left(\tau_{f(0)}^{-1} \circ f, \tau_{g(0)}^{-1} \circ g\right) \\ = & |f(0) - g(0)| + d_{AC}(f,g) \end{aligned}$$

so $d_\infty(f,g) + d_{AC}(f,g) \leqslant 2\ d(f,g)$. This implies d and $d_\infty + d_{AC}$ induce the same topology on G. $\square$

For what remains, let

$$\mathcal{T} = \{\tau_r \mid r \in \mathbb{R}\}$$

be the subgroup of $AC_{\mathbb{Z}}^{loc}(\mathbb{R})$ consisting of real translations.

As an aside we note that the topology induced by the pseudometric d_{AC} on $AC_{\mathbb{Z}}^{loc}(\mathbb{R})$ is not a group topology. To see this, for $k \in K_*$ consider the cosets $\mathcal{T} \circ k$ and $k \circ \mathcal{T}$. By Proposition 2.6 the right coset $\mathcal{T} \circ k$ has d_{AC}–diameter 0. On the other hand, if $k \circ \mathcal{T}$ has d_{AC}–diameter 0 then the fundamental theorem of Lebesgue integration implies the homeomorphism k is also a homomorphism of $(\mathbb{R}, +)$, and so k must be the identity $\mathbb{R} \to \mathbb{R}$. This says $k \circ \mathcal{T}$ has

d_{AC}–diameter 0 if and only if k is the identity. From this it follows that inversion in $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ exchanges subsets with d_*–diameter 0 and subsets with positive d_{AC}–diameter, and so inversion is not a homeomorphism with the topology induced by d_{AC}.

We set

$$\mathsf{K}_\circ = \left\{\mathsf{k} \in \mathrm{AC}_+\left(\mathbb{S}^1\right) \mid \mathsf{k}(1) = 1\right\}$$

and note that $\mathsf{k} \mapsto \tilde{\mathsf{k}}|_{\mathrm{I}}$ defines an isomorphism of groups $\mathsf{K}_\circ \to \mathrm{AC}_+(\mathrm{I})$ which preserves d_{AC}. It follows that d_{AC} is a right-invariant metric on $\mathsf{K}_\circ$ which induces a Polish group topology, and the Polish groups $\mathsf{K}_\circ$ and $\mathrm{AC}_+(\mathrm{I})$ are isomorphic. In Proposition 5.4 we extend the topology on $\mathsf{K}_\circ$ to a Polish group topology on $\mathrm{AC}_+\left(\mathbb{S}^1\right)$. Alternatively, one may define the same Polish group topology on $\mathrm{AC}_+\left(\mathbb{S}^1\right)$ by identifying this group with the quotient of $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ by the closed normal subgroup consisting of integer translations.

Proposition 5.4. *Set* $\mathsf{H} = \mathbb{S}^1$ *and* $\mathsf{G} = \mathrm{AC}_+\left(\mathbb{S}^1\right)$ *and let* $\Omega : \mathsf{H} \times \mathsf{K}_\mathsf{G} \to \mathsf{G}$ *be the bijection from Notation 2.9. Then the unique topology on* G *which makes* Ω *a homeomorphism also makes* G *a Polish group, and the metric* $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$ *is compatible with this topology.*

Proof. In the present notation $\mathsf{K}_\circ = \mathsf{K}_\mathsf{G}$. Let

$$\mathsf{K}_\infty = \left\{\mathsf{k} \in \mathrm{Homeo}_+\left(\mathbb{S}^1\right) \mid \mathsf{k}(1) = 1\right\}$$

let

$$\Phi : \mathsf{K}_\mathsf{G} \times \mathsf{H} \to \mathsf{K}_\infty \times \mathsf{H}$$

be the inclusion, and let

$$\Psi : \mathsf{K}_\infty \times \mathsf{H} \to \mathsf{H}$$

be evaluation. Lemma 2.5 implies that the inclusion $\mathsf{K}_\mathsf{G} \to \mathsf{K}_\infty$ is continuous so also Φ is continuous, and by Proposition 2.13 (with $\mathsf{H} = \mathbb{S}$ and $\mathsf{G} = \mathrm{Homeo}_+\left(\mathbb{S}^1\right)$) Ψ is continuous. Evaluation $\mathsf{K}_\mathsf{G} \times \mathsf{H} \to \mathsf{H}$ is the composition $\Psi \circ \Phi$, and so it is continuous. The function in Lemma 5.2 is continuous and descends to a continuous function $\mathsf{K}_* \times \mathbb{S}^1 \to \mathsf{K}_*$, so

$$\mathsf{K}_\mathsf{G} \times \mathsf{H} \to \mathsf{K}_\mathsf{G}, (k, r) \mapsto \lambda_{k(r)}^{-1} \circ k \circ \lambda_r$$

is continuous. By Proposition 2.13 the topology that makes Ω a homeomorphism also makes G a Polish group. A compatible metric d on G is given by

$$d(f, g) = d_{\mathbb{S}^1}(f(1), g(1)) + d_{\mathrm{AC}}(f, g)$$

for all $f, g \in \mathsf{G}$.

The argument that d and $d_\infty + d_{\mathrm{AC}}$ induce the same topology on G works similarly as in the proof of Proposition 5.3. □

The application of Corollary 4.3 with $\mathsf{G} = \mathrm{AC}_\mathbb{Z}^{\mathrm{loc}}(\mathrm{I})$ is basically the same as with $\mathsf{G} = \mathrm{Homeo}_\mathbb{Z}(\mathbb{R})$ of Example 4.7. With $\mathsf{G} = \mathrm{AC}_\mathbb{Z}^{\mathrm{loc}}(\mathbb{R})$ the isotropy subgroup K_G is isomorphic to the coarsely bounded group $\mathrm{AC}_+(\mathrm{I})$ so $\mathbb{R} \to \mathrm{AC}_\mathbb{Z}^{\mathrm{loc}}(\mathbb{R}), r \mapsto \tau_r$ is a quasi-isometry. By applying the corollary with $\mathsf{G} = \mathrm{AC}_+(\mathbb{S}^1)$ we get a quasi-isometry $\mathbb{S}^1 \to \mathrm{AC}_+(\mathbb{S}^1)$. As $\mathbb{S}^1$ is compact it is coarsely bounded and so $\mathrm{AC}_+(\mathbb{S}^1)$ is coarsely bounded as well. Thus we have completed the proof of Theorem 1.3.

In the general theory of (6), if G is a Polish group which is generated by a coarsely bounded subset then there is always a metric on G which is simultaneously compatible with the topology, right-invariant, and realizes the quasi-isometry type of G. In closing we complete the proof that $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$ is such a metric on $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$.

Proposition 5.5. *The metric* $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$ *on* $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ *is a representative of the quasi-isometry type of* $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$*.*

Proof. For all $\mathsf{r}, \mathsf{s} \in \mathbb{R}$

$$\mathsf{d}_\infty(\tau_\mathsf{r}, \tau_\mathsf{s}) + \mathsf{d}_{\mathrm{AC}}(\tau_\mathsf{r}, \tau_\mathsf{s}) = |\mathsf{r} - \mathsf{s}|$$

so $\mathsf{r} \mapsto \tau_\mathsf{r}$ defines an isometric embedding of $\mathbb{R}$ with its standard metric into $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ with the metric $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$. As $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R}) = \mathcal{T} \circ \mathsf{K}_*$ and K_* is bounded in $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$ it follows that the isometric embedding of $\mathbb{R}$ into $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ is coarsely onto, and so the metric $\mathsf{d}_\infty + \mathsf{d}_{\mathrm{AC}}$ on $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$ represents the quasi-isometry of type of $\mathrm{AC}^{\mathrm{loc}}_{\mathbb{Z}}(\mathbb{R})$. $\square$

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