Strong Generators in $\mathbf{D_{perf}}(\mathrm{X})$ for Schemes with a Separator

by

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THESIS

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"One thing Grothendieck said was that one should never try to prove anything that is not almost obvious" - Allyn Jackson, *"Comme Appelé du Néant— As If Summoned from the Void: The Life of Alexandre Grothendieck"*.

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For my old friends and family, living in Brasil and Canada, and the new ones made in the United States.

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CONTRIBUTIONS OF AUTHORS

After brief introduction in Chapter 5, we prove some properties about the derived category $\mathbf{D_{qc}}(\mathsf{X})$ needed prove Lemma 5.0.2 and Proposition 5.0.3

Done by the author is Proposition 5.0.6, which states that we can pullback a compact generator from a quasicompact, separated scheme $\mathsf{X}_{\mathsf{sep}}$ to X being quasicompact quasiseparated scheme via the separator morphism, granted the scheme X satisfies Hypothesis 5.0.5. This hypothesis motivated Definition 5.0.4, about how the open affines $\mathsf{V} \subset \mathsf{X}$ from a cover of X should contain the closed subset of non separated points Z_{V}.

Later, Proposition 5.0.7 is proved, this forms the hypothesis that is used to prove the main theorem.

Finally, the author concludes with the main Theorem 5.0.8, which proves the assertion made that $\mathbf{D_{perf}}(\mathsf{X})$ is regular if and only if X can be covered by affines of finite global dimension, given that X admits a separator and satisfies Hypothesis 5.0.5.

In Chapter 6 we briefly discuss further directions for this theory.

Most of the work done in this thesis is based on the article (26, Jatoba) published online by the author of this thesis at arXiv with free access.

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CHAPTER 1

INTRODUCTION

This chapter is based on the article (26, Jatoba).

1.1 History

In 2003, Bondal and Van den Bergh in (2, Theorem 2.2), defined what it means for an object to (strongly) generate a triangulated category. The definitions was inspired by the close relation between certain types of triangulated categories having a strong generator and being saturated, i.e., that every contravariant cohomological functor of finite type to vector spaces is representable. Categories that admit a strong generator were then called *regular*.

Moreover, in the same article Bondal and Van den Bergh showed that whenever X is a smooth variety, $\mathbf{D}_{\mathrm{perf}}(\mathsf{X})$ is regular if and only if, X can be covered by open affine subschemes $\mathsf{Spec}(\mathsf{R}_i)$ with each R_i of finite global dimension. It was then asked by Bondal and Van den Bergh if one could generalize the condition over the scheme to be quasicompact and separated.

We should point out that this theorem started with the affine case proved by Kelly (3), see also Street (4). The result was rediscovered by Christensen (5) and later Rouquier (7).

Over the next decade, several steps followed in this direction. First, the case where X is regular and of finite type over a field k was proved by both Orlov (6, Theorem 3.27), and Rouquier (7), Theorem 7.38,. This last paper from Rouquier is also responsible for the generality of the following important theorem.

Theorem 1.1.1 (Rouquier)**.** *Let* R *be a neotherian, commutative ring. Let* $\mathcal{T}$ *be a regular triangulated category proper over* R*, and suppose that* $\mathcal{T}$ *is idempotent complete. Then an* R*-linear functor* $\mathsf{H} : \mathcal{T} \to \mathsf{R} - \mathsf{Mod}$ *is representable if and only if*

i) H is homological, and

ii) for any object $\mathsf{X} \in \mathcal{T}$*, the direct sum* $\oplus_{i=-\infty}^{\infty} \mathsf{H}(\Sigma^i \mathsf{X})$ *is a finite* R*-module.*

This motivate finding examples of regular, idempotent complete triangulated categories proper over a noetherian ring R. In particular, it is a well-known fact that the category $\mathbf{D}_{\text{perf}}(\mathsf{X})$, for X a quasicompact, quasiseparated scheme, is idempotent complete.

In 2017, Neeman (1) proved the Bondal and Van den Bergh conjecture, i.e.

Theorem 1.1.2 (Neeman)**.** *Let* X *be a quasicompact, separated scheme. Then* $\boldsymbol{D}_{\textit{perf}}(\mathsf{X})$ *is regular if and only if,* X *can be covered by open affine subschemes* $\mathsf{Spec}(\mathsf{R}_i)$*, with each* R_i *of finite global dimension.*

Remark 1.1.3. One direction of the Theorem 1.1.2 has been proven in full generality. If $\mathbf{D}_{\text{perf}}(\mathsf{X})$ is regular, one may show that if $\mathsf{U} = \mathsf{Spec}(\mathsf{R})$ is any open affine subscheme of X, then R is of finite global dimension. This claim follows by Thomason and Trobaugh (8), which shows that the restriction functor $j^* : \mathbf{D}_{\text{perf}}(\mathsf{X}) \to \mathbf{D}_{\text{perf}}(\mathsf{U})$ is the idempotent completion of the Verdier quotient map, that is, there is a factorization of j^* by a Verdier localization functor $\mathsf{V} : \mathbf{D}_{\text{perf}}(\mathsf{X}) \to \mathsf{S}^{-1}\mathbf{D}_{\text{perf}}(\mathsf{X})$, where the $\mathsf{Ker}(\mathsf{V})$ is the strictly full saturated subcategory whose objects are 0 outside of U, and then the idempotent completion$\tilde{}$: $\mathsf{S}^{-1}\mathbf{D}_{\text{perf}}(\mathsf{X}) \to \mathsf{S}^{-1}\widetilde{\mathbf{D}_{\text{perf}}}(\mathsf{X}) = \mathbf{D}_{\text{perf}}(\mathsf{U})$.

If $G \in \mathbf{D}_{\mathrm{perf}}(X)$ is a strong generator, then $j^*G \in \mathbf{D}_{\mathrm{perf}}(U)$ is also a strong generator. By (7, Theorem 7.25), this implies that R must be of finite global dimension.

One might ask if the separated condition could be weakened to quasiseparated. As shown above, one of the main applications involves idempotent complete triangulated categories, and $\mathbf{D}_{\mathrm{perf}}(X)$ is an idempotent complete triangulated category, for X a quasicompact, quasiseparated scheme.

This paper gives one step in this direction, extending Theorem 1.1.2. We show that for quasicompact, quasiseparated scheme that admits a *separator* - with some extra condition for the subscheme of non separated points - the theorem holds.

Theorem 1.1.4. *Let X be a quasicompact, quasiseparated scheme that admits a separator $f : X \to X_{sep}$ and a cover by affines U_λ such that the closed subscheme of non separated points Z_{U_λ} is contained in $f^{-1}(f(U_\lambda))$. Then $\boldsymbol{D}_{perf}(X)$ is regular if and only if, X can be covered by open affine subschemes $\mathrm{Spec}(R_i)$ with each R_i of finite global dimension.*

A separator is a morphism $f : X \to X_{sep}$ with some universal property from a quasicompact, quasiseparated scheme X to a particular quasicompact separated scheme X_{sep}, introduced by Ferrand and Khan in (9) and discussed here in Chapter 3. It encapsulates the idea of "gluing" open sets from a quasiseparated scheme in such a way that the image is separated. If the scheme is already separated, the separator is an isomorphism. Because of this, for any open affine $U \subset X$, a point $x \in U$ and the induced morphism $f^{-1}(f(U)) \to f(U)$, one may have a fiber of x being more than just x, i.e., $\{x\} \subsetneq f^{-1}(f(U))$. We call x a *non separated point* of U and the schematically closure in X of all non separated points of U we call Z_U. The hypothesis is that Z_U must be in $f^{-1}(f(U))$. More on that will be discussed in Chapter 5.

One direction of the proof of theorem 1.1.4 is identical to the Remark 1.1.3, so it remains to show that $\mathbf{D}_{\mathrm{perf}}(X)$ is regular if X can be covered by affines of finite global dimension. With the assumption of the existence of a $\mathsf{separator}$, the main idea is to pullback the strong generator from the separated scheme and show that it is again a strong generator in $\mathbf{D_{qc}}(X)$. Not all quasiseparated schemes admits a separator, but several examples may be found in (9).

CHAPTER 2

BACKGROUND

This work will need some background knowledge on algebraic geometry and category theory and is based on the article (26).

2.1 Algebraic Geometry

We start the introduction of the algebraic geometry theory. Mostly of what is here can be found in (10, Sections 4, 5) or in (9, Apendix A).

2.1.1 Morphisms of schemes

We start by showing some definitions and properties of what we will use throughout this work. We assume the reader is familiar with the concepts of what is a *scheme*, some topological properties - such as: what is an *open affine, an open subscheme, a closed subscheme*, being *quasicompact* - what is a *morphism of schemes* and the *diagonal morphism.* For those definitions, we refer to books such as Hartshorne (11), Görtz and Wedhorn (12), Atiyah (13) and Eisenbud (14).

Proposition 2.1.1. *Let* $\mathsf{f} : \mathsf{X} \to \mathsf{S}$ *be a morphism of schemes and* $\Delta_{\mathsf{f}} : \mathsf{X} \to \mathsf{X} \times_{\mathsf{S}} \mathsf{X}$ *the diagonal morphism. Then the diagonal morphism is an immersion, that is, there exist an open* $\mathsf{U} \subset \mathsf{X} \times_{\mathsf{S}} \mathsf{X}$ *containing the image* $\Delta_{\mathsf{f}}(\mathsf{X})$ *such that* Δ_{f} *induces a closed immersion* $\mathsf{X} \to \mathsf{U}$.

Definition 2.1.2. Let $f : X \to S$ be a morphism of schemes. We say that f is *separated* if the diagonal morphism Δ_f is a closed immersion, that is, that the image of $\Delta_f(X) \subset X \times_S X$ is closed. A scheme X is called *separated* if the morphism $X \to \mathrm{Spec}(\mathbf{Z})$ is separated.

Example 2.1.3. Affine schemes are separated (10, 5.2.2). Moreover, any open subscheme of a separated scheme is separated.

Definition 2.1.4. Let $f : X \to S$ be a morphism of schemes. The morphism f is called *quasiseparated* if the diagonal morphism is *quasicompact*. A scheme X is called *quasiseparated* if the morphism $X \to \mathrm{Spec}(\mathbf{Z})$ is quasiseparated.

Lemma 2.1.5. *Let $f : X \to S$ be a morphism of schemes. Then:*

- *If X is a separated scheme, then all morphisms $f : X \to S$ are separated.*

- *If X is a quasiseparated scheme, then all morphisms $f : X \to S$ are quasiseparated.*

- *If S is a separated scheme and f is a separated morphism, then X is separated.*

- *If S is a quasiseparated scheme and f is a quasiseparated morphism, then X is quasiseparated.*

Proof. Consider the following Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] & X \times_S X \arrow[r, "\delta"] \arrow[d] & X \times X \arrow[d, "f\times f"] \\ & S \arrow[r, "\Delta_S"] & S \times S. \end{tikzcd}$$

The first two statements follows from the fact that both δ and Δ_S are immersions. For the last two statements, suppose Δ_S, and δ, are closed immersion (or for the quasiseparated case, both are quasicompact). Therefore, Δ_f has the same property, as the composition $\delta \circ \Delta_f = \Delta_X$ has the respective property.

$\square$

It is a known fact that for $S = \mathsf{Spec}(R)$ for some ring R, then $f : X \to S$ is separated if and only if, for every open affine $U, V \subset X$ the intersection $U \cap V$ is also an affine open, the induced $R-$algebra morphism of sections $\phi_{UV} : \Gamma(U) \otimes_R \Gamma(V) \to \Gamma(U \cap V)$ is surjective.

Now that we have the tools at our disposal, we can start discussing the properties of the morphisms we will be working with. Recall that the main idea is to use properties of morphisms to obtain a desired family of schemes on some derived category of $\mathcal{O}_X-$modules. The morphism we will be working with will be called a separator, which is a quasiseparated, quasicompact morphism that is a *local isomorphism.* What is left for us to define and discuss is what is a local isomorphism and we will follow (10, 4.4) and (9, Apendix A).

Definition 2.1.6. A morphism of schemes $f : X \to Y$ is called *local isomorphism* if for every point $p \in X$, there exist an open $U \subset X$ such that $p \in U$ and f induces an open immersion $U \to Y$.

The idea behind a *local isomorphism* is exactly of gluing together two open sets, that is, if U and V are opens of X, then the induced morphisms of f to each open is an isomorphism of U to $f(U)$ and V to $f(V)$ and their image are "glued" along the open $f(U) \cap f(V)$ which contain $f(U \cap V)$.

By (10, 6.2.1), any local isomorphism is open, flat and locally of finite presentation. A sufficient and necessary condition is that if $f : X \to Y$ is locally of finite presentation and for every point $p \in X$, the morphism between the stalks $\theta_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is an isomorphism, then f is a local isomorphism.

We also have the following proposition

Proposition 2.1.7 ((9, Proposition A.3.1))**.** *Let* $f : X \to Y$ *be a separated local isomorphism. If* f *induces an injection over all maximum points of* X*, then* f *is an open immersion.*

Definition 2.1.8 ((10, Definition 5.4))**.** A morphism $f : X \to Y$ is said to be *schematically dense* if $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is injective.

If a morphism $f : X \to Y$ is schematically dense, then it is *dense*, i.e., the image $f(X)$ is dense in Y. Any immersion that is schematically dense is an open morphism. Any closed immersion that is schematically dense is an isomorphism.

Let $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ be a quasicompact, quasiseparated morphism. The direct image $\mathsf{f}_*(\mathcal{O}_\mathsf{X})$ is a quasicoherent $\mathcal{O}_\mathsf{X}$-algebra. The ideal $\mathcal{I}$ from the canonical morphism $\mathcal{O}_\mathsf{Y} \to \mathsf{f}_*(\mathcal{O}_\mathsf{X})$ is also quasicoherent. $\mathcal{I}$ defines a closed subscheme $\mathsf{Z} \subset \mathsf{Y}$ which motivates the following definition.

Definition 2.1.9. The schematic closure of a quasicompact, quasiseparated morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$, or of X in Y, is the closed $\mathsf{Z} \subset \mathsf{Y}$ defined by the ideal $\mathcal{I}$. There is a factorization

$$\mathsf{X} \xrightarrow{\nu} \mathsf{Z} \xrightarrow{u} \mathsf{Y}$$

where u is a closed immersion and ν is schematically dense.

An important property of schematic closure is that if the morphism is quasicompact, quasiseparated, then the schematic closure commutes with flat base change. Another known property of schematic closure is the universal property which we state without proof as follows:

Lemma 2.1.10 (Uniqueness of Schematically Closure)**.** *Consider the following commutative square of schemes*

$$\begin{tikzcd} \mathsf{X} \arrow[r, "\nu"] \arrow[d, "\nu'"] & \mathsf{Z} \arrow[d, "u"] \\ \mathsf{Z}' \arrow[r, "u'"] & \mathsf{Y}. \end{tikzcd}$$

where u and u' are closed immersion defined by $\mathcal{I}$ and $\mathcal{I}'$ ideals of $\mathcal{O}_\mathsf{Y}$ respectively. If the morphism ν is schematically dense, then $\mathcal{I}' \subset \mathcal{I}$ and there exist a morphism $w : \mathsf{Z} \to \mathsf{Z}'$ making the two triangles commute.

We end this section by giving two different ways to check when the diagonal Δ_f is schematically dense, which will be used in Chapter 3.

Proposition 2.1.11 ((9, Proposition 2.2.1))**.** *Let* $f : X \to Y$ *be a quasiseparated local isomorphism. Then the diagonal* $\Delta_f : X \to X \times_Y X$ *is schematically dense if and only if, for all open* $U \subset X$ *such that the restriction of* f *is an open immersion on* Y*, the morphism* $U \to f^{-1}(f(U))$ *is schematically dense.*

Proposition 2.1.12. *Let* X *be a scheme. Let* $U \subset X$ *be an open subscheme. If the inclusion morphism* $U \to X$ *is quasicompact, then U is scheme theoretically dense in X if and only if the scheme theoretic closure of U in X is X.*

With that, we conclude the algebraic geometry part of this section.

2.2 Category Theory

2.2.1 Derived Category

The following sections will be devoted to explaining the background in category theory needed to understand this work. We start with the derived category of an abelian category $\mathcal{A}$ which will be a category where we invert a collection of morphisms. The notation used in this section follows the one of (15).

Definition 2.2.1 ((16, Gabriel and Zisman))**.** Let $\mathcal{C}$ be a category and $\mathcal{S}$ a collection of morphism in $\mathcal{C}$. Then, there exist a category $S^{-1}\mathcal{C}$ and a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ such that $\mathrm{Obj}(\mathcal{C}) = \mathrm{Obj}(S^{-1}\mathcal{C})$

(i) If $s \in S \subset \mathrm{Mor}(\mathcal{C})$, then $F(s)$ is invertible.

(ii) Any functor $F'' : \mathcal{C} \to \mathcal{B}$ with $F''(S)$ contained in the isomorphisms of $\mathcal{B}$, factors uniquely as $\mathcal{C} \xrightarrow{F} S^{-1}\mathcal{C} \xrightarrow{F'} \mathcal{B}$.

Care must be taken when dealing with categories of the type $S^{-1}\mathcal{C}$, since morphisms are composable strings, whose pieces are either an element of $\mathrm{Mor}(\mathcal{C})$ or the inverse of some element of S. This increases the size of $\mathrm{Mor}(S^{-1}\mathcal{C})$, which does not need to be small.

We refer to (17, Hartshorne) or (18, Verdier) for the original presentations, or for a more recent approach (19, Gelfand and Manin), (20, Kashiwara and Schapira) or (21, Weibel).

Definition 2.2.2. Let $\mathcal{A}$ be an abelian category. The derived category $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$ is defined as follows:

a) The objects are cochain complexes in $\mathcal{A}$, that is diagrams in $\mathcal{A}$ of the form

$$\ldots \to A^{n-1} \xrightarrow{d_{n-1}} A^{n} \xrightarrow{d_n} A^{n+1} \to \ldots$$

where the composites of the morphisms $d_i : A_i \to A_{i+1}$ all vanish for all $i \in \mathbf{Z}$, i.e., $d_{i+1} \circ d_i = 0$.

b) Morphisms in $\mathbf{D}^{\mathfrak{C}'}_{\mathfrak{C}}(\mathcal{A})$ are homotopy classes of cochain maps $f^\bullet : A^\bullet \to B^\bullet$, that is a commutative diagram

$$\begin{tikzcd} A^i \arrow[r, "d_i"'] \arrow[d, "f^i"'] & A^{i+1} \arrow[d, "f^{i+1}"] \\ B^i \arrow[r, "d_i"] & B^{i+1} \end{tikzcd}$$

satisfying the homotopy restriction and such that the objects $\mathsf{A}^\bullet$ and $\mathsf{B}^\bullet$ are in $\mathbf{D}_{\mathfrak{C}}^{\mathfrak{C}'}(\mathcal{A})$ and moreover we formally invert *quasi-isomorphism*, i.e. the maps that induces isomorphisms in cohomology.

The subscript $\mathfrak{C}$ and superscript $\mathfrak{C}'$ stand for conditions.

Example 2.2.3. Now let X be a scheme. Consider the abelian category of sheaves of $\mathcal{O}_\mathsf{X}$-modules. Then $\mathbf{D}(\mathcal{O}_\mathsf{X}\text{-modules})$ is the *derived category of sheaves of $\mathcal{O}_\mathsf{X}$-modules*, which we shorten as $\mathbf{D}(\mathsf{X})$ for convenience.

We then construct some more derived category by adding restrictions to $\mathbf{D}(\mathsf{X})$.

Example 2.2.4. Let $\mathbf{D}_{\mathbf{qc}}(\mathsf{X})$ be the derived category of objects with quasicoherent cohomology, that is, the objects in $\mathbf{D}_{\mathbf{qc}}(\mathsf{X})$ are the objects of $\mathbf{D}(\mathsf{X})$ (the cochain complexes of $\mathcal{O}_\mathsf{X}$-modules) with the condition that the cohomology must be quasicoherent.

Example 2.2.5. Let $\mathbf{D}_{\mathbf{perf}}(\mathsf{X})$ be the derived category where the objects are the perfect complexes. A cochain complex of $\mathcal{O}_\mathsf{X}$-modules is *perfect* if it is locally isomorphic to a bounded complex of vector bundles.

2.2.2 Triangulated Category

The derived category is an example of a *triangulated category*, which is already endowed with extra desired structures, such as the shifting endofunctor and that short exact sequences form long exact sequence in cohomology. The theory of triangulated categories is rich and we shall be brief for the purpose of this work, since we will not be using the specifics of triangulated category theory, but more about the consequences that being triangulated implies. For instance, since the derived categories we will be working with are triangulated categories, we will have a description of what a compact object is and moreover the compact objects will be exactly the perfect complexes.

Definition 2.2.6. A triangulated category $\mathcal{T}$ is an additive category with the following extra structure:

(i) An invertible additive endofunctor $[1] : \mathcal{T} \to \mathcal{T}$.

(ii) A collection of *exact triangles*, also called distinguished triangles, which in our case are diagrams in $\mathcal{T}$ of the form

$$\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z} \xrightarrow{h} \mathsf{X}[1]$$

This should satisfies the following axioms:

- **[TR1:]** Any triangle isomorphic to a distinguished triangle is a distinguished triangle. For any object $\mathsf{X} \in \mathcal{T}$ the diagram

$$0 \xrightarrow{0} \mathsf{X} \xrightarrow{\mathrm{id}} \mathsf{X} \xrightarrow{0} 0$$

  is an exact triangle. Any morphism $f : \mathsf{X} \to \mathsf{Y}$ may be completed to an exact triangle

$$\mathsf{X} \xrightarrow{f} \mathsf{Y} \xrightarrow{g} \mathsf{Z} \xrightarrow{h} \mathsf{X}[1].$$

- **[TR2:]** Any rotation of an exact triangle is exact. That is:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is an exact triangle if and only if the following triangle also is

$$Y \xrightarrow{-g} Z \xrightarrow{-h} X[1] \xrightarrow{-f[1]} Y[1].$$

- **[TR3+4:]** Given a commutative diagram, where the rows are exact triangles,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] & X[1] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

we may complete it to a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d, dashed] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd} .$$

Moreover: we can do it in such a way that the following is an exact triangle

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -g & 0 \\ b & f' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -h & 0 \\ c & g' \end{pmatrix}} X[1] \oplus Z' \xrightarrow{\begin{pmatrix} -f[1] & 0 \\ a[1] & h'[1] \end{pmatrix}} Y[1] \oplus X'[1].$$

To show that the derived category is a triangulated category is a long, although simple, problem. We will show what is the endofunctor $[1]$, but for the exact triangles and why they satisfies the axioms **TR1** to **TR4** we referred to (15).

Example 2.2.7. The endofunctor $[1]$ in $\mathbf{D}(X)$, also called the shifting or suspension functor, is the functor $[1] : A^\bullet \to A[1]^\bullet$ defined as taking the cochain complex

$$\ldots \to A^{-2} \xrightarrow{d_{-2}} A^{-1} \xrightarrow{d_{-1}} A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \to \ldots$$

to

$$\ldots \to A^{-1} \xrightarrow{-d_{-1}} A^0 \xrightarrow{-d_0} A^1 \xrightarrow{-d_1} A^2 \xrightarrow{-d_2} A^3 \to \ldots$$

That is, we shift the complex to the left by one, $A[1]^n = A^{n+1}$, and the maps all change sign. For the morphisms, $(f[1])^\bullet$ also follows the shift, not changing the map degree wise.

This defines $[1]$ only for the cochain maps. To extend to arbitrary morphisms in $\mathbf{D}(X)$, we must use the universal property of the localization of the derived category.

Remember that $\mathbf{D}(X) = S^{-1}\mathcal{C}$, where $\mathcal{C}$ is the category with same objects as $\mathbf{D}(X)$, morphisms are cochain maps up to homotopy and S is the class of morphisms that induces isomorphisms in cohomology, namely quasi-isomorphism. The shifting functor $[1]$ takes S to itself, since if s is a quasi-isomorphism, then $s[1]$ will also be a quasi-isomorphism. We consider the composition $\mathcal{C} \xrightarrow{[1]} \mathcal{C} \xrightarrow{F} S^{-1}\mathcal{C}$. By the universal property of the localization there exists an unique map, represented by the dotted arrow below, such that the following diagram commute

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "[1]"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "F"] \\
S^{-1}\mathcal{C} \arrow[r, dashed] & S^{-1}\mathcal{C}
\end{tikzcd} .$$

Therefore, we define $[1] : \mathbf{D}(X) \to \mathbf{D}(X)$ to be this unique map.

Notice that the same construction gives us the shifting functor for the other two examples of derived category: $\mathbf{D_{perf}}(X)$ and $\mathbf{D_{qc}}(X)$.

From now on, we will also call exact triangles or distinguish triangles as just *triangles*. We also will define another endofunctor as following:

Definition 2.2.8. Let $\mathcal{T}$ be a triangulated category. The endofunctor $[n]$ will be defined as $[n] = [1]^n : \mathcal{T} \to \mathcal{T}$.

For the introduction of a compact generator we shall define what is a triangulated full subcategory.

Definition 2.2.9. A *full subcategory* $\mathcal{S} \subset \mathcal{T}$ is called triangulated if $0 \in \mathcal{S}$, if $\mathcal{S}[1] = S$, and if, whenever $X, Y \in \mathcal{S}$ and there exists in $\mathcal{T}$ a triangle $X \to Y \to Z \to X[1]$, we must also have $Z \in \mathcal{S}$. The subcategory $\mathcal{S}$ is *thick* if it is triangulated, as well as closed in $\mathcal{T}$ under direct summands.

With the needed structures of triangulated category defined, we move to compact generators. Compact generators are the foundation of this work, as stated in the Section 1.1.

Definition 2.2.10. Let $\mathcal{T}$ be a triangulated category with coproducts. An object $G \in \mathcal{T}$ is compact if the functor $\mathrm{Hom}(G, -)$ respects coproducts, i.e., the map

$$\coprod_{i\in I} \mathrm{Hom}_{\mathcal{T}}(G, E_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}\left(G, \coprod_{i\in I} E_i\right)$$

is a bijection, for any set I and objects $E_i \in \mathrm{Ob}(\mathcal{T})$ parametrized by $i \in I$. A set of compact objects $\{G_i, i \in I\}$ is said to generate the category $\mathcal{T}$ if the following equivalent conditions hold:

(i) If $X \in \mathcal{T}$ is an object, and if $\mathrm{Hom}(G_i, X[n]) = 0$ for all $i \in I$ and all $n \in Z$, then $X = 0$.

(ii) If a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ is closed under coproducts and contains the objects $\{G_i, i \in I\}$, then $S = T$.

If the category T contains a set of compact generators it is called *compactly generated.*

The equivalence from the Definition 2.2.10 above is not obvious, although it is an standard result. In particular, we will be focusing on categories where the class of compact generators $\{G_i, i \in I\}$ is only a singleton $\{G\}$. We show now that the categories we have introduced before are compactly generated.

Example 2.2.11. Consider the affine scheme $X = \mathrm{Spec}(R)$ for some ring R. We will call $\mathbf{D_{qc}}(X) = \mathbf{D_{qc}}(R) = \mathbf{D_{qc}}(R-\mathrm{mod})$ where we view the abelian category $\mathcal{O}_X - \mathrm{mod}$ as $R - \mathrm{mod}$.

The category $\mathbf{D_{qc}}(R)$ has coproducts: one can show that the coproduct of a family of cochains parametrized by $\lambda \in \Lambda$

$$\ldots \to A_\lambda^{-2} \to A_\lambda^{-1} \to A_\lambda^{0} \to A_\lambda^{1} \to A_\lambda^{2} \to \ldots$$

is

$$\ldots \to \coprod_{\lambda \in \Lambda} A_\lambda^{-2} \to \coprod_{\lambda \in \Lambda} A_\lambda^{-1} \to \coprod_{\lambda \in \Lambda} A_\lambda^{0} \to \coprod_{\lambda \in \Lambda} A_\lambda^{1} \to \coprod_{\lambda \in \Lambda} A_\lambda^{2} \to \ldots$$

If we consider the functor that takes $R-$mod to the zero degree cochain and look at the image of R - viewed as R-mod - we obtain the object in $\mathbf{D_{qc}}(R)$

$$\ldots \to 0^{-2} \to 0^{-1} \to R^0 \to 0^1 \to 0^2 \to \ldots$$

which we will also call as $R \in \mathbf{D_{qc}}(R)$.

The Hom functor $\mathrm{Hom}(R, -)$ is isomorphic in $\mathbf{D_{qc}}(R)$ to the 0-th degree cohomology functor, that is $\mathrm{Hom}_{\mathbf{D}(R)}(R, -) = \mathrm{H}^0(-)$. It is known that $\mathrm{H}^0(-)$ respects coproducts. Hence, R is a compact object.

We observe that for any object $X \in \mathbf{D_{qc}}(R)$, $\mathrm{Hom}_{\mathbf{D_{qc}}(R)}(R, X[n]) = \mathrm{H}^n(X)$, for all $n \in \mathbf{Z}$. Hence, if $\mathrm{H}^n(X) = 0$ for all $n \in \mathbf{Z}$, X is acyclic, i.e. is quasi-isomorphic to 0. Therefore, $X = 0$ in $\mathbf{D_{qc}}(R)$ and R is a compact generator by Definition 2.2.10.

So we conclude that $\mathbf{D_{qc}}(R)$ is compactly generated by the single compact object $R \in \mathbf{D_{qc}}(R)$.

It should be clear that the formula used in $\mathbf{D_{qc}}(R)$ for coproducts does not work for $\mathbf{D_{perf}}(X)$, since the arbitrary coproducts may not be perfect anymore. In fact, that is not an isolated case, $\mathbf{D_{perf}}(X)$ does not have coproducts. Conversely, $\mathbf{D_{qc}}(R)$ has coproducts and in fact is compactly generated by R.

Even more can be said about $\mathbf{D_{qc}}(\mathsf{X})$. If X is a quasicompact, quasiseparated scheme, then the category $\mathbf{D_{qc}}(\mathsf{X})$ also has a single compact generator. This is a theorem, proved in (2, Theorem 3.1.1(ii)).

Remark 2.2.12. Let $\mathcal{T}$ be a triangulated category with coproducts. We denote by $\mathcal{T}^c$ the full subcategory whose objects are the compact objects in $\mathcal{T}$. In the particular case where $\mathcal{T} = \mathbf{D_{qc}}(\mathsf{X})$, for X quasicompact quasiseparated, the category $\mathcal{T}^c$ is exactly $\mathbf{D_{perf}}(\mathsf{X})$ of Example 2.2.5. This is not obvious, but can be found in (2, Theorem 3.1.1(i))

Example 2.2.11 is another good illustration of the big picture that encapsulates the philosophy behind this work. First some result about compact objects of the derived category of an affine scheme was proven. Since compact objects in that category are precisely the perfect complexes in $\mathbf{D_{qc}}(\mathsf{R})$, it was expected to extend the result to the case where the scheme is quasicompact, quasiseparated. This is because the "ideal" place where results of perfect complexes should exist is for quasicompact, quasiseparated schemes, once the affine case is proven. This is only an idea, not a rule.

CHAPTER 3

SEPARATOR MORPHISM

This chapter is based on the article (26).

A separator of a morphism $f : T \to S$ is another morphism h, which is universal among morphisms from T to separated S-schemes E. This section will follow (9), which contains in-depth explanations and further properties of separators and local isomorphisms.

Definition 3.0.1. Let $f : T \to S$ a morphism of schemes. A *separator* of f, or a *separator through* f, is a morphism of S-schemes $h : T \to E$, with E separated over S, such that the following propreties are satisfied:

i) h is a quasicompact, quasiseparated, surjective local isomorphism (2.1.6) and

ii) the diagonal morphism Δ_h is schematically dense (2.1.8).

If $S = \mathsf{Spec}(\mathbf{Z})$, we call h a separator of T. We observe that

(a) Morphism $f : T \to S$ that admits a separator is quasiseparated. This follows from the fact that the diagonal Δ_f factors as

$$T \xrightarrow{\Delta_h} T \times_E T \xrightarrow{u} T \times_S T$$

where u is the induced morphism from the base change. Since Δ_h is quasicompact, h is quasiseparated and u is a closed immersion, the composition is quasicompact.

(b) If T is integral, than property *(i)* of Definition 3.0.1 implies property *(ii)* of Definition 3.0.1.

The separator has several desired properties, all of which have an in-depth explanation in (9). We are only interested in the following:

Proposition 3.0.2. *Let* $\mathsf{f} : \mathsf{T} \to \mathsf{S}$ *be a morphism and* $\mathsf{T} \xrightarrow{\mathsf{h}} \mathsf{E} \xrightarrow{g} \mathsf{S}$ *a separator of* f.

i) Let U *be an open of* T *that is separated over* S. *Then the restriction of* h *induces an isomorphism of* U *to* $\mathsf{h}(\mathsf{U})$. *In particular,* $\mathsf{h}(\mathsf{U})$ *is open and if* T *is already separated,* h *is an isomorphism.*

ii) (Universal Property) For all S*-morphisms* $\mathsf{h}' : \mathsf{T} \to \mathsf{E}'$ *with* E' *separated over* S*, there exists a unique* S*-morphism* $\mathsf{u} : \mathsf{E} \to \mathsf{E}'$ *such that* $\mathsf{h}' = \mathsf{u}\mathsf{h}$.

Proof.

i) Let U be an open of T that is separated over S. First notice the morphism $\mathsf{U} \to \mathsf{E}$ induced from h is a separated morphism, since both U and E are, and h is quasi-separated. Second, by the definition of a separator, Δ_{h} is schematically dense. By 2.1.11, this implies that the restriction of h to all maximal points of T is injective. Hence by Proposition 2.1.7, h is an open immersion.

ii) Let $\mathsf{h}' : \mathsf{T} \to \mathsf{E}'$ be an S-morphism with E' separated over S. Then, there exists a commutative diagram

$$\begin{tikzcd} T \arrow[r, "\Delta_h"] \arrow[d, "\Delta_{h'}"'] & T \times_E T \arrow[d, "\phi"] \arrow[dl, dashed, "\exists w"] \\ T \times_{E'} T \arrow[r, "\phi'"'] & T \times_S T \end{tikzcd}$$

where the morphisms ϕ, ϕ' are closed immersions, since both E and E' are separated over S. Since Δ_h is schematic dense by assumption, the conditions on both ϕ and ϕ', the requirements for the existence of w are met by the Uniqueness of Schematic Closure 2.1.10.

Hence the diagram

$$\begin{tikzcd} T \times_E T \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "w"'] & T \arrow[r, "h"] \arrow[d, equal] & E \arrow[d, dashed, "u"] \\ T \times_{E'} T \arrow[r, shift left] \arrow[r, shift right] & T \arrow[r, "h'"'] & E' \end{tikzcd}$$

commutes, and the result follows.

$\square$

We give a simple, but important example which will guide our intuition throughout this work.

Definition 3.0.3. Let $U = Spec(k[t])$ and $V = Spec(k[s])$. Let $O_1 = D(t)$ and $O_2 = D(s)$ be open subschemes of U and V respectively. The affine line with double origin, named as $\mathbb{A}^1_d$, is the scheme obtained by glueing U and V over $O_1 \cong O_2$ via the isomorphism $k[t, \frac{1}{t}] \cong k[u, \frac{1}{u}]$ given by $t \mapsto u$.

Consider the affine line with double origin, which is quasicompact and quasiseparated. One may ask the question: “What morphism of schemes from $\mathbb{A}^1_{\mathsf{d}}$ should be universal over separated schemes?”. A nice guess would be the projection morphism to the affine line. We shall give a reasoning for why this should be the case.

Example 3.0.4. It is well known that $\mathbb{A}^1_{\mathsf{d}}$ is quasicompact and quasiseparated. There is an induced projection map $\pi : \mathbb{A}^1_{\mathsf{d}} \to \mathbb{A}^1$ that maps both origin from $\mathbb{A}^1_{\mathsf{d}}$ to the origin of $\mathbb{A}^1 = \mathsf{Spec}(\mathsf{k}[x])$. This map is clearly quasicompact, quasiseparated (since both schemes are) and surjective (is the left inverse of the open immersion $\mathfrak{i} : \mathbb{A}^1 \to \mathbb{A}^1_{\mathsf{d}}$). It is also clear that any point has an open (affine) neighbourhood that is isomorphic to the image $\pi(\mathsf{U})$, as we can take the open to be the same as either U or V from the definition 3.0.3. So π is a local isomorphism.

It remains to show that the diagonal Δ_π is schematically dense. But this comes from the fact that the inclusion of $\Delta_\pi(\mathbb{A}^1_{\mathsf{d}})$ in $\mathbb{A}^1_{\mathsf{d}} \otimes_{\mathbb{A}^1} \mathbb{A}^1_{\mathsf{d}}$ is quasicomapct and that $\mathbb{A}^1_{\mathsf{d}} \otimes_{\mathbb{A}^1} \mathbb{A}^1_{\mathsf{d}}$ is an affine line with quadruple origin. Now, the schematic closure of $\Delta_\pi(\mathbb{A}^1_{\mathsf{d}})$ is $\mathbb{A}^1_{\mathsf{d}} \otimes_{\mathbb{A}^1} \mathbb{A}^1_{\mathsf{d}}$, so Δ_π is schematic dense, by Proposition 2.1.12.

Another way is to use Proposition 2.1.11. The map π defined as the left inverse of the inclusion $\mathfrak{i} : \mathbb{A}^1 \to \mathbb{A}^1_{\mathsf{d}}$ satisfies the hypothesis of the proposition. Moreover, we may take U to be affine and check that $\mathsf{U} \to \pi^{-1}(\pi(\mathsf{U}))$ is schematically dense. Again, we use Proposition 2.1.12, since the closure of any open containing one origin must contain the other, and if U is an open not containing any of the origins, then π is the identity map. Hence, the diagonal is schematically dense. Therefore π is the separator of $\mathbb{A}^1_{\mathsf{d}}$.

We give another point of view on the same example.

Remark 3.0.5. Recall the Valuation Criteria for Separateness. Let $f : X \to S$ be a morphism of schemes. Assume that f is quasicompact and quasiseparated. Given any commutative solid diagram

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where A is a valuation ring with field of fractions K,there exists a unique dotted arrow making the diagram commute if and only if f is separated.

Example 3.0.6. We show that any map from $\mathbb{A}^1_d$ to a separated scheme must factor through $\mathbb{A}^1$. As stated in the above remark 3.0.5, the key is to use the valuation criteria. Assume we have a morphism $f : \mathbb{A}^1_d \to S$, for S some separated scheme. Then, since $\mathbb{A}^1_d$ is quasiseparated and quasicompact, there should have at least two different maps h and g making the following diagram commute

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathbb{A}^1_d \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "h", "g"'] & \mathsf{Spec}(\mathbf{Z}) \end{tikzcd}$$

Further composing this diagram with f, we obtain

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathbb{A}^1_d \arrow[r, "f"] \arrow[d] & S \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "h", "g"'] & \mathsf{Spec}(\mathbf{Z}) \arrow[r, "\cong"] & \mathsf{Spec}(\mathbf{Z}) \end{tikzcd}$$

Since S is separated, the two images of the point $\mathsf{Spec}(\mathsf{K})$ in $\mathbb{A}^1_{\mathsf{d}}$ induced by g and h should be equal after composing f, i.e., $\mathsf{f} \circ \mathsf{g}(\mathsf{Spec}(\mathsf{K})) = \mathsf{f} \circ \mathsf{h}(\mathsf{Spec}(\mathsf{K}))$ where we made a slight abuse of notation for the maps $\mathsf{Spec}(\mathsf{K}) \to \mathbb{A}^1_{\mathsf{d}}$. In particular, the image of the two origins should coincide in S. Therefore the map f should factor through $\pi : \mathbb{A}^1_{\mathsf{d}} \to \mathbb{A}^1$.

We conclude that $\pi : \mathbb{A}^1_{\mathsf{d}} \to \mathbb{A}^1$ *could* be the separator. Notice that this is not enough to show that it is indeed the separator. Later we will give an example of a morphism that satisfies the universal property, but is *not* a separator.

The good part of this example is that it gives a clear idea of a separator, both computationally as given in Example 3.0.4 and through intuition as given by Example 3.0.6. Moreover, we should not forget that the idea of a separator is to glue together two affines over the intersection, which is basically how $\mathbb{A}^1_{\mathsf{d}}$ is defined. Hence to get the separator of $\mathbb{A}^1_{\mathsf{d}}$ we should "just" finish gluing the two affines.

Finally, it is important to understand when a separator exists. The following theorem will give some criteria to work with

Theorem 3.0.7. *Let* $\mathsf{f} : \mathsf{T} \to \mathsf{S}$ *be a quasiseparated morphism, and let* $\mathsf{T}_1 \subset \mathsf{T} \times_{\mathsf{S}} \mathsf{T}$ *be the schematic closure of the diagonal morphism* $\Delta_{\mathsf{f}} : \mathsf{T} \to \mathsf{T} \times_{\mathsf{S}} \mathsf{T}$*. Then,* f *admits a separator* h *if and only if, every irreducible component of* T *is locally finite over* S *- i.e., every point of* T *has an open neighborhood which is disjoint to all but finitely many irreducible components of* T *- and both the composition morphisms induced by the projections*

$$T_1 \to T \times_S T \rightrightarrows T$$

are flat and of finite type.

The proof is in (9, Theorem 5.1.1)

Example 3.0.8. The schematic closure T_1 of $\Delta(\mathbb{A}^1_d)$ over $f : \mathbb{A}^1_d \to \mathsf{Spec}(\mathbf{Z})$ is the affine line with quadruple origin. Each projection is of finite type, so remains to show that it is flat which is a local condition.

Let U_1, V_1 be the affine cover of the first $\mathbb{A}^1_d$ that comes from the definition of how $\mathbb{A}^1_d$ is constructed in 3.0.3 and U_2,V_2 the second copy $\mathbb{A}^1_d$. Since $U_i \times V_j$, for $i, j = \{1, 2\}$ is a cover for $\mathbb{A}^1_d \times \mathbb{A}^1_d$, we should check flatness for projection for each one of the four opens of the cover. But by definition of how $\mathbb{A}^1_d$ is constructed, each one of $T_1 \cap (U_i \times V_j)$ is isomorphic to $\mathbb{A}^1$, each intersection selecting one of the four origins. It is clear that the projections are flat for those cases.

Therefore, $f : \mathbb{A}^1_d \to \mathsf{Spec}(\mathbf{Z})$ has a separator. Notice that this does not tell us what map is the separator, it only gives us that the separator exists.

Corollary 3.0.9. *Let T be a quasiseparated S-scheme where each irreducible component is locally finite. Then T admits a separator $h : T \to E$ if and only if, for all affine opens U, V of T, the scheme $U \cup V$ admits a separator.*

Proof. For the necessary condition, it suffices to show that T has a cover by affine opens U_λ such that the union of every two opens in the cover $U_\lambda \cup U_\mu$ admits a separator.

First, let $U, V \subset T$ be affine opens. Since T is quasi-separated, the intersection of any affine open with $U \cup V$ is quasi-compact. Recall that a subset Z of a topological space X is said to be *retrocompact* if $Z \cap U$ is quasi-compact for every quasi-compact open subset U of X. So $U \cup V$ is retrocompact in T.

Hence, it suffices to show that for all retrocompact open $U \subset T$, $h(U)$ is open and the morphism $U \to h(U)$ is a separator of U. That $h(U)$ is open in E follows from the fact that U, by hypothesis, is retrocompact. It remains to show that $h(U)$ is separated over S. Since h is a local isomorphism, we have the induced isomorphism $h' : U \to h(U)$, which induces the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Delta_{h'}"] \arrow[d, "i"'] & U \times_{h(U)} U \arrow[r, "u"] & U \times_E U \arrow[d, "i\times i"] \\ T \arrow[r, "\Delta_h"'] & T \times_E T \arrow[r, equal] & T \times_E T \end{tikzcd}$$

where u is an isomorphism, since $h(U) \to E$ is an immersion. Since $i \times i$ is an open immersion and Δ_h is quasi-compact and schematic dense, $\Delta_{h'}$ is also schematically dense. Finally, for h' to be a separator, it remains to show that it is quasi-compact. But h' can be expressed as the composition of two quasi-compact morphisms, i.e.,

$$U \xrightarrow{i'} h^{-1}(h(U)) \xrightarrow{h} h(U)$$

where i' is the open immersion induced by the inclusion i. Therefore the condition is necessary.

Next, notice that $(U \cup V) \times (U \cup V) \subset T \times T$ is the union of four canonical opens, namely, $U \times U, V \times V, U \times V, V \times U$. Let T_1 be the schematic closure of the diagonal in $T \times T$. Then both $U \times U$ and $V \times V$ are isomorphic via the projection to U and V respectively in T, hence flat and of finite type. It suffices to work with $U \times V$. Let $W = T_1 \cap (U \times V)$. By Theorem 3.0.7, both projections $d_1 : W \to U$ and $d_0 : W \to V$ are flat and of finite type. Since T is quasi-separated, the open immersions $U \to T$ and $V \to T$ are (flat and) of finite type. The open sets $U \times V$, with U and V affines, cover $T \times T$, so the two projections of T_1 to T are flat and of finite type and, again, from Theorem 3.0.7, the corollary follows. □

Since we can not use the Corollary 3.0.9 on the $\mathbb{A}^1_d$ case, as $\mathbb{A}^1_d$ is already the union of two known affines, we shall show without proof some examples taken from (9), Ferrand and Kahn.

Example 3.0.10. Here is a list of examples that do admit a separator:

i) Every regular locally Neotherian scheme of dimension 1 admits a separator, for instance if T is a Neotherian dedekind scheme over $\mathtt{Spec}(\mathbf{Z})$ (9)[5.3.3].

ii) If $f : T \to S$ is étale of finite presentation and S is normal, then f admits a separator.

iii) Any normal scheme of finite type over a Noetherian ring admits an open subscheme containing all points of codimension 1 and this subscheme has a separator (9, Example 6.1.1).

Sections 7 and 8 of (9) have some examples of schemes that do not admit a separator.

Example 3.0.11. The following are examples of schemes that do not admits a separator.

i) (9, Example 7.3.1) Let P be a discrete valuation ring with field of fraction K. Let $a \mapsto \overline{a}$ be the quotient map from P to the residual field $k = P/\mathfrak{m}$. Suppose there exists *two different* field extensions k_1 and k_2 such that $[k : k_1] = [k : k_2] = 2$. Let $R_i \subset P$, for $i = 1, 2$ formed by $a \in P$ such that $\overline{a} \in k_1$ and $\overline{a} \in k_2$. One can show that following properties

- The integral closure of A_i is P, for $i = 1, 2$.

- The homomorphism $A_1 \otimes A_2 \to P$, defined by $a \otimes b \mapsto ab$ is surjective.

Let T be the scheme defined by gluing $U_i = \mathsf{Spec}(A_i)$ over the open $U_0 = \mathsf{Spec}(K)$. Let x_1 and x_2 be the two closed points such that $j_{x_i}(\mathcal{O}_{x_i}) = A_i$, where $j_x : \mathcal{O}_x \to K$ is the (injective) homomorphism induced by restriction. By construction, both local rings A_i are isomorphic (in P) to each other, but they are not equal.

Define T_1 to be the schematic closure of the diagonal and $d_0, d_1 : T_1 \rightrightarrows T$ be the projections. Then d_0 and d_1 are flat if and only if for every couple of points x, y of T, the local rings $j_x(\mathcal{O}_x)$ being isomorphic to $j_y(\mathcal{O}_y)$ implies that they are equal (9, Example 7.2.1). Hence, by what was shown in the previous paragraph, one of the diagonals is not flat. Therefore T does not admits a separator. After further inspection, one can show that $i_T : T \to \mathsf{Spec}(\Gamma(T))$ is the map that factors any other map from T to a separated scheme. By the universal property of the separator, i_T should be the separator of T. But what fails is that i_t is neither a local isomorphism, nor it is flat.

ii) There exist a morphism $\mathsf{T} \to \mathsf{S}$ smooth, quasicompact, with S being regular affine and of dimension $\mathsf{1}$ that does not admits a separator. (9, Example 8.2.1)

iii) There exist an *étale* morphism $\mathsf{f} : \mathsf{T} \to \mathsf{S}$, with S local integral noetherian scheme of dimension $\mathsf{1}$ that does not admits a separator. (9, Example 8.3.1).

CHAPTER 4

DERIVED CATEGORY

4.1 Strong Generators of a Triangulated Category

We begin with some definitions, terminology and key properties of a strongly generated category. Most of what is written here follows the first few chapters of (1).

Definition 4.1.1. Let $\mathcal{T}$ be a triangulated category and $\mathsf{G} \in \mathcal{T}$ an object. The full subcategory $\langle \mathsf{G} \rangle_n \subset \mathcal{T}$ is defined inductively as follows:

i) $\langle \mathsf{G} \rangle_1$ is the full subcategory consisting of all direct summands of finite coproducts of suspensions of G.

ii) For $n > 1$, $\langle \mathsf{G} \rangle_n$ is the full subcategory consisting of all objects that are direct summand of an object y, where y fits into a triangle $x \to y \to z$, with $x \in \langle \mathsf{G} \rangle_1$ and $z \in \langle \mathsf{G} \rangle_{n-1}$.

Definition 4.1.2. Let G be an object in a triangulated category $\mathcal{T}$. Then G is said to be a *classical generator* if $\mathcal{T} = \cup_{n=1}^{\infty} \langle \mathsf{G} \rangle_n$ and a *strong generator* if there exists an $n \in \mathbf{Z}_{\geq 1}$ with $\mathcal{T} = \langle \mathsf{G} \rangle_n$.

Definition 4.1.3. A triangulated category $\mathcal{T}$ is called *regular* or *strongly generated* if a strong generator exists.

Remark 4.1.4.

- One might also say that a regular category $\mathcal{T}$ is built from G in finitely many steps.

- In (1), a general discussion about different properties of triangulated category, such as being *proper* or *idempotent complete* follows. It also gives insight about the importance of studying such objects.

Definition 4.1.5. Let $\mathcal{T}$ be a triangulated category with coproducts, $\mathsf{G} \in \mathcal{T}$ an object and $A < B$ integers. Then $\overline{\langle \mathsf{G} \rangle}_n^{[A,B]} \subset \mathcal{T}$ is the full subcategory defined inductively as follows:

i) $\overline{\langle \mathsf{G} \rangle}_1^{[A,B]}$ is the full subcategory of T consisting of all direct summands of arbitrary coproducts of objects in the set $\{\Sigma^{-i}\mathsf{G}, A \leq i \leq B\}$.

ii) $\overline{\langle \mathsf{G} \rangle}_n^{[A,B]}$ is the full subcategory consisting of all objects that are direct summand of an object y, where y fits into a triangle $x \to y \to z$, with $x \in \overline{\langle \mathsf{G} \rangle}_1^{[A,B]}$ and $z \in \overline{\langle \mathsf{G} \rangle}_{n-1}^{[A,B]}$.

The difference between the categories $\langle \mathsf{G} \rangle_n$ and $\overline{\langle \mathsf{G} \rangle}_n^{[A,B]}$ is that $\overline{\langle \mathsf{G} \rangle}_n^{[A,B]}$ allows arbitrary coproducts, but restrict the allowed suspensions to a fixed range from A to B.

4.2 Operations between subcategories

There are several ways to create a new subcategory from others. Some of them will be defined in this section, which follows (1).

Definition 4.2.1. Let $\mathcal{T}$ be a triangulated category with $\mathcal{A}$ and $\mathcal{B}$ two subcategories of $\mathcal{T}$. Then:

i) $\mathcal{A} \star \mathcal{B}$ is the full subcategory of all objects y in T for which there exist a triangle $x \to y \to z$ with $x \in \mathcal{A}$ and $z \in \mathcal{B}$.

ii) $\mathsf{add}(\mathcal{A})$ is the full subcategory containing all finite coproducts of objects in $\mathcal{A}$.

iii) If $\mathcal{T}$ is closed under coproducts, then $\mathsf{Add}(\mathcal{A})$ is the full subcategory containing all (set-indexed) coproducts of objects in $\mathcal{A}$

iv) If $\mathcal{A}$ is also a full subcategory, then $\mathsf{smd}(\mathcal{A})$ is the full subcategory of all direct summands of objects in $\mathcal{A}$.

Note that the empty coproduct is 0, hence $0 \in \mathsf{add}(\mathcal{A}) \subset \mathsf{Add}(\mathcal{A})$ for any $\mathcal{A}$.

Definition 4.2.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{A}$ a subcategory. Define:

i) $\mathsf{coprod}_1(\mathcal{A}) := \mathsf{add}(\mathcal{A})$, and inductively as $\mathsf{coprod}_{n+1}(\mathcal{A}) := \mathsf{coprod}_1(\mathcal{A}) \star \mathsf{coprod}_n(\mathcal{A})$.

ii) $\mathsf{Coprod}_1(\mathcal{A}) := \mathsf{Add}(\mathcal{A})$, and inductively as $\mathsf{Coprod}_{n+1}(\mathcal{A}) := \mathsf{Coprod}_1(\mathcal{A}) \star \mathsf{Coprod}_n(\mathcal{A})$.

iii) $\mathsf{coprod}(\mathcal{A}) := \cup_{n=1}^{\infty} \mathsf{coprod}_n(\mathcal{A})$.

iv) If $\mathcal{T}$ has coproducts, define $\mathsf{Coprod}(\mathcal{A})$ to be the smallest strictly full subcategory of $\mathcal{T}$ containing $\mathcal{A}$ and satisfying

$$\mathsf{Add}(\mathsf{Coprod}(\mathcal{A})) \subset \mathsf{Coprod}(\mathcal{A}) \text{ and } \mathsf{Coprod}(\mathcal{A}) \star \mathrm{Coprod}(\mathcal{A}) \subset \mathsf{Coprod}(\mathcal{A}).$$

Remark 4.2.3. The diagram

$$\begin{array}{ccc} \mathsf{coprod}_n(\mathcal{A}) & \hookrightarrow & \mathsf{coprod}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathsf{Coprod}_n(\mathcal{A}) & \hookrightarrow & \mathsf{Coprod}(\mathcal{A}) \end{array}$$

commutes. Moreover, the associativity of the $\star$ operation gives that

$$\mathsf{coprod}_m(\mathcal{A}) \star \mathsf{coprod}_n(\mathcal{A}) = \mathsf{coprod}_{m+n}(\mathcal{A}),$$

$$\mathsf{Coprod}_m(\mathcal{A}) \star \mathsf{Coprod}_n(\mathcal{A}) = \mathsf{Coprod}_{m+n}(\mathcal{A}).$$

It can also be shown that $\mathsf{Coprod}_1(\mathsf{Coprod}_n(\mathcal{A})) = \mathsf{Add}(\mathsf{Coprod}_n(\mathcal{A})) = \mathsf{Coprod}_n(\mathcal{A})$, and $\mathsf{Coprod}_n(\mathsf{Coprod}_m(\mathcal{A})) \subset \mathsf{Coprod}_{nm}(\mathcal{A})$.

The following lemma may be found in (1, Lemma 1.7) and will be used once to prove the next corollary.

Lemma 4.2.4. *Let $\mathcal{T}$ be a triangulated category with coproducts, $\mathcal{T}^c$ be the subcategory of compact objects in $\mathcal{T}$, and let $\mathcal{B}$ be a subcategory of $\mathcal{T}^c$. Then*

(i) For $x \in \mathsf{Coprod}_n(\mathcal{B})$ and $s \in \mathcal{T}^c$, any map $s \to x$ factors as $s \to b \to x$ with $b \in \mathsf{coprod}_n(\mathcal{B})$.

(ii) For $x \in \mathsf{Coprod}(\mathcal{B})$ and $s \in \mathcal{T}^c$, any map $s \to x$ factors as $s \to b \to x$ with $b \in \mathsf{coprod}(\mathcal{B})$.

Proof. (1, Lemma 1.7) □

Corollary 4.2.5. *Let $\mathcal{T}$ be a triangulated category with coproducts, $\mathcal{T}^c$ be the subcategory of compact objects in $\mathcal{T}$, and let $\mathcal{B}$ be a subcategory of $\mathcal{T}^c$. Then*

(i) Any compact object in $\mathsf{Coprod}_n(\mathcal{B})$ belongs to $\mathsf{smd}(\mathsf{coprod}_n(\mathcal{B}))$.

(ii) Any compact object in $\mathsf{Coprod}(\mathcal{B})$ belongs to $\mathsf{smd}(\mathsf{coprod}(\mathcal{B}))$.

Proof. Let x be a compact object in $\mathsf{Coprod}_n(\mathcal{B})$. The identity map $1\colon x \to x$ is a morphism from the compact object x to $x \in \mathsf{Coprod}_n(\mathcal{B})$. By the previous Lemma 4.2.4, the morphism factors through an object $b \in \coprod_n(\mathcal{B})$. Thus x is a direct summand of b and the results follows.

The same proof holds be removing the subscript n, which proves item (ii). □

The next three results follow from these definitions with proofs found in the background section from (1).

Lemma 4.2.6. *Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{B}$ be an arbitrary subcategory. Then*

$$\mathsf{Coprod}_n(\mathcal{B}) \subset \mathsf{smd}(\mathsf{Coprod}_n(\mathcal{B})) \subset \mathsf{Coprod}_{2n}(\mathcal{B} \cup \Sigma\mathcal{B}).$$

Remark 4.2.7. Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{B} \subset \mathcal{T}$ be a subcategory. For any pair of integers $m \le n$ define

$$\mathcal{B}[m, n] = \bigcup_{i=-n}^{-m} \Sigma^i \mathcal{B}.$$

Corollary 4.2.8. *For integers* $N > 0, A \leq B$ *the identity* $\overline{\langle G \rangle}_N^{[A,B]} = \mathsf{smd}(\mathsf{Coprod}_N(G[A,B]))$ *always holds. Furthermore, one has the inclusions:*

$$\mathsf{Coprod}_N(G[A,B]) \subset \overline{\langle G \rangle}_N^{[A,B]} \subset \mathsf{Coprod}_{2N}(G[A-1,B]).$$

From these two results, one concludes that regarding finiteness conditions there is no loss in generality when working with $\mathsf{Coprod}_n(G[A,B])$ instead of $\overline{\langle G \rangle}_N^{[A,B]}$, and smd will not change the finiteness of the category generated by $G[A,B]$. So we may work with $\mathsf{Coprod}_n(G[A,B])$, which behaves well with smd and the $\star$ operations.

We end this section with a brief discussion about the $\mathsf{Coprod}_n(G[A,B])$ subcategory. As stated in [4.1.5], $\mathsf{Coprod}_n(G[A,B])$ is a full subcategory of T. Together with Remark 4.2.7, we may let the suspensions free by considering $\mathsf{Coprod}_n(G[-\infty,\infty])$. This means that any object $x \in \mathsf{Coprod}_n(G[-\infty,\infty])$ factors through $\mathsf{Coprod}_n(G[A,B])$ for integers A and B. As usual, it is possible that $\mathcal{T} = \mathsf{Coprod}_n(G[-\infty,\infty])$, which motivates the following definition.

Definition 4.2.9. Let $\mathcal{T}$ be a triangulated category with coproducts and $G \in \mathcal{T}$ an object in $\mathcal{T}$. Then, $\mathcal{T}$ is said to be *fast generated by* G if $\mathcal{T} = \mathsf{Coprod}_n(G[-\infty,\infty])$ for some $n \in \mathbf{N}$.

When G is a compact object, Corollary 4.2.8 tells us that if $\mathcal{T} = \overline{\langle G \rangle}_n^{[-\infty,\infty]}$, then $\mathcal{T}$ is fast generated. In this paper, we will always consider the case when G is a compact generator.

4.3 The $\mathbf{D_{qc}}(\mathsf{X_{sep}})$ case

Let $\mathsf{X_{sep}}$ be a quasicompact separated scheme. One may consider the category $\mathbf{D_{qc}}(\mathsf{X_{sep}})$, which is the unbounded derived category of cochain complexes sheaves of $\mathcal{O}_{\mathsf{X_{sep}}}-$modules with quasicoherent cohomology, and let $\mathbf{D_{perf}}(\mathsf{X})$ be the subcategory of compact objects.

Although the main result is about $\mathbf{D_{perf}}(\mathsf{X})$, the next result is the reason why we may work over the bigger triangulated category $\mathbf{D_{qc}}(\mathsf{X})$, which contains coproducts for X quasicompact, quasiseparated and moreover is compactly generated.

Proposition 4.3.1. *Let* X *be a quasicompact, quasiseparated scheme and* $\mathsf{G} \in \boldsymbol{D_{perf}}(\mathsf{X})$ *be a compact generator of* $\boldsymbol{D_{qc}}(\mathsf{X})$*. If* $\boldsymbol{D_{qc}}(\mathsf{X})$ *is fast generated by* G*, then* G *strongly generates* $\boldsymbol{D_{perf}}(\mathsf{X})$.

Proof. Let $\mathbf{D_{qc}}(\mathsf{X}) = \mathtt{Coprod}_n(\mathsf{G}[-\infty,\infty])$. Consider $\mathcal{B} = \{\Sigma^i \mathsf{G}, i \in \mathbf{Z}\}$. Then Corollary 4.2.5 gives that $\mathbf{D_{perf}}(\mathsf{X}) = \mathtt{smd}(\mathtt{coprod}_n(\mathcal{B}))$, which implies that G strongly generates $\mathbf{D_{perf}}(\mathsf{X})$. $\square$

The path should be clear by now. With the conditions of Theorem 1.1.4, if we show that $\mathbf{D_{qc}}(\mathsf{X})$ is fast generated by a compact generator, then by the above Proposition 4.3.1 the main result will follow.

We finish this section with two more results from (1) stated without proof.

Theorem 4.3.2 (Neeman)**.** *Let* $\mathsf{j} : \mathsf{V} \to \mathsf{X_{sep}}$ *be an open immersion of quasicompact, separated schemes, and let* G *be a compact generator for* $\boldsymbol{D_{qc}}(\mathsf{X_{sep}})$*. If* H *is any compact object* *of* $\boldsymbol{D}_{\boldsymbol{qc}}(\mathsf{V})$*, and we are given integers* $\mathsf{n}, \mathsf{a} \leq \mathsf{b}$*, then there exist integers* $\mathsf{N}, \mathsf{A} \leq \mathsf{B}$ *so that* $\mathit{Coprod}_{\mathsf{n}}(\boldsymbol{R}\mathsf{j}_*\mathsf{H}[\mathsf{a},\mathsf{b}]) \subset \mathit{Coprod}_{\mathsf{N}}(\mathsf{G}[\mathsf{A},\mathsf{B}])$*.*

Proof. (1, Theorem 6.2) □

Theorem 4.3.3 (Neeman)**.** *Let* $\mathsf{X}_{\mathsf{sep}}$ *be a quasicompact separated scheme. If* $\mathsf{X}_{\mathsf{sep}}$ *can be covered by affine subschemes* $\mathsf{Spec}(\mathsf{R}_{\mathsf{i}})$ *with each* R_{i} *of finite global dimension, then there exists a compact generator* G *that fast generates* $\boldsymbol{D}_{\boldsymbol{qc}}(\mathsf{X}_{\mathsf{sep}})$*, i.e.,* $\boldsymbol{D}_{\boldsymbol{qc}}(\mathsf{X}_{\mathsf{sep}}) = \mathsf{Coprod}_{\mathsf{n}}(\mathsf{G}[-\infty,\infty])$*.*

Proof. (1, Theorem 2.1) □

CHAPTER 5

GENERATORS FOR SCHEMES WITH SEPARATOR

This chapter is based on the article (26).

In this section we apply the constructions and results from the previous Chapters to conclude that $\mathbf{D_{qc}}(X)$ is fast generated as defined in 4.2.9, which by Proposition 4.3.1 will conclude this work.

Throughout this section, assume X to be a quasicompact, quasiseparated scheme with separator $f : X \to X_{sep}$. Without lost of generality, X may be written as $X = U \cup V$ with U and V quasicompact open subschemes of X. Let V to be affine and Z be the closed complement of U on X, i.e., $Z = X \backslash U$. Then $Z \subset V$ and we have the commuting diagram

$$\begin{tikzcd} Z \arrow[d, "c"] & \\ V \arrow[d, "i"] \arrow[dr, "j"] & \\ X \arrow[r, "f"] & X_{sep} \end{tikzcd}$$

where $c : Z \to V$ is a closed immersion and $i : V \to X$, $j : V \to X_{sep}$ are open immersions.

Remark 5.0.1. Some remarks about this chapter:

- Throughout this section, the index $[A, B]$ is omitted, as the range itself is not relevant for almost all proofs, only that it is finite. That means that $\mathrm{Coprod}_N(G[A, B])$ for some integers $A < B$ is written as $\mathrm{Coprod}_N(G)$.

- Unless otherwise specified, G will be a compact strong generator of $\mathbf{D_{qc}}(\mathsf{X}_{\mathsf{sep}})$, which exists by Theorem 4.3.3.

- Unless otherwise specified, all functors are derived.

Lemma 5.0.2. *Assume $\mathsf{X}_{\mathsf{sep}}$ to be a quasicompact, separated scheme and $\mathsf{V} \subset \mathsf{X}_{\mathsf{sep}}$ an open subscheme. For $\mathsf{P} \in \boldsymbol{D_{perf}}(\mathsf{V})$, let $\mathsf{j} : \mathsf{V} \to \mathsf{X}_{\mathsf{sep}}$ be the open immersion and G the compact strong generator of $\boldsymbol{D_{qc}}(\mathsf{X}_{\mathsf{sep}})$. Then the pushforward $\mathsf{j}_*\mathsf{P}$ is in Coprod$_{\mathsf{N}}(\mathsf{G})$.*

Proof. It follows from Theorem 4.3.2 that $\mathsf{j}_*(\mathsf{P}) \in \mathrm{Coprod}_{\mathsf{N}}(\mathsf{G})$. ☐

Proposition 5.0.3. *Let $\mathsf{P} \in \boldsymbol{D_{qc}}(\mathsf{V} on\ \mathsf{Z})$, $\mathsf{i} : \mathsf{V} \to \mathsf{X}$ and $\mathsf{j} : \mathsf{V} \to \mathsf{X}_{\mathsf{sep}}$ be the open immersions. Then $\mathsf{i}_*\mathsf{P}$ is a retract of $\mathsf{f}^*\mathsf{j}_*\mathsf{P}$.*

Proof. First, we show that $\mathsf{f}^*\mathsf{j}_*\mathsf{P}$ is supported on $\mathsf{Z}\coprod W$, by viewing Z as $\mathsf{Z} = \mathsf{X}\backslash\mathsf{U}$ and W is some closed subset of $\mathsf{U} \subset \mathsf{X}$. Now it suffices to show that the pullback of $\mathsf{f}^*\mathsf{j}_*\mathsf{P}$ to the intersection $\mathsf{U} \cap \mathsf{V}$ is zero.

Since the diagram

$$\begin{tikzcd} & \mathsf{V} \arrow[d, "\mathsf{i}"] \arrow[dr, "\mathsf{j}"] & \\ \mathsf{U} \cap \mathsf{V} \arrow[ur, "\mathsf{k}"] \arrow[r, "\mathsf{l}"] & \mathsf{X} \arrow[r, "\mathsf{f}"] & \mathsf{X}_{\mathsf{sep}} \end{tikzcd}$$

is commutative (where every map, except f is an open immersion), $\mathsf{Z} \nsubseteq \mathsf{U} \cap \mathsf{V}$, using the counit equivalence map one can see that

$$\begin{aligned} l^*f^*j_*P &\cong k^*j^*j_*P \\ &\cong k^*P \\ &\cong 0. \end{aligned}$$

Hence, $f^*j_*P \simeq R \oplus S$, with S supported on $W \subset U$ and R supported on $Z \subset V$. Finally, since V is separated and f is a local isomorphism on separated open subschemes, the restriction of f^*j_*P to V is i_*P, i.e., $i_*P \simeq R$.

Therefore $f^*j_*P \simeq i_*P \oplus S$ and the result follows. □

The next goal is to show that the pullback of a compact generator via a separator is again a compact generator. This will be done in several steps. First, notice that the isomorphism over separated opens property from the separator induces locally the notion of "*non separated points*". Those are the points in an open affine for which the separator is not an isomorphism.

Definition 5.0.4. Let X be quasicompact and quasiseparated. Suppose X admits a separator $f : X \to Y$ and $V \subset X$ be an open affine. Define Z_V as the closure of the set $\{x \in V; f^{-1}f(x) \neq \{x\}\}$, which we will call the *non separated points of V.*

We shall give some information about Z_V. First, notice that by the definition of the separator 3.0.1, the diagonal morphism Δ_f is schematically dense. Hence, we have that there exist an affine cover of X such that for any open affine U of the cover, the open immersion $U \to f^{-1}(f(U))$

is schematically dense (9, 2.2.1) . Since U and $f^{-1}f(U)$ are quasicompact, the open immersion is quasicompact. Hence, the scheme theoretical closure $\overline{U}$ in $f^{-1}(f(U))$ is $f^{-1}(f(U))$.

This implies that if there is a point $p \in Z_V$ that is not in V, then p is not a specialization of any point in V. We wish to not have such pathology, although we could not prove that this does not happen in full generality. Therefore, throughout this work, we add the following hypothesis to the separator.

Hypothesis 5.0.5. *Assume that the separator $f : X \to Y$ has the property that for some affine cover $\{V_\lambda\}$ of X, $Z_{V_\lambda} \subset f^{-1}(f(V_\lambda))$ for every V_λ.*

Recall that the localization sequence for $\mathbf{D_{qc}}(X)$ holds true for X quasicompact and quasiseparated, i.e, for $U \subset X$ quasicompact open and Z the closed complement, one have

$$\mathbf{D_{qc}}(X \textbf{ on } Z) \to \mathbf{D_{qc}}(X) \to \mathbf{D_{qc}}(U)$$

Moreover, $G \in \mathbf{D_{qc}}(X)$ is a compact generator if and only if, for any $F \in \mathbf{D_{qc}}(X)$, $\mathrm{Hom}_{\mathbf{D_{qc}}(X)}(G, F[n]) = 0$ implies $F[n] = 0$ for all $n \in \mathbf{Z}$.

With that in mind, it is possible to prove that the pullback via a separator of a compact generator is a compact generator.

Proposition 5.0.6. *Let $f : X \to Y$ be a separator satisfying condition 5.0.5. If $G \in \boldsymbol{D_{perf}}(Y)$ is a compact generator of $\boldsymbol{D_{qc}}(Y)$, then $f^*G \in \boldsymbol{D_{perf}}(X)$ is a compact generator of $\boldsymbol{D_{qc}}(X)$.*

Proof. Since X is quasicompact and quasiseparated, it suffices to show that the restriction of $f^*(G)$ is a generator for any affine open and use induction on the number of affines for any quasicompact open subscheme of X.

For the affine case consider the commutative diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ V \arrow[ur, hook, "i"] \arrow[r, "\mathrm{Id}"] & V \arrow[r, hook, "f|_V"'] & Y \end{tikzcd}$$

where $V \hookrightarrow X$ is an open affine.

The restriction to V is indeed a generator, since $f\big|_V$ is an isomorphism onto $f(V)$, i.e,

$$i^*f^*(G) = (f \circ i)^*(G) = (f\big|_V)^*G = G_{\mathbf{D_{qc}}(V)}$$

where $G_{\mathbf{D_{qc}}(V)}$ is the restriction of G to $\mathbf{D_{qc}}(V)$.

Notice this remains true if we replace V by any separated open subscheme of V, a fact that will be used soon.

Next, one proceeds with the case of some quasicompact, quasiseparated open subscheme, say $U \subset X$, by induction on the number of affines covering U. The case where U can be covered by only one affine is exactly the case above. So we may assume U may be covered by n affines and the property is true for any quasicompact, quasiseparated subscheme covered by up to $n-1$ affines.

Let V be some open affine from the cover of U and let W be the union of the other $n-1$ affines, i.e. $U = V \cup W$. Let $Z_V \subset V$ be the closed subscheme of non separated points of V. Let $L = V \cap W$. We have three morphisms $i : V \hookrightarrow U$, $j : W \hookrightarrow U$ and $k : L \hookrightarrow U$.

Let $F \in \mathbf{D_{qc}}(U)$ and consider the square

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D_{qc}}(U)}(f^*G, F) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{D_{qc}}(W)}(j^*f^*G, j^*(F)) \arrow[d] \\ \mathrm{Hom}_{\mathbf{D_{qc}}(V)}(i^*f^*G, i^*(F)) \arrow[r] & \mathrm{Hom}_{\mathbf{D_{qc}}(L)}(k^*f^*G, k^*(F)) \end{tikzcd} \tag{5.1}$$

in the derived category, where an abuse of notation was used for $f = f\big|_U$.

The goal is to show that the restriction of f^*G to each category is a compact generator. By the induction hypothesis, the restrictions of f^*G to W, V are generators in each respective derived category.

For the purpose of the proof, one may assume that $W \cap Z_V = \emptyset$, as one may consider the complement of $(Z_V)^c = U \backslash Z_V$ and define $\hat{W} = W \cap (Z_V)^c$. Indeed, with $\hat{j} : \hat{W} \hookrightarrow W$ the open immersion, it suffices to show that the further restriction $\hat{j}^* j^* f^* G$ is again a generator. But this comes from the fact that $\hat{j}_*$ is fully faithful, so $0 = \mathrm{Hom}(\hat{j}^* j^* f^* G, H) = \mathrm{Hom}(j^* f^* G, \hat{j}_* H)$ implies that $\hat{j}_* H = 0$, that is $H = 0$. Hence, even though $\hat{W}$ may be covered by more than $n-1$ affines, one may replace W for $\hat{W}$ and still get the same square of Homs as before, with the restriction of f^*G being a compact generator for $\hat{W}$.

Therefore, without loss of generality assume that $U = V \cup W$, with $W \cap Z_V = \emptyset$. In particular, $L \cap Z_V = \emptyset$. Now L is an open separated subscheme of the affine V, hence the restriction of f^*G to L is again a generator. Assume that $\mathrm{Hom}_{\mathbf{D_{qc}}(U)}(f^*G, F[n]) = 0$ for all $n \in \mathbf{Z}$. Then, the proof will follow if $F[n] = 0$.

By adjunction, $f_*F[n] = 0$ for all $n \in \mathbf{Z}$. Consider the localization sequence

$$\mathbf{D_{qc}}(U \textbf{ on } Z^L) \to \mathbf{D_{qc}}(U) \to \mathbf{D_{qc}}(L)$$

in the derived category over $k : L \hookrightarrow U$ with Z^L the complement of L, which induces the triangle

$$M \to F \to k_*k^*F,$$

for $M \in \mathbf{D_{qc}}(U \textbf{ on } Z)$ Applying f_*, one obtains the triangle

$$f_*M \to f_*F \to f_*k_*k^*F.$$

By the hypothesis, the middle term is zero, and hence $f_*M[1] \simeq f_*k_*k^*F$. The claim is that $k^*F = 0$. If that was not the case, the support of k^*F would not be empty, which would imply the existence of a point $p \in L$ such that $p \in \mathrm{Supph}(k^*F)$.

But L has no non separated points. In particular $\mathsf{f}^{-1}(\mathsf{f}(\mathsf{L})) = \mathsf{L}$, which implies that $\mathsf{f}_*\mathsf{k}_*\mathsf{k}^*\mathsf{F} = (\mathsf{f} \circ \mathsf{k})_*\mathsf{k}^*\mathsf{F}$ is supported on $\mathsf{f}(\mathsf{L}) \simeq \mathsf{L}$.

On the other hand $\mathsf{f}_*\mathsf{M}$ is not supported on $\mathsf{f}(\mathsf{L}) \simeq \mathsf{L}$. Therefore $\mathsf{k}^*\mathsf{F} \simeq 0$.

Going back to the square of morphism Equation 5.1, the top left term is 0 by hypothesis and the bottom right is also zero, since $\mathsf{k}^*\mathsf{F} = 0$. Hence the whole square is zero. That means that each restriction of F is zero, i.e., $\mathsf{j}^*(\mathsf{F}) = \mathsf{i}^*(\mathsf{F}) = \mathsf{k}^*(\mathsf{F}) = 0$ Using another pullback square, now for $\mathbf{D_{qc}}(\mathsf{U})$, one may glue each restriction back to F. Hence $\mathsf{F} = 0$ as desired. □

To prove the main theorem, a standard induction argument over the covering of X will be used. The following proposition will provide the induction hypothesis needed. The notation of subschemes and morphisms will follow the diagram shown in the beginning of this section.

Proposition 5.0.7. *Let X be a quasicompact and quasiseparated scheme that admits a separator $\mathsf{f} : \mathsf{X} \to \mathsf{X}_{\mathsf{sep}}$ satisfying condition 5.0.5. Assume X can be covered by affine subschemes $\mathsf{Spec}(\mathsf{R}_i)$ with each R_i of finite global dimension. Moreover, let $\mathsf{X} = \mathsf{U} \cup \mathsf{V}$ with U and V open subschemes of X and assume V to be affine. Consider the diagram*

$$\begin{tikzcd} \mathsf{U} \arrow[d, "u"] \arrow[dr, "o"] & \\ \mathsf{X} \arrow[r, "f"] & \mathsf{X}_{\mathsf{sep}} \end{tikzcd}$$

where $\mathsf{u} : \mathsf{U} \to \mathsf{X}$ is the open immersion and $\mathsf{o} : \mathsf{U} \to \mathsf{X}_{\mathsf{sep}}$ the induced map.

Let G be the strong generator of $\boldsymbol{D_{qc}}(\mathsf{X}_{\mathsf{sep}})$. Then $\mathsf{u}_*\mathsf{o}^*\mathsf{G} \in \mathsf{Coprod}_n(\mathsf{f}^*\mathsf{G})$.

Proof. First, by Proposition 3.0.2, X being covered by affine subschemes of finite global dimension implies X_{sep} also can be covered by affines with the same properties. Hence, by Theorem 4.3.3 there exists a G that fast generates $\mathbf{D_{qc}}(X_{sep})$.

Let G be the generator of $\mathbf{D_{qc}}(X_{sep})$ as above. One can fit f^*G into a triangle

$$Q \longrightarrow f^*G \longrightarrow u_*u^*f^*G \simeq u_*o^*G \ ,$$

so it suffices to show that $Q \in \mathrm{Coprod}_N(f^*G)$ for some $N \in \mathbf{Z}$.

Let Q be as above. Since Q vanishes on U, and V is assumed to be affine, by Thomason-Trobaugh there exists a closed subscheme $Z \subset V$ and $P \in \mathbf{D_{qc}}(V)$ such that $Q \simeq i_*P$.

By Theorem 4.3.3 there exists a fast generator $G' \in \mathbf{D_{qc}}(V)$. Hence, there exists $M \in \mathbf{Z}$, such that $P \in \mathrm{Coprod}_M(G')$.

So $Q \simeq i_*P$ is in $i_*\mathrm{Coprod}_M(G') \subseteq \mathrm{Coprod}_M(i_*G')$.

But, by Proposition 5.0.3, i_*G' is a retract of f^*j_*G', which again by Theorem 4.3.2 is in $\mathrm{Coprod}_N(f^*G)$ for some N. Hence $Q \in \mathrm{Coprod}_{MN}(f^*G)$, proving that u_*o^*G is indeed generated by f^*G. □

Now we move to the main Theorem 1.1.4. The goal is to show that one may pullback a fast generator via a separator and obtain a fast generator. By Proposition 4.3.1, this implies Theorem 1.1.4.

Theorem 5.0.8. *Let X be a quasicompact and quasiseparated scheme that admits a separator $f : X \to X_{sep}$ satisfying condition 5.0.5 and let G be a compact fast generator of $\boldsymbol{D_{qc}}(X_{sep})$.*

Assume X can be covered by affine subschemes $\mathsf{Spec}(R_i)$ with each R_i of finite global dimension. Then there exists an object H in $\boldsymbol{D_{perf}}(X)$ that fast generates $\boldsymbol{D_{qc}}(X)$.

Proof. First, we notice that by Proposition 5.0.6, f^*G is already a compact generator. Hence, it suffices to show that it is a fast generator. We proceed by induction on the number of affines in the cover of X to show that f^*G is indeed a fast generator.

The case $n = 1$ means that X is affine, hence separated. Therefore, the separator f is an isomorphism and $f^*G = G$.

Assume the theorem holds for any scheme which admits a cover by up to n affines $\mathsf{Spec}(R_i)$, each R_i with finite global dimensions. Suppose that X can be covered by $n+1$ affines $U_i = \mathsf{Spec}(R_i)$, each R_i with finite global dimensions, i.e., $X = \bigcup_{i=1}^{n+1} U_i$.

Let $U = \bigcup_{i=1}^{n} U_i$ and $V = U_{n+1}$, so $X = U \cup V$. Assume we are in the same situation as the previous diagrams, i.e., the following diagrams:

$$\begin{tikzcd} V \arrow[d, "i"'] \arrow[dr, "j"] & \\ X \arrow[r, "f"] & X_{sep}. \end{tikzcd} \qquad \begin{tikzcd} U \arrow[d, "u"'] \arrow[dr, "o"] & \\ X \arrow[r, "f"] & X_{sep}. \end{tikzcd}$$

Let $G \in \mathbf{D_{qc}}(X_{sep})$ be a fast generator. By Lemma 5.0.6, the restriction of f^*G to U, i.e. $u^*f^*(G) = o^*(G)$ is a compact generator. By induction hypothesis, there exists G' that fast generates $\mathbf{D_{qc}}(U)$. Since G' is compact and is in the subcategory generated by coproducts of $o^*(G)$, without loss of generality we may take o^*G to be the fast generator of $\mathbf{D_{qc}}(U)$. Now, by Proposition 5.0.7, there exist N such that $u_*o^*G \in \mathrm{Coprod}_N(f^*G)$.

In similar fashion, since V is an affine open from X, we may take j^*G as a fast generator of $\mathbf{D_{qc}}(V)$.

Using another localization sequence, one obtain the triangle

$$H \longrightarrow f^*G \longrightarrow i_*i^*f^*G = i_*j^*G$$

in $\mathbf{D_{qc}}(X)$ for H not supported in V. That implies that there exist some $P \in \mathbf{D_{qc}}(U)$ such that $H = u_*P$. By the previous paragraph, $\mathbf{D_{qc}}(U)$ is fast generated by $o^*(G)$, which implies that $H \in \mathsf{Coprod}_L(u_*o^*G)$, for some $L > 0$. Since $u_*o^*G \in \mathsf{Coprod}_N(f^*G)$, one obtains that $H \in \mathsf{Coprod}_{LN}(f^*G)$. Therefore, there exists $M > LN > 0$ such that $i_*j^*G \in \mathsf{Coprod}_M(f^*G)$.

Let $T = U \cap V$ and $t : T \to U$ be the inclusion. Then, any object $F \in \mathbf{D_{qc}}(X)$ fits in the triangle

$$u_*[t_*t^*u^*\Sigma^{-1}F] \longrightarrow F \longrightarrow u_*[u^*F] \oplus i_*[i^*F].$$

Thus, F belongs to $[u_*(\mathbf{D_{qc}}(U)] \star [u_*\mathbf{D_{qc}}(U) \oplus i_*\mathbf{D_{qc}}(V)]$ which is contained in

$$\mathsf{Coprod}_{MN}(f^*G) \star \mathsf{Coprod}_{MN}(f^*G) = \mathsf{Coprod}_{2MN}(f^*G)$$

Therefore $\mathbf{D_{qc}}(X)$ is fast generated and the result follows. $\square$

Theorem 5.0.8 together with Remark 1.1.3 prove the main Theorem 1.1.4, restated below

Theorem 5.0.9. *Let* X *be a quasicompact, quasiseparated scheme that admits a separator satisfying hypothesis 5.0.5. Then* $\boldsymbol{D}_{perf}(\mathsf{X})$ *is regular if and only if* X *can be covered by open affine subschemes* $\mathsf{Spec}(\mathsf{R_i})$ *with each* $\mathsf{R_i}$ *of finite global dimension.*

CHAPTER 6

FURTHER DIRECTIONS

In this final section we consider what should be the next extension of the constructions and results shown so far.

6.1 $\mathbf{D_{qc}}(\mathsf{X})$ for X quasicompact and quasiseparated

Although this work has given positive results for a family of schemes that are quasicompact and quasiseparated, the original question *"Is $\boldsymbol{D_{perf}}(\mathsf{X})$ strongly generated for any X quasicompact, quasiseparated and covered by affines, each of finite global dimension?"* is still unknown.

The idea that properties on derived category involving perfect objects should be extended to schemes that are quasicompact and quasiseparated is still what pushes this theory a bit further.

Important to note that Neeman (1) extended the result using that schemes can be approximated by the limit of neotherian schemes, as shown by Thomason and Trobaugh (8). By 2.1.5, if X is separated, then all maps are separated and therefore S is also separated. Neeman proceeds by approximating the schemes with an affine neotherian scheme - just like in Kelly(3) - hence the final scheme must be separated.

For this particular reason, Neeman's results implies the extension to *separated* schemes and the proof can not be easily adapted to quasiseparated schemes.

This work extend the result almost solely using properties of schemes and not using much of derived category. It is possible that the result might follow with the use of more categorical tools. Nevertheless it is good to know that the most basic examples, such as $\mathbb{A}^1_{\mathsf{d}}$, has a positive answer for the extension.

Another point is that the *separator* is somewhat "too strong" and the proof might be true in general if we loose a bit on the restriction of the map. Maybe one could have a family of morphisms such that the direct sum of the pullback of each morphism is a fast generator and the separator is the particular case where the family of morphism is a singleton.

There is a lot of work ahead in this theory and it is by no means near the end.

6.2 Extensions to other Derived Categories

Everything done here was for the particular case of $\mathbf{D_{perf}}(\mathsf{X})$. But the fact that $\mathbf{D_{qc}}(\mathsf{X})$ can be *fast generated* actually creates consequences further than only for $\mathbf{D_{perf}}(\mathsf{X})$. In fact, Neeman's paper (1) also proves that another category also may have a strong generator, namely $\mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X})$, as long as some extra conditions are satisfied.

Section 8 of (15) is devoted to explaining why, under some conditions on X, $\mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X}) \cong \mathbf{D_{perf}}(\mathsf{X})$, which would imply that $\mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X})$ is also regular. In fact the following is shown in the same paper.

Definition 6.2.1. Suppose X is a noetherian scheme, finite-dimensional, reduced and irreducible. A *regular alteration* of X is a generically finite, surjective morphism $\tilde{\mathsf{X}} \to \mathsf{X}$ with $\tilde{\mathsf{X}}$ regular.

Theorem 6.2.2 ((15, Theorem 6.11)). *Let* X *be a separated, noetherian, finite-dimensional scheme, and assume that every closed, reduced, irreducible subscheme of* X *has a regular alteration. Then the category* $\boldsymbol{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X})$ *is strongly generated.*

Regular alterations have some known theorems and are much more general than resolution of sigularities. For some work on regular alterations see de Jong [(24), (25)].

Theorem 6.2.2 is not supposed to be obvious, not even for the affine case. For a more detailed explanation about the history of this theorem and the consequences that puzzles algebraist to this day, we recommend the reading of (15, Historical Survey 6.12).

A natural question would be if the results from this paper - which already extended regularity in $\mathbf{D_{perf}}(\mathsf{X})$ for some family of quasicompact, quasiseparated schemes - could also extend for $\mathbf{D}^{\mathsf{b}}_{\mathsf{coh}}(\mathsf{X})$.

6.3 Triangulated Categories

Recently, Neeman in (15) studied when a triangulated category could be *approximable*. The idea behind is motived by how we can approximate *points* in metric spaces as limits of Cauchy sequences of simpler points. To better understand this topic, we briefly define some structures on a triangulated category. We start by defining what a $\mathsf{t}-$structure is.

Definition 6.3.1. A t–structure on a triangulated category T is a pair of full subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ satisfying:

(i) $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$

(ii) $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq 0}[1]$

(iii) $\mathrm{Hom}(\mathcal{T}^{\leq 0}[1], \mathcal{T}^{\geq 0}) = 0$

(iv) Every object $Y \in \mathcal{T}$ admits a triangle $A \to B \to C$ with $A \in \mathcal{T}^{\leq 0}[1]$ and $C \in \mathcal{T}^{\geq 0}$.

Let $\mathcal{T}$ be a category with $t-$structure. Neeman uses the $t-$structure as the "metric" to say when two *objects* are *close* inspired by the idea that two objects are close if they agree up to a small "difference". Hence, we say that x is close to y, for $x, y \in \mathcal{T}$, if there exists in $\mathcal{T}$ a triangle $x \to y \to z$, with $z \in \mathcal{T}^{-n}$ for some large n. Obviously this is not a metric, since it is not symmetric, maybe a map from y to x doesn't exist. Moreover, different $t-$structures leads to different "metrics" which leads to the concept of equivalent $t-$structures.

But to approximate via a metric, just like Taylor series uses polynomials to approximate functions, we also need to specify what would be our simpler objects. For several reasons, we will use a compact generator G. This all converge to the following definition of when a triangulated category $\mathcal{T}$ is called *approximable.*

Definition 6.3.2. A triangulated category $\mathcal{T}$ is said to be *approximable* if it has coproducts and there exists

(i) a compact generator G

(ii) a $t-$structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$

and these $t-$structure and generator can be chosen to satisfy

(iii) For some $n > 0$ we give $G[n] \in \mathcal{T}^{\leq <0}$ and $\mathrm{Hom}(G[-n], \mathcal{T}^{\leq 0}) = 0$.

(iv) In the metric induced by the $t-$structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, every object in $\mathcal{T}^{\leq 0}$ can be expressed as the limit of a sequence whose terms belong to $\cup_n \overline{\langle G \rangle}_n^{[-n,n]}$

More relations between compact generators and being approximable may be seen at (23), specially Remark 3.3. In (22), further discussion regarding $\mathbf{D_{perf}}(\mathsf{X})$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathsf{X})$ as one determining the other via a $\mathbf{t}-$structure construction is made, wrapping up everything discussed so far.

It turns out that because $\mathbf{D_{qc}}(\mathsf{X})$ is *fast generated* for X quasicompact, *separated*, then $\mathbf{D_{qc}}(\mathsf{X})$ is approximable. It would be then expected that this work would extend this result to X being quasicompact, quasiseparated satisfying Hypothesis 5.0.5 .

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