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ON MATRICES WITH SIGNED NULL-SPACES*

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Abstract. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A. A matrix A has *signed null-space* provided there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$ for each $\widetilde{A} \in \mathcal{Q}(A)$. Some properties of matrices with signed null-spaces are investigated.

Key words. totally L-matrices, signed compounds, signed null-spaces

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1. Introduction.

The *sign* of a real number a is defined by

$$\operatorname{sign}(a) = \begin{cases} -1 & \text{if } a < 0, \\ \ \ 0 & \text{if } a = 0, \text{ and} \\ \ \ 1 & \text{if } a > 0. \end{cases}$$

A *sign pattern* is a $(0, 1, -1)$-matrix. The *sign pattern of a matrix* A is the matrix obtained from A by replacing each entry with its sign. We denote by $\mathcal{Q}(A)$ the set of all matrices with the same sign pattern as A.

Let A be an m by n matrix and b an m by 1 vector. The linear system $Ax = b$ has *signed solutions* provided there exists a collection $\mathcal{S}$ of n by 1 sign patterns such that the set of sign patterns of the solutions to $\widetilde{A}x = \tilde{b}$ is $\mathcal{S}$ for each $\widetilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. This notion generalizes that of a sign-solvable linear system (see [1] and references therein). The linear system, $Ax = b$, is *sign-solvable* provided each linear system $\widetilde{A}x = \tilde{b}$ ($\widetilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$) has a solution and all solutions have the same sign pattern. Thus $Ax = b$ is sign-solvable if and only if $Ax = b$ has signed solutions and the set $\mathcal{S}$ has cardinality 1.

The matrix A has *signed null-space* provided $Ax = 0$ has signed solutions. Thus A has signed null-space if and only if there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of vectors in the null-space of $\widetilde{A}$ is $\mathcal{S}$ for each $\widetilde{A} \in \mathcal{Q}(A)$. An *$L$-matrix* is a matrix A, with the property that each matrix in $\mathcal{Q}(A)$ has linearly independent rows. A square L-matrix is a *sign-nonsingular* (SNS)-matrix. A *totally L-matrix* is an m by n matrix such that each m by m submatrix is an SNS-matrix. It is known that totally L-matrices have signed null-spaces [3]. We also have the fact as a corollary of some results in this paper. Thus matrices with signed null-spaces generalize totally L-matrices.

A vector is *mixed* if it has a positive entry and a negative entry. A matrix is *row-mixed* if each of its rows is mixed. A *signing* is a nonzero diagonal $(0, 1, -1)$-matrix.

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A signing is *strict* if each of its diagonal entries is nonzero. A matrix B is *strictly row-mixable* provided there exists a strict signing D such that BD is row-mixed.

In this paper, some properties of matrices with signed null-spaces are investigated, and we show that there exists an m by n matrix A with signed null-space such that A contains a totally L-matrix with m rows as its submatrix and the columns of A are distinct up to multiplication by -1 for any $n \in \{m, m+1, \ldots, 2m\}$.

We use the following standard notation throughout the paper. If k is a positive integer, then $\langle k\rangle$ denotes the set $\{1, 2, \ldots, k\}$. Let A be an m by n matrix. If α is a subset of $\{1, 2, \ldots, m\}$ and β is a subset of $\{1, 2, \ldots, n\}$, then $A[\alpha|\beta]$ denotes the submatrix of A determined by the rows whose indices are in α and the columns whose indices are in β. We sometimes use $A[*|\beta]$ instead of $A[\langle m\rangle|\beta]$. The submatrix complementary to $A[\alpha|\beta]$ is denoted by $A(\alpha|\beta)$. In particular, $A(-|\beta)$ denotes the submatrix obtained from A by deleting columns whose indices are in β. We write $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ for the n by n diagonal matrix whose (i, i)-entry is d_i. Let $J_{m,n}$ denote the m by n matrix, all of whose entries are 1, and let e_i denote the column vector, all of whose entries are 0 except for the ith entry, which is 1.

2. Matrices with signed null-space. We say that an m by n matrix $A = [a_{ij}]$ *contains a mixed cycle* provided there exist distinct $i_1, i_2, \ldots, i_k$ and distinct $j_1, j_2, \ldots, j_k$ such that

$$a_{i_t,j_t}a_{i_t,j_{t+1}} < 0 \text{ for } t = 1, \ldots, k-1 \text{ and } a_{i_k,j_k}a_{i_k,j_1} < 0.$$

An m by n $(0, 1, -1)$-matrix has *signed* m*th compound* provided each of its m by m submatrices having term rank m is an SNS-matrix.

We make use of the following results of matrices with signed null-spaces.

THEOREM 2.1 (see [3]). *Let A be a strictly row-mixable m by n matrix. Then the following three conditions are equivalent.*

(a) *A has signed null-space.*

(b) *A has term rank m, and its mth compound is signed.*

(c) *AD has no mixed cycle for each strict signing such that AD is row-mixed.*

THEOREM 2.2 (see [2], [3]). *If a strictly row-mixable matrix A has signed null-space, then there exist matrices B and C (possibly with no rows) and nonzero vectors b and c such that B and C are strictly row-mixable matrices with signed null-spaces,*

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

have signed null-spaces, and, up to permutation of rows and columns,

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix}.$$

The converse also holds.

Let A be an m by n $(0, 1, -1)$-matrix. The matrix B is *conformally contractible* to A provided there exists an index k such that the rows and columns of B can be permuted so that B has the form

$$\left[\begin{array}{c|c|c} A[\langle m\rangle|\langle n\rangle \setminus \{k\}] & x & y \\ \hline 0 \quad \cdots \quad 0 & 1 & -1 \end{array}\right],$$

where $x = [x_1, \ldots, x_m]^T$ and $y = [y_1, \ldots, y_m]^T$ are $(0, 1, -1)$-vectors such that $x_iy_i \geq 0$ for $i = 1, 2, \ldots, m$, and the sign pattern of $x + y$ is the kth column of A.

Let B be conformally contractible to A. It is known that A has signed null-space if and only if B has signed null-space [3]. All matrices we consider from now on are assumed to be $(0,1,-1)$-matrices.

THEOREM 2.3 (see [4]). *Let an m by n matrix A have a k by $k+1$ submatrix B whose complementary submatrix in A has term rank $m-k$. If there is a matrix B^* obtained from B by replacing some nonzero entries with 0's if necessary such that $J_{2,3}$ is the zero pattern of a matrix obtained from B^* by a sequence of conformal contractions, then A does not have signed null-space.*

Let M be an m by n strictly row-mixable matrix of the form

$$M = \begin{bmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ & 1 & 1 \end{bmatrix}. \tag{2.1}$$

PROPOSITION 2.4. *M has signed null-space if and only if*

$$A = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & M & & & \vdots \\ & & & & & 0 \\ \hline 0 & \cdots & 0 & 1 & -1 & 1 \end{array}\right]$$

has signed null-space.

Proof. Let M have signed null-space, and let C be any $m+1$ by $m+1$ submatrix of A. If C contains the last column of A, then $C(m+1|m+1)$ is an m by m submatrix of M. Hence $C(m+1|m+1)$ is an SNS-matrix, or $C(m+1|m+1)$ has identically zero determinant by Theorem 2.1. Thus C is an SNS-matrix, or C has identically zero determinant. Hence we may assume that C does not contain the last column of A. If C contains neither the $n-1$th column nor the nth column, then clearly C has identically zero determinant. If C contains only one of the $(n-1)$th column or the nth column, then $C(m+1|m+1)$ is an m by m submatrix of M. Hence $C(m+1|m+1)$ is an SNS-matrix, or $C(m+1|m+1)$ has identically zero determinant. Therefore, C is an SNS-matrix, or C has identically zero determinant. Let C contain both the $(n-1)$th column and the nth column of A. Then $C(m+1|m+1)$ is an SNS-matrix, or $C(m+1|m+1)$ has identically zero determinant. If $C(m+1|m+1)$ has identically zero determinant, then there exists an s by t zero submatrix of $C(m+1|m+1)$ such that $s+t=m+1$. From this, it is easy to show that C has a p by q zero submatrix such that $p+q=m+2$; i.e., C has identically zero determinant. Let $C(m+1|m+1)$ be an SNS-matrix. Since C is conformally contractible to $C(m+1|m+1)$, C is also an SNS-matrix. Thus the $(m+1)$th compound of A is signed. Since M has signed null-space, the term rank of M is m, and hence the term rank of A is $m+1$. Thus A has signed null-space by Theorem 2.1. The converse is trivial. ∎

We say that A is a *single extension* of M in Proposition 2.4. Proposition 2.4 means that a strictly row-mixable matrix has signed null-space if and only if its single extension has signed null-space.

Let

$$G = \begin{bmatrix} & & & & & 0 \\ & & * & & & \vdots \\ & & & & & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \end{bmatrix}$$

be an m by n matrix, and let

$$H = \left[\begin{array}{cccccc|cc} & & G & & & & O & \\ \hline 0 & \cdots & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right].$$

PROPOSITION 2.5. *The m by n strictly row-mixable matrix G has signed null-space if and only if H has signed null-space.*

Proof. Let G have signed null-space, and let $C = [c_{ij}]$ be an $m+2$ by $m+2$ submatrix of H. That is, $C = H[*|\beta]$ for some $\beta \subset \langle n+2\rangle$. If $n+2 \in \beta$, then $H[\langle m+1\rangle|\beta \setminus \{n+2\}]$ is an SNS-matrix, or it has identically zero determinant since $H(m+2|n+2)$ is a single extension of G. Hence C is an SNS-matrix, or C has identically zero determinant. Similarly, we can show that C is an SNS-matrix or C has identically zero determinant if $n+1 \in \beta$. Hence we may assume that β contains neither $n+1$ nor $n+2$. Then it is easy to show that C has identically zero determinant if β contains at most two among $n-2,\ n-1$, and n. Let $\{n-2,\ n-1,\ n\} \subset \beta$. Then $H[\langle m\rangle|\beta \setminus \{n-1, n\}]$ is an SNS-matrix or it has identically zero determinant since G has signed null-space. If $H[\langle m\rangle|\beta \setminus \{n-1, n\}]$ has identically zero determinant, then clearly C has identically zero determinant. Let $H[\langle m\rangle|\beta \setminus \{n-1, n\}]$ be an SNS-matrix. Then $H[\langle m-1\rangle|\beta \setminus \{n-2, n-1, n\}]$ is an SNS-matrix since $c_{mm} = 1$. Since C is in the form of

$$\left[\begin{array}{c|ccc} H[\langle m-1\rangle|\beta \setminus \{n-2,n-1,n\}] & & * & \\ \hline & 1 & 1 & 1 \\ O & 0 & 1 & -1 \\ & 1 & 0 & -1 \end{array}\right]$$

and $C[m, m+1, m+2|m, m+1, m+2]$ is also an SNS-matrix, C is an SNS-matrix. The converse is trivial. □

We say that H is a *double extension* of G in Proposition 2.5. That G should have a row with exactly three ones is necessary in Proposition 2.5. For example, let

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Then B is a double extension of A that has signed null-space. But $B[1,2,3,4|1,2,3,4]$ is a mixed submatrix of A, and hence B does not have signed null-space.

COROLLARY 2.6. *Every totally L-matrix has signed null-space.*

Proof. From Propositions 2.4 and 2.5, we have the result. □

PROPOSITION 2.7. *Let A be a strictly row-mixable m by n matrix with no duplicate columns up to multiplication by -1. If A has signed null-space and is not conformally contractible to a matrix, then it has at least two rows with exactly three nonzero entries.*

Proof. Without loss of generality, we may assume that each row of A has at least three nonzero entries and A has no zero column. Notice that $m \geq 2$ comes from the condition. We prove the result by induction on m. Trivially, we have the result for $m = 2$. Let $m \geq 3$. By Theorem 2.2, A can be rearranged as

$$A = \begin{bmatrix} B & O \\ b & c \\ O & C \end{bmatrix},$$

where matrices B and C (possibly with no rows) are strictly row-mixable matrices which have signed null-spaces, and vectors b and c are nonzero.

$$\begin{bmatrix} B \\ b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c \\ C \end{bmatrix}$$

also have signed null-spaces. Let $A[\alpha|\beta] = \left[\begin{smallmatrix} B \\ b \end{smallmatrix}\right]$ and $A[\gamma|\delta] = \left[\begin{smallmatrix} c \\ C \end{smallmatrix}\right]$ such that $|\alpha| = k,\ |\beta| = s,\ |\gamma| = l$, and $|\delta| = t$. Then $k + l - 1 = m$ and $s + t = n$.

Let $k > 1$ and $l > 1$. If $A[\alpha|\beta]$ has one of the unit vectors $\pm e_k$ as a column, then we can assume that $A[\alpha|\beta]$ is of the form

$$\begin{bmatrix} B' & O \\ b' & 1 \end{bmatrix}.$$

Let B' have no duplicate columns up to multiplication by -1. By induction, B' and hence A have at least two rows with exactly three nonzero entries. Thus we are done. Therefore, we assume that B' has duplicate columns up to multiplication by -1. Then $b' \neq 0$. If b' has at least two nonzero entries, then $A[\alpha|\beta]$ is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1. Since $A[\alpha|\beta]$ is not conformally contractible to a matrix, B has at least one row with exactly three nonzero entries. Let b' have exactly one nonzero entry. Let the columns $1, 2$ of B' be a pair of duplicate columns up to multiplication by -1, and let p be the number of nonzero entries in the column 1 of B'. Let D be a strict signing such that $M = B'D$ is row-mixed. Since B' has signed null-space, M has no mixed cycle, and hence the columns 1 and 2 of M must be identical or $p = 1$. If $p \geq 2$, then the matrix M' obtained from M by multiplying the column 2 by -1 has a mixed cycle. Thus M' is a row-mixed matrix with signed null space, which is impossible by Theorem 2.1. Hence $p = 1$. Therefore, every duplicate column of B' is of the form e_i or $-e_i$ for some i. Hence B' has only one pair of duplicate columns, which are e_i or $-e_i$ for some $i (< k)$. The matrix obtained from B' by deleting one of the duplicate columns, which are e_i or $-e_i$, satisfies the conditions of the hypothesis if its ith row has at least three nonzero entries. This implies that B has at least one row with exactly three nonzero entries. Let $C' = A[\gamma|\{s\} \cup \delta]$. Similarly, C' has a row i with exactly three nonzero entries for some $i (\neq 1)$. Hence C has at least one row with exactly three nonzero entries. Therefore, A has at least two rows with exactly three nonzero entries. Similarly, in the case in which $A[\gamma|\delta]$ has one of the unit vectors $\pm e_1$ as a column, we have the result. Assume that $A[\alpha|\beta]$ and $A[\gamma|\delta]$ do not have the unit vectors $\pm e_k$ and $\pm e_1$, respectively, as columns. Since b is nonzero, the k by $s + 1$ matrix B^* obtained from $A[\alpha|\beta]$ by adding e_k as a column is a strictly row-mixable matrix with no duplicate columns up to multiplication by -1. Since B has signed null-space, B^* also has signed null-space. Applying the similar method above to B^*, we can derive that B has at least one row with exactly three nonzero entries. Similarly, C also has at least one row with exactly three nonzero entries. Hence we have the result when $k > 1$ and $l > 1$.

Let $k = 1$. Then $s = 1$ since the columns of A are distinct up to multiplication by -1. Hence we may assume that $A = [a_{ij}]$ is of the form

$$\begin{bmatrix} 1 & c \\ O & C \end{bmatrix}.$$

If C has no duplicate columns up to multiplication by -1, then we have the result for C by induction, and hence we have the result for A. Let C have duplicate columns up to multiplication by -1. Then the duplicate columns of C are of the form e_i or $-e_i$ for some i, as we have shown before. This implies that the number of identical columns of C up to multiplication by -1 is at most 3. Therefore, we may assume that the zero pattern of A is of the form

$$\left[\begin{array}{c|ccc|ccc|ccc|c}
1 & u & \cdots & u & v & \cdots & v & w & \cdots & w & 0 \text{ or } 1 \\ \hline
 & x & & & & & & & & & \\
 & & \ddots & & & & & & & & \\
 & & & x & & & & & & & \\
 & & & & v & & & & & & \\
 & & & & & \ddots & & & & & S \\
 & & & & & & v & & & & \\
 & & & & & & & v & & & \\
 & & & & & & & & \ddots & & \\
 & & & & & & & & & v & \\ \hline
 & & & & & & & & & & T
\end{array}\right],$$

where $u = (1, 1, 0),\ v = (1, 1),\ w = (1, 0)$, and $x = (1, 1, 1)$, and the unspecified entries are zero. Let ϵ be the set of indices of columns in A corresponding to $\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right]$. Then we may also assume that $A[\gamma \setminus \{1\}|\epsilon]$ has no duplicate columns up to multiplication by -1, and the columns are also different from the ones of $A(1|\epsilon)$ up to multiplication by -1. If $\left[\begin{smallmatrix} S \\ T \end{smallmatrix}\right]$ is vacuous, we are done since $l \geq 3$ and every row but the first row of A has at least three nonzero entries. Let only T be vacuous. Notice that each column of S has at least two nonzero entries. Hence each row of S has at most one nonzero entry. For, suppose that a row of S has two nonzero entries. Since the columns of $A[\gamma \setminus \{1\}|\epsilon]$ are distinct up to multiplication by -1, we may assume that there exists a submatrix of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & * & * \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 & * & * \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & * \\ 0 & 0 & 1 & * & 1 \end{bmatrix},$$

where * is 0 or 1. By Theorem 2.3, A does not have signed null-space. This is a contradiction. Next, suppose that a row r of $A[\gamma \setminus \{1\}|\langle n \rangle]$ has four nonzero entries. Since each row of S has at most one nonzero entry and each column of S has at least two nonzero entries, we have a submatrx of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & 1 & * \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

which is also impossible by Theorem 2.3. Hence each row of $A[\gamma\setminus\{1\}|\langle n\rangle]$ has exactly three nonzero entries. Thus we have the result when T is vacuous. Let T be nonvacuous. Notice that the submatrix of A corresponding to T is a strictly row-mixable matrix with signed null-space. Let $T^{'}$ be the matrix obtained from T by deleting zero columns. Then we may assume that T is of the form $[O\ T^{'}]$. If the submatrix $A^{'}$ of A corresponding to $T^{'}$ has no duplicate columns up to multiplication by -1, then $A^{'}$ has at least two rows with exactly three nonzero entries by induction. Hence we have the result. Suppose that $A^{'}$ has duplicate columns up to multiplication by -1. It is easy to show that such columns of $A^{'}$ have exactly one nonzero entry as we have shown above. We want to show that the number of identical columns of $A^{'}$ is at most three. Suppose that there are four identical columns in $A^{'}$ up to multiplication by -1. We may assume that the zero pattern of the submatrix consisting of such duplicate columns of $A^{'}$ is of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ & O & & \end{bmatrix}.$$

Since $A[\gamma\setminus\{1\}|\epsilon]$ has no duplicate columns up to multiplication by -1, we may assume that $A[\gamma\setminus\{1\}|\epsilon]$ has a submatrix whose zero pattern is

$$\begin{bmatrix} 1 & * & * \\ * & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

where * is 0 or 1. Hence we can have a submatrix N of A whose zero pattern is

$$\begin{bmatrix} 1 & 1 & * & * & * \\ 1 & 0 & 1 & * & * \\ 0 & 1 & * & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 & * & * & * \\ 1 & 0 & 0 & 1 & * & * \\ 0 & 1 & 0 & * & 1 & * \\ 0 & 0 & 1 & * & * & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

where * is 0 or 1. By Theorem 2.3, A does not have signed null-space. This is a contradiction. Thus we can assume that $T^{'}$ is of the form

$$\begin{bmatrix} T_1^{'} & T_2^{'} \\ O & T_3^{'} \end{bmatrix},$$

where $T_1^{'}$ is a block diagonal matrix whose diagonal blocks are either [1 1] or [1 1 1], and the submatrix of A corresponding to $\left[\begin{smallmatrix} T_2^{'} \\ T_3^{'} \end{smallmatrix}\right]$ has no duplicate columns up to multiplication by -1. Continuing this process, we can assume that T is of the form

$$\begin{bmatrix} T_1 & & * \\ O & \ddots & \\ & & T_q \end{bmatrix},$$

where $T_i = [O\ T_i^{'}]$ for $i = 1, 2, \ldots, q$ and $T_i^{'}$ are block diagonal matrices whose diagonal blocks are either [1 1] or [1 1 1] for $i = 1, 2, \ldots, q-1$.

Let λ_i be the set of indices of rows in A corresponding to T_i. Let ϵ_i and δ_i be the set of indices of nonzero columns in A and zero columns in A corresponding to T_i, respectively. It is easy to show that each row of $A[\lambda_i|\epsilon_i \cup \delta_{i+1}]$ has at most three nonzero entries for $i = 1, 2, \ldots, q-1$ by a method similar to that used in the case in which only T is vacuous. If the submatrix $A_q^{'}$ of A corresponding to $T_q^{'}$ has no duplicate columns up to multiplication by -1, then $A_q^{'}$ satisfies the hypothesis. Hence we have the result. If $A_q^{'}$ has duplicate columns up to multiplication by -1, then we may assume that $T_q^{'} = [T_q^{''}\ T_q^{'''}]$, where $T_q^{''}$ is a block diagonal matrix whose diagonal blocks are [1 1] or [1 1 1]. As we have shown above in the case in which T is vacuous, each row of $T_q^{'}$ has exactly three nonzero entries. If $T_q^{'}$ has at least two rows, then we are done.

Thus we may assume that $T_q^{'} = [1\ 1\ 1]$. Then $A[\langle m-1\rangle|n-2, n-1, n]$ cannot have a row whose zero pattern is equal to $(1,1,1)$ because, if so, then A has $J_{2,3}$ as a submatrix, and this is impossible by Theorem 2.3. If $A[m-1|n-2, n-1, n] = O$, then we are done. Hence we may assume that the zero pattern of $A[m-1|n-2, n-1, n]$ is either [1 1 0] or [1 0 0].

Let the zero pattern of $A[m-1|n-2, n-1, n]$ be [1 1 0]. If the rth row of $A[\langle m-2\rangle|n-2, n-1, n]$ has the zero pattern $(1,1,0)$ for some r, then there exist distinct $i_1, i_2, \ldots, i_k$ and distinct $j_1, j_2, \ldots, j_k$ such that $a_{i_1,j_1}, a_{i_2,j_1}, \ldots, a_{i_k,j_k}$ are nonzero, where $i_1 = 1, i_k = r$, and $j_k = n-2$. There also exist distinct $p_1, p_2, \ldots, p_t$ and distinct $q_1, q_2, \ldots, q_t$ such that $a_{p_1,q_1}, a_{p_2,q_1}, \ldots, a_{p_t,q_t}$ are nonzero, where $p_1 = 1,\ p_t = m-1$, and $q_t = n-2$. Choosing some entries from these entries, we have a matrix which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$. This is impossible by Theorem 2.3. We can apply a method similar to that used above to show that $A[\langle m-2\rangle|n] = O$. Hence each row of $A[\langle m-2\rangle|n-2, n-1, n]$ has a zero pattern of the forms $(0,0,0)$, $(1,0,0)$, or $(0,1,0)$. Let $T_{q-1}^{'}$ have at least two rows. It is easy to show that, if each row of $A[\lambda_{q-1}|\epsilon_{q-1} \cup \delta_q \cup \epsilon_q]$ has at least four nonzero entries, we have a submatrix of A which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$ by the method just used above. By Theorem 2.3, it is impossible. Hence some row of $A[\lambda_{q-1}|\epsilon_{q-1} \cup \delta_q \cup \epsilon_q]$ has exactly three nonzero entries. Thus we have the result when $T_{q-1}^{'}$ has at least two rows. Therefore, we may assume that $T_{q-1}^{'}$ is either [1 1] or [1 1 1]. Notice that $T_q = T_q^{'} = [1\ 1\ 1]$.

Let $T_{q-1}^{'} = [1\ 1\ 1]$. If $A[\langle m-2\rangle|n-2, n-1, n] \neq O$, then we can show that there exists a submatrix of A which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$. This is impossible. Hence we may assume that $A[\langle m-2\rangle|n-2, n-1, n] = O$. Then $A[\langle m-1\rangle|\langle n-3\rangle]$ has at least two rows with exactly three nonzero entries by induction. Hence we are done. Let $T_{q-1}^{'} = [1\ 1]$. Notice that $A[\langle m-2\rangle|n-4, n-3]$ has no submatrix whose zero pattern is $J_{2,2}$ by Theorem 2.1. That is, all rows of $A[\langle m-2\rangle|n-4, n-3]$ except for one row have at least one zero entry. Since the conformal contraction of $A[\langle m-1\rangle|\langle m-3\rangle]$ on the last row has signed null-space, $A[\langle m-1\rangle|\langle n-3\rangle]$ has at least one row with exactly three nonzero entries. Thus we have the result if $A[\langle m-2\rangle|n-2, n-1, n] = O$. Let $A[\langle m-2\rangle|n-2, n-1, n] \neq O$. Since we are done if the $(m-2)$nd row of A has exactly three nonzero entries, we may assume that the $(m-2)$nd row of A has at least four nonzero entries. Deleting the cases in which a contradiction occurs, we may assume that the zero pattern of $A[m-2, m-1, m|n-6, n-5, n-4, n-3, n-2, n-1, n]$ is

$$\left[\begin{array}{cccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

It is easy to show that $A[\langle m-2\rangle|n-1,n]=O$ by using a method similar to that used above. If the columns of $A[m-2,m-1|n-4,n-2]$ are identical up to multiplication by -1, then it is easy to find a strict signing D such that AD is a row-mixed matrix and $A[m-2,m-1|n-4,n-2]D$ contains a mixed cycle. This is impossible by Theorem 2.1. Hence the columns of $A[\langle m-1\rangle|\langle n\rangle]$ are not identical up to multiplication by -1. Therefore, $A[\langle m-1\rangle|\langle n-2\rangle]$ satisfies the hypothesis. Thus we have the result when the zero pattern of $A[m-1|n-2,n-1,n]$ is [1 1 0].

In the case in which the zero pattern of $A[m-1|n-2,n-1,n]$ is [1 0 0], the last row of $A[\langle m-1\rangle|\langle n-3\rangle]$ must have two or three nonzero entries. If it has two nonzero entries, then we are done. Let it have three nonzero entries. Then we have a submatrix of A which is conformally contractible to a matrix whose zero pattern is $J_{2,3}$ if $A[\langle m-2\rangle|n-2,n-1,n]\neq O$ by a method similar to that used above. Hence we have $A[\langle m-2\rangle|n-2,n-1,n]=O$. Therefore, $A[\langle m-1\rangle|\langle n-3\rangle]$ has at least two rows with exactly three nonzero entries by induction. Thus we have the result for $k=1$. Similarly, we have the same result for $l=1$. ∎

3. Matrices containing totally *L*-matrices. Let A be a matrix with signed null-space. A is a *maximal matrix with signed null-space* if any matrix obtained from A by replacing a zero entry with a nonzero entry does not have signed null-space.

LEMMA 3.1. *An m by $m+2$ totally L-matrix is a maximal matrix with signed null-space.*

Proof. Let A be an m by $m+2$ totally L-matrix. Let A^* be an m by $m+2$ matrix obtained from A by replacing a zero entry with 1 or -1. Notice that every m by m submatrix of A^* has term rank m. Since A^* has a row with four nonzero entries, A^* is not a totally L-matrix. Therefore, there exists an m by m submatrix of A^* that is not an SNS-matrix. Hence A^* does not have signed null-space by Theorem 2.1. ∎

LEMMA 3.2. *Let A be an m by $m+2$ totally L-matrix, and let $\mathbf{x}$ be an m by 1 column vector which has at least two nonzero entries. Then $B=[A\ \mathbf{x}]$ does not have signed null-space.*

Proof. We will prove the result by induction on m. The statement is clear for $m=2$. We may assume that

$$B=[b_{ij}]=\left[\begin{array}{c|c|c} M' & \begin{matrix} O \\ I_2 \end{matrix} & \mathbf{x} \end{array}\right],$$

where I_2 is the identity matrix of order 2. If $b_{m-1,m+3}=0$ or $b_{m,m+3}=0$, say, $b_{m,m+3}=0$, then $B(m|m+2)$ does not have signed null-space by induction. Hence we have the result by Theorem 2.3. Therefore, we may assume that the last two positions of $\mathbf{x}$ have nonzero entries. Since a totally L-matrix is a maximal matrix with signed null-space, $B(-|m+2)$ does not have signed null-space. Hence B does not have signed null-space. ∎

We say that an m by $m+2$ totally L-matrix contains k *double-extensions* (or $m-2k-2$ *single-extensions*) if A is obtained from

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

by a sequence of $m-2k-2$ single-extensions and k double-extensions up to row and column permutations and multiplication of rows and columns by -1.

PROPOSITION 3.3. *Let A be an m by n matrix with signed null-space whose* *columns are nonzero and distinct up to multiplication by* -1*. If* A *contains an* m *by* $m+2$ *totally* L*-matrix with* k *double-extensions, then* $n \leq 2m-2k$*.*

Proof. We will prove the result by induction on k. Let T_k be an m by $m+2$ totally L-matrix with k double-extensions contained in A. Notice that each column of A which does not correspond to T_k has exactly one zero entry by Lemma 3.2. If $k=0$, then it is known [1] that T_k has a signed rth compound for each $r=1,2,\ldots,m$. Hence we can have the identity matrix I_m as a submatrix of A. Since T_0 has exactly two columns with exactly one nonzero entry, $n \leq m+2+(m-2)=2m=2m-2k$. Let $k \neq 0$. By Proposition 2.4 and Lemma 3.2, we may assume that A is of the form

$$\left[\begin{array}{c|cc|ccc} A_1 & \multicolumn{2}{c|}{A_2} & \multicolumn{3}{c}{O} \\ \hline \multirow{2}{*}{A_3} & 1 & -1 & 0 & 0 & 0 \\ & 1 & 1 & -1 & 0 & 0 \\ \hline \multirow{2}{*}{O} & 0 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

Then $A(m-1,m|n-1,n)$ has signed null-space, and it contains an $m-2$ by m totally L-matrix with $k-1$ double-extensions. The columns of $A(m-1,m|n-1,n)$ are distinct up to multiplication by -1 because, if not, then A_3 has a column of the forms $(0,1)^T$ or $(0,-1)^T$, say, $(0,1)^T$. Then A has a submatrix

$$B=\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \tag{3.1}$$

which is not an SNS-matrix. Since A contains an m by $m+2$ totally L-matrix, the complementary submatrix to B in A has term rank $m-4$. Hence A does not have signed null-space by Theorem 2.1. This is a contradiction. Therefore, $n-2 \leq 2(m-2)-2(k-1)=2m-2k-2$ by induction. Thus we have $n \leq 2m-2k$. $\square$

Let l be the number of single-extensions contained in A. Then we have $l=m-2k-2$. Hence we can restate the result of Proposition 3.3 in terms of l: $n \leq m+l+2$.

Corollary 3.4. *Let* T *be an* m *by* $m+2$ *totally* L*-matrix which contains no single-extensions. Then there is no* m *by* n *matrix* A *with signed null-space such that* A *contains* T *properly, and the columns of* A *are nonzero and distinct up to multiplication by* -1*.*

Proof. Let A be an m by n matrix with signed null-space, and let A contain T. Since T contains no single-extensions, $l=0$. Hence $n \leq m+l+2=m+2$. Hence $A=T$. $\square$

Let M be an m by n matrix of the form in (2.1) with signed null-space, and let A be the $m+1$ by $n+2$ matrix such that

$$A=\left[\begin{array}{ccccc|cc} & & & & & 0 & 0 \\ & & M & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & 1 & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & -1 & 1 \end{array}\right].$$

Since $A(-|n+2)$ is conformally contractible to M, $A(-|n+2)$ has signed null-space. Since M has signed null-space, A has signed null-space by Theorem 2.1. Let T_k be an m by $m+2$ totally L-matrix with k double-extensions. Let $\{i_1,i_2,\ldots,i_l\}$ with

$i_1 < i_2 < \cdots < i_l$ be the set of indices of rows used when single-extensions are constructed in T_k. Notice that T_k does not contain any e_{i_j}, $j = 1, 2, \ldots, l$. The remark above and Proposition 2.5 imply that

$$T = [T_k \ e_{i_1} \ e_{i_2} \ \ldots \ e_{i_l}] \tag{3.2}$$

is an m by $2m - 2k$ matrix whose columns are distinct up to multiplication by -1, and it has signed null-space.

Let j be the index of a row of T_k used when a double-extension is done, and suppose that T_k does not have e_j as a column. $[T_k \ e_j]$ has a submatrix of the form in (3.1), and hence it does not have signed null-space, as we have shown in the proof of Proposition 3.3.

Let $\mathcal{T}_k$ be the set of all matrices of the form in (3.2). Notice that columns of $A \in \mathcal{T}_k$ are nonzero and distinct up to multiplication by -1. We can express the m by n matrices A with $n = 2m - 2k$ in Proposition 3.3 in terms of elements of $\mathcal{T}_k$.

PROPOSITION 3.5. *In Proposition* 3.3, $n = 2m - 2k$ *if and only if there exists a permutation matrix* Q *such that* A *is equal to* TQ *up to multiplication of rows and columns by* -1 *for some* $T \in \mathcal{T}_k$.

Proof. Let A be an m by n matrix such that $A = TQ$ for some permutation matrix Q and $T \in \mathcal{T}_k$. Then $m = 2k+l+2$, and hence $n = m+2+l = m+2+(m-2-2k) = 2m - 2k$. Conversely, let A be an m by $2m - 2k$ matrix satisfying the conditions in Proposition 3.3. Let T_k be an m by $m + 2$ totally L-matrix with k double-extensions contained in A. Then there exists a permutation matrix Q and strict signings D, E such that $DAQE$ is a submatrix of matrix T of the form in (3.2) by Lemma 3.2 and the remark above. Since T is an m by $2m - 2k$ matrix, $A = DTQ^{-1}E$. Since $T \in \mathcal{T}_k$, we have the result. ∎

COROLLARY 3.6. *Let* m *be a positive integer with* $m \geq 2$*, and let* n *be any integer in* $\{m, m+1, \ldots, 2m\}$*. Then there exists an* m *by* n *matrix* A *with signed null-space such that* A *contains a totally* L*-matrix with* m *rows as its submatrix and the columns of* A *are nonzero and distinct up to multiplication by* -1.

Proof. Let n be any integer in $\{m, m+1, \ldots, 2m\}$. If $n \leq m + 2$, then we can take an m by n totally L-matrix as such a matrix A. If $n > m + 2$, there exists an m by $m + 2$ totally L-matrix T_{n-m-2} with $n - m - 2$ single-extensions. Hence there exists an m by n matrix $A \in \mathcal{T}_{n-m-2}$ which contains T_{n-m-2} by the remark above. ∎

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