

# K<sub>3</sub>-Satulator

By Roy Oursler

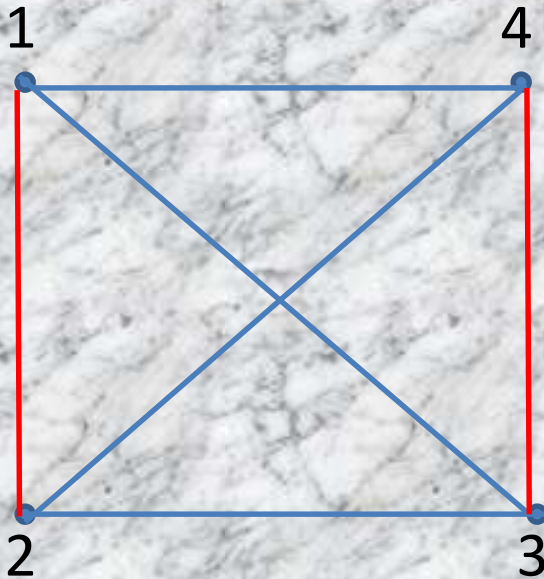
# What is $K_3$ -Saturator

- $K_3$ -Saturator is a combinatorial game
- A combinatorial game is a two player game where both of the players have perfect information, there are no chance based moves, and the game's outcome is either a win or a loss.
- In combinatorial games, one of the players has a winning strategy from the beginning of play, my goal was to find who would be the winner in different games of  $K_3$ -Saturator.
- This game was partially explored by Michael Ferrara, Michael Jacobson, and Angela Harris in their article *The Game of F-Saturator* in *Discrete Applied Mathematics* pages 189 through 197.

# How to Play

- **Setup:** There are two players which start off with some points drawn on a piece of paper.
- **Play:** Each player takes turns connecting two points together.
- **Goal:** The players do not want to connect two points together that will form a triangle.

# Example Game

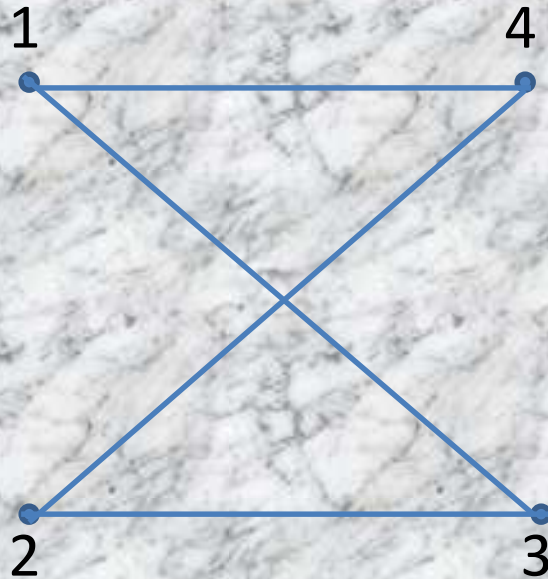


Note: In this game a triangle is created when connecting three points together, lines crossing do not create a triangle. The three connected points are a type of graph called a  $K_3$

# Useful Definitions

- Graph- A graph is a set of points called vertices that are connected by lines called edges.
- H-Saturated Graph- A H-saturated graph is formed from two graphs, graph H and graph G. Graph G is H saturated if G contains no copy of H and the addition of any edge to G creates a copy of H
- This is the reason for this game to be named  $K_3$  saturator.

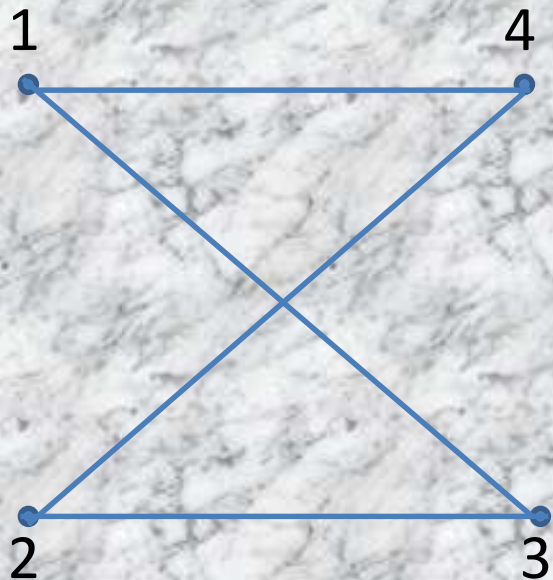
# Example Game



This is an example of a  $K_3$  saturated graph since no line can be added that won't form one of the triangles.

# Winning Strategy

- For the example game, the 2<sup>nd</sup> player can always win it.



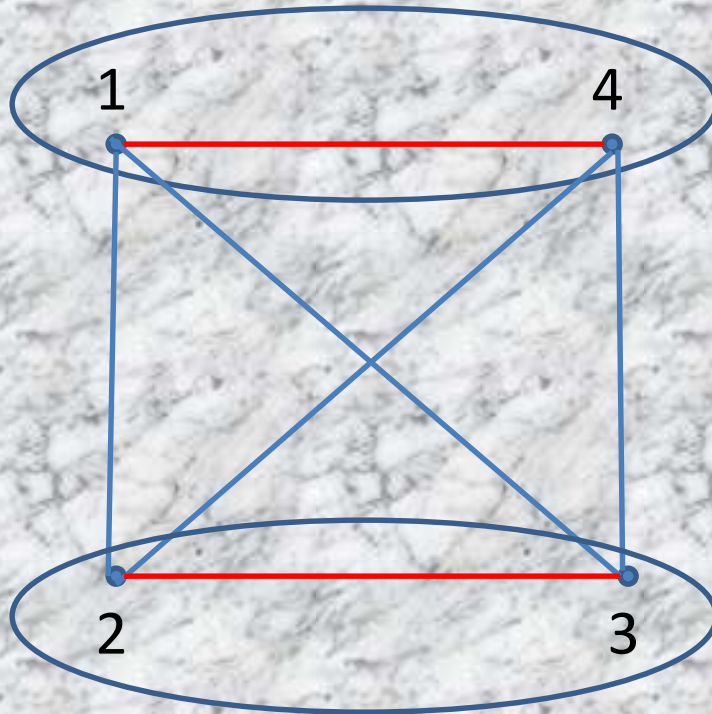
# A Revision of the Game(version 2)

- Play the basic game with an even number of points.
- Divide the points into pairs which are poisoned pairs, or pairs which can't be played between.



# Example Game

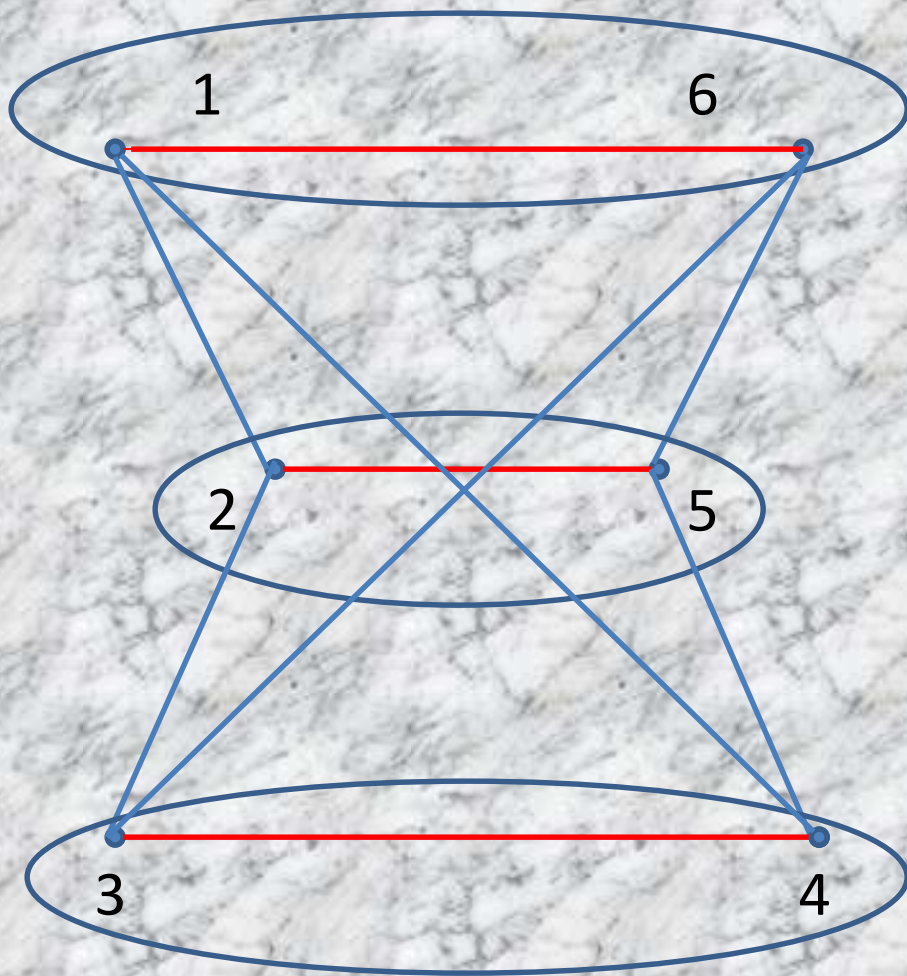
The winner of this modified game will always be the 2nd player



Red lines are the moves which aren't allowed.

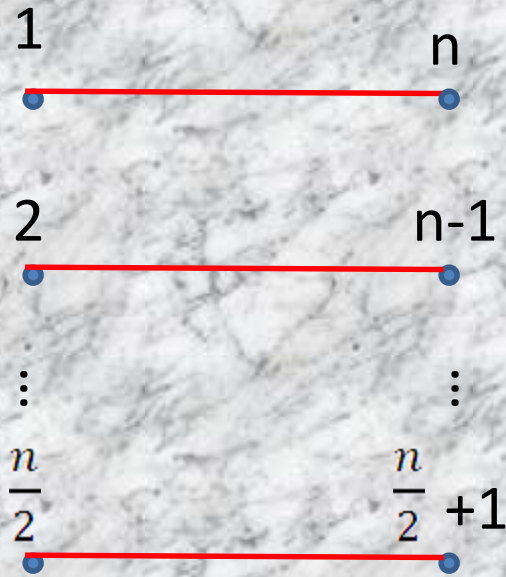
The strategy for the 2nd player is to mirror the 1<sup>st</sup> player's moves between the poisoned pairs.

# Example Game



# Proof of the Winning Strategy

First, set the graph up so that it can be labeled in the shown fashion.



The connections between any two points  $i$  and  $j$  is written as  $\{i, j\}$ .

This means when the 1<sup>st</sup> player connects the point  $\{i, j\}$  the second player connects the points  $\{n-i+1, n-j+1\}$  where  $n$  is the size of the graph.

# Proof of Winning Strategy

Theorem:

If there is a graph  $G$  of even order  $n$ , which does not have mirror symmetry, and one edge  $g$  can be added to  $G$  that makes the graph have mirror symmetry, then either a  $K_3$  graph is not formed within  $G$ , or a  $K_3$  graph already existed within  $G$ .

In this game, the second player's strategy will always be creating mirror symmetry while the 1<sup>st</sup> player is always destroying the symmetry, so if a  $K_3$  forms, that  $K_3$  will have been created by the first player.

# Proof of Winning Strategy

- Assume the graph  $G$  is in a position such that a  $K_3$  will be formed when the edge  $g$  is added to  $G$ . Assume this  $K_3$  consists of the edges  $\{a,b\}$ ,  $\{b,c\}$ , and  $\{a,c\}$  where  $a$ ,  $b$ , and  $c$  are points of the graph. In this position, there are 3 cases to deal with.

# Proof of Winning Strategy

## Case 1

Assume that the edge  $g$  is a reflection of itself. This edge does not affect mirror symmetry which implies that  $G$  already had mirror symmetry. This is a contradiction therefore  $g$  doesn't exist.

## Case 2

Assume that none of the edges  $\{a,b\}$ ,  $\{b,c\}$ , and  $\{a,c\}$  are reflections of each other.

Due to the assumption that any two of the edges  $\{a,b\}$ ,  $\{b,c\}$ , and  $\{a,c\}$  aren't reflections of each other, then the  $K_3$   $\{n-a+1, n-b+1\}$ ,  $\{n-b+1, n-c+1\}$ , and  $\{n-a+1, n-c+1\}$  must already have existed before  $g$  was added to the graph.

# Proof of Winning Strategy

## Case 3

Assume that of the edges in the  $K_3$  consisting of  $\{a,b\}$ ,  $\{b,c\}$ , and  $\{a,c\}$ , and  $\{a,c\}$  is a reflection of one of the other edges.

This implies that  $\{a,c\} = \{n-a+1, n-b+1\}$  or  $\{n-b+1, n-a+1\}$ .

Assume  $\{a,c\} = \{n-a+1, n-b+1\}$

Then  $a = n-a+1$

Then  $2a+1 = n$

This implies  $n$  is odd a contradiction.

Assume  $\{a,c\} = \{n-b+1, n-a+1\}$ .

Then  $a=n-b+1$  and  $c=n-a+1$

Then  $c+(n-b+1)=n-a+1+a$

Then  $c-b+n+1=n+1$

Then  $c=b$ , a contradiction.

Therefore a  $K_3$  didn't form.

# Proof of Winning Strategy

Therefore the theorem holds.

So the second player will always win in the modified game

Using this theorem and a few others, I can prove the winner on games of size 4,6,and 8.



# Useful Definitions

In order to prove the winner on graphs of size 4,6 and 8 there are a few definitions which need to be known.

A connected pair is formed when points in a pair are connected by an edge.

An unconnected pair is formed when a pair is not connected but still can be connected

A poisoned pair is formed when a pair cannot become a connected pair without losing the game.

# Game(Version 3)

- Now for the game we will start the game off with some number of poisoned, connected and unconnected pairs.
- Also, play between the poisoned pairs and the unconnected pairs is not allowed.
- For the original game, all the pairs would be unconnected pairs.
- The strategy for this game is the same as in the version 2 with the extra rule that the winning player will keep the number of unconnected pairs even.

# Example Setup



This is a game of size 6 where we are starting with a poisoned pair, a connected pair, and an unconnected pair

# Proof

- The follow theorems prove that the number of poisoned pairs created during a game is an even number.
- This allows for an unconnected pair to be a the deciding move in this game.

# Proof

Theorem :Play between connected pairs or poisoned pairs with the rules of this game will not create any additional poisoned pairs.

Proof

Because edges cannot be removed in this game, connected pairs will always be connected pairs and poisoned pairs will always be poisoned pairs. Therefore the number of poisoned pairs cannot be increased with those moves.

# Proof

Theorem : Play between a connected pair and an unconnected pair will never create a single poisoned pair with the strategy being used.

Proof

Assume that there is a connected pair  $\{a,b\}$  and an unconnected pair  $\{c,d\}$  and that a connection exists that turns the pair  $\{c,d\}$  into a poisoned pair.

This implies that the point  $c$  and the point  $d$  are connected to the point  $a$  and the point  $b$ . Because of the strategy used in this game. This creates 2 K3, so either the game is lost by the losing player, or no poisoned pairs are formed

# Proof

Theorem: Play between two unconnected pairs with the strategy for this game will always result in either no poisoned pairs being formed or 2 poisoned pairs being formed.

Proof:

Case 1: Assume the pair  $\{a,b\}$  and  $\{c,d\}$  being played between have not been played between before.

If the losing player connects  $a$  to  $c$ , then the winning player connects  $b$  to  $d$  or if  $a$  is connected to  $d$ , then  $b$  is connected to  $c$ . In either case, no change is made to the state of the pairs  $\{a,b\}$  or  $\{b,c\}$  so no poisoned pair is formed.

Case 2: Assume that the pair  $\{a,b\}$  and  $\{c,d\}$  have been played between before.

This implies that after the next round of playing, the points will be connected so that  $a$  and  $b$  are connected to both  $c$  and  $d$ . This implies that the pair  $\{a,b\}$  is a poisoned pair and that  $\{c,d\}$  is also a poisoned pair. Therefore two poisoned pairs were formed.

# Proof

- All of those proofs show that so long as play between poisoned and unconnected pairs cannot happen, there will be an even number of poisoned pairs.
- Since the winning strategy has a response to any move between pairs, then the deciding move is based off of the number of unconnected pairs, which for the winner will always be an even number after their turn since there will eventually be 0 unconnected pairs after their turn.
- The rules of this game version can be forced into the original game on games of size 4,6,and 8 by the moves one of the players makes.



# Proof

- Since an even number of poisoned pairs will be formed in a game, the winning player will be the 2<sup>nd</sup> player if the number of unconnected pairs is divisible by 2, and the 1<sup>st</sup> player will win otherwise.
- This leads to the 2<sup>nd</sup> player winning on games of size 4,8 and the 1<sup>st</sup> player winning on a game of size 6.

# Final Results

- I could not prove who would win any of the original game except on graphs of size 4,6,and 8.
- I did create a script using a program called magma which solved for the winner of the game using brute force methods, and these were the results:

<b>Graph Size</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>	<b>11</b>
Winning Player	2	2	2	1	2	2	2	1	2

- I could only get to graphs of size 11 because the number of possible moves in a game grew extremely quickly.

# Final Results

Graph Size	3	4	5	6	7	8	9	10	11
Winning Player	2	2	2	1	2	2	2	1	2

This table suggests that the 1<sup>st</sup> player will only win on number satisfying  $n = 2m+2$ .

I believe that this pattern will continue, but have not found a way to prove that conjecture.

# Acknowledgements

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Wyoming Epscor

Any Questions