

## Exercises for Chapter 15

**Exercise 15.1** Let

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

Give the characteristic equation and find the eigenvalues and a corresponding set of eigenvectors.

**Exercise 15.2** Let

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 2 \\ 4 & 0 & 2 \end{pmatrix}.$$

Give the characteristic equation and find the eigenvalues and a corresponding set of eigenvectors.

**Exercise 15.3** Let

$$\mathbf{A} = \begin{pmatrix} -\frac{6}{4} & \frac{6}{4} \\ -\frac{12}{4} & \frac{11}{4} \end{pmatrix}.$$

Let  $\lambda_1$  and  $\lambda_2$  be the eigenvalues with eigenvectors  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . Set  $\mathbf{X}$  as the matrix with columns  $\mathbf{x}_1$  and  $\mathbf{x}_2$  and  $\mathbf{\Lambda}$  the diagonal matrix with  $\lambda_1$  and  $\lambda_2$  on the main diagonal. Use the formula  $\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$  to show that  $\mathbf{A}^k \rightarrow 0$  as  $k \rightarrow \infty$ .

**Exercise 15.4** Let  $\mathbf{A}$  be an  $n \times n$  matrix. Suppose that the sum of the absolute values of the entries in any column is no larger than 1 (for each  $j$ ,  $\sum_{i=1}^n |a_{ij}| \leq 1$ ). Show that all eigenvalues are equal to or less than 1 (in absolute value).

**Exercise 15.5** Show that a square matrix  $\mathbf{A}$  is non-singular if and only if all eigenvalues are not zero.

**Exercise 15.6** Consider the matrix:

$$\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix},$$

$$\mathbf{A} - \lambda \mathbf{I} = \begin{pmatrix} 2 - \lambda & 5 \\ 4 & 3 - \lambda \end{pmatrix}.$$

Therefore  $|\mathbf{A} - \lambda \mathbf{I}| = -14 - 5\lambda + \lambda^2$  (which factors to  $(\lambda + 2)(\lambda - 7)$ ). The characteristic equation is:

$$p(\lambda) = \lambda^2 - 5\lambda - 14 = 0.$$

According to the Cayley–Hamilton theorem, the matrix satisfies the characteristic polynomial

$$p(\mathbf{A}) = \mathbf{A}^2 - 5\mathbf{A} - 14\mathbf{I} = 0.$$

Show that this condition is satisfied for this matrix.

**Exercise 15.7** Let  $p(\lambda) = a_2\lambda^2 + a_1\lambda + a_0 = 0$  be the characteristic equation of a  $2 \times 2$  matrix  $\mathbf{A}$ . From the Cayley–Hamilton theorem,  $p(\mathbf{A}) = a_2\mathbf{A}^2 + a_1\mathbf{A} + a_0\mathbf{I} = 0$ . This implies that  $\mathbf{A}^2$  may be computed as  $\mathbf{A}^2 = -\frac{a_1}{a_2}\mathbf{A} - \frac{a_0}{a_2}\mathbf{I}$ . Illustrate this calculation with the matrix:

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}.$$

Show how this procedure may be used to solve for  $\mathbf{A}^k$  in terms of  $\mathbf{A}$  and  $\mathbf{I}$ .

**Exercise 15.8** Show how the Cayley–Hamilton theorem may be used to solve for  $\mathbf{A}^k$  in terms of  $\mathbf{A}$  and  $\mathbf{I}$  when  $\mathbf{A}$  is a  $2 \times 2$  matrix. Show also how it may be used to find the inverse of  $\mathbf{A}$ .

**Exercise 15.9** Consider:

$$\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of  $\mathbf{B}$ .

**Exercise 15.10** Find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

**Exercise 15.11** For complex matrices, the inverse is defined in the usual way. If  $\mathbf{A}$  be a square matrix with complex entries (a complex valued matrix),  $\mathbf{A}$  is invertible if there is a matrix  $\mathbf{B}$  with  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ , and  $\mathbf{B}$  is called the inverse of  $\mathbf{A}$  and denoted  $\mathbf{A}^{-1}$ . Calculate the determinant and inverse of:

$$\mathbf{A} = \begin{pmatrix} 2+2i & 4+i \\ 3+2i & 5+2i \end{pmatrix}.$$

