CHAPTER 73 MEAN AND ROOT MEAN SQUARE VALUES

EXERCISE 286 Page 778

1. Determine the mean value of (a) $y = 3\sqrt{x}$ from $x = 0$ to $x = 4$

(b) $y = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$ (c) $y = 4e^t$ from $t = 1$ to $t = 4$

(a) Mean value, $\bar{y} = \frac{1}{4-0}\int_0^4 y\,dx = \frac{1}{4}\int_0^4 3\sqrt{x}\,dx = \frac{1}{4}\int_0^4 3x^{1/2}\,dx = \frac{1}{4}\left[\frac{3x^{3/2}}{3/2}\right]_0^4 = \frac{1}{2}\left[\sqrt{x}^3\right]_0^4$

$= \frac{1}{2}\left(\sqrt{4^3}\right) = \frac{1}{2}\left(2^3\right) = \frac{1}{2}(8) = 4$

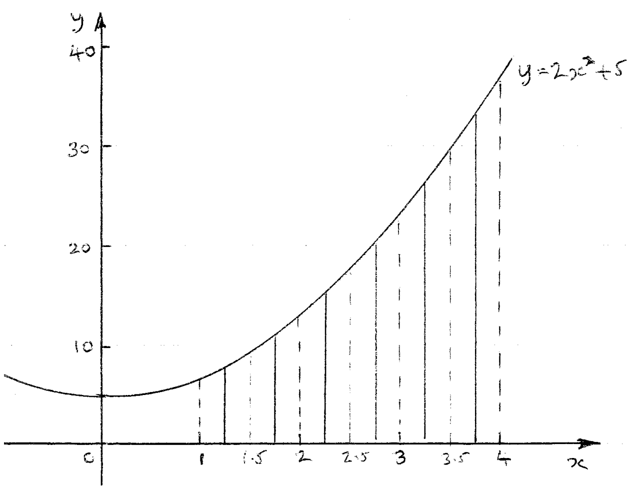
(b) Mean value, $\bar{y} = \frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} y\,d\theta = \frac{4}{\pi}\int_0^{\pi/4}\sin 2\theta\,d\theta = \frac{4}{\pi}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/4} = -\frac{2}{\pi}\left[\cos 2\left(\frac{\pi}{4}\right) - \cos 0\right]$

$= -\frac{2}{\pi}(0-1) = \frac{2}{\pi}$ or 0.637

(c) Mean value, $\bar{y} = \frac{1}{4-1}\int_1^4 y\,dx = \frac{1}{3}\int_1^4 4e^t\,dt = \frac{1}{3}\left[4e^t\right]_1^4 = \frac{4}{3}\left[e^4 - e^1\right] = 69.17$

2. Calculate the mean value of $y = 2x^2 + 5$ in the range $x = 1$ to $x = 4$ by (a) the mid-ordinate rule and (b) integration.

(a) A sketch of $y = 2x^2 + 5$ is shown below.



Using 6 intervals each of width 0.5 gives mid-ordinates at x = 1.25, 1.75, 2.25, 2.75, 3.25 and 3.75, as shown in the diagram

x	1.25	1.75	2.25	2.75	3.25	3.75
$y = 2x^2 + 5$	8.125	11.125	15.125	20.125	26.125	33.125

Area under curve between x = 1 and x = 4 using the mid-ordinate rule with 6 intervals

≈ (width of interval)(sum of mid-ordinates)

≈ (0.5)(8.125 + 11.125 + 15.125 + 20.125 + 26.125 + 33.125)

≈ (0.5)(113.75) = 56.875

and mean value = $\dfrac{\text{area under curve}}{\text{length of base}} = \dfrac{56.875}{4-1} = \dfrac{56.875}{3}$ = 18.96

(b) By integration, $\bar{y} = \dfrac{1}{4-1}\int_1^4 y\,\mathrm{d}x = \dfrac{1}{3}\int_1^4 2x^2 + 5\ \mathrm{d}x = \dfrac{1}{3}\left[\dfrac{2x^3}{3} + 5x\right]_1^4 = \dfrac{1}{3}\left[\left(\dfrac{128}{3} + 20\right) - \left(\dfrac{2}{3} + 5\right)\right]$

$= \dfrac{1}{3}(57)$ = 19

This is the precise answer and could be obtained by an approximate method as long as sufficient intervals were taken

3. The speed v of a vehicle is given by: $v = (4t + 3)$ m/s, where t is the time in seconds. Determine the average value of the speed from t = 0 to t = 3 s.

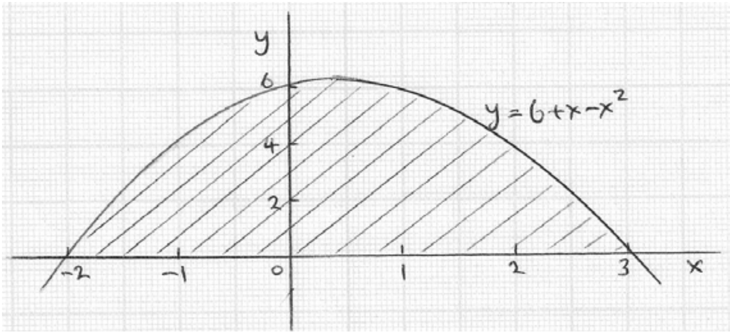
Average speed, $\bar{v} = \dfrac{1}{3-0}\int_0^3 (4t+3)\,\mathrm{d}t = \dfrac{1}{3}\left[2t^2 + 3t\right]_0^3 = \dfrac{1}{3}\left[(18+9) - (0)\right] = \dfrac{1}{3}(27)$ = 9 m/s

4. Find the mean value of the curve $y = 6 + x - x^2$ which lies above the x-axis by using an approximate method. Check the result using integration.

$6 + x - x^2 = (3 - x)(2 + x)$ and when y = 0 (i.e. the x-axis), then $(3 - x)(2 + x) = 0$

from which, x = 3 and x = –2 Hence the curve $y = 6 + x - x^2$ cuts the x-axis at x = –2 and at x = 3 and at x = 0, y = 6

A sketch of $y = 6 + x - x^2$ is shown below.



Mean value, $\overline{y} = \frac{1}{3--2}\int_{-2}^{3} y\,\mathrm{d}x = \frac{1}{5}\int_{-2}^{3}\left(6+x-x^2\right)\mathrm{d}x = \frac{1}{5}\left[6x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{-2}^{3}$

$$= \frac{1}{5}\left[\left(18+4.5-9\right)-\left(-12+2+\frac{8}{3}\right)\right]$$

$$= \frac{1}{5}\left[\left(13.5\right)-\left(-7.3333\right)\right]$$

$$= 4.17$$

5. The vertical height h km of a missile varies with the horizontal distance d km, and is given by $h = 4d - d^2$. Determine the mean height of the missile from $d = 0$ to $d = 4$ km.

Mean height, $\overline{h} = \frac{1}{4-0}\int_{0}^{4}\left(4d-d^2\right)\mathrm{d}d = \frac{1}{4}\left[2d^2-\frac{d^3}{3}\right]_{0}^{4} = \frac{1}{4}\left[\left(32-\frac{64}{3}\right)-(0)\right]$ = 2.67 km

6. The velocity v of a piston moving with simple harmonic motion at any time t is given by:

$v = c\sin\omega t$, where c is a constant. Determine the mean velocity between $t = 0$ and $t = \frac{\pi}{\omega}$

Mean velocity, $\overline{v} = \frac{1}{\frac{\pi}{\omega}-0}\int_{0}^{\pi/\omega}\left(c\sin\omega t\right)\mathrm{d}t = \frac{\omega}{\pi}\left[-\frac{c\cos\omega t}{\omega}\right]_{0}^{\pi/\omega} = -\frac{c}{\pi}\left[\left(\cos\omega\left(\frac{\pi}{\omega}\right)\right)-(\cos 0)\right]$

$$= -\frac{c}{\pi}(-1-1) = \frac{2c}{\pi}$$

EXERCISE 287 Page 779

1. Determine the r.m.s. values of: (a) $y = 3x$ from $x = 0$ to $x = 4$

(b) $y = t^2$ from $t = 1$ to $t = 3$ (c) $y = 25 \sin \theta$ from $\theta = 0$ to $\theta = 2\pi$

(a) R.m.s value = $\sqrt{\left\{\frac{1}{4-0}\int_0^4 y^2\,dx\right\}} = \sqrt{\left\{\frac{1}{4}\int_0^4 (3x)^2\,dx\right\}} = \sqrt{\left\{\frac{9}{4}\int_0^4 x^2\,dx\right\}} = \sqrt{\left\{\frac{9}{4}\left[\frac{x^3}{3}\right]_0^4\right\}}$

$$= \sqrt{\left\{\frac{9}{4}\left[\frac{64}{3}\right]\right\}} = \sqrt{\left\{\frac{144}{3}\right\}} = \frac{12}{\sqrt{3}} = 6.928$$

(b) R.m.s value = $\sqrt{\left\{\frac{1}{3-1}\int_1^3 y^2\,dx\right\}} = \sqrt{\left\{\frac{1}{2}\int_1^3 (t^2)^2\,dx\right\}} = \sqrt{\left\{\frac{1}{2}\int_1^3 t^4\,dx\right\}} = \sqrt{\left\{\frac{1}{2}\left[\frac{t^5}{5}\right]_1^3\right\}}$

$$= \sqrt{\left\{\frac{1}{2}\left[\frac{243}{5}-\frac{1}{5}\right]\right\}} = \sqrt{\left\{\frac{1}{2}\left[\frac{242}{5}\right]\right\}} = \sqrt{\left\{\frac{121}{5}\right\}} = \frac{11}{\sqrt{5}} = 4.919$$

(c) R.m.s value =

$$\sqrt{\left\{\frac{1}{2\pi-0}\int_0^{2\pi} (25\sin\theta)^2\,d\theta\right\}} = \sqrt{\left\{\frac{25^2}{2\pi}\int_0^{2\pi} \sin^2\theta\,d\theta\right\}} = \sqrt{\left\{\frac{25^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta)\,d\theta\right\}}$$

$$= \sqrt{\left\{\frac{25^2}{4\pi}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi}\right\}} = \sqrt{\left\{\frac{25^2}{4\pi}[(2\pi-0)-(0)]\right\}} = \sqrt{\left\{\frac{25^2}{2}\right\}} = \frac{25}{\sqrt{2}} \text{ or } 17.68$$

2. Calculate the r.m.s. values of: (a) $y = \sin 2\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$

(b) $y = 1 + \sin t$ from $t = 0$ to $t = 2\pi$ (c) $y = 3 \cos 2x$ from $x = 0$ to $x = \pi$

(note that $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$, from Chapter 44).

(a) R.m.s value = $\sqrt{\left\{\frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} \sin^2 2\theta\,d\theta\right\}} = \sqrt{\left\{\frac{4}{\pi}\int_0^{\pi/4} \frac{1}{2}(1-\cos 4\theta)\,d\theta\right\}} = \sqrt{\left\{\frac{2}{\pi}\left[\theta-\frac{\sin 4\theta}{4}\right]_0^{\pi/4}\right\}}$

$$= \sqrt{\left\{\frac{2}{\pi}\left[\left(\frac{\pi}{4}-0\right)-(0)\right]\right\}} = \sqrt{\left\{\frac{1}{2}\right\}} = \frac{1}{\sqrt{2}} \text{ or } 0.707$$

(b) R.m.s value = $\sqrt{\left\{\frac{1}{2\pi-0}\int_0^{2\pi}(1+\sin t)^2\,\mathrm{d}t\right\}} = \sqrt{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sin t+\sin^2 t\right)\mathrm{d}t\right\}}$

$$= \sqrt{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left[1+2\sin t+\frac{1}{2}(1-\cos 2t)\,\mathrm{d}t\right]\right\}} = \sqrt{\left\{\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{3}{2}+2\sin t-\frac{1}{2}\cos 2t\right)\mathrm{d}t\right\}}$$

$$= \sqrt{\left\{\frac{1}{2\pi}\left[\frac{3t}{2}-2\cos t-\frac{\sin 2t}{4}\right]_0^{2\pi}\right\}} = \sqrt{\left\{\frac{1}{2\pi}\left[(3\pi-2-0)-(0-2-0)\right]\right\}}$$

$$= \sqrt{\left\{\frac{1}{2\pi}(3\pi)\right\}} = \sqrt{\frac{3}{2}} = 1.225$$

(c) R.m.s value = $\sqrt{\left\{\frac{1}{\pi-0}\int_0^{\pi}(3\cos 2x)^2\,\mathrm{d}x\right\}} = \sqrt{\left\{\frac{9}{\pi}\int_0^{\pi}\cos^2 2x\,\mathrm{d}x\right\}} = \sqrt{\left\{\frac{9}{\pi}\int_0^{\pi}\frac{1}{2}(1+\cos 4x)\,\mathrm{d}x\right\}}$

$$= \sqrt{\left\{\frac{9}{2\pi}\left[x+\frac{\sin 4x}{4}\right]_0^{\pi}\right\}} = \sqrt{\left\{\frac{9}{2\pi}\left[(\pi+0)-(0)\right]\right\}} = \sqrt{\frac{9}{2}} \text{ or } 2.121$$

[Note that $\cos 2x = 2\cos^2 x - 1$ and $\cos 4x = 2\cos^2 2x - 1$

from which, $\cos^2 2x = \frac{1}{2}(1+\cos 4x)$]

3. The distance p of points from the mean value of a frequency distribution are related to the variable q by the equation $p = \frac{1}{q} + q$. Determine the standard deviation (i.e. the r.m.s. value), correct to 3 significant figures, for values from $q = 1$ to $q = 3$

Standard deviation = r.m.s. value = $\sqrt{\left\{\frac{1}{3-1}\int_1^3\left(\frac{1}{q}+q\right)^2 dq\right\}} = \sqrt{\left\{\frac{1}{2}\int_1^3\left(\frac{1}{q^2}+2+q^2\right)dq\right\}}$

$$= \sqrt{\left\{\frac{1}{2}\left[\frac{q^{-1}}{-1}+2q+\frac{q^3}{3}\right]_1^3\right\}} = \sqrt{\left\{\frac{1}{2}\left[\left(-\frac{1}{2}+6-9\right)-\left(-1+2+\frac{1}{3}\right)\right]\right\}}$$

$$= \sqrt{\left\{\frac{1}{2}(13.3333)\right\}} = 2.58$$

4. A current, $i = 30\sin 100\pi t$ amperes is applied across an electric circuit. Determine its mean and r.m.s. values, each correct to 4 significant figures, over the range $t = 0$ to $t = 10$ ms.

Mean value = $\dfrac{1}{10\times10^{-3}-0}\int_0^{10\times10^{-3}}\left(30\sin100\pi t\right)\mathrm{d}t = 100\left[-\dfrac{30}{100\pi}\cos100\pi t\right]_0^{10\times10^{-3}}$

$$= -\frac{100(30)}{100\pi}\left[\cos(100\pi\times10\times10^{-3})-\cos0\right]$$

$$= -\frac{30}{\pi}\left[\cos\pi-\cos0\right] = -\frac{30}{\pi}\left[-1-1\right] = \frac{60}{\pi}$$

= 19.10 A

r.m.s. value = $\sqrt{\dfrac{1}{10\times10^{-3}-0}\int_0^{10\times10^{-3}}30^2\sin^2100\pi t\,\mathrm{d}t} = \sqrt{(100)(30)^2\int_0^{10\times10^{-3}}\dfrac{1}{2}(1-\cos200\pi t)\,\mathrm{d}t}$

since cos 2A = 1 – $2\sin^2 A$ from which, $\sin^2 A = \dfrac{1}{2}(1-\cos2A)$

$$= \sqrt{\frac{(100)(30)^2}{2}\left[t-\frac{\sin200\pi t}{200\pi}\right]_0^{10\times10^{-3}}} = \sqrt{\frac{(100)(30)^2}{2}\left[\left(10\times10^{-3}-\frac{\sin2\pi}{200\pi}\right)-(0-\sin0)\right]}$$

$$= \sqrt{\frac{(100)(30)^2}{2}\left[10\times10^{-3}\right]} = \sqrt{\frac{30^2}{2}} = \frac{30}{\sqrt{2}} = 21.21 \text{ A}$$

5. A sinusoidal voltage has a peak value of 340 V. Calculate its mean and r.m.s. values, correct to 3 significant figures.

For a sine wave,

mean value = $\dfrac{2}{\pi}\times\text{peak value} = \dfrac{2}{\pi}\times340$ = 216 V

and r.m.s. value = $\dfrac{1}{\sqrt{2}}\times\text{peak value} = \dfrac{1}{\sqrt{2}}\times340$ = 240 V

6. Determine the form factor, correct to 3 significant figures, of a sinusoidal voltage of maximum value 100 volts, given that form factor = $\dfrac{\text{r.m.s. value}}{\text{average value}}$

For a sine wave,

$$\text{average value} = \frac{2}{\pi}\times\text{peak value} = \frac{2}{\pi}\times 100 = 63.66 \text{ V}$$

$$\text{r.m.s. value} = \frac{1}{\sqrt{2}}\times\text{peak value} = \frac{1}{\sqrt{2}}\times 100 = 70.71 \text{ V}$$

and $$\text{form factor} = \frac{\text{r.m.s. value}}{\text{average value}} = \frac{70.71}{63.66} = 1.11$$

7. A wave is defined by the equation:

$$v = E_1 \sin\omega t + E_3 \sin 3\omega t$$

where E_1, E_3 and ω are constants.

Determine the r.m.s. value of v over the interval $0 \le t \le \frac{\pi}{\omega}$

$$\text{r.m.s. value} = \sqrt{\frac{1}{\frac{\pi}{\omega}-0}\int_0^{\frac{\pi}{\omega}} \left(E_1\sin\omega t + E_3 \sin 3\omega t\right)^2 \mathrm{d}(\omega t)}$$

$$= \sqrt{\frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}}\left(E_1^{\ 2}\sin^2\omega t + 2E_1E_3\sin\omega t\sin 3\omega t + E_3^{\ 2}\sin^2 3\omega t\right)\mathrm{d}(\omega t)}$$

$$\int_0^{\frac{\pi}{\omega}}\sin^2\omega t\,\mathrm{d}t = \int_0^{\frac{\pi}{\omega}}\frac{1-\cos 2\omega t}{2}\mathrm{d}t = \left[\frac{t}{2}-\frac{\sin 2\omega t}{4\omega}\right]_0^{\frac{\pi}{\omega}} = \left[\left(\frac{\pi}{2\omega}-0\right)-(0-0)\right] = \frac{\pi}{2\omega}$$

$$\int_0^{\frac{\pi}{\omega}}\sin^2 3\omega t\,\mathrm{d}t = \int_0^{\frac{\pi}{\omega}}\frac{1-\cos 6\omega t}{2}\mathrm{d}t = \left[\frac{t}{2}-\frac{\sin 6\omega t}{12\omega}\right]_0^{\frac{\pi}{\omega}} = \left[\left(\frac{\pi}{2\omega}-0\right)-(0-0)\right] = \frac{\pi}{2\omega}$$

$$\int_0^{\frac{\pi}{\omega}}\sin\omega t\sin 3\omega t\,\mathrm{d}t = \int_0^{\frac{\pi}{\omega}}\sin 3\omega t\sin\omega t\,\mathrm{d}t = \int_0^{\frac{\pi}{\omega}}-\frac{1}{2}\left(\cos 4\omega t - \cos 2\omega t\right)\mathrm{d}t$$

$$= -\frac{1}{2}\left[\frac{\sin 4\omega t}{4\omega}-\frac{\sin 2\omega t}{2\omega}\right]_0^{\frac{\pi}{\omega}} = -\frac{1}{2}\left[(0-0)-(0-0)\right] = 0$$

Hence, $$\text{r.m.s. value} = \sqrt{\frac{\omega}{\pi}\left[\frac{\pi}{2\omega}E_1^{\ 2}+\frac{\pi}{2\omega}E_3^{\ 2}\right]} = \sqrt{\left(\frac{E_1^{\ 2}}{2}+\frac{E_3^{\ 2}}{2}\right)}$$

$$= \sqrt{\left(\frac{E_1^{\ 2}+E_3^{\ 2}}{2}\right)}$$