

# FORMULAE/REVISION HINTS FOR SECTION I

## DIFFERENTIAL CALCULUS

### Standard derivatives

$y$ or $f(x)$	$\frac{dy}{dx}$ or $f'(x)$
$ax^n$	$anx^{n-1}$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$
$\tan ax$	$a \sec^2 ax$
$\sec ax$	$a \sec ax \tan ax$
$\operatorname{cosec} ax$	$-a \operatorname{cosec} ax \cot ax$
$\cot ax$	$-a \operatorname{cosec}^2 ax$
$e^{ax}$	$ae^{ax}$
$\ln ax$	$\frac{1}{x}$
$\sinh ax$	$a \cosh ax$
$\cosh ax$	$a \sinh ax$
$\tanh ax$	$a \operatorname{sech}^2 ax$
$\operatorname{sech} ax$	$-a \operatorname{sech} ax \tanh ax$
$\operatorname{cosech} ax$	$-a \operatorname{cosech} ax \coth ax$
$\coth ax$	$-a \operatorname{cosech}^2 ax$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\sin^{-1} f(x)$	$\frac{f'(x)}{\sqrt{1 - [f(x)]^2}}$
$\cos^{-1} \frac{x}{a}$	$\frac{-1}{\sqrt{a^2 - x^2}}$

$\cos^{-1}f(x)$	$\frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$
$\tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$
$\tan^{-1}f(x)$	$\frac{f'(x)}{1+[f(x)]^2}$
$\sec^{-1}\frac{x}{a}$	$\frac{a}{x\sqrt{x^2-a^2}}$
$\sec^{-1}f(x)$	$\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$
$\operatorname{cosec}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{x^2-a^2}}$
$\operatorname{cosec}^{-1}f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$
$\cot^{-1}\frac{x}{a}$	$\frac{-a}{a^2+x^2}$
$\cot^{-1}f(x)$	$\frac{-f'(x)}{1+[f(x)]^2}$
$\sinh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2+a^2}}$
$\sinh^{-1}f(x)$	$\frac{f'(x)}{\sqrt{[f(x)]^2+1}}$
$\cosh^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{x^2-a^2}}$
$\cosh^{-1}f(x)$	$\frac{f'(x)}{\sqrt{[f(x)]^2-1}}$
$\tanh^{-1}\frac{x}{a}$	$\frac{a}{a^2-x^2}$
$\tanh^{-1}f(x)$	$\frac{f'(x)}{1-[f(x)]^2}$
$\operatorname{sech}^{-1}\frac{x}{a}$	$\frac{-a}{x\sqrt{a^2-x^2}}$

$\operatorname{sech}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{1-[f(x)]^2}}$
$\operatorname{cosech}^{-1} \frac{x}{a}$	$\frac{-a}{x\sqrt{x^2+a^2}}$
$\operatorname{cosech}^{-1} f(x)$	$\frac{-f'(x)}{f(x)\sqrt{[f(x)]^2+1}}$
$\operatorname{coth}^{-1} \frac{x}{a}$	$\frac{a}{a^2-x^2}$
$\operatorname{coth}^{-1} f(x)$	$\frac{f'(x)}{1-[f(x)]^2}$

### Product rule

When  $y = uv$  and  $u$  and  $v$  are functions of  $x$ , then:

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$


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### Quotient rule

When  $y = \frac{u}{v}$  and  $u$  and  $v$  are functions of  $x$ , then:

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$


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### Function of a function

If  $u$  is a function of  $x$ , then:  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

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### Parametric differentiation

If  $x$  and  $y$  are both functions of  $\theta$ , then:

$$\frac{dy}{dx} = \frac{dy}{d\theta} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{d\theta} \left( \frac{dy}{dx} \right)$$


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### Implicit function

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$


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### Maximum and minimum values

If  $y = f(x)$  then  $\frac{dy}{dx} = 0$  for stationary points.

Let a solution of  $\frac{dy}{dx} = 0$  be  $x = a$ ; if the value of  $\frac{d^2y}{dx^2}$  when  $x = a$  is:

- *positive*, the point is a *minimum*
  - *negative*, the point is a *maximum*.
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### Points of inflexion

(i) Given  $y = f(x)$ , determine  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$

(ii) Solve the equation  $\frac{d^2y}{dx^2} = 0$

(iii) Test whether there is a change of sign occurring in  $\frac{d^2y}{dx^2}$ . This is achieved by substituting into

the expression for  $\frac{d^2y}{dx^2}$  first a value of  $x$  just less than the solution and then a value just

greater than the solution.

(iv) A point of inflexion has been found if  $\frac{d^2y}{dx^2} = 0$  **and** there is a change of sign.

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### Velocity and acceleration

If distance  $x = f(t)$ , then

velocity  $v = f'(t)$  or  $\frac{dx}{dt}$  and acceleration  $a = f''(t)$  or  $\frac{d^2x}{dt^2}$

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### Tangents and normals

The equation of tangent to curve  $y = f(x)$  at the point  $(x_1, y_1)$  is:

$$y - y_1 = m(x - x_1) \quad \text{where } m = \text{gradient of curve at } (x_1, y_1)$$

The equation of normal to curve  $y = f(x)$  at the point  $(x_1, y_1)$  is:

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

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### Partial differentiation

*Total differential* If  $z = f(u, v, \dots)$ , then the total differential,

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \dots$$

*Rate of change* If  $z = f(u, v, \dots)$  and  $\frac{du}{dt}, \frac{dv}{dt}, \dots$  denote the rate of change of  $u, v, \dots$

respectively, then the rate of change of  $z$ ,

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \dots$$

*Small changes* If  $z = f(u, v, \dots)$  and  $\delta x, \delta y, \dots$  denote small changes in  $x, y, \dots$  respectively,

then

the corresponding change,

$$\delta z \approx \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y + \dots$$

*To determine maxima, minima and saddle points for functions of two variables*

Given  $z = f(x, y)$ ,

(i) determine  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$

(ii) for stationary points,  $\frac{\partial z}{\partial x} = 0$  and  $\frac{\partial z}{\partial y} = 0$

(iii) solve the simultaneous equations  $\frac{\partial z}{\partial x} = 0$  and  $\frac{\partial z}{\partial y} = 0$  for  $x$  and  $y$ , which gives the

coordinates of the stationary points,

(iv) determine  $\frac{\partial^2 z}{\partial x^2}$ ,  $\frac{\partial^2 z}{\partial y^2}$  and  $\frac{\partial^2 z}{\partial x \partial y}$

(v) for each of the coordinates of the stationary points, substitute values of  $x$  and  $y$  into

$\frac{\partial^2 z}{\partial x^2}$ ,  $\frac{\partial^2 z}{\partial y^2}$  and  $\frac{\partial^2 z}{\partial x \partial y}$  and evaluate each

(vi) evaluate  $\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$  for each stationary point

(vii) substitute the values of  $\frac{\partial^2 z}{\partial x^2}$ ,  $\frac{\partial^2 z}{\partial y^2}$  and  $\frac{\partial^2 z}{\partial x \partial y}$  into the equation

$$\Delta = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 - \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) \text{ and evaluate}$$

(viii)(a) if  $\Delta > 0$  then the stationary point is a **saddle point**

(b) if  $\Delta < 0$  and  $\frac{\partial^2 z}{\partial x^2} < 0$ , then the stationary point is a **maximum point**

(c) if  $\Delta < 0$  and  $\frac{\partial^2 z}{\partial x^2} > 0$ , then the stationary point is a **minimum point**