

FORMULAE/REVISION HINTS FOR SECTION J

INTEGRAL CALCULUS

Standard integrals

y	$\int y dx$
ax^n	$a \frac{x^{n+1}}{n+1} + c$ (except where $n = -1$)
$\cos ax$	$\frac{1}{a} \sin ax + c$
$\sin ax$	$-\frac{1}{a} \cos ax + c$
$\sec^2 ax$	$\frac{1}{a} \tan ax + c$
$\operatorname{cosec}^2 ax$	$-\frac{1}{a} \cot ax + c$
$\operatorname{cosec} ax \cot ax$	$-\frac{1}{a} \operatorname{cosec} ax + c$
$\sec ax \tan ax$	$\frac{1}{a} \sec ax + c$
e^{ax}	$\frac{1}{a} e^{ax} + c$
$\frac{1}{x}$	$\ln x + c$
$\tan ax$	$\frac{1}{a} \ln(\sec ax) + c$
$\cos^2 x$	$\frac{1}{2} \left(x + \frac{\sin 2x}{2} \right) + c$
$\sin^2 x$	$\frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + c$
$\tan^2 x$	$\tan x - x + c$
$\cot^2 x$	$-\cot x - x + c$
$\frac{1}{\sqrt{(a^2 - x^2)}}$	$\sin^{-1} \frac{x}{a} + c$
$\sqrt{(a^2 - x^2)}$	$\frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{(a^2 - x^2)} + c$

$$\frac{1}{a^2 + x^2}$$

$$\frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\frac{1}{\sqrt{(x^2 + a^2)}}$$

$$\sinh^{-1} \frac{x}{a} + c \text{ or } \ln \left[\frac{x + \sqrt{(x^2 + a^2)}}{a} \right] + c$$

$$\sqrt{(x^2 + a^2)}$$

$$\frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{x}{2} \sqrt{(x^2 + a^2)} + c$$

$$\frac{1}{\sqrt{(x^2 - a^2)}}$$

$$\cosh^{-1} \frac{x}{a} + c \text{ or } \ln \left[\frac{x + \sqrt{(x^2 - a^2)}}{a} \right] + c$$

$$\sqrt{(x^2 - a^2)}$$

$$\frac{x}{2} \sqrt{(x^2 - a^2)} - \frac{a^2}{2} \cosh^{-1} \frac{x}{a} + c$$

$t = \tan \frac{\theta}{2}$ substitution

To determine $\int \frac{1}{a \cos \theta + b \sin \theta + c} d\theta$,

$$\text{let } \sin \theta = \frac{2t}{1+t^2}, \cos \theta = \frac{1-t^2}{1+t^2} \text{ and } d\theta = \frac{2dt}{1+t^2}$$

Integration by parts

If u and v are both functions of x , then:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Reduction formulae

$$\int x^n e^x dx = I_n = x^n e^x - nI_{n-1}$$

$$\int x^n \cos x dx = I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}$$

$$\int_0^\pi x^n \cos x dx = I_n = -n\pi^{n-1} - n(n-1)I_{n-2}$$

$$\int x^n \sin x dx = I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}$$

$$\int \sin^n x \, dx = I_n = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}$$

$$\int \cos^n x \, dx = I_n = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = I_n = \frac{n-1}{n} I_{n-2}$$

$$\int \tan^n x \, dx = I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\int (\ln x)^n \, dx = I_n = x(\ln x)^n - n I_{n-1}$$

With reference to Figure FJ1

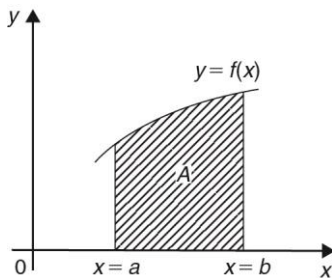


Figure FJ1

$$\text{Area } A = \int_a^b y \, dx$$

$$\text{Mean value} = \frac{1}{b-a} \int_a^b y \, dx$$

$$\text{R.m.s. value} = \sqrt{\left\{ \frac{1}{b-a} \int_a^b y^2 \, dx \right\}}$$

$$\text{Volume of solid of revolution} = \int_a^b \pi y^2 \, dx \quad \text{about the } x\text{-axis}$$

Centroids

With reference to Figure FJ2

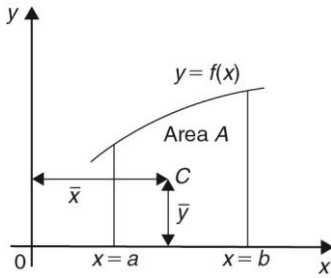


Figure FJ2

$$\bar{x} = \frac{\int_a^b xy \, dx}{\int_a^b y \, dx} \quad \text{and} \quad \bar{y} = \frac{\frac{1}{2} \int_a^b y^2 \, dx}{\int_a^b y \, dx}$$

Theorem of Pappus

With reference to Figure FJ2, when the curve is rotated one revolution about the x -axis between the limits $x = a$ and $x = b$, the volume V generated is given by:

$$V = 2\pi A \bar{y}$$

Second moment of area and radius of gyration

Shape	Position of axis	Second moment of area, I	Radius of gyration, k
Rectangle length l breadth b	(1) Coinciding with b	$\frac{bl^3}{3}$	$\frac{l}{\sqrt{3}}$
	(2) Coinciding with l	$\frac{lb^3}{3}$	$\frac{b}{\sqrt{3}}$
	(3) Through centroid, parallel to b	$\frac{bl^3}{12}$	$\frac{l}{\sqrt{12}}$
	(4) Through centroid, parallel to l	$\frac{lb^3}{12}$	$\frac{b}{\sqrt{12}}$
Triangle Perpendicular height h base b	(1) Coinciding with b	$\frac{bh^3}{12}$	$\frac{h}{\sqrt{6}}$
	(2) Through centroid, parallel to base	$\frac{bh^3}{36}$	$\frac{h}{\sqrt{18}}$
	(3) Through vertex, parallel to base	$\frac{bh^3}{4}$	$\frac{h}{\sqrt{2}}$

Circle radius r	(1) Through centre, perpendicular to plane (i.e. polar axis)	$\frac{\pi r^4}{2}$	$\frac{r}{\sqrt{2}}$
	(2) Coinciding with diameter	$\frac{\pi r^4}{4}$	$\frac{r}{2}$
	(3) About a tangent	$\frac{5\pi r^4}{4}$	$\frac{\sqrt{5}}{2}r$
Semicircle radius r	Coinciding with diameter	$\frac{\pi r^4}{8}$	$\frac{r}{2}$

Parallel axis theorem

If C is the centroid of area A in Figure FJ3, then

$$Ak_{BB}^2 = Ak_{GG}^2 + Ad^2 \quad \text{or} \quad k_{BB}^2 = k_{GG}^2 + d^2$$

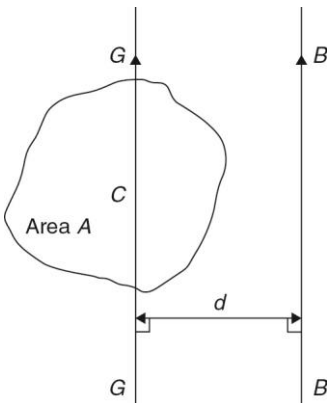


Figure FJ3

Perpendicular axis theorem

If OX and OY lie in the plane of area A in Figure FJ4, then

$$Ak_{OZ}^2 = Ak_{OX}^2 + Ak_{OY}^2 \quad \text{or} \quad k_{OZ}^2 = k_{OX}^2 + k_{OY}^2$$

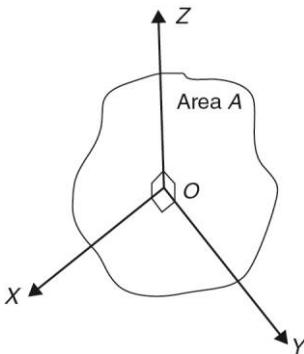


Figure FJ4

Numerical integration

Trapezoidal rule

$$\int y dx \approx \left(\begin{array}{c} \text{width of} \\ \text{interval} \end{array} \right) \left[\frac{1}{2} \left(\begin{array}{c} \text{first + last} \\ \text{ordinates} \end{array} \right) + \text{sum of remaining ordinates} \right]$$

Mid-ordinate rule

$$\int y dx \approx (\text{width of interval})(\text{sum of mid-ordinates})$$

Simpson's rule

$$\int y dx \approx \frac{1}{3} \left(\begin{array}{c} \text{width of} \\ \text{interval} \end{array} \right) \left[\left(\begin{array}{c} \text{first + last} \\ \text{ordinate} \end{array} \right) + 4 \left(\begin{array}{c} \text{sum of even} \\ \text{ordinates} \end{array} \right) + 2 \left(\begin{array}{c} \text{sum of remaining} \\ \text{odd ordinates} \end{array} \right) \right]$$
