

FORMULAE/REVISION HINTS FOR SECTION K

DIFFERENTIAL EQUATIONS

First-order differential equations

Separation of variables

$$\text{If } \frac{dy}{dx} = f(x) \text{ then } y = \int f(x) dx$$

$$\text{If } \frac{dy}{dx} = f(y) \text{ then } \int dx = \int \frac{dy}{f(y)}$$

$$\text{If } \frac{dy}{dx} = f(x).f(y) \text{ then } \int \frac{dy}{f(y)} = \int f(x) dx$$

Homogeneous equations

If $P \frac{dy}{dx} = Q$, where P and Q are functions of both x and y of the same degree throughout (i.e. a

homogeneous first-order differential equation), then:

(i) Rearrange $P \frac{dy}{dx} = Q$ into the form $\frac{dy}{dx} = \frac{Q}{P}$

(ii) Make the substitution $y = vx$ (where v is a function of x), from which, by the product rule,

$$\frac{dy}{dx} = v(1) + x \frac{dv}{dx}$$

(iii) Substitute for both y and $\frac{dy}{dx}$ in the equation $\frac{dy}{dx} = \frac{Q}{P}$

(iv) Simplify, by cancelling, and then separate the variables and solve using the $\frac{dy}{dx} = f(x).f(y)$

method

(v) Substitute $v = \frac{y}{x}$ to solve in terms of the original variables.

Linear first order

If $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only (i.e. a linear first-order differential equation), then

- (i) determine the integrating factor, $e^{\int P dx}$
- (ii) substitute the integrating factor (I.F.) into the equation

$$y(\text{I.F.}) = \int (\text{I.F.}) Q dx$$

- (iii) determine the integral $\int (\text{I.F.}) Q dx$

Numerical solutions of first-order differential equations

Euler's method: $y_1 = y_0 + h(y')_0$

Euler–Cauchy method: $y_{P_1} = y_0 + h(y')_0$

and $y_{C_1} = y_0 + \frac{1}{2}h[(y')_0 + f(x_1, y_{P_1})]$

Runge–Kutta method: To solve the differential equation $\frac{dy}{dx} = f(x, y)$ given the initial condition $y = y_0$ at $x = x_0$ for a range of values of $x = x_0 + h, x_0 + 2h, \dots$

- (i) Identify x_0 , y_0 and h , and values of x_1, x_2, x_3, \dots
- (ii) Evaluate $k_1 = f(x_n, y_n)$ starting with $n = 0$
- (iii) Evaluate $k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$
- (iv) Evaluate $k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right)$
- (v) Evaluate $k_4 = f(x_n + h, y_n + hk_3)$
- (vi) Use the values determined from steps (ii) to (v) to evaluate:

$$y_{n+1} = y_n + \frac{h}{6}\{k_1 + 2k_2 + 2k_3 + k_4\}$$

- (vii) Repeat steps (ii) to (vi) for $n = 1, 2, 3, \dots$

Second-order differential equations

If $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$ (where a, b and c are constants), then:

- (i) rewrite the differential equation as $(aD^2 + bD + c)y = 0$
- (ii) substitute m for D and solve the auxiliary equation $am^2 + bm + c = 0$
- (iii) if the roots of the auxiliary equation are:

(a) **real and different**, say $m = \alpha$ and $m = \beta$, then the general solution is

$$y = Ae^{\alpha x} + Be^{\beta x}$$

(b) **real and equal**, say $m = \alpha$ twice, then the general solution is

$$y = (Ax + B)e^{\alpha x}$$

(c) **complex**, say $m = \alpha \pm j\beta$, then the general solution is

$$y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

- (iv) given boundary conditions, constants A and B can be determined and the particular solution obtained.
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If $a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$ then:

- (i) rewrite the differential equation as $(aD^2 + bD + c)y = 0$
 - (ii) substitute m for D and solve the auxiliary equation $am^2 + bm + c = 0$
 - (iii) obtain the complimentary function (C.F.), u , as per (iii)
 - (iv) to find the particular integral, v , first assume a particular integral which is suggested by $f(x)$, but which contains undetermined coefficients (see Table 82.1, page 854 for guidance)
 - (v) substitute the suggested particular integral into the original differential equation and equate relevant coefficients to find the constants introduced
 - (vi) the general solution is given by $y = u + v$
 - (vii) given boundary conditions, arbitrary constants in the C.F. can be determined and the particular solution obtained
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Higher derivatives

y	$y^{(n)}$
e^{ax}	$a^n e^{ax}$
$\sin ax$	$a^n \sin\left(ax + \frac{n\pi}{2}\right)$
$\cos ax$	$a^n \cos\left(ax + \frac{n\pi}{2}\right)$
x^a	$\frac{a!}{(a-n)!} x^{a-n}$
$\sinh ax$	$\frac{a^n}{2} \left\{ \left[1 + (-1)^n \right] \sinh ax + \left[1 - (-1)^n \right] \cosh ax \right\}$
$\cosh ax$	$\frac{a^n}{2} \left\{ \left[1 - (-1)^n \right] \sinh ax + \left[1 + (-1)^n \right] \cosh ax \right\}$
$\ln ax$	$(-1)^{n-1} \frac{(n-1)!}{x^n}$

Leibniz's theorem

To find the n th derivative of a product $y = uv$:

$$y^{(n)} = (uv)^{(n)} = u^{(n)}v + nu^{(n-1)}v^{(1)} + \frac{n(n-1)}{2!}u^{(n-2)}v^{(2)} + \frac{n(n-1)(n-2)}{3!}u^{(n-3)}v^{(3)} + \dots$$

Power series solutions of second-order differential equations

(a) Leibniz–Maclaurin method

- (i) Differentiate the given equation n times, using the Leibniz theorem
- (ii) Rearrange the result to obtain the recurrence relation at $x = 0$
- (iii) Determine the values of the derivatives at $x = 0$, i.e. find $(y)_0$ and $(y')_0$
- (iv) Substitute in the Maclaurin expansion for $y = f(x)$
- (v) Simplify the result where possible and apply the boundary condition (if given).

(b) *Frobenius method*

- (i) Assume a trial solution of the form: $y = x^c \{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_r x^r + \dots\}$ $a_0 \neq 0$
 - (ii) Differentiate the trial series to find y' and y''
 - (iii) Substitute the results in the given differential equation
 - (iv) Equate coefficients of corresponding powers of the variable on each side of the equation: this enables index c and coefficients a_1, a_2, a_3, \dots from the trial solution to be determined.
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Bessel's equation

The solution of $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \nu^2)y = 0$

is:
$$y = Ax^\nu \left\{ 1 - \frac{x^2}{2^2(\nu+1)} + \frac{x^4}{2^4 \times 2!(\nu+1)(\nu+2)} - \frac{x^6}{2^6 \times 3!(\nu+1)(\nu+2)(\nu+3)} + \dots \right\}$$
$$+ Bx^{-\nu} \left\{ 1 + \frac{x^2}{2^2(\nu-1)} + \frac{x^4}{2^4 \times 2!(\nu-1)(\nu-2)} + \frac{x^6}{2^6 \times 3!(\nu-1)(\nu-2)(\nu-3)} + \dots \right\}$$

or, in terms of **Bessel functions** and **gamma functions**:

$$y = AJ_\nu(x) + BJ_{-\nu}(x) = A \left(\frac{x}{2}\right)^\nu \left\{ \frac{1}{\Gamma(\nu+1)} - \frac{x^2}{2^2(1!)\Gamma(\nu+2)} + \frac{x^4}{2^4(2!)\Gamma(\nu+4)} - \dots \right\}$$
$$+ B \left(\frac{x}{2}\right)^{-\nu} \left\{ \frac{1}{\Gamma(1-\nu)} - \frac{x^2}{2^2(1!)\Gamma(2-\nu)} + \frac{x^4}{2^4(2!)\Gamma(3-\nu)} - \dots \right\}$$

In general terms: $J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!) \Gamma(\nu+k+1)}$ and $J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} (k!) \Gamma(k-\nu+1)}$

and in particular: $J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{n!} - \frac{1}{(n+1)!} \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)(n+2)!} \left(\frac{x}{2}\right)^4 - \dots \right\}$

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \dots$$

and $J_1(x) = \frac{x}{2} - \frac{x^3}{2^3(1!)(2!)} + \frac{x^5}{2^5(2!)(3!)} - \frac{x^7}{2^7(3!)(4!)} + \dots$

Legendre's equation

The solution of $(1-x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + k(k+1)y = 0$

is:
$$y = a_0 \left\{ 1 - \frac{k(k+1)}{2!}x^2 + \frac{k(k+1)(k-2)(k+3)}{4!}x^4 - \dots \right\}$$
$$+ a_1 \left\{ x - \frac{(k-1)(k+2)}{3!}x^3 + \frac{(k-1)(k-3)(k+2)(k+4)}{5!}x^5 - \dots \right\}$$

Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2 - 1)^n}{dx^n}$$
