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***Solutions Manual of
Optimal and Robust
Control: Advanced Topics
with MATLAB®***



1

Solutions manual

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1.1 Introduction

This manual is complementary to the book “Optimal and robust control: Advanced Topics with MATLAB®”, CRC Press, referred in the following to as the textbook. It contains the solutions to the exercises given at the end

of each chapter. Although some solutions are given in analytical forms, for most of them the use of MATLAB® is required. The detailed procedures are all explained in the textbook, so that major emphasis is here given to the solution flow.

1.2 Solutions of exercises of Chapter 2

1.2.1 Exercise 2.1

Exercise: Apply the vectorization method to solve the Lyapunov equation:

$$A^T P + P A = -I$$

with

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -5 & -1 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution: The solution of this exercise follows MATLAB® exercise 2.3 of the textbook. We have first to calculate matrix $M = I \otimes A^T + A^T \otimes I$ (where I is

the 3×3 identity matrix) and then to solve $P = -M^{-1}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. One gets:

$$P = \begin{bmatrix} 1.7045 & 0.2973 & -0.3649 \\ 0.2973 & 0.1494 & -0.0506 \\ -0.3649 & -0.0506 & 0.4577 \end{bmatrix}$$

1.2.2 Exercise 2.2

Exercise: Given the nonlinear system $\dot{x} = x^3 - 8x^2 + 17x + u$ calculate the equilibrium points for $u = -10$ and study their stability.

Solution: The equilibrium points are given by: $\dot{x} = 0$, which yields:

$$x^3 - 8x^2 + 17x - 10 = 0$$

Since

$$\begin{aligned} x^3 - 8x^2 + 17x - 10 &= x^3 - x^2 - 7x^2 + 7x + 10x - 10 = \\ &= (x - 1)(x^2 - 7x + 10) = (x - 1)(x - 2)(x - 5) \end{aligned}$$

the equilibrium points are $\bar{x}_1 = 1$, $\bar{x}_2 = 2$ and $\bar{x}_3 = 5$.

Their stability is now calculated by linearizing the system around each of these equilibrium, i.e., by evaluating $\left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}}$.

Since $\left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}_1=1} = 4$, \bar{x}_1 is unstable.

Since $\left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}_2=2} = -3$, \bar{x}_2 is stable.

Since $\left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}_3=5} = 12$, \bar{x}_3 is unstable.

1.2.3 Exercise 2.3

Exercise: Given the nonlinear system

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_2 \\ \dot{x}_2 = -2x_1 - x_2 + x_2^3 \end{cases}$$

analyze numerically the stability of the equilibrium point $\bar{x} = 0$.

Solution: Define the suitable MATLAB function to integrate the system dynamics as

```
function xdot=nonlinsysEx2_3(t,x)
    xdot=[-x(1)+2*x(2);
          -2*x(1)-x(2)+x(2)^3];
end
```

and calculate the trajectory of the system for different initial conditions with the command

```
>> [T,Y]=ode45(@nonlinsys3_2,[0:0.01:20],[x1 x2])
```

where **x1** and **x2** are the initial conditions. It can be seen that for values of the initial conditions around the equilibrium point the system converges, otherwise choosing initial conditions far from the origin leads the system to instability. Thus, the equilibrium is locally asymptotically stable.

To characterize numerically this behavior, the following code can be used:

```
warning off;
```

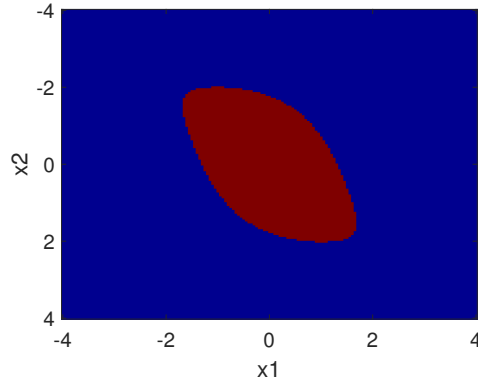


FIGURE 1.1

Exercise 2.3: basin of attraction of the equilibrium $\bar{x} = 0$.

```
bas=zeros(161);
ind1=0;
for x1=-4:.05:4,
    ind1=ind1+1;
    ind2=0;
    for x2=-4:.05:4,
        ind2=ind2+1;
        [T,Y]=ode45(@nonlinsys3_2,[0:0.01:20],[x1 x2]);
        if (abs(Y(end,1))>1e-2 || abs(Y(end,2))>1e-2)
            bas(ind1,ind2)=0;
        else bas(ind1,ind2)=1;
        end
    end
end
>> imagesc(-4:0.05:4,-4:0.05:4,bas)
```

The code generates a map, reported in Figure 1.1, where the red area indicates the basin of attraction of the equilibrium $\bar{x} = 0$, i.e. those initial conditions for which the system is locally asymptotically stable.

1.2.4 Exercise 2.4

Exercise: Study the stability of system with transfer function

$$G(s) = \frac{s^2 + 2s + 2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

with $1 \leq a_1 \leq 3$, $4 \leq a_2 \leq 7$, $1 \leq a_3 \leq 2$, $0.5 \leq a_4 \leq 2$.

Solution: We can study the stability of this *uncertain* system by applying the Kharitonov criterion. Let us consider therefore:

$$\begin{aligned} D_1(s) &= s^4 + s^3 + 7s^2 + 2s + 0.5 \\ D_2(s) &= s^4 + 3s^3 + 4s^2 + s + 2 \\ D_3(s) &= s^4 + 3s^3 + 7s^2 + s + 0.5 \\ D_4(s) &= s^4 + s^3 + 4s^2 + 2s + 2 \end{aligned}$$

By calculating the roots of these polynomial (using MATLAB® command `roots`), we find that $D_1(s)$, $D_3(s)$ and $D_4(s)$ have all roots with negative real part, while $D_2(s)$ has two positive real part roots. Therefore, we cannot state that for any value of the parameters the uncertain system is stable.

1.2.5 Exercise 2.5

Exercise: Given the polynomial $p(s, a) = s^4 + 5s^3 + 8s^2 + 8s + 3$ with $a = [3 \ 8 \ 8 \ 5]$, find $p(s, b)$ with $b = [(b_0^-, b_0^+) \ \dots \ (b_3^-, b_3^+)]$ so that the polynomial class $p(s, b)$ is Hurwitz.

Solution: It is possible to solve this problem numerically by exploiting the Kharitonov criterion. Let us reduce the coefficients in vector a by the same quantity δ . With the following MATLAB code the roots of the obtained polynomial can be analysed:

```
for delta=0:0.1:10
    b=[1 5-delta 8-delta 8-delta 3-delta];
    plot(real(roots(v)),imag(roots(v)),'b. ')
    pause
    hold on
end
```

We notice that for $\delta > -3$ the polynomial is Hurwitz. Therefore, we can fix $\delta < 3$ and verify that the polynomials $p_1(s) = s^4 + b_0^+ s^3 + b_1^+ s^2 + b_2^+ s + b_3^+$, $p_2(s) = s^4 + b_0^- s^3 + b_1^+ s^2 + b_2^- s + b_3^+$, and $p_3(s) = s^4 + b_0^+ s^3 + b_1^- s^2 + b_2^+ s + b_3^-$ are Hurwitz with $b_0^\pm = 5 \pm \delta$, $b_1^\pm = 8 \pm \delta$, $b_2^\pm = 8 \pm \delta$, $b_3^\pm = 3 \pm \delta$.

1.2.6 Exercise 2.6

Exercise: Study the stability of the system $G(s) = \frac{s^2 + 3s + 2}{s^4 + q_1 s^3 + 5s^2 + q_2 s + q_3}$ with parameters $q_1 \in [1, 3]$, $q_2 \in [5, 10]$, $q_3 \in [2, 18]$.

Solution: As for exercise 2.4, we can apply the Kharitonov criterion. Consider:

$$\begin{aligned} D_1(s) &= s^4 + s^3 + 5s^2 + 10s + 2 \\ D_2(s) &= s^4 + 3s^3 + 5s^2 + 5s + 18 \\ D_3(s) &= s^4 + 3s^3 + 5s^2 + 5s + 2 \\ D_4(s) &= s^4 + s^3 + 5s^2 + 10s + 18 \end{aligned}$$

The analysis of the roots of these polynomials reveal that only $D_3(s)$ has all the roots with negative real part. Therefore, there exist values of the parameters for which the system is unstable.

1.3 Solutions of exercises of Chapter 3

1.3.1 Exercise 3.1

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$A = \begin{bmatrix} -2 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 \\ -2 & 1 & -1 & -1 & 0 \\ -4 & 1 & 2 & -1 & -2 \end{bmatrix}; B = C^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: We first calculate the controllability and observability matrices:

$$M_c = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & -3 & -1 & -9 & -19 \\ 1 & -4 & 14 & -12 & 84 \end{bmatrix}; M_o^T = \begin{bmatrix} 1 & -6 & 32 & -104 & 384 \\ 1 & 4 & -26 & 94 & -362 \\ 1 & 4 & 10 & 34 & 94 \\ 1 & -2 & 4 & -8 & 16 \\ 1 & -2 & 4 & -8 & 16 \end{bmatrix}$$

Both have rank equal to 4 and in both cases the first four columns are linearly independent. Therefore, we can determine X_r as the subspace spanned by the first four columns of M_c and X_o as the subspace spanned by the first four columns of M_o^T . We then determine X_{nr} as the null space of M_c^T (MATLAB® command `Xnr=null(Mc')`) and X_{no} as the null space of M_o (MATLAB® command `Xno=null(Mo)`). We get: $X_{nr} = \begin{bmatrix} -0.7071 & 0.7071 & 0 & 0 & 0 \end{bmatrix}^T$ and $X_{no} = \begin{bmatrix} 0 & 0 & 0 & -0.7071 & 0.7071 \end{bmatrix}^T$.

We now determine X_A as the intersection of X_r and X_{no} . This can be done in MATLAB[®] with the following commands:

```
>> N=null([Xr -Xno])
>> [m,n]=size(Xr)
>> Xa=Xr*N(1:n,:)
```

We obtain that X_A is the subspace spanned by the vector $\mathbf{Xa} = \begin{bmatrix} 0 & 0 & 0 & -0.6542 & 0.6542 \end{bmatrix}^T$ (in fact, $X_A = X_{no}$).

We then calculate \bar{X}_B as the intersection of X_r and $(X_{nr} + X_o)$. Since in this case $X_{nr} \subset X_o$ we use the commands:

```
>> N=null([Xr -Xo])
>> [m,n]=size(Xr)
>> Xb=Xr*N(1:n,:)
```

$$\text{and get } \mathbf{Xb} = \begin{bmatrix} -0.2684 & 0.5936 & -0.4116 \\ -0.2684 & 0.5936 & -0.4116 \\ 4.1648 & -2.1798 & -4.8176 \\ -1.0047 & 0.5700 & -0.5190 \\ -1.0047 & 0.5700 & -0.5190 \end{bmatrix}.$$

Since $X_{nr} \subset X_o$, $\bar{X}_C = \emptyset$.

Finally, we calculate \bar{X}_D as the intersection of X_{nr} and X_o with the commands:

```
>> N=null([Xnr -Xo])
>> [m,n]=size(Xnr)
>> Xd=Xnr*N(1:n,:)
```

We get $\mathbf{Xd} = \begin{bmatrix} 0.7064 & -0.7064 & 0 & 0 & 0 \end{bmatrix}^T$ (in fact, $X_D = X_{nr}$).

We can thus define the state transformation matrix T as $T = [\mathbf{Xa} \ \mathbf{Xb} \ \mathbf{Xd}]$ and obtain the Kalman decomposition (we report the exact output of the MATLAB[®] procedure):

$$\tilde{A} = \left[\begin{array}{c|cccc} -1.0000 & 11.4958 & -6.7769 & -9.6240 & -1.0799 \\ -0.0000 & -5.7606 & 4.8221 & -0.1974 & 1.7245 \\ -0.0000 & -6.3240 & 5.3374 & 0.8976 & -0.5861 \\ -0.0000 & -4.7122 & 3.1111 & 2.4232 & 1.4628 \\ \hline 0.0000 & 0.0000 & -0.0000 & -0.0000 & -3.0000 \end{array} \right]; \tilde{B} = \left[\begin{array}{c} 0.0000 \\ 0.0636 \\ 1.2236 \\ -0.7062 \\ 0.0000 \end{array} \right]$$

$$\tilde{C} = \left[\begin{array}{cccc|c} 0.0000 & 1.6187 & 0.1474 & -6.6788 & -0.0000 \end{array} \right]$$

1.3.2 Exercise 3.2

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 \\ -2 & 1 & -1 & -1 & 0 \\ -4 & 1 & 2 & -1 & -2 \end{bmatrix}; B = C^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Following the same approach of the exercise previously discussed or using the MATLAB® command `minreal` (as discussed in MATLAB® exercise 3.1 reported in the textbook), we get the following Kalman decomposition (we report the exact output of the MATLAB® procedure):

$$\tilde{A} = \left[\begin{array}{cccc|c} -0.8507 & -2.8620 & 2.9046 & -1.2457 & -0.0000 \\ 1.0153 & -0.3045 & 2.8071 & -0.5976 & -0.0000 \\ 0.3160 & 2.4929 & 0.8342 & 1.2658 & -0.0000 \\ -0.3077 & -1.4424 & 0.9940 & -0.6791 & -0.0000 \\ \hline 0.9304 & 2.3019 & 0.5671 & 1.0070 & -1.0000 \end{array} \right]; \tilde{B} = \left[\begin{array}{c} 0.5513 \\ -0.4312 \\ -1.1159 \\ 1.8069 \\ \hline 0.0000 \end{array} \right]$$

$$\tilde{C} = [\ 0.5513 \quad -0.4312 \quad -1.1159 \quad 1.8069 \mid 0.0000 \]$$

which shows that the system has a fourth-order controllable and observable part and a first-order controllable and unobservable part.

1.3.3 Exercise 3.3

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & -3 & 2 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; C = [\ 0 \quad 1 \quad 1 \quad 0 \]$$

Solution: By applying the procedure discussed in exercise 3.1, we get:

$$X_A = \text{span} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \bar{X}_B = \text{span} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \bar{X}_C = \text{span} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \bar{X}_D = \text{span} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

This leads to the following state transformation matrix:

$$T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In this case, the Kalman decomposition is obtained by just a reordering of the original state variables. It is:

$$\tilde{A} = \left[\begin{array}{c|c|c|c} 2 & -3 & 1 & 1 \\ \hline 0 & 2 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array} \right]; \tilde{B} = \left[\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array} \right]$$

$$\tilde{C} = [0 \mid 1 \mid 0 \mid 1]$$

All the four parts (each of order 1) exist.

1.3.4 Exercise 3.4

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; C = [1 \quad 1 \quad 1 \quad 1]$$

Solution: The system is in diagonal form, from which it is immediate to derive that the system is fully observable, but not fully controllable (the order of the controllable part is 2).

We can apply the `minreal` command to obtain the Kalman decomposition. In this case too, the Kalman decomposition is obtained by a reordering of the original state variables:

$$\tilde{A} = \left[\begin{array}{c|c|c|c} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array} \right]; \tilde{B} = \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right]$$

$$\tilde{C} = [1 \quad 1 \mid 1 \quad 1]$$

1.3.5 Exercise 3.5

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution: This system too is in diagonal form. By applying the `minreal` command, the following Kalman decomposition is obtained:

$$\tilde{A} = \left[\begin{array}{cc|cc} -2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 \end{array} \right]; \tilde{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \tilde{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{Even in this case, the state transformation matrix } T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

represents a reordering of the original state variables.

1.3.6 Exercise 3.6

Exercise: Calculate the Kalman decomposition for the system with state-space matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & -3 & 4 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}; C = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

Solution: Use the MATLAB® command `minreal` command to get the following Kalman decomposition:

$$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & -3 & 4 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array} \right]; \tilde{B} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

The system is divided in two parts: a controllable and observable part of order 3 and an uncontrollable and observable part of order 1.

1.4 Solutions of exercises of Chapter 4

1.4.1 Exercise 4.1

Exercise: Calculate the singular value decomposition of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0.3 & -3 \\ -1 & -7 & 3 & -7 & -2 \\ 1 & 0.5 & 2 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Solution: Using the MATLAB[®] command `svd` (or equivalently `sqrt(eig(A*A'))`), one gets: $\sigma_1 = 10.6665$, $\sigma_2 = 3.5652$, $\sigma_3 = 2.4450$, $\sigma_4 = 1.6960$.

1.4.2 Exercise 4.2

Exercise: Calculate the singular value decomposition of $A = \begin{bmatrix} 2-j & j & 1 \\ -j & 3j & 1 \\ 7 & 1 & 6j \end{bmatrix}$.

Solution: The fact that matrix A is complex does not change the way in which singular values are computed. One can use either the MATLAB[®] command `svd` or `sqrt(eig(A'*A))` to get $\sigma_1 = 9.5679$, $\sigma_2 = 3.3196$ and $\sigma_3 = 1.1981$. In fact, the MATLAB[®] command `A'` for a complex matrix returns the conjugate transpose.

1.4.3 Exercise 4.3

Exercise: Calculate the condition number of matrix $A = \begin{bmatrix} -1 & 0.5 & 3 \\ 0.1 & 7 & 1 \\ 3 & -4 & -5 \end{bmatrix}$.

Solution: The matrix is invertible. Since $\sigma_1 = 9.3438$ and $\sigma_3 = 0.7104$, $h = \sigma_1/\sigma_3 = 13.1530$.

1.4.4 Exercise 4.4

Exercise: Calculate the inverse of $A = \begin{bmatrix} 2 & 0 & 2 & 2 \\ 2 & 5 & -7 & -11 \\ 0 & -2 & 6 & -7 \\ 0 & 2 & -6 & -7 \end{bmatrix}$.

Solution: We calculate the inverse of A , exploiting its singular value decomposition with the MATLAB® commands $[U, S, V] = \text{svd}(A)$ and $V * \text{inv}(S) * U'$:

$$A^{-1} = \begin{bmatrix} 0.6667 & -0.1667 & 0.0179 & 0.4345 \\ -0.5000 & 0.5000 & -0.0893 & -0.8393 \\ -0.1667 & 0.1667 & 0.0536 & -0.3631 \\ -0.0000 & -0.0000 & -0.0714 & -0.0714 \end{bmatrix}$$

1.4.5 Exercise 4.5

Exercise: Calculate the eigenvalues of

$$A = \begin{bmatrix} 2.5000 & -1.0000 & -3.6416 & -1.6416 \\ -21.9282 & 2.0359 & 13.7439 & 6.2080 \\ 13.4641 & -1.7679 & -8.1927 & -2.9248 \\ -13.4641 & 1.7679 & 11.3343 & 6.0664 \end{bmatrix}$$

and verify that they satisfy equation (4.3).

Solution: The eigenvalues of A are $\lambda_1 = 0.5000 + 0.9999i$, $\lambda_2 = 0.5000 - 0.9999i$, $\lambda_3 = -1.7320$ and $\lambda_4 = 3.1416$, and in modulus: $|\lambda_1| = 1.1179$, $|\lambda_2| = 1.1179$, $|\lambda_3| = 1.7320$ and $|\lambda_4| = 3.1416$. Since $\sigma_1 = 36.5160$ and $\sigma_4 = 0.0575$, the eigenvalues of A satisfy equation (4.3).

1.5 Solutions of exercises of Chapter 5

1.5.1 Exercise 5.1

Exercise: Given the continuous-time LTI system with state matrices

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

calculate the gramians and study system controllability and observability.

Solution: Let us consider $W_o = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$, where x , y and z indicate three unknowns (recall that the gramian is symmetrical), and solve the Lyapunov equation for the observability gramian:

$$\begin{aligned} \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} &= - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} -3y & -3z \\ x-2y & y-2z \end{bmatrix} + \begin{bmatrix} -3y & x-2y \\ -3z & y-2z \end{bmatrix} &= - \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ \Rightarrow \begin{cases} -6y = -1 \\ -3z + x - 2y = 1 \\ 2y - 4z = -1 \end{cases} \Rightarrow W_o = \begin{bmatrix} \frac{7}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \end{aligned}$$

Since the gramian is positive definite, the system is observable.

Let us now indicate $W_c = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the controllability gramian:

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & -3 \\ 1 & -2 \end{bmatrix} &= - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \Rightarrow \begin{cases} 2y = 0 \\ z - 3x - 2y = 0 \\ -6y - 4z = -1 \end{cases} \Rightarrow W_c = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \end{aligned}$$

W_c is positive definite and so the system is controllable.

1.5.2 Exercise 5.2

Exercise: Calculate analytically the singular values of the system with transfer function $G(s) = \frac{s+3}{(s+1)(s+7)}$.

Solution: Consider $G(s) = \frac{s+3}{s^2+8s+7}$ (which is stable, controllable and observable) and a minimal state-space realization of this transfer function given by:

$$A = \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [3 \quad 1]$$

Let us indicate $W_o = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the observability gramian:

$$\begin{aligned}
\begin{bmatrix} 0 & -7 \\ 1 & -8 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix} &= - \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \\
\Rightarrow \begin{bmatrix} -7y & -7z \\ x-8y & y-8z \end{bmatrix} + \begin{bmatrix} -7y & x-8y \\ -7z & y-8z \end{bmatrix} &= - \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \\
\Rightarrow \begin{cases} -14y = -9 \\ -7z + x - 8y = -3 \\ 2y - 16z = -1 \end{cases} \Rightarrow W_o = \begin{bmatrix} \frac{22}{7} & \frac{9}{14} \\ \frac{9}{14} & \frac{1}{7} \end{bmatrix}
\end{aligned}$$

Let us now indicate $W_c = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the controllability gramian:

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & -7 \\ 1 & -8 \end{bmatrix} &= - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
\Rightarrow \begin{cases} 2y = 0 \\ z - 7x - 8y = 0 \\ -14y - 16z = -1 \end{cases} \Rightarrow W_c = \begin{bmatrix} \frac{1}{112} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}
\end{aligned}$$

Thus, $W_c W_o = \begin{bmatrix} \frac{22}{784} & \frac{9}{1568} \\ \frac{9}{224} & \frac{1}{112} \end{bmatrix}$. Finally, the singular values are calculated as $\sigma_i = \sqrt{\lambda_i(W_c W_o)}$. We get: $\sigma_1 = 0.1909$ and $\sigma_2 = 0.0234$.

1.5.3 Exercise 5.3

Exercise: Calculate analytically the singular values of the system with transfer function $G(s) = \frac{s^2+1}{s^2+s+1}$.

Solution: The system is stable and minimal. Therefore, we can consider a minimal state-space realization of $G(s)$ given by:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [0 \quad -1]; D = 1$$

The singular values do not depend on D, since this matrix does not appear in the equations of the gramians. As in the previous exercise, let us indicate $W_o = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the observability gramian:

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2y = 0 \\ -z + x - y = 0 \\ 2y - 2z = -1 \end{cases} \Rightarrow W_o = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Let us now indicate $W_c = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the controllability gramian:

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2y = 0 \\ z - x - y = 0 \\ -2y - 2z = -1 \end{cases} \Rightarrow W_c = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The system is open-loop balanced with singular values $\sigma_1 = \sigma_2 = \frac{1}{2}$.

1.5.4 Exercise 5.4

Exercise: Calculate the singular values of the system with transfer function $G(s) = -10 + \frac{60s}{s^2+3s+2}$.

Solution: The system is stable and minimal. Therefore, we can consider a minimal state-space realization of $G(s)$ given by:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [0 \quad 60]; D = -10$$

As in the previous exercise, let us indicate $W_o = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the observability gramian:

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 3600 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 4y = 0 \\ x - 3y - 2z = 0 \\ 2y - 6z = -3600 \end{cases} \Rightarrow W_o = \begin{bmatrix} 1200 & 0 \\ 0 & 600 \end{bmatrix}$$

Let us now indicate $W_c = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$ and solve the Lyapunov equation for the controllability gramian:

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x & y \\ y & z \end{bmatrix} + \begin{bmatrix} x & y \\ y & z \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 2y = 0 \\ z - 2x - 3y = 0 \\ -4y - 6z = -1 \end{cases} \Rightarrow W_c = \begin{bmatrix} \frac{1}{12} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

Note that the gramians are diagonal, but not equal each other, so the system is not open-loop balanced. Since $W_c W_o = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$, the system singular values are $\sigma_1 = \sigma_2 = 10$. In fact, this system (see Chapter 6 of the textbook) is an all-pass system and $\sigma_1 = \sigma_2 = |k|$ where $k = -10$ as it can be immediately noticed if $G(s)$ is rewritten as: $G(s) = -10 \frac{(s-1)(s-2)}{(s+1)(s+2)}$.

1.5.5 Exercise 5.5

Exercise: Calculate the open-loop balanced realization of system with state-space matrices:

$$A = \begin{bmatrix} -0.5 & -1 & 0 & 0 \\ 1 & -0.5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}; B = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}; C = [0 \quad 1 \quad -1 \quad 1]$$

Solution: The system is in Jordan form, from which it is immediate to derive that it is i) stable; ii) controllable and iii) observable. We can thus use the MATLAB® command `balreal` to get the following open-loop balanced realization:

$$\bar{A} = \begin{bmatrix} -0.2013 & -0.7577 & 0.5968 & 0.04224 \\ 0.7577 & -0.2968 & 1.57 & 0.07057 \\ 0.5968 & -1.57 & -3.994 & -0.4918 \\ 0.04224 & -0.07057 & -0.4918 & -3.508 \end{bmatrix}; \bar{B} = \begin{bmatrix} -0.5671 \\ 0.5013 \\ 0.9624 \\ 0.05952 \end{bmatrix}$$

$$\bar{C} = [-0.5671 \quad -0.5013 \quad 0.9624 \quad 0.05952]; \bar{D} = 0$$

1.6 Solutions of exercises of Chapter 6

1.6.1 Exercise 6.1

Exercise: Given the system with transfer function

$$G(s) = 1 - \frac{56s^6 + 2676s^4 + 8736s^2 + 1600}{(s+10)^2(s+2)^3(s+1)^2}$$

determine an open-loop balanced realization and a suitable reduced order model.

Solution: It is immediate to verify that the system is stable and minimal. By using the MATLAB[®] command `balreal`, the open-loop balanced realization can be calculated:

$$\bar{A} = \begin{bmatrix} -0.3859 & -0.6288 & -1.362 & 0.4492 & 3.747 & 0.4233 & 0.6692 \\ -1.5 & -2.937 & -3.843 & 1.016 & 10.17 & 0.7449 & -1.799 \\ -1.397 & -3.767 & -4.931 & 2.825 & 13.46 & 1.022 & 2.183 \\ 0.501 & 1.605 & 0.5712 & -0.5849 & -6.447 & 0.1351 & 0.5077 \\ 0.7759 & 2.306 & 2.711 & 0.8786 & -13.25 & -0.3517 & 2.183 \\ -0.02296 & 0.3594 & 0.4087 & -0.6279 & -1.994 & -0.1038 & 6.373 \\ 2.325 & 10.06 & 8.518 & -4.193 & -19.73 & -7.926 & -5.806 \end{bmatrix};$$

$$\bar{B} = \begin{bmatrix} -0.8785 \\ -2.423 \\ -3.14 \\ 1.082 \\ 5.148 \\ 0.4557 \\ 3.408 \end{bmatrix};$$

$$\bar{C} = [0.8785 \quad 2.423 \quad 3.14 \quad -1.082 \quad -5.148 \quad -0.4557 \quad -3.408];$$

$$\bar{D} = 1$$

with system singular values $\sigma_1 = \dots = \sigma_7 = 1$. Since the system singular values are all equal, an open-loop reduced order model cannot be obtained. In fact, if we rewrite the transfer function as $G(s) = \frac{(s-10)^2(s-2)^3(s-1)^2}{(s+10)^2(s+2)^3(s+1)^2}$, it is immediate to verify that this system (see Chapter 6 of the textbook) is all-pass.

1.6.2 Exercise 6.2

Exercise: Calculate a reduced order model of the system $G(s) = 200 \frac{(s+10)(s^2+s+1)}{(s+5)^3(s+4)(s+2)^2}$.

Solution: We have first to verify that the system is stable and minimal. After this, the open-loop balanced realization can be calculated. It is given by:

$$\bar{A} = \begin{bmatrix} -0.7922 & -3.456 & 0.02344 & 1.122 & -0.9986 & -0.1104 \\ 3.456 & -1.611 & 0.08211 & 1.408 & -2.172 & -0.2168 \\ 0.02344 & -0.08211 & -0.001352 & -1.116 & 0.08479 & 0.01083 \\ -1.122 & 1.408 & 1.116 & -1.701 & 4.737 & 0.3989 \\ -0.9986 & 2.172 & 0.08479 & -4.737 & -9.828 & -2.126 \\ 0.1104 & -0.2168 & -0.01083 & 0.3989 & 2.126 & -9.067 \end{bmatrix};$$

$$\bar{B} = \begin{bmatrix} -1.231 \\ 1.273 \\ 0.02146 \\ -0.7294 \\ -0.8026 \\ 0.08574 \end{bmatrix};$$

$$\bar{C} = \begin{bmatrix} -1.231 & -1.273 & 0.02146 & 0.7294 & -0.8026 & -0.08574 \end{bmatrix};$$

$$\bar{D} = 0$$

and the system singular values are $\sigma_1 = 0.9569$, $\sigma_2 = 0.5033$, $\sigma_3 = 0.1704$, $\sigma_4 = 0.1564$, $\sigma_5 = 0.0328$ and $\sigma_6 = 0.0004$. They suggest to build a reduced order model of order 4 (for instance with direct truncation). The transfer function of this reduced order model is: $\tilde{G}(s) = \frac{-0.6368s^3 + 15.2s^2 + 6.776s + 16.47}{s^4 + 4.106s^3 + 17.85s^2 + 15.28s + 17.61}$.

1.6.3 Exercise 6.3

Exercise: Given the system $G(s) = \frac{s+1}{s^4 + 4.3s^3 + 7.92s^2 + 7.24s + 2.64}$ calculate the reduced order models with direct truncation and singular perturbation approximation and compare the models obtained.

Solution: The transfer function can be rewritten as:

$$G(s) = \frac{s+1}{(s+1.2)(s+1.1)(s^2+2s+2)}$$

which clearly shows that the system is stable and minimal. The singular values are $\sigma_1 = 0.2709$, $\sigma_2 = 0.0932$, $\sigma_3 = 0.0117$, $\sigma_4 = 0.0001$. Therefore, we can consider reduced order models of order 2. With direct truncation, we get: $G_{DT}(s) = \frac{-0.05094s + 0.3114}{s^2 + 1.153s + 0.876}$. With singular perturbation approximation, we get: $G_{SP}(s) = \frac{0.02329s^2 - 0.143s + 0.4618}{s^2 + 1.723s + 1.219}$. The two models can be compared, for instance, in terms of their step response (MATLAB® command `step`).

1.6.4 Exercise 6.4

Exercise: Given the continuous-time system with state-space matrices:

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 \\ -3 & -5 & 6 & 1 \\ 0 & -1 & -5 & 0 \\ -4 & -5 & -7 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}$$

calculate the reduced order models with direct truncation and singular perturbation approximation and the error and compare the rise and settling time of the reduced order models and original system.

Solution: It is a stable ($\lambda_{1,2} = -4.5173 \pm 3.2742i$, $\lambda_{3,4} = -1.9827 \pm 0.3420i$) and minimal (both M_c and M_o have full rank) system. Therefore, we can calculate the open-loop balanced realization and get the following singular values: $\sigma_1 = 0.3420$, $\sigma_2 = 0.0238$, $\sigma_3 = 0.0160$ and $\sigma_4 = 0.0006$.

Since $\sigma_1 \gg \sigma_2$, we can consider reduced order models of order 1. We get with direct truncation $G_t(s) = \frac{4.164}{s+6.088}$ and with singular perturbation approximation $G_p(s) = \frac{0.01442s+3.907}{s+5.594}$.

We calculate the error between approximated model and original model as in MATLAB® exercise 6.1 (MATLAB® command `normhinf`). We get for $G_t(s)$ an error equal to $E_t = 0.0326$ and for $G_p(s)$ an error equal to $E_p = 0.0460$. Both satisfy equation (6.14), since $2(\sigma_2 + \sigma_3 + \sigma_4) = 0.0806$.

The rise time and the settling time can be calculated using the MATLAB® tool `ltiview`. They are reported in Table 1.6.4.

TABLE 1.1

Rise and settling time for original system and reduced order models of exercise 4.

	rise time	settling time
original system	0.4s	1.26s
$G_t(s)$	0.361s	0.643s
$G_p(s)$	0.393s	0.7s

1.6.5 Exercise 6.5

Exercise: Given system $G(s) = \frac{s}{s^4+3s^3+4.1s^2+4.1s+0.2}$, choose the order of the reduced model to guarantee an error between nominal model and approximation not larger than 0.02.

Solution: By using the MATLAB® command `ltiview`, we can rewrite $G(s)$ as $G(s) = \frac{s}{(s+1.949)(s+0.05132)(s^2+s+2)}$ and observe that it is a stable and minimal system. It has singular values given by: $\sigma_1 = 0.2445$, $\sigma_2 = 0.1475$, $\sigma_3 = 0.1122$ and $\sigma_4 = 0.0152$. Since $2\sigma_4 > 0.02$, the specific of an error not larger than 0.02 cannot be satisfied.

1.6.6 Exercise 6.6

Exercise: Determine, if possible, the system with eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$ and with singular values $\sigma_1 = 5$ and $\sigma_2 = 2$.

Solution: Consider the open-loop parameterization with three unknowns: b_1 , b_2 and $s_1 s_2$ (which can be either +1 or -1). If $s_1 s_2 = 1$, we get:

$$A_1 = \begin{bmatrix} -\frac{b_1^2}{10} & -\frac{b_1 b_2}{7} \\ -\frac{b_1 b_2}{7} & -\frac{b_2^2}{4} \end{bmatrix}$$

while if $s_1 s_2 = -1$ we get:

$$A_{-1} = \begin{bmatrix} -\frac{b_1^2}{10} & \frac{b_1 b_2}{3} \\ -\frac{b_1 b_2}{3} & -\frac{b_2^2}{4} \end{bmatrix}$$

The characteristic polynomial of A_1 is $\varphi_1(\lambda) = \lambda^2 + \left(\frac{b_1^2}{10} + \frac{b_2^2}{4}\right)\lambda + b_1^2 b_2^2 \left(\frac{1}{40} - \frac{1}{49}\right)$, while that of A_{-1} is $\varphi_{-1}(\lambda) = \lambda^2 + \left(\frac{b_1^2}{10} + \frac{b_2^2}{4}\right)\lambda + b_1^2 b_2^2 \left(\frac{1}{40} + \frac{1}{9}\right)$. We equate the two polynomials with $\varphi(\lambda) = \lambda^2 + 3\lambda + 2$, so that, if $s_1 s_2 = 1$, we have:

$$\begin{cases} \frac{b_1^2}{10} + \frac{b_2^2}{4} = 3 \\ b_1^2 b_2^2 \left(\frac{1}{40} - \frac{1}{49}\right) = 2 \end{cases}$$

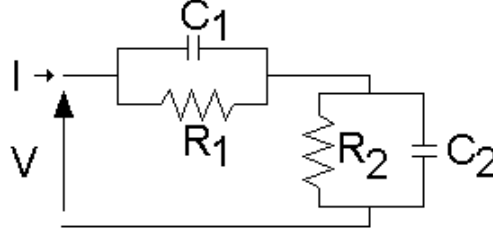
while, if $s_1 s_2 = -1$, we have:

$$\begin{cases} \frac{b_1^2}{10} + \frac{b_2^2}{4} = 3 \\ b_1^2 b_2^2 \left(\frac{1}{40} + \frac{1}{9}\right) = 2 \end{cases}$$

If $s_1 s_2 = 1$ the system has no real solution, while if $s_1 s_2 = -1$ we get two solutions: $b_1 = 1.1309$ and $b_2 = 3.3895$ or $b_1 = 5.3592$ and $b_2 = 0.7153$. Therefore, we have two systems that satisfy the given constraints:

$$A = \begin{bmatrix} -0.1279 & 1.2777 \\ -1.2777 & -2.8721 \end{bmatrix}; B = \begin{bmatrix} 1.1309 \\ 3.3895 \end{bmatrix}; C = [\ 1.1309 \quad -3.3895 \]$$

and

**FIGURE 1.2**

Circuit for exercise 6.8.

$$A = \begin{bmatrix} -2.8721 & 1.2777 \\ -1.2777 & -0.1279 \end{bmatrix}; B = \begin{bmatrix} 5.3592 \\ 0.7153 \end{bmatrix}; C = \begin{bmatrix} 5.3592 & -0.7153 \end{bmatrix}$$

1.6.7 Exercise 6.7

Exercise: Calculate analytically the Cauchy index of the system $G(s) = \frac{s+1}{s(s^2+s+1)}$.

Solution: By applying the Cauchy index definition (Definition 16 of the text-book) we have to study $G(\sigma) = \frac{\sigma+1}{\sigma^3+\sigma^2+\sigma}$, which has one jump from $-\infty$ to $+\infty$, so $I_c = 1$. We can also calculate the index by calculating the Hankel matrix.

$$H = M_o M_c = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

The eigenvalues of H are $\lambda_1 = -1.2470$, $\lambda_2 = 0.4450$ and $\lambda_3 = 1.8019$. Thus, $I_c = 1$. Also note that, since A has one zero eigenvalue, it is not possible to compute W_{co} .

1.6.8 Exercise 6.8

Exercise: Given the system in Figure 1.2 with $R_1 = R_2 = 1$, $C_1 = 1$ and $C_2 = \frac{1}{2}$ determine the transfer function and the Cauchy index.

Solution: The transfer function of the system is the series of two impedances, each given by the parallel of a resistor with a capacitor:

$$G(s) = \frac{R_1}{sC_1R_1 + 1} + \frac{R_2}{sC_2R_2 + 1} = \frac{3s + 4}{s^2 + 3s + 2}$$

It is a relaxation system, so $I_c = n = 2$ as it can be verified by calculating the Hankel matrix:

$$H = M_o M_c = \begin{bmatrix} 4 & 3 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$$

which has both eigenvalues positive ($\lambda_1 = 0.1690$, $\lambda_2 = 11.8310$).

1.6.9 Exercise 6.9

Exercise: Write down an example of a relaxation system and verify the value of the Cauchy index.

Solution: Let us consider for instance a third-order system given by:

$$G(s) = \frac{1}{s+2} + \frac{1}{s+5} + \frac{1}{s+7} = \frac{3s^2 + 28s + 59}{s^3 + 14s^2 + 59s + 70}$$

To verify that $I_c = 3$, we determine the minimal realization:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -70 & -59 & -14 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = [59 \quad 28 \quad 3]$$

and compute the cross-gramian W_{co} from the Lyapunov equation (we generi-

cally indicate the unknowns as $W_{co} = \begin{bmatrix} w_1 & w_2 & w_3 \\ w_4 & w_5 & w_6 \\ w_7 & w_8 & w_9 \end{bmatrix}$):

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -70 & -59 & -14 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 \\ w_4 & w_5 & w_6 \\ w_7 & w_8 & w_9 \end{bmatrix} + \begin{bmatrix} w_1 & w_2 & w_3 \\ w_4 & w_5 & w_6 \\ w_7 & w_8 & w_9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -70 & -59 & -14 \end{bmatrix} = \\ = - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 59 & 28 & 3 \end{bmatrix} \end{aligned}$$

$$\Rightarrow \begin{cases} -70w_3 + w_4 = 0 \\ w_1 - 59w_3 + w_5 = 0 \\ w_2 - 14w_3 + w_6 = 0 \\ -70w_6 + w_7 = 0 \\ w_4 - 59w_6 + w_8 = 0 \\ w_5 - 14w_6 + w_9 = 0 \\ -70w_1 - 59w_4 - 14w_7 - 70w_9 = -59 \\ -70w_2 - 59w_5 + w_7 - 14w_8 - 59w_9 = -28 \\ -70w_3 - 59w_6 + w_8 - 28w_9 = -3 \end{cases}$$

$$\Rightarrow W_{co} = \begin{bmatrix} 0.1622 & 0.0630 & 0.0058 \\ 0.4074 & 0.1812 & 0.0185 \\ 1.2963 & 0.6852 & 0.0780 \end{bmatrix}$$

Since the eigenvalues of W_{co} are $\lambda_1 = 0.3954$, $\lambda_2 = 0.0258$ and $\lambda_3 = 0.0003$, then $I_c = 3$.

1.6.10 Exercise 6.10

Exercise: Calculate the singular values of system $G(z) = \frac{3+2z+5z^2+6z^3}{z^4}$.

Solution: $G(z)$ is a FIR filter. The Hankel matrix is $H = \begin{bmatrix} 6 & 5 & 2 & 3 \\ 5 & 2 & 3 & 0 \\ 2 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$.

The eigenvalues of H are $\lambda_1 = -2.9670$, $\lambda_2 = -1.5766$, $\lambda_3 = 1.5793$ and $\lambda_4 = 10.9643$ and, thus, $\sigma_1 = 10.9643$, $\sigma_2 = 2.9670$, $\sigma_3 = 1.5793$ and $\sigma_4 = 1.5766$.

1.6.11 Exercise 6.11

Exercise: Consider an open-loop balanced representation of a discrete-time system. Derive a model of order $r < n$ and verify that its singular values do not coincide with the first r singular values of the original system.

Solution: Let us consider the system in Exercise 6.10 which admits the following state-space representation

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = [3 \quad 2 \quad 5 \quad 6]. \quad (1.1)$$

Let us calculate an open-loop balance representation using MATLAB with the command

```
[Ab,Bb,Cb,S,T]=dbalreal(A,B,C);
```

The singular values are those reported in the previous example, i.e. $\sigma_1 = 10.964$, $\sigma_2 = 2.967$, $\sigma_3 = 1.579$, and $\sigma_4 = 1.577$. We fix $r = 2$ and determine a reduced order model through direct truncation with the commands

```
Abr=Ab(1:2,1:2);
```

```
Bbr=Bb(1:2,1:2);
```

```
Cbr=Cb(1:2,1:2);
```

The singular values of the reduced order model are calculated using the command

```
[Abrb,Bbrb,Cbrb,S1,Tr]=dbalreal(Abr,Bbr,Cbr);
```

as $\bar{\sigma}_1 = 10.777 < \sigma_1$ and $\bar{\sigma}_2 = 2.479 < \sigma_2$.

1.7 Solutions of exercises of Chapter 7

1.7.1 Exercise 7.1

Exercise: Given the system $G(s) = \frac{s+2}{s^2+4s+6}$ calculate the optimal controller that minimizes the index defined by $Q = C^T C$ in these three cases: $r = r_1 = 1$, $r = r_2 = 0.01$ and $r = r_3 = 20$. Verify the performance of the control law obtained.

Solution: We consider a minimal realization of the system in canonical control form:

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -4 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

and calculate the linear quadratic regulator through the MATLAB® command `[K,E,L]=lqr(A,B,C'*C,r)` with different values of r . The results are summarized in Table 1.7.1 where the optimal eigenvalues are also compared with those obtained by the approximated Letov formulas discussed in Sec. 7.8 of the textbook.

1.7.2 Exercise 7.2

Exercise: Given the system with transfer function $G(s) = \frac{2s-1}{s(s-1)}$ calculate the optimal eigenvalues with respect to r , if the index to optimize is $J =$

TABLE 1.2

Linear quadratic regulator and optimal eigenvalues for exercise 1.

	K	optimal eigenvalues	appr. opt. eigenvalues
$r = r_1$	$\begin{bmatrix} 0.3246 & 0.2011 \end{bmatrix}$	$\lambda_{1,2} = -2.1005 \pm 1.3829i$	—
$r = r_2$	$\begin{bmatrix} 14.8806 & 8.0732 \end{bmatrix}$	$\lambda_1 = -9.9812, \lambda_2 = -2.0920$	$\lambda_1 = -10, \lambda_2 = -2$
$r = r_3$	$\begin{bmatrix} 0.0166 & 0.0104 \end{bmatrix}$	$\lambda_{1,2} = -2.0052 \pm 1.4127i$	$\lambda_{1,2} = -2 \pm 1.4142i$

$\int_0^\infty (\mathbf{y}^T \mathbf{y} + \mathbf{u}^T r \mathbf{u}) dt$. Then calculate the characteristic values of the system fixing r so that the optimal eigenvalue is $\lambda = -\sqrt{2}$.

Solution: The system is minimal and we can apply the Letov theorem so that the optimal poles are the negative real part solutions of:

$$s^4 + s^2 \left(-1 - \frac{4}{r} \right) + \frac{1}{r} = 0$$

which yields:

$$s_{1,2} = -\sqrt{\frac{1 + \frac{4}{r} \pm \sqrt{1 + \frac{16}{r^2} + \frac{4}{r}}}{2}}$$

To find that value of r so that one of the optimal poles is $-\sqrt{2}$, we can equate the polynomial $s^4 + s^2 \left(-1 - \frac{4}{r} \right) + \frac{1}{r}$ with $(s^2 - 2)(s^2 - b^2)$ where b is a parameter. This yields $r = \frac{7}{2}$.

1.7.3 Exercise 7.3

Exercise: Design the linear quadratic regulator ($r = 2$) for the system $G(s) = \frac{s^2 + 2s + 1}{s^3 - s^2 + 5s + 3}$ and calculate the optimal eigenvalues.

Solution: We consider a minimal realization of the system in canonical control form:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -5 & 1 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

and calculate the linear quadratic regulator through the MATLAB® command $[\mathbf{K}, \mathbf{E}, \mathbf{L}] = \text{lqr}(\mathbf{A}, \mathbf{B}, \mathbf{C}' * \mathbf{C}, 2)$. We get: $\mathbf{K} = \begin{bmatrix} 0.0822 & 1.7706 & 3.2452 \end{bmatrix}$ and $\lambda_{1,2} = -0.8600 \pm 2.2644i$ and $\lambda_3 = -0.5253$.

1.7.4 Exercise 7.4

Exercise: Given the system with state-space matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

design the linear quadratic regulator with $Q = C^T C$ and $r = 5$.

Solution: The system is controllable since it is in canonical control form. We can use the MATLAB® command `K=lqr(A,B,C'*C,5)` to calculate the linear quadratic regulator. We get $K = \begin{bmatrix} 6.0332 & 2.0332 \end{bmatrix}$.

1.7.5 Exercise 7.5

Exercise: Given the system with state-space matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

design the linear quadratic regulator with $Q = I$ and $r = 7$.

Solution: First, the controllability matrix has to be calculated (with the MATLAB® command `ctrb(A,B)`), since it is full rank, the system is controllable. We can then use the MATLAB® command `K=lqr(A,B,eye(3),7)` to get: $K = \begin{bmatrix} 0.2623 & -0.4658 & -7.0161 \end{bmatrix}$.

1.8 Solutions of exercises of Chapter 8

1.8.1 Exercise 8.1

Exercise: Given matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -8 & -8 & -5 \end{bmatrix} \quad (1.2)$$

determine $\|\Delta A\|_{max}$ so that $A + \Delta A$ is a matrix with eigenvalues with negative real part.

Solution: Observe that A has all eigenvalues with negative real part. So, we can apply equation (8.10) of the textbook. To calculate $\|(sI - A)^{-1}\|_{\infty}$ in MATLAB[®], we determine the H_{∞} norm of a fictitious system with $B = C = I$. We get: $\|(sI - A)^{-1}\|_{\infty} = 4.5463$. Thus $\|\Delta A\|_{max} = 1/4.5463 = 0.22$. As an example consider a perturbation $\|\Delta A\| = 0.20I$ and note that $A + \Delta A$ has all eigenvalues with negative real part.

1.8.2 Exercise 8.2

Exercise: Given the system with transfer function

$$G(s) = \frac{1}{(1-s)(s+2)^2(s+0.5)^2}$$

determine a balanced closed-loop realization and a suitable reduced order model. Then verify that the model is closed-loop stable.

Solution: We consider a minimal realization of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 3.25 & -3.25 & -4 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; C = [-1 \quad 0 \quad 0 \quad 0 \quad 0]$$

and use the command `balrealMcC` introduced in Section 8.4 of the textbook. The following characteristic values are obtained: $\mu_1 = 47.0192$, $\mu_2 = 0.4970$, $\mu_3 = 0.0971$, $\mu_4 = 0.0072$, $\mu_5 = 0.0002$. We consider a second-order ($r = 2$) reduced model (calculated through direct truncation). Its transfer function is: $G(s) = \frac{0.0934s - 0.1466}{s^2 - 0.8983s - 0.1016}$. We then verify that the closed-loop system is stable by imposing $\bar{P}_2 = 0$ and calculating the closed-loop eigenvalues (see also MATLAB[®] exercise 8.2 of the textbook). They are $\lambda_{1,2} = -2.0165 \pm 0.1153i$, $\lambda_{3,4} = -0.5876 \pm 0.2284i$ and $\lambda_5 = -0.8628$ and so the closed-loop system is stable.

1.8.3 Exercise 8.3

Exercise: Calculate the characteristic values for the system with transfer function $G(s) = \frac{2s}{s^2 + 1}$.

Solution: We consider a minimal realization of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [0 \quad 2]$$

and solve the CARE and FARE equations. This yields:

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}; \Pi = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix};$$

Since $P\Pi = I$, the characteristic values are $\mu_1 = \mu_2 = 1$.

1.8.4 Exercise 8.4

Exercise: Calculate the closed-loop balanced realization for the system with transfer function $G(s) = \frac{s+1}{s^5+7s^2+6s+5}$. Design a reduced order regulator and observer making sure that the closed-loop system is asymptotically stable.

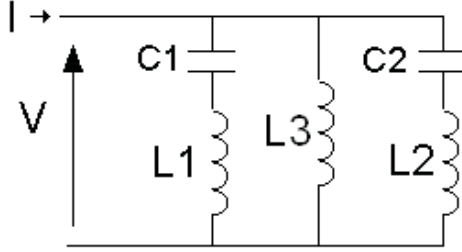
Solution: First of all, we consider a minimal realization of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -5 & -6 & -7 & 0 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; C = [1 \quad 1 \quad 0 \quad 0 \quad 0]$$

and use the command `balrealncc` introduced in Section 8.4 of the textbook to find the closed-loop balanced realization and calculate the system characteristic values: $\mu_1 = 126.9094$, $\mu_2 = 41.4091$, $\mu_3 = 0.1344$, $\mu_4 = 0.0496$ and $\mu_5 = 0.0016$. We then calculate the right hand term of equation (8.14). Since $f(A_c) = 1/8.64$ and $\|\tilde{B}\tilde{B}^T\| = 0.2657$, r has to be selected such that $\mu_{r+1} \leq 0.4361$. Thus, we can choose $r = 2$ and obtain a second-order reduced model with direct truncation. Finally, we can verify that the closed-loop system with the reduced order controller is stable (see also MATLAB® exercise 8.2 of the textbook) by calculating the closed-loop eigenvalues which are all with negative real part: $\lambda_{1,2} = -1.2618 \pm 1.6939i$, $\lambda_{3,4} = -0.4008 \pm 0.6989i$ and $\lambda_5 = -1.7371$.

1.8.5 Exercise 8.5

Exercise: Given the system with transfer function $G(s) = \frac{s}{s^2+1} + \frac{s^2+1}{s(s^2+2)}$ calculate the characteristic values and synthesize the system with circuit components.

**FIGURE 1.3**

Circuit synthesizing $G(s) = \frac{s}{s^2+1} + \frac{s^2+1}{s(s^2+2)}$.

Solution: $G(s)$ can be rewritten as $G(s) = \frac{2s^4+4s^2+1}{s^5+3s^3+2s}$, from which a minimal realization is derived:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -3 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; C = [1 \quad 0 \quad 4 \quad 0 \quad 2]$$

We now solve the CARE and FARE equations:

$$P = \begin{bmatrix} 2 & 0 & 3 & 0 & 1 \\ 0 & 5 & 0 & 3 & 0 \\ 3 & 0 & 9 & 0 & 4 \\ 0 & 3 & 0 & 2 & 0 \\ 1 & 0 & 4 & 0 & 2 \end{bmatrix}; \Pi = \begin{bmatrix} 2 & 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & -3 & 0 \\ -2 & 0 & 3 & 0 & -5 \\ 0 & -3 & 0 & 5 & 0 \\ 3 & 0 & -5 & 0 & 9 \end{bmatrix};$$

Since $P\Pi = I$, the characteristic values are $\mu_1 = \dots = \mu_5 = 1$. In fact, the system is a loss-less one, being given by the series of two loss-less systems.

We synthesize the system by considering $G(s) = \frac{I(s)}{V(s)}$ and implementing the parallel of two circuital blocks, the first synthesizing $\frac{I_1(s)}{V(s)} = \frac{s}{s^2+1}$ and the second $\frac{I_2(s)}{V(s)} = \frac{s^2+1}{s(s^2+2)}$. The circuit is shown in Figure 1.3. The transfer function, given by $\frac{I(s)}{V(s)} = \frac{sC_1}{1+s^2C_1L_1} + \frac{1+s^2C_2L_2+s^2C_2L_3}{sL_3(1+s^2C_2L_2)}$, matches $G(s)$ if $C_1 = L_1 = 1$, $C_2 = 1/4$, $L_2 = L_3 = 2$.

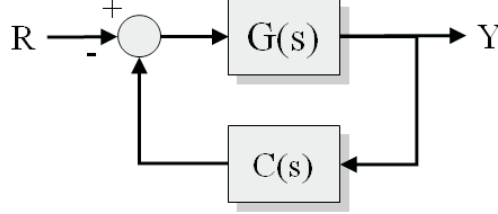


FIGURE 1.4
Block scheme for exercise 8.6.

1.8.6 Exercise 8.6

Exercise: Given the system with transfer function $G(s) = \frac{1}{s-1}$ calculate the linear state regulator, optimal observer and the compensator $C(s)$ (see Figure 1.4), using the CARE and FARE equations.

Solution: Consider a minimal realization of $G(s)$: $A = B = C = 1$. The CARE is: $2p - p^2 + 1 = 0$ and so $P = 1 + \sqrt{2}$. The FARE is exactly the same equations and so $\Pi = 1 + \sqrt{2}$. Since $B = C = 1$, $K = 1 + \sqrt{2}$ (linear state regulator) and $H = 1 + \sqrt{2}$ (optimal observer). We now calculate $C(s)$ which is given by $C(s) = K(sI - A)^{-1}H = \frac{(1+\sqrt{2})^2}{s+1+2\sqrt{2}}$. Finally, we calculate the closed-loop transfer function $W(s) = \frac{G(s)}{1+C(s)G(s)} = \frac{s+1+2\sqrt{2}}{(s+\sqrt{2})^2}$. Note that the closed-loop poles are those of the optimal regulator and optimal observer (in this case they coincide).

1.8.7 Exercise 8.7

Exercise: Calculate, if possible, the system that has characteristic values $\rho_1 = 5$ e $\rho_2 = 1$ and eigenvalues $\lambda_1 = -5$ and $\lambda_2 = -1$.

Solution: The closed-loop parameterization for a SISO system is given by:

$$a_{ii} = \frac{\mu_i^2 - 1}{2\mu_i} b_i^2$$

$$a_{ij} = \frac{\mu_j(1 + \mu_i^2) - s_i s_j \mu_i(1 + \mu_j^2)}{\mu_i^2 - \mu_j^2} b_i b_j$$

with $b_i = s_i c_i$.

Let us consider two different cases: $s_1 s_2 = 1$ and $s_1 s_2 = -1$.

In the first case, we get:

$$A_1 = \begin{bmatrix} 2.4b_i^2 & \frac{2}{3}b_1b_2 \\ \frac{2}{3}b_1b_2 & 0 \end{bmatrix}$$

while for $s_1s_2 = -1$ we get:

$$A_{-1} = \begin{bmatrix} 2.4b_i^2 & \frac{3}{2}b_1b_2 \\ -\frac{3}{2}b_1b_2 & 0 \end{bmatrix}$$

In both cases (as it is evident from the characteristic polynomial, $\varphi_1(\lambda) = \lambda^2 - 2.4b_1^2 - \frac{4}{9}b_1^2b_2^2$ or $\varphi_{-1}(\lambda) = \lambda^2 - 2.4b_1^2 + \frac{9}{4}b_1^2b_2^2$) the system is unstable for each value of b_1 and b_2 and, thus, it cannot have $\lambda_1 = -5$ and $\lambda_2 = -1$.

1.8.8 Exercise 8.8

Exercise: Verify that the characteristic values of an all-pass system are all equal to k .

Solution: Consider for instance $G(s) = 2\frac{(1-s)(5-s)}{(1+s)(5+s)} = 2 - \frac{24}{s^2+6s+5}$. Consider now a minimal realization of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [0 \quad -24]$$

and solve the CARE and FARE equations for not-strictly proper systems (equations (B.5) and (B.6)). We get:

$$P = \begin{bmatrix} 240 & 0 \\ 0 & 48 \end{bmatrix}; \Pi = \begin{bmatrix} 1/60 & 0 \\ 0 & 1/12 \end{bmatrix};$$

and so $\mu_1 = \mu_2 = 2$.

1.8.9 Exercise 8.9

Exercise: Design a reduced order model for system $G(s) = \frac{s+1}{s^3+5s^2+7s-2}$.

Solution: Consider a minimal realization of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -7 & -5 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; C = [1 \quad 1 \quad 0]$$

and solve the CARE and FARE equations. Then, compute the characteristic values as $\mu_i = \sqrt{\lambda_i(\Pi P)}$. One gets: $\mu_1 = 3.9444$, $\mu_2 = 0.0309$, and $\mu_3 = 0.0144$. Consider thus a first-order reduced model (obtained for instance with direct truncation) and verify (following the procedure of MATLAB® exercise

8.2 of the textbook) that the closed-loop system is stable. In fact, the closed-loop eigenvalues with the reduced order approximation are: $\lambda_1 = -0.2691$, $\lambda_{2,3} = -2.6238 \pm 1.1960i$.

1.8.10 Exercise 8.10

Exercise: Given the system

$$G(s) = \frac{1}{(1-s)(s+2)^2(s+0.5)^2}$$

determine a closed-loop balanced realization and a reduced order model. Verify then that the reduced order model is stable in closed-loop.

Solution: Consider a minimal realization of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 4 & 3.25 & -3.25 & -4 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}; C = [-1 \quad 0 \quad 0 \quad 0 \quad 0]$$

and solve the CARE and FARE equations. Then, compute the characteristic values as $\mu_i = \sqrt{\lambda_i(\Pi P)}$. One gets: $\mu_1 = 47.019$, $\mu_2 = 0.4970$, $\mu_3 = 0.097$, $\mu_4 = 0.007$ and $\mu_5 = 0.0002$. Consider thus a first-order reduced model and calculate the closed-loop eigenvalues with the reduced order approximation of the gain vector which are $\lambda_1 = -2.1073$, $\lambda_2 = -1.8383$, $\lambda_3 = -1.1593$, and $\lambda_{4,5} = -0.4493 \pm 0.4407j$. Hence the closed-loop system is stable.

1.9 Solutions of exercises of Chapter 9

1.9.1 Exercise 9.1

Exercise: Given the continuous-time system with transfer function $G(s) = \frac{\alpha}{s^3 + s^2 + 4s + 4}$ calculate for which values of α the system is bounded-real.

Solution: $G(s)$ can be rewritten as $G(s) = \frac{\alpha}{(s+1)(s^2+4)}$ which clearly shows

that $G(s)$ has a pair of imaginary poles. Therefore, the system is not asymptotically stable. There are no values of α for which the system is bounded-real.

1.9.2 Exercise 9.2

Exercise: Given the continuous-time system with transfer function $G(s) = \frac{4}{s^2 + \alpha s + 4}$ calculate for which values of α the system is bounded-real.

Solution: The system is second-order. Recall that a second-order system written as $G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ has a resonance peak equal to $M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$ for $0 < \xi < \frac{1}{\sqrt{2}}$. Since in our case $\alpha = 4\xi$, we can conclude that $G(s)$ is bounded real if $\alpha \geq 2\sqrt{2}$.

1.9.3 Exercise 9.3

Exercise: Given the continuous-time system with state-space matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \quad C = [1 \quad 0 \quad 0]$$

calculate for which values of α the system is bounded-real.

Solution: The system is in control canonical form. Its transfer function is $G(s) = \frac{\alpha}{s^3 + 2s^2 + 3s + 5} = \frac{\alpha}{(s+1.844)(s^2+0.1563s+2.712)}$. The system is asymptotically stable. Consider $G(s)/\alpha$. The maximum of the Bode magnitude plot of system $G(s)/\alpha$ occurs at $\omega = 1.64 \text{ rad/s}$ and is equal to 1.575. Therefore, $\|G(s)\|_\infty = 1.575\alpha$ and so $G(s)$ is bounded real if $|\alpha| \leq 1/1.575$.

1.9.4 Exercise 9.4

Exercise: Given the continuous-time system with state-space matrices:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}; \quad B = C^T = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix}$$

calculate for which values of α the system is strictly bounded-real.

Solution: This is a stable system in diagonal form. Its transfer function is

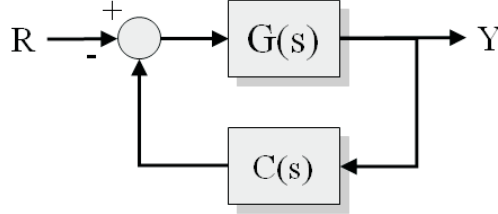


FIGURE 1.5

Block scheme for exercise 9.5.

$G(s) = \frac{\alpha}{s+1} + \frac{\alpha}{s+2} + \frac{\alpha}{s+0.5}$, which can be rewritten as $G(s) = \alpha \frac{3s^2+7s+3.5}{s^3+3.5s^2+3.5s+1}$ or $G(s) = \alpha \frac{3(s+1.608)(s+0.7257)}{(s+2)(s+1)(s+0.5)}$. Consider the system $G(s)/\alpha$. Its Bode magnitude plot has a maximum (equal to 3.5) for $\omega \rightarrow 0$. Thus, $\|G(s)\|_\infty = 3.5\alpha$ and so $G(s)$ is bounded real if $|\alpha| \leq 1/3.5$.

1.9.5 Exercise 9.5

Exercise: For the feedback system shown in Figure 1.5 with $G(s) = \frac{1}{s^2}$ prove, if possible, that there is a stable first order compensator which makes the system closed-loop passive. Note: do not make any cancellations.

Solution: Given a generic first-order $C(s) = \frac{s+\alpha}{s+\beta}$, the closed-loop transfer function is $W(s) = \frac{G(s)}{1+C(s)G(s)} = \frac{s+\beta}{s^3+\beta s^2+s+\alpha}$. Since the relative degree of $W(s)$ is 2, the system cannot be passive.

1.9.6 Exercise 9.6

Exercise: Given the system with transfer function $G(s) = \frac{s+1}{(s+2)(s+3)}$ calculate the Cauchy index of the system with two different analytical methods. Then calculate the energy associated with the impulse response. Finally, using the bounded-real lemma, verify analytically if $G(s)$ is bounded-real.

Solution: Let us rewrite $G(s)$ as $G(s) = \frac{-1}{s+2} + \frac{2}{s+3}$. Thus, since the two residuals have opposite signs, $I_c = 0$.

Let us now consider a minimal realization of the system given by:

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [1 \quad 1]$$

and calculate

$$M_c = \begin{bmatrix} 0 & 1 \\ 1 & -5 \end{bmatrix}; M_o = \begin{bmatrix} 1 & 1 \\ -6 & -4 \end{bmatrix}; H = M_o M_c = \begin{bmatrix} 1 & -4 \\ -4 & 14 \end{bmatrix};$$

Since the eigenvalues of H are $\lambda_1 = -0.1322$ and $\lambda_2 = 15.1322$, then $I_c = 0$.

The energy associated to the impulse response is calculated as $E = B^T W_o^2 B$ and is $E = 0.1167$.

To prove that the system is bounded-real, we calculate the Hamiltonian matrix as in equation (10.32) of the textbook:

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -5 & 0 & 1 \\ -1 & -1 & 0 & 6 \\ -1 & -1 & -1 & 5 \end{bmatrix}$$

The characteristic polynomial of H is $\varphi(\lambda^4) = \lambda^4 - 12\lambda^2 + 35$, whose roots are $\lambda_{1,2} = \pm\sqrt{7}$ and $\lambda_{3,4} = \pm\sqrt{5}$.

1.9.7 Exercise 9.7

Exercise: Determine for which values of α the system with transfer function $G(s) = \frac{\alpha}{2s^3 + s^2 + 4s + 5}$ is bounded-real using two different methods.

Solution: Rewrite $G(s)$ as $G(s) = \frac{0.5}{(s+1)(s^2 - 0.5s + 2.5)}$ and observe that the system is not asymptotically stable. Thus, it is not bounded-real.

1.9.8 Exercise 9.8

Exercise: Given the system with transfer function $G(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+4)}$, propose, if possible, a passive electric circuit realization.

Solution: Note preliminarily that $G(s)$ is lossless. Consider the system in Figure 9.7(b) of the textbook and calculate the transfer function $F(s) = \frac{I(s)}{V(s)}$. This is given by:

$$F(s) = \frac{sC_1(1 + s^2L_2C_2 + s^2L_1C_2)}{s^4C_1C_2L_1L_2 + s^2(C_1L_1 + C_2L_2 + C_2L_1) + 1}$$

Let us rewrite $G(s)$ as $G(s) = \frac{s\frac{1}{2}(\frac{s^2}{2}+1)}{(s^2+1)(\frac{s^2}{4}+1)}$ and equate $G(s)$ with $F(s)$ to get: $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{9}$, $L_1 = \frac{3}{2}$ and $L_2 = 3$.

1.9.9 Exercise 9.9

Exercise: Given the system $G(s) = \frac{1}{(s+1)^2}$ determine if there is an R such that the optimal closed-loop system (with functional $J = \int_0^\infty (y^T y + u^T R u) dt$) is passive.

Solution: Recall that the state feedback does not change the zeros of the system, i.e., the closed-loop system has the same zeros of the open-loop system. Therefore, the closed-loop systems will have a numerator of degree zero, while a denominator of degree two. Therefore, the closed-loop system will never be passive, since the difference of the degrees is greater than one.

1.9.10 Exercise 9.10

Exercise: Write down an example of a third-order lossless system and verify that its characteristic values are equal to one.

Solution: Consider for instance $G(s) = \frac{s^2+1}{s^3+4s}$. To verify that the characteristic values are all equal to one, consider a minimal realization given by:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; C = [1 \quad 0 \quad 1]$$

and solve the CARE and FARE. We get:

$$P = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}; \Pi = \begin{bmatrix} 1/3 & 0 & -1/3 \\ 0 & 1/3 & 0 \\ -1/3 & 0 & 4/3 \end{bmatrix}$$

Since $P\Pi = I$, the system has all characteristic values equal to one.

1.9.11 Exercise 9.11

Exercise: Determine for which values of z_1 and p_1 the continuous-time system with transfer function $G(s) = \frac{s+z_1}{s^2(s+p_1)}$ is negative-imaginary.

Solution: Since $G(s)$ has two poles at the origin, we should consider Definition 25 of Chapter 9. Condition 1) requires that there are no poles in the open right half of the complex plane, from which we derive that p_1 must be positive, i.e., $p_1 > 0$. Considering condition 4), then $\lim_{s \rightarrow 0} s^2 G(s) = \frac{z_1}{p_1}$ must be positive, and thus $z_1 > 0$. Finally, to meet condition 2), we must have that:

$$j(G(j\omega) - G(-j\omega)) = \frac{2(p_1 - z_1)}{\omega(\omega^2 + p_1^2)} > 0$$

that yields: $0 < z_1 < p_1$.

1.9.12 Exercise 9.12

Exercise: Given the system in positive feedback configuration (as in Figure 1.6) with $G_1(s) = \frac{40}{s+40}$ and $G_2(s) = \frac{5}{s^2+5s+\alpha}$, find for which values of $\alpha > 0$ the closed-loop system is stable.

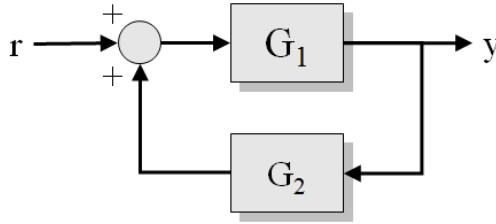


FIGURE 1.6

Positive-feedback configuration with negative-imaginary $G_1(s)$ and $G_2(s)$.

Solution: Since $G_1(s)$ is negative-imaginary and $G_2(s)$ is so for $\alpha > 0$, we can leverage the results of Theorem 27 of Chapter 9 to solve the problem. Notice that $G_1(s)$ and $G_2(s)$ satisfy the hypotheses of the theorem, so we only need to check that $\rho(G_1(0)G_2(0)) < 1$. This yields $5/\alpha < 1$ and so $\alpha > 5$.

1.9.13 Exercise 9.13

Exercise: Consider the continuous-time system with transfer function $G(s) = \frac{1}{s^2+3s+4}$. If possible, derive an electric circuit realization and a mechanical one.

Solution: For the mechanical realization, one can refer to Example 9.2 of Chapter 9. Selecting $M = 1$, $k = 4$, $b = 3$, and $y = x$, we obtain a mechanical system with a transfer function $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2+sb+k} = \frac{1}{s^2+3s+4}$.

For the electric circuit realization, we can start with the configuration of Figure 1.7 and determine $Z(s)$ to meet the requirement of the problem. The circuit has a transfer function equal to $G(s) = \frac{1}{1+sCZ(s)}$. Therefore, $Z(s)$ must satisfy:

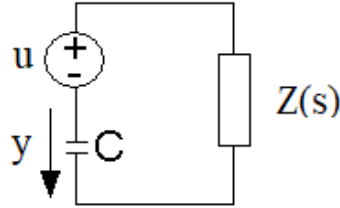


FIGURE 1.7

Electrical circuit that can implement a negative-imaginary transfer function.

$$1 + sCZ(s) = s^2 + 3s + 4$$

and then

$$Z(s) = \frac{s^2 + 3s + 4}{sC} = \frac{s}{C} + \frac{3}{C} + \frac{4}{sC}$$

which corresponds to the circuit of Figure 1.8 with an inductor $L = \frac{1}{C}$, a resistor $R = \frac{3}{C}$ and a capacitor $C_1 = \frac{C}{3}$. Here C (the value of the capacitor where the output is measured) remains a free parameter.

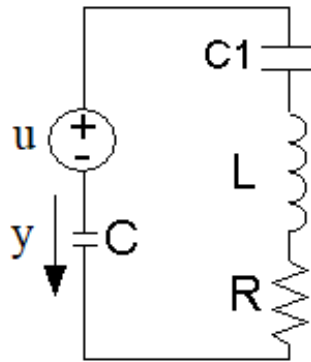


FIGURE 1.8

Electrical circuit that can be modeled by $G(s) = \frac{1}{s^2 + 3s + 4}$ selecting $L = \frac{1}{C}$, a resistor $R = \frac{3}{C}$ and a capacitor $C_1 = \frac{C}{3}$.

1.10 Solutions of exercises of Chapter 10

1.10.1 Exercise 10.1

Exercise: Given the continuous-time system with transfer function $G(s) = \frac{7-s}{s^2+7s+10}$, find $F(s)$ such that $G(s) + F(s)$ is positive-real.

Solution: The system $G(s) = \frac{7-s}{s^2+7s+10}$ is not positive-real. In fact, one has:

$$G(j\omega) = \frac{7(10 - \omega^2) - 7\omega^2}{(10 - \omega^2)^2 + 49\omega^2} + j \frac{\omega(\omega^2 - 10) - 49\omega}{(10 - \omega^2)^2 + 49\omega^2}$$

such that the real part is not always positive for $\omega > 0$. Now, notice that $G(s)$ may be rewritten as

$$G(s) = \frac{3}{s+2} - \frac{4}{s+5}$$

Hence, if we consider for instance $F(s) = \frac{8}{s+5}$, then we will get

$$G(s) + F(s) = \frac{3}{s+2} + \frac{4}{s+5}$$

that is positive-real, as it is a relaxation system.

1.10.2 Exercise 10.2

Exercise: Given the continuous-time system with transfer function $G(s) = \frac{7-s}{s^2+7s+10}$, find $F(s)$ such that $G(s) + F(s)$ is negative-imaginary.

Solution: The system $G(s)$ is not negative-imaginary, as the imaginary part of $G(j\omega)$ is given by:

$$G_I(j\omega) = \frac{\omega^3 - 59\omega}{(10 - \omega^2)^2 + 49\omega^2}$$

a function which is not always negative for $\omega > 0$.

Let us fix $\alpha = 2$. We have then to study the function $4\frac{G_I(j\omega)}{\omega} + G_I(j\omega)\omega$. This study can be done numerically or analytically, finding that the maximum of this function for $\omega > 0$ is one. At this point we can select $F(s) = \frac{k}{s+2}$ with $k > 1$. Let us select for instance $k = 2$ to obtain

$$G(s) + F(s) = \frac{s + 17}{s^2 + 7s + 10}$$

This is a negative-imaginary function as it can be verified by the Nyquist plot or by calculating its imaginary part:

$$G_I(j\omega) + F_I(j\omega) = \frac{-\omega^3 - 109\omega}{(10 - \omega^2)^2 + 49\omega^2}$$

1.10.3 Exercise 10.3

Exercise: Given a system $G(s) = \frac{s}{(s+5)^2}$, find a forward action $F(s)$ such that the system $G(s) + F(s)$ is negative-imaginary. Use $F(s) = F_1(s) = \frac{k}{s+2}$ and repeat the exercise with $F(s) = F_2(s) = \frac{k}{s}$. Then, compare the results.

Solution: Let us first calculate $G(j\omega)$:

$$G(j\omega) = \frac{j\omega}{(5 - j\omega)^2} = -\frac{10\omega^2}{(25 + \omega^2)^2} + j\frac{\omega(25 - \omega^2)}{(25 + \omega^2)^2}$$

from which it follows that $G_I(j\omega) = \frac{\omega(25 - \omega^2)}{(25 + \omega^2)^2}$.

If $F(s) = F_1(s) = \frac{k}{s+2}$ is used, then the minimum value of k to obtain a negative-imaginary system is given by:

$$k = \max_{\omega > 0} \frac{(4 + \omega^2)(25 - \omega^2)}{(25 + \omega^2)^2} \simeq 6.2$$

Otherwise, when $F(s) = F_2(s) = \frac{k}{s}$ is used, then the minimum value of k is:

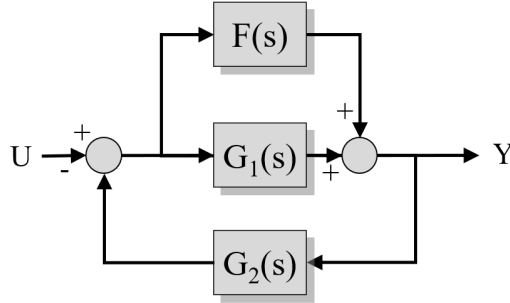
$$k = \max_{\omega > 0} \frac{\omega^2(25 - \omega^2)}{(25 + \omega^2)^2} \simeq 4.3$$

Notice that, in the second case, a lower gain is required, but an imaginary-system with a pole in the origin (i.e., not asymptotically stable) is obtained.

1.10.4 Exercise 10.4

Exercise: Consider the system in Figure 1.9 with $G_1(s) = \frac{s-2}{s^2+2s+1}$ and $G_2(s) = \frac{1}{s+5}$. Find a transfer function $F(s)$ such that the closed-loop system is stable.

Solution: Since in Figure 1.9 the feedback is negative and $G_2(s) = \frac{1}{s+5}$ is

**FIGURE 1.9**

Block scheme for exercise 10.4.

strictly positive-real, then we can select $F(s)$ such that $G_1(s) + F(s)$ is positive-real and then obtain a stable closed-loop system in virtue of Theorem 22 of Chapter 9.

Let us then consider $G_1(s) + F(s)$ and calculate its real part:

$$G_{1,R}(j\omega) + F_R(j\omega) = \frac{k\omega^2 + 4\omega^2 + k - 2}{(1 + \omega^2)^2}$$

Selecting $k > 2$ guarantees that $G_1(s) + F(s)$ is positive-real and then the closed-loop system is stable.

1.10.5 Exercise 10.5

Exercise: Consider the system in Figure 1.10 with $G_1(s) = \frac{s-2}{s^2+2s+1}$ and $G_2(s) = \frac{1}{s+5}$. Find a transfer function $F(s)$ such that the closed-loop system is stable.

Solution: In Figure 1.10 the feedback is positive, so (taking into account that $G_2(s) = \frac{1}{s+5}$ is also strictly negative-imaginary) we can select $F(s)$ such that $G_1(s) + F(s)$ is positive-real and then check the condition on the static gain $\rho((G_1(0) + F(s))G_2(0)) < 1$.

We have that $G_{1,I}(j\omega) = \frac{\omega(5-\omega^2)}{(1+\omega^2)^2}$. Considering $F(s) = \frac{k}{s+\alpha}$, then $G_{1,I}(j\omega) + F_I(j\omega) = \frac{(5-\omega^2)(\alpha^2+\omega^2)}{(1+\omega^2)^2}$. Now, setting for instance $\alpha = 2$, this function has a maximum equal to 20. So, $k > 20$. On the other hand, from the condition on the static gain we derive that $k < 14$, thus $\alpha = 2$ cannot be used. Instead, if we select $\alpha = 1$, the maximum of $G_{1,I}(j\omega) + F_I(j\omega) = \frac{(5-\omega^2)(\alpha^2+\omega^2)}{(1+\omega^2)^2}$ is equal to 5. On the other hand, from the condition on the static gain one

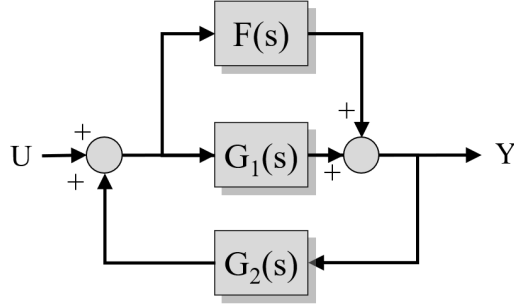


FIGURE 1.10
Block scheme for exercise 10.5.

has $k < 7$, such that the problem can be solved selecting $F(s) = \frac{k}{s+1}$ with $5 < k < 7$.

1.11 Solutions of exercises of Chapter 11

1.11.1 Exercise 11.1

Exercise: Calculate analytically the H_∞ norm for the system with transfer function $G(s) = \frac{2s+1}{s+2}$.

Solution: The system $G(s)$ can be written as $G(s) = \frac{2s+1}{s+2} = \frac{2(s+2)-3}{s+2} = 2 - \frac{3}{s+2}$. Hence, $|G(j\omega)| = 2 - \frac{3}{\sqrt{\omega^2+4}}$ whose maximum value corresponds to $\|G(s)\|_\infty = 2$.

1.11.2 Exercise 11.2

Exercise: Given the continuous-time system with state-space matrices:

$$A = -1; \quad B = \begin{bmatrix} 1 & 2 \end{bmatrix}; \quad C = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

determine analytically the H_∞ norm.

Solution: The method based on the bounded-real lemma can be applied. The Hamiltonian matrix associated to the Riccati equation is:

$$H = \begin{bmatrix} -1 & \frac{5}{\gamma^2} \\ -25 & 1 \end{bmatrix}$$

whose characteristic polynomial is $p(\lambda) = \lambda^2 + \frac{125}{\gamma^2} - 1$. The H_∞ norm can be obtained solving $1 - \frac{125}{\gamma^2} = 0$, hence $\gamma = \sqrt{125} \approx 11.18$.

1.11.3 Exercise 11.3

Exercise: Given the continuous-time system with state-space matrices:

$$A = -3; \quad B = \begin{bmatrix} 1 & 3 \end{bmatrix}; \quad C = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

determine analytically the H_∞ norm.

Solution: The method based on the bounded-real lemma can be applied also in this case. The Hamiltonian matrix associated to the Riccati equation is:

$$H = \begin{bmatrix} -3 & \frac{10}{\gamma^2} \\ -117 & 3 \end{bmatrix}$$

whose characteristic polynomial is $p(\lambda) = \lambda^2 + \frac{1170}{\gamma^2} - 9$. The H_∞ norm can be obtained solving $9 - \frac{1170}{\gamma^2} = 0$, hence $\gamma = \sqrt{\frac{1170}{9}} \approx 11.40$.

1.11.4 Exercise 11.4

Exercise: Given the system

$$\begin{cases} \dot{x}_1 = x_1 + u + 2w \\ y = x_1 + w \\ z = 2x_1 + 2u \end{cases}$$

calculate analytically the compensator $C(s)$ with the H_∞ control technique.

Solution: The H_∞ controller can be designed with the MATLAB[®] function `hinfric` through the following command:

```
>> [gopt,C] = hinfric(P,[1 1])
```

where P is the process to be controlled defined as a `ltisys` object. The controller $C(s) = -173.2 \frac{1}{s+89.58}$ allows to obtain a performance $\|T_{z_\infty w}\|_\infty < 4.02$.

1.11.5 Exercise 11.5

Exercise: Calculate the performance of the H_∞ control for the system:

$$\begin{cases} \dot{x} = 2x + 0.1w + u \\ z = 2x + 0.7u \\ y = -x + 2w \end{cases} \quad (1.3)$$

Solution: The H_∞ controller can be designed with the MATLAB® function `hinfli` through the following command:

```
>> [gopt,C] = hinfli(P,[1 1])
```

where P is the process to be controlled defined as a `ltisys` object. The controller $C(s) = 26160 \frac{1}{s+9159}$ allows to obtain a performance $\|T_{z_\infty w}\|_\infty < 4.002$.

1.12 Solutions of exercises of Chapter 12**1.12.1 Exercise 12.1**

Exercise: Using LMI techniques, determine the H_∞ norm of the system with transfer function $G(s) = \frac{s+1}{(s+2)(s+3)}$.

Solution: Choosing the following state-space representation of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = [1 \quad 1]$$

the LMI problem for H_∞ norm calculation can be written as:

$$\begin{aligned} &P > 0 \\ &\begin{bmatrix} A^T P + P A + C^T C & P B \\ B^T P & 0 \end{bmatrix} \leq \alpha^2 \begin{bmatrix} \varepsilon I & 0 \\ 0 & I \end{bmatrix} \end{aligned}$$

which is a generalized eigenvalue minimization problem. The following MATLAB® code allows to define and solve the LMI problem and to derive $\|G(s)\|_\infty = \sqrt{\alpha^2}$:

```
>> setlmis([]);
>> P=lmivar(1,[2 1]);
>> lmiterm([-1 1 1 P],1,1);
>> lmiterm([2 1 1 P],A',1,'s');
```



```

>> lmiterm([2 1 1 0],C'*C);
>> lmiterm([2 2 1 P],B',1);
>> lmiterm([2 2 2 0],0);
>> lmiterm([-2 1 1 0],e*1);
>> lmiterm([-2 2 1 0],0);
>> lmiterm([-2 2 2 0],1);
>> hinfnormLMI=getlmis;
>> [tmin,xfeas]=gevp(hinfnormLMI,1);

```

where $e = 0.001$. The value of α^2 is $tmin = 0.0481$. Hence $\|G(s)\|_\infty = 0.2193$.

1.12.2 Exercise 12.2

Exercise: Using LMI techniques, determine if the system with transfer function $G(s) = 10 \frac{s+1}{(s^2+5s+10)}$ is bounded-real or not.

Solution: Choosing the following state-space representation of $G(s)$:

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 10 & 10 \end{bmatrix}$$

the LMI problem for the bounded-real lemma can be written as:

$$\begin{bmatrix} P > 0 \\ \begin{bmatrix} A^T P + P A + C^T C & P B \\ B^T P & -I \end{bmatrix} \leq 0 \end{bmatrix}$$

which is a linear objective minimization problem where the trace of the solution P has to be minimized. The following MATLAB® code allows to define the LMI problem, if the problem is feasible, then the system is bounded-real.

```

>> setlmis([]);
>> P=lmivar(1,[2 1]);
>> lmiterm([-1 1 1 P],1,1);
>> lmiterm([2 1 1 P],A',1,'s');
>> lmiterm([2 1 1 0],C'*C);
>> lmiterm([2 2 1 P],B',1);
>> lmiterm([2 2 2 0],-1);
>> brlLMI=getlmis;
>> c=mat2dec(brlLMI,eye(2));
>> [copt,xopt]=mincx(brlLMI,c,[1e-5 0 0 0 0]);

```

The LMI problem is unfeasible and, hence, the system $G(s)$ is not bounded-real. This result can be verified calculating the H_∞ norm of $G(s)$ with the MATLAB® command `normhinf` obtaining $\|G(s)\|_\infty = 2.1021 > 1$.

1.12.3 Exercise 12.3**Exercise:** Given the system:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u} \\ \mathbf{z}_\infty = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{11}\mathbf{w} + \mathbf{D}_{12}\mathbf{u} \\ \mathbf{z}_2 = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w} + \mathbf{D}_{22}\mathbf{u} \\ \mathbf{y} = \mathbf{C}_3\mathbf{x} + \mathbf{D}_{31}\mathbf{w} + \mathbf{D}_{32}\mathbf{u} \end{cases}$$

with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0.2 & -1 \end{bmatrix}$; $\mathbf{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\mathbf{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\mathbf{C}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}$; $\mathbf{C}_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}$; $\mathbf{C}_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}$; $\mathbf{D}_{11} = \mathbf{D}_{12} = 0$; $\mathbf{D}_{21} = 0$; $\mathbf{D}_{22} = 1$; $\mathbf{D}_{31} = 1$ and $\mathbf{D}_{32} = 0$, design a multiobjective control and verify the obtained performance. Consider then the objective $\|T_{z_\infty w}\|_\infty \leq 0.5$ and $\|T_{z_2 w}\|_2 \leq 2.5$. Is it possible to design a controller satisfying the specifications?

Solution: The multiobjective controller can be designed using the MATLAB® function `hinfmix`:

```
>> [gopt,h2opt,K] = hinfmix(P,[1 1 1],[0 0 1 1]);
```

where P is the `ltisys` object defining the process to be controlled. The multiobjective controller is $K(s) = -2.19 \frac{s+1.42}{s^2+3.19s+3.91}$ with performances $\|T_{z_\infty w}\|_\infty \leq 0.78$ and $\|T_{z_2 w}\|_2 \leq 1.26$.

In order to impose an upper bound to $\|T_{z_\infty w}\|_\infty$ and $\|T_{z_2 w}\|_2$, the following MATLAB® command can be used:

```
>> [gopt,h2opt,K] = hinfmix(P,[1 1 1],[0.5 2.5 1 1]);
```

obtaining a compensator $K(s) = -3.13 \frac{s+1.495}{s^2+4.13s+5.48}$ whose performance satisfies the imposed constraints: $\|T_{z_\infty w}\|_\infty \leq 0.4991$ and $\|T_{z_2 w}\|_2 \leq 1.49$.

1.12.4 Exercise 12.4**Exercise:** Calculate the compensator (regulator and observer) which stabilizes the system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

with $\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} b_{1,i} \\ b_2 \\ b_3 \end{bmatrix}$; $\mathbf{C} = \mathbf{B}^T$; $\lambda_1 = -1$; $\lambda_2 = 1$; $\lambda_3 = 5$; $b_{1,1} = 4$; $b_{1,2} = -1$; $b_2 = 1$ and $b_3 = 1$.

Solution: The simultaneous stabilization problem can be formulated in terms of linear matrix inequalities and solved with the MATLAB® function `feasp` through the following commands:

```
>> setlmis([]);
>> P=lmivar(1,[3 1]);
```

```

>> Q=lmivar(2,[1,3]);
>> lmiterm([-1 1 1 P],1,1);
>> lmiterm([2 1 1 P],A1,1,'s');
>> lmiterm([2 1 1 Q],B1,-1,'s');
>> lmiterm([3 1 1 P],A1,1,'s');
>> lmiterm([3 1 1 Q],B2,-1,'s');
>> simstab=getlmis;
>> [tmin,xfeas]=feasp(simstab);
>> Pvalue=dec2mat(simstab,xfeas,P);
>> Qvalue=dec2mat(simstab,xfeas,Q);
>> k=Qvalue*inv(Pvalue);

```

Since A is diagonal and $B = C^T$, the observer gain are $h = k^T$.

1.12.5 Exercise 12.5

Exercise: Using the LMI approach, find the control law that stabilizes simultaneously the two systems:

$$A_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}; B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 3 \end{bmatrix}; B_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Solution: The problem can be formulated in terms of linear matrix inequalities and solved with the MATLAB[®] function `feasp` through the following commands:

```

>> setlmis([]);
>> P=lmivar(1,[3 1]);
>> Q=lmivar(2,[1,3]);
>> lmiterm([-1 1 1 P],1,1);
>> lmiterm([2 1 1 P],A1,1,'s');
>> lmiterm([2 1 1 Q],B1,-1,'s');
>> lmiterm([3 1 1 P],A2,1,'s');
>> lmiterm([3 1 1 Q],B2,-1,'s');
>> simstab=getlmis;
>> [tmin,xfeas]=feasp(simstab);
>> Pvalue=dec2mat(simstab,xfeas,P);
>> Qvalue=dec2mat(simstab,xfeas,Q);
>> k=Qvalue*inv(Pvalue);

```

To verify the obtained controller, it is sufficient to calculate the closed-loop eigenvalues:

```

>> eigenv1=eig(A1-B1*k);
>> eigenv2=eig(A2-B2*k);

```

1.13 Solutions of exercises of Chapter 13

1.13.1 Exercise 13.1

Exercise: Given the system $G(s) = \frac{s+5}{s^2+2s+2}$ determine the class of stabilizing compensators. Then determine a closed-loop compensator that assures zero error of the unit step response.

Solution: The system $G(s)$ has two stable poles, hence the class of stabilizing compensators for stable plants can be used:

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

where $Q(s) \in \mathbb{H}^\infty$ can be chosen according to specifications. In particular, calculating the final value of the unit step response of the closed-loop system:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sF(s) \frac{1}{s} = Q(0)P(0) = 1$$

we derive that $Q(0) = \frac{1}{P(0)} = \frac{2}{5}$ is the only constrain for the parametric function: Hence, we can select $Q(s) = \frac{2}{5}$.

1.13.2 Exercise 13.2

Exercise: Given the system $G(s) = \frac{s+5}{s^2-2s+2}$ determine the class of stabilizing compensators. Then determine a closed-loop compensator that assures zero error of the unit step response.

Solution: The system $G(s)$ has two unstable poles, hence the class of stabilizing compensators for unstable plants can be used:

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)}$$

where $Q(s) \in \mathbb{H}^\infty$ can be chosen according to specifications, while $N(s)$, $M(s)$, $X(s)$, and $Y(s)$ are a coprime factorization of $G(s)$. We can select $N(s) = \frac{s+5}{(s+1)^2}$ and $M(s) = \frac{s^2-2s+2}{(s+1)^2}$, and calculate $X(s)$ and $Y(s)$ in order to obtain a coprime factorization. Assuming $X(s) = k \frac{s+\alpha}{(s+1)^2}$ and $Y(s) = \frac{(s+\beta)(s+\gamma)}{(s+1)^2}$, it follows

$$\begin{aligned}
& N(s)X(s) + M(s)Y(s) = \\
& = \frac{s^4 + (\beta + \gamma - 2)s^3 + \delta s^2 + (2\beta + 2\gamma + 5k - 2\beta\gamma + k\alpha)s + 2\beta\gamma + 5k\alpha}{(s+1)^4}
\end{aligned}$$

with $\delta = (k - 2\gamma - 2\beta + \beta\gamma + 2)$. Equating coefficients of the numerator and denominator polynomials, the four unknowns can be calculated as $k = 4.08$, $\alpha = -1.12$, $\beta = 3 + 1.71j$, and $\gamma = 3 - 1.71j$.

In order to choose the parametric function $Q(s)$, the final value of the unit step response of the closed-loop system can be calculated:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sF(s) \frac{1}{s} = (X(0) + M(0)Q(0))N(0) = 1$$

Hence $Q(0) = \frac{1}{M(0)N(0)} - \frac{X(0)}{M(0)} = 2.38$ is the only constrain for the parametric function, which can be $Q(s) = 2.38$.

1.13.3 Exercise 13.3

Exercise: Given the system $G(s) = \frac{1}{s-2}$ determine the class of the stable and stabilizing compensators. Then determine a closed-loop compensator that assures a pole in $p = -5$.

Solution: The system $G(s)$ satisfies the parity interlacing property, hence it can be stabilized with a stable compensator. The class of stable and stabilizing compensators is:

$$C(s) = \frac{U(s) - M(s)}{N(s)}$$

where $U(s)$ is a unit function which can be chosen according to specifications, while $N(s)$ and $M(s)$ are a coprime factorization of $G(s)$. We can select $N(s) = \frac{1}{s+1}$ and $M(s) = \frac{s-2}{s+1}$, obtaining

$$C(s) = U(s)(s+1) - (s-2)$$

In order to have a realizable compensator the maximum degree coefficient of the two terms in $C(s)$ must be equal, this leads to the condition $U(\infty) = M(\infty) = 1$. Furthermore the closed-loop transfer function can be calculated as

$$F(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\frac{U(s)-M(s)}{N(s)} \frac{N(s)}{M(s)}}{1 + \frac{U(s)-M(s)}{N(s)} \frac{N(s)}{M(s)}} = \frac{U-M}{U}$$

This implies that the poles of $F(s)$ are the poles of $M(s)$ and the zeros of $U(s)$. Choosing $U(s) = \frac{s+5}{s+\alpha}$, with $\alpha > 0$, allows to impose a pole in $p = -5$ to $F(s)$.

1.13.4 Exercise 13.4

Exercise: Given the system $G(s) = \frac{s-1}{(s-2)(s-5)}$ determine the class of stable and stabilizing compensators.

Solution: The system $G(s)$ satisfies the parity interlacing property, hence it can be stabilized with a stable compensator. Also in this case the class of stable and stabilizing compensators is:

$$C(s) = \frac{U(s) - M(s)}{N(s)}$$

where $U(s)$ is a unit function, while $N(s)$ and $M(s)$ are a coprime factorization of $G(s)$. We can select $N(s) = \frac{s-1}{(s+2)^2}$ and $M(s) = \frac{(s-2)(s-5)}{(s+2)^2}$, obtaining

$$C(s) = \frac{U(s)(s+2)^2 - (s-2)(s-5)}{s-1}$$

In this case, we have to ensure that the term $s-1$ at the denominator is simplified. This means that $U(1) = M(1) = \frac{4}{9}$. Furthermore, in order to obtain a realizable compensator $U(\infty) = M(\infty) = 1$. Choosing $U(s) = k \frac{s+\alpha}{s+\beta}$, the first condition implies $k \frac{1+\alpha}{1+\beta} = \frac{4}{9}$, while the second condition corresponds to $k = 1$. All the values $\frac{1+\alpha}{1+\beta} = \frac{4}{9}$ satisfy the two conditions, e.g. for $\alpha = 3$ and $\beta = 8$ the stable compensator is $C(s) = \frac{6(s+11.33)(s+1)^2}{(s+8)(s+2)^2}$.

1.13.5 Exercise 13.5

Exercise: Given the system with transfer function $G(s) = \frac{4}{s-3}$ determine the coprime factorization. Then determine the class of stabilizing compensators with unit step response equal to 1. Finally, calculate the energy associated to the impulse response for the closed-loop system.

Solution: The coprime factorization of $G(s)$ can be obtained choosing arbitrarily $N(s) = \frac{4}{s+1}$ and $M(s) = \frac{s-3}{s+1}$ and calculating $X(s) = \frac{k}{s+1}$ and $Y(s) = \frac{s+\alpha}{s+1}$ according to $N(s)X(s) + M(s)Y(s) = \frac{4k+(s-3)(s+\alpha)}{s^2+2s+1} = 1$. Hence, $\alpha - 3 = 2$ and $4k - 3\alpha = 1$ from which $X(s) = \frac{4}{s+1}$ and $Y(s) = \frac{s+5}{s+1}$ follows.

The class of stabilizing compensators is

$$C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)}$$

with $Q(s) \in \mathbb{H}^\infty$. Calculating the final value of the step response of the closed-loop system $F(s) = (X(s) + M(s)Q(s))N(s)$:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sF(s) \frac{1}{s} = (X(0) + M(0)Q(0))N(0) = 1$$

Hence, $Q(0) = \frac{5}{4}$. Choosing for example $Q(s) = \frac{5}{4}$, the compensator is $C(s) = \frac{1.25s+0.25}{s}$ and the closed-loop system is $F(s) = \frac{5s+1}{(s+1)^2}$.

The impulse response of the closed-loop system is of the type:

$$F(s) = \frac{R_1}{s+1} + \frac{R_2}{(s+1)^2} \Rightarrow y(t) = R_1 e^{-t} + R_2 t e^{-t}$$

with $R_1 = 5$, and $R_2 = -4$. The energy of the impulse response is defined by $E = \int_0^\infty y(t)^2 dt = 6.5$.

1.13.6 Exercise 13.6

Exercise: Determine the right and left unitary coprime factorization of the system $G(s) = \frac{1}{(s-1)^2}$.

Solution: Being $G(s)$ a SISO system, the right and left coprime factorization are equal. The functions $N(s)$ and $M(s)$ are defined by the following realization matrices:

$$N(s) = \left[\begin{array}{c|c} A+BF & B \\ \hline C & 0 \end{array} \right], M(s) = \left[\begin{array}{c|c} A+BF & B \\ \hline F & I \end{array} \right]$$

or equivalently:

$$\tilde{N}(s) = \left[\begin{array}{c|c} (A+HC)^T & C^T \\ \hline B^T & 0 \end{array} \right], \tilde{M}(s) = \left[\begin{array}{c|c} (A+HC)^T & C^T \\ \hline H^T & I \end{array} \right]$$

where $F = -B^T P_1$, and $H = -P_2 C^T$ with P_1 and P_2 the solutions of the CARE and FARE associated to $G(s)$. In order to derive the solutions of the CARE and FARE, consider the following state-space realization:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

The solution P_1 of the CARE can be calculated assuming

$$P_1 = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

obtaining:

$$A^T P_1 + P_1 A - P_1 B^T B^T P_1 + C^T C = 0$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} +$$

$$-\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

This matrix equation leads to an algebraic nonlinear system of three equations in the three unknowns:

$$\begin{cases} -2p_2 - p_2^2 + 1 = 0 \\ p_1 + 2p_2 - p_3 - p_2p_3 = 0 \\ 2p_2 + 4p_3p_3^2 = 0 \end{cases}$$

Solving the system, the solution P of the CARE is:

$$P = \begin{bmatrix} 2(1 + \sqrt{1 + \sqrt{2}}) & \sqrt{2} - 1 \\ \sqrt{2} - 1 & 2 + \sqrt{2(1 + \sqrt{2})} \end{bmatrix}$$

and $F = \begin{bmatrix} -\sqrt{2} + 1 & -2 - \sqrt{2(1 + \sqrt{2})} \end{bmatrix}$.

The same procedure can be applied to solve the FARE and derive the vector $H = \begin{bmatrix} -4.197 \\ -\sqrt{2} + 1 \end{bmatrix}$.

The two functions of the coprime factorization are $N(s) = \tilde{N}(s) = \frac{1}{s^2 + 2.197s + 1.414}$, and $M(s) = \tilde{M}(s) = \frac{s^2 - 2s + 1}{s^2 + 2.197s + 1.414}$.

1.13.7 Exercise 13.7

Exercise: Calculate the unitary coprime factorization of the system with transfer function $G(s) = \frac{s+1}{s^2+3s+1}$.

Solution: In order to derive the functions $N(s)$ and $M(s)$, the solutions of the CARE and FARE associated to $G(s)$ have to be calculated. Consider the following state-space realization:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The solution P_1 of the CARE and P_2 of the FARE can be calculated according to the procedure adopted in the previous exercise or by using the MATLAB® function `are`:

```
>> F=-B'*are(A,B*B',C'*C)
>> H=-are(A',C'*C,B*B')*C'
```

The unitary coprime factorization is given by the functions $N(s) = \frac{s+1}{(s+2.78)(s+0.51)}$, $M(s) = \frac{(s+2.62)(s+0.38)}{(s+2.78)(s+0.51)}$, $X(s) = 0.1 \frac{s+1.87}{(s+2.78)(s+0.51)}$, and $Y(s) = \frac{(s+2.97)(s+0.61)}{(s+2.78)(s+0.51)}$.

1.13.8 Exercise 13.8

Exercise: Calculate a compensator that stabilizes simultaneously the two plants with transfer function $G_1(s) = \frac{1}{s-1}$ and $G_2(s) = \frac{1}{s-2}$.

Solution: The two systems can be simultaneously stabilized if the system $\Delta(s) = G_1(s) - G_2(s) = \frac{-1}{(s-1)(s-2)}$ is strongly stabilizable, i.e. $\Delta(s)$ satisfies the parity interlacing property. This is indeed the case and we can proceed with the solution of the problem. In particular, a state-feedback controller with $k > 2$ allows to simultaneously stabilize the two plants, in fact: $A_1 - B_1k = 1 - k < 0$ and $A_2 - B_2k = 2 - k < 0$, where $A_1 = 1$, $A_2 = 2$, and $B_1 = B_2 = 1$.

1.13.9 Exercise 13.9

Exercise: Given the systems

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}_i \mathbf{x} + \mathbf{B}_i \mathbf{u} \\ \mathbf{y} = \mathbf{C}_i \mathbf{x} \end{cases}$$

with $\mathbf{A}_i = \begin{bmatrix} \alpha_i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$; $\mathbf{B}_i = \mathbf{C}_i^T = \begin{bmatrix} 1 \\ -1 \\ \theta_i \end{bmatrix}$ and with $\alpha_i = i$ and $\theta_i = i + 1$ for $i = \{1, 2\}$, determine if it is possible to simultaneously stabilize them and, if so, design the linear state regulator and observer so that the two systems are asymptotically stable. Verify the result.

Solution: The problem can be formulated in terms of linear matrix inequalities and solved with the MATLAB® function `feasp` through the following commands:

```
>> setlmis([]);
>> P=lmivar(1,[3 1]);
>> Q=lmivar(2,[1,3]);
>> lmiterm([-1 1 1 P],1,1);
>> lmiterm([2 1 1 P],A1,1,'s');
>> lmiterm([2 1 1 Q],B1,-1,'s');
>> lmiterm([3 1 1 P],A2,1,'s');
>> lmiterm([3 1 1 Q],B2,-1,'s');
>> simstab=getlmis;
>> [tmin,xfeas]=feasp(simstab);
>> Pvalue=dec2mat(simstab,xfeas,P);
>> Qvalue=dec2mat(simstab,xfeas,Q);
>> k=Qvalue*inv(Pvalue);
```

To verify the obtained controller, it is sufficient to calculate the closed-loop eigenvalues:

```
>> eigenv1=eig(A1-B1*k);
>> eigenv2=eig(A2-B2*k);
```

1.13.10 Exercise 13.10

Exercise: Calculate, if possible, a control law $u = -Kx$ that can simultaneously stabilize the two systems with state-space matrices:

$$A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}; \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{7} & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -7 \end{bmatrix};$$

$$B_1^T = B_2^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \end{bmatrix}^T.$$

Solution: The problem can be formulated in terms of linear matrix inequalities and solved with the procedure described in the previous exercise leading to a compensator with gains

$$k = \begin{bmatrix} -0.19 & -240.07 & 259.42 & -1.29 & 0 \end{bmatrix}$$

1.14 Solutions of exercises of Chapter 14

1.14.1 Exercise 14.1

Exercise: Using the procedure introduced in Chapter 14, calculate the inverse

of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & a \end{bmatrix}$ with $a = 0.1$, $a = 0.01$, and $a = 0.001$.

Evaluate the time needed to reach the steady-state in the three cases.

Solution: The exercise can be solved in MATLAB with the following commands

```
A=[1 0 0;0 2 0; 0 0 a]
[T,Y]=ode45(@prob1_MatrixInversion,[0 10000],rand(numel(A),1),[],A);
plot(T,Y)
end
```

As it possible to determine by inspecting the resulting plots, for $a = 0.1$ the steady-state is reached at $t \approx 100\text{s}$, for $a = 0.01$ at $t \approx 9000\text{s}$, while for $a = 0.001$ a higher integration time is needed (approximatively 10^6s).

1.14.2 Exercise 14.2

Exercise: Calculate the eigenvalues of the matrix A in equation (14.13) perturbing the matrix N with a gaussian noise. Evaluate the effect of the perturbation on the steady-state matrix.

Solution: Let us rewrite the MATLAB function reported in Section 14.1.2 as

```
function dxdt=prob2_MatrixEigenvalues(t,x,N)
n2=length(x);
H=reshape(x,sqrt(n2),sqrt(n2));
dxdt=H*(H*N-N*H)-(H*N-N*H)*H;
dxdt=dxdt(:);
end
```

where the matrix N is now a further input of the MATLAB function.

Let us define in MATLAB the matrix A in Eq. (14.13)

```
A=[0.6930 0.7707 0.1762; 0.2608 0.0757 0.5628; 0.1127 0.5856 0.1239];
```

and calculate the eigenvalues including a perturbation on matrix N

```
N=diag([sqrt(3):-1:1])+0.01*randn;
[T,Y]=ode45(@Prob2_MatrixEigenvalues,[0:0.01:100],A,[],N);
```

As it is possible to verify the steady-state is not a diagonal matrix, but a matrix which is similar to matrix A, i.e. they share the same set of eigenvalues.

1.14.3 Exercise 14.3

Exercise: Calculate the eigenvectors of the matrix A in equation (14.16) perturbing the matrix N with a gaussian noise. Evaluate the effect of the perturbation on the steady-state matrix.

Solution: Let us rewrite the MATLAB function reported in Section 14.1.3 as

```
function dxdt=prob3_MEigenv(t,x,A,N)
n2=length(x);
Theta=reshape(x,sqrt(n2),sqrt(n2));
dxdt=A*Theta*N-Theta*N*Theta'*A*Theta;
dxdt=dxdt(:);
end
```

where the matrix N is now a further input of the MATLAB function.

Let us define in MATLAB the matrix A in equation (14.16)

```
>> A=[7.0975 -8.4650 5.8477 -6.3906 -0.9032;
-8.4650 25.2876 -19.9587 2.3167 15.7190;
```

```

5.8477    -19.9587    16.8034    1.0059   -14.2585;
-6.3906     2.3167     1.0059    24.2867   -12.0941;
-0.9032    15.7190   -14.2585   -12.0941    29.2483];

```

and calculate the eigenvalues including a perturbation on matrix N

```

N=diag([sqrt(5):-1:1])+0.01*randn;
[T,Y]=ode45(@Prob2_MatrixEigenvalues,[0:0.01:100],A,[],N);
>> Q=reshape(Y(end,:),length(A),length(A));

```

As it is possible to verify the steady-state matrix Q does not contains the eigenvectors of matrix A. Even a small perturbation of the matrix N can make it non-diagonal, thus violating the hypothesis on which the method relies and yielding a steady-state matrix Q that does not solve the eigenvectors problem.

1.14.4 Exercise 14.4

Exercise: Determine the controllability and observability gramians for the continuous-time LTI system with state-space realization

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix} \quad (1.4)$$

Solution: Following the procedure described in Section 14.1.4, it is possible to solve the exercise with the commands

```

>> A=[0 1; -3 -2];
>> B=[0; 1];
>> C=[-1 1];
>> [T,Y]=ode45(@CtrbGram,[0:0.01:10],eye(length(A)),[],A,B);
>> Wc=reshape(Y(end,:),length(A),length(A));
>> [T,Y]=ode45(@ObsvGram,[0:0.01:10],eye(length(A)),[],A,C);
>> Wo=reshape(Y(end,:),length(A),length(A));

```

getting $W_c = \begin{bmatrix} 0.0833 & 0 \\ 0 & 0.25 \end{bmatrix}$ and $W_o = \begin{bmatrix} 2.334 & 0.1667 \\ 0.1667 & 0.3333 \end{bmatrix}$.

1.14.5 Exercise 14.5

Exercise: Calculate the singular values of the system with transfer function $G(s) = \frac{s^2 - 5s + 4}{s^2 + 5s + 4}$.

Solution: Following the procedure described in Section 14.1.4, it is possible to solve the exercise with the commands

```

>> A=[0 1; -4 -5];
>> B=[0; 1];
>> C=[0 -10];
>> D=1;

```

```
>> [T,Y]=ode45(@CtrbGram,[0:0.01:10],eye(length(A)),[],A,B);
>> Wc=reshape(Y(end,:),length(A),length(A));
>> [T,Y]=ode45(@ObsvGram,[0:0.01:10],eye(length(A)),[],A,C);
>> Wo=reshape(Y(end,:),length(A),length(A));
```

getting $W_c = \begin{bmatrix} 0.025 & 0 \\ 0 & 0.1 \end{bmatrix}$ and $W_o = \begin{bmatrix} 40 & 0 \\ 0 & 10 \end{bmatrix}$. The singular values can be calculated as

```
>> eig(Wc*Wo)
```

which are both one, as expected since the system is all-pass.

1.14.6 Exercise 14.6

Exercise: Calculate an open-loop balanced realization for the system with state-space matrices:

$$\begin{aligned} A &= \begin{bmatrix} -0.5 & -1 & 0 & 0 \\ 1 & -0.5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}; \\ C &= \begin{bmatrix} 0 & 1 & -1 & 1 \end{bmatrix} \end{aligned} \quad (1.5)$$

Solution: We can follow the procedure described in Section 14.2 as

```
>> [T,Y]=ode45(@ObsvGram,[0:0.01:10],reshape(eye(4),16,1),[],A,C);
>> Wo=reshape(Y(end,:),4,4);
>> [T,Y]=ode45(@CtrbGram,[0:0.01:10],reshape(eye(4),16,1),[],A,B);
>> Wc=reshape(Y(end,:),4,4);
>> [T,Y]=ode45(@PinfSystem,[0:0.01:10],reshape(eye(4),16,1),[],Wo,Wc);
>> Pinf=reshape(Y(end,:),4,4);
>> Tt=Pinf^(-0.5);
>> [T,Y]=ode45(@prob1_MatrixInversion,[0:0.01:10],rand(numel(Tt),1),[],Tt);
>> TtInv=reshape(Y(end,:),4,4);
>> At=TtInv*A*Tt;
>> Bt=TtInv*B;
>> Ct=C*Tt;
>> [T,Y]=ode45(@CtrbGram,[0:0.01:10],reshape(eye(4),16,1),[],At,Bt);
>> Wct=reshape(Y(end,:),4,4);
>> [T,Y]=ode45(@ObsvGram,[0:0.01:10],reshape(eye(4),16,1),[],At,Ct);
>> Wot=reshape(Y(end,:),4,4);
>> [T,Y]=ode45(@prob3_MEigenv,[0:0.01:50],reshape(eye(4),16,1),[],Wct);
>> Q=reshape(Y(end,:),4,4);
>> Tbal=Tt'*Q;
>> Abal=inv(Tbal)*A*Tbal;
>> Bbal=inv(Tbal)*B;
>> Cbal=C*Tbal;
```

yielding the open-loop balanced realization

$$\begin{aligned} A_b &= \begin{bmatrix} -0.2013 & 0.7577 & 0.5968 & 0.0422 \\ -0.7577 & -0.2989 & -1.5695 & -0.0706 \\ 0.5968 & 1.5695 & -3.9936 & -0.4918 \\ 0.0422 & 0.0706 & -0.4918 & -3.5082 \end{bmatrix}; \quad B_b = \begin{bmatrix} -0.5671 \\ -0.5013 \\ 0.9624 \\ 0.0595 \end{bmatrix}; \\ C_b &= \begin{bmatrix} -0.5671 & 0.5013 & 0.9624 & 0.0595 \end{bmatrix} \end{aligned} \quad (1.6)$$

1.15 Solutions of exercises of Chapter 15

1.15.1 Exercise 15.1

Exercise: Propose an approximation for $G(s) = e^{-s\tau}$ with $\tau = 1$, $\tau = 0.5$ and $\tau = 0.1$ by using the Padé method and numerically compute the H_∞ norm of the error between $G(s)$ and its approximation.

Solution: The exercise can be solved in MATLAB by using the function `pade` to determine the approximation $H(s)$ of the desired order. In principle, it is possible to use the function `normhinf` to estimate the H_∞ norm but this function generates numerical errors when used for systems with internal delays, as it is for $G(s) - H(s)$. Therefore, we may use the following commands:

```
for tau=[1 0.5 0.1]
    err=zeros(20,1);
    for n=1:20
        [nn,dd]=pade(tau,n);
        ind=0;
        errOmega=zeros(2000,1);
        for o=0.01:0.01:20
            ind=ind+1;
            vv=[];
            for k=n:-1:0
                vv=[vv (1j*o)^k];
            end
            H=sum(nn.*vv(1:end))./sum(dd.*vv(1:end));
            G=exp(-1j*o*tau);
            errOmega(ind)=abs(G-H);
        end
    end
end
```

```

end
err(n)=max(errOmega);
end
plot(1:20,err)
hold on
end

```

The results are shown in Fig. 1.11, where the three curves illustrate the H_∞ norm of the error dynamics for increasing order of the Padé approximation.

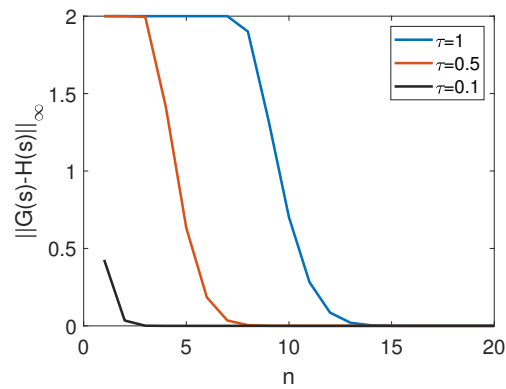


FIGURE 1.11

H_∞ norm of the error dynamics $G(s) - H(s)$ for increasing order of the Padé approximation and for $\tau = 1$ (blue curve), $\tau = 0.5$ (red curve), and $\tau = 0.1$ (black curve).

1.15.2 Exercise 15.2

Exercise: Determine, if possible, the compensator $C(s)$ that stabilizes the closed-loop system of Figure 1.12 with $G(s) = \frac{k}{s}$, $k = 1$ and $\tau = 2$.

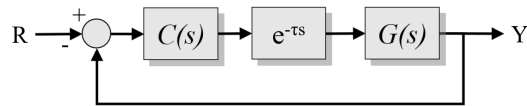


FIGURE 1.12

Block scheme for Exercise 15.22.

Solution: This problem can be solved recalling the results obtained in Example 15.1. In this case we have $k = 1$ and $\tau = 2$, therefore we can select $C(s) = K$ deriving the condition $Kk\tau < \frac{\pi}{2}$ which leads to $K < \frac{\pi}{2k\tau} = \frac{\pi}{4}$.

1.15.3 Exercise 15.3

Exercise: Consider the block scheme in Figure 1.13 with $G(s) = \frac{1}{s+\alpha}$. Determine the value of α such that the system is asymptotically stable.

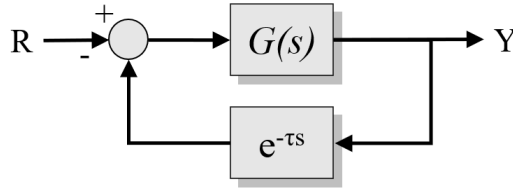


FIGURE 1.13

Block scheme for Exercises 15.3 and 15.4.

Solution: Recall that system $G(s)$ is positive-real for $\alpha > 0$. Therefore in virtue of the small-gain theorem, the closed-loop system is asymptotically stable for $\alpha > 0$.

1.15.4 Exercise 15.4

Exercise: Consider the system in Figure 1.13 with $G(s) = \frac{1}{(s+1)^3}$ and determine for which values of τ the system is asymptotically stable.

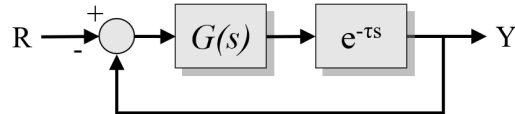
Solution: Since the closed-loop transfer function is $F(s) = \frac{1}{(s+1)^3 + e^{-s\tau}}$, the quasi-polynomial $p(s, \tau)$ is

$$p(s, \tau) = p_0(s) + p_1(s)e^{-s\tau} = (s+1)^3 + e^{-s\tau}$$

with $p_0(s) = (s+1)^3$ and $p_1(s) = 1$. Note that for $\tau = 0$ the roots lie in the left half-plane. We now apply the direct method discussed in Section 15.5, which consists in studying the roots of the equation $|p_1(j\omega)|^2 - |p_0(j\omega)|^2 = 0$, which leads to $-\omega^2(\omega^4 + 3\omega^2 + 3) = 0$ which admits no strictly positive real solutions. Therefore the system is delay-independent stable.

1.15.5 Exercise 15.5

Exercise: With reference to system in Figure 1.14 with $G(s) = \frac{k}{(s+1)^3}$ and $\tau = 0.1$, determine for which values of k the feedback system is asymptotically stable.

**FIGURE 1.14**

Block scheme for Exercise 15.5.

Solution: The gain margin of system $F(s) = \frac{1}{(s+1)^3}e^{-s0.1}$ can be calculated as $m_G = 6.2373$. In order to have a closed-loop asymptotically stable system, the gain k should not exceed this value. Therefore, we have that for $k < 6.2373$ the feedback system is asymptotically stable.