

Workbook to Accompany  
Political Economy and Policy Analysis

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# Chapter 1

## Overview and introduction

This workbook accompanies the textbook *Political Economy and Policy Analysis* by Antonio Merlo. Chapters 2 through 14 contain exercises for each corresponding chapter of the textbook. Each exercise is meant either to illustrate the concepts presented in the main text, to apply the models developed there, or to analyze related work presented in the discussion section at the end of each chapter of the textbook. Chapter 15 contains suggested solutions to the exercises.

The aim of this workbook is to help the reader become acquainted with the techniques and reasonings presented in *Political Economy and Policy Analysis*.



## Chapter 2

# Basic tools of microeconomics

1. Consider an economy with two individuals,  $A$  and  $B$ , and two private goods,  $x_1$  and  $x_2$ . The preferences of each individual are described by the following utility functions:

$$U^A(x_1^A, x_2^A) = 40x_1^A + 60x_2^A - (x_1^A)^2 - (x_2^A)^2$$

and

$$U^B(x_1^B, x_2^B) = 60x_1^B + 40x_2^B - (x_1^B)^2 - (x_2^B)^2.$$

Goods can be transformed 1 for 1 (*i.e.*,  $MRT = 1$ ). The total endowments of goods in the economy are  $\bar{x}_1 = 30$  and  $\bar{x}_2 = 10$ .

- (a) Suppose that the social welfare function is

$$W[U^A(x_1^A, x_2^A), U^B(x_1^B, x_2^B)] = U^A(x_1^A, x_2^A) + U^B(x_1^B, x_2^B).$$

Find the allocations that satisfy the efficiency conditions and the allocation that satisfies the equity conditions.

- (b) Suppose that the social welfare function is

$$W[U^A(x_1^A, x_2^A), U^B(x_1^B, x_2^B)] = 0.4U^A(x_1^A, x_2^A) + 0.6U^B(x_1^B, x_2^B).$$

Find the allocations that satisfy the efficiency conditions and the allocation that satisfies the equity conditions.

2. Consider an economy with two individuals,  $A$  and  $B$ , and two private goods,  $x_1$  and  $x_2$ . The preferences of each individual are described by the following utility functions:

$$U^A(x_1^A, x_2^A) = x_1^A(x_2^A + 1)$$

and

$$U^B(x_1^B, x_2^B) = x_1^B(x_2^B + 1).$$

Goods can be transformed 1 for 1 (*i.e.*,  $MRT = 1$ ). The total endowments of goods in the economy are  $\bar{x}_1 = 10$  and  $\bar{x}_2 = 20$ .

- (a) Suppose that the social welfare function is

$$W[U^A(x_1^A, x_2^A), U^B(x_1^B, x_2^B)] = 0.5 \ln U^A(x_1^A, x_2^A) + 0.5 \ln U^B(x_1^B, x_2^B).$$

Find the allocations that satisfy the efficiency conditions and the allocation that satisfies the equity conditions.

- (b) Suppose that the social welfare function is

$$W[U^A(x_1^A, x_2^A), U^B(x_1^B, x_2^B)] = 0.75 \ln U^A(x_1^A, x_2^A) + 0.25 \ln U^B(x_1^B, x_2^B).$$

Find the allocations that satisfy the efficiency conditions and the allocation that satisfies the equity conditions.

3. Consider an economy with two individuals,  $A$  and  $B$ , and two private goods,  $x_1$  and  $x_2$ . The preferences of each individual are described by the following utility functions:

$$U^A(x_1^A, x_2^A) = \frac{1}{2} \ln x_1^A + \frac{1}{2} \ln x_2^A$$

and

$$U^B(x_1^B, x_2^B) = \frac{1}{2} \ln x_1^B + \frac{1}{2} \ln x_2^B$$

The endowments of good  $x_1$  and good  $x_2$  for each individual are denoted by  $(\bar{x}_1^A, \bar{x}_2^A)$  and  $(\bar{x}_1^B, \bar{x}_2^B)$  and are as follows:

$$(\bar{x}_1^A, \bar{x}_2^A) = (1, 2) \quad , \quad (\bar{x}_1^B, \bar{x}_2^B) = (2, 1).$$

- (a) Suppose there is no market for good  $x_2$ , that is, individuals cannot trade good  $x_2$ . Does this situation violate any of the conditions for the First Fundamental Theorem of Welfare Economics?

- (b) Derive the equilibrium allocation of the two goods when there is no market for good  $x_2$ .
- (c) Is the equilibrium allocation when there is no market for good  $x_2$  Pareto optimal?
- (d) Is there an alternative endowment  $(\bar{x}_1^A, \bar{x}_2^A, \bar{x}_1^B, \bar{x}_2^B)$  such that  $\bar{x}_1^A + \bar{x}_1^B = 3$  and  $\bar{x}_2^A + \bar{x}_2^B = 3$  that is Pareto optimal?
- (e) If markets for both goods exist, what is the competitive equilibrium for this economy?
- (f) If markets for both goods exist, is the competitive equilibrium efficient?
4. Consider a game where there are two players,  $A$  and  $B$ . Player  $A$  can choose  $L$  or  $R$  and player  $B$  can choose  $Y$  or  $N$ . If player  $A$  chooses  $L$  and player  $B$  chooses  $Y$  the payoff of player  $A$  is equal to 9 and the payoff of player  $B$  is equal to 3; If player  $A$  chooses  $L$  and player  $B$  chooses  $N$  the payoff of player  $A$  is equal to 4 and the payoff of player  $B$  is equal to 0; If player  $A$  chooses  $R$  and player  $B$  chooses  $Y$  the payoff of player  $A$  is equal to 0 and the payoff of player  $B$  is equal to 4; If player  $A$  chooses  $R$  and player  $B$  chooses  $N$  the payoff of player  $A$  is equal to 8 and the payoff of player  $B$  is equal to 5.
- (a) Suppose that the players have to choose their actions simultaneously. Find the Nash equilibria of the game and determine whether they are Pareto optimal.
- (b) Suppose that player  $A$  moves first and player  $B$  second. Find the subgame perfect equilibrium of the game and determine whether it is Pareto optimal.
- (c) Suppose that player  $B$  moves first and player  $A$  second. Find the subgame perfect equilibrium of the game and determine whether it is Pareto optimal.
5. Consider a game where there are two players,  $A$  and  $B$ . Player  $A$  can take an action  $a > 0$  and player  $B$  can take an action  $b > 0$ . The players' payoffs depend on the actions of both players and are equal to

$$U^A(a, b) = [100 - 2(a + b)]a$$

and

$$U^B(a, b) = [100 - 2(a + b)]b.$$

- (a) Suppose that the players have to choose their actions simultaneously. Find the Nash equilibrium of the game.
- (b) Suppose that player  $A$  moves first and player  $B$  second. Find the subgame perfect equilibrium of the game.
6. Consider a game where there are two players,  $A$  and  $B$ . Player  $A$  can choose among three actions,  $TOP(T)$ ,  $MIDDLE(M)$ ,  $BOTTOM(B)$ , and player  $B$  can choose among three actions  $LEFT(L)$ ,  $CENTER(C)$ ,  $RIGHT(R)$ . The payoffs that players get as a result of the two players' actions are as summarized in the following matrix wherein player  $A$ 's utility is listed first and player  $B$ 's second.

A \ B	L	C	R
T	4,2	1,-1	5,0
M	6,2	0,0	2,3
B	1,3	0, $x$	3,1

- (a) Does player  $A$  have a strictly dominated strategy?
- (b) Does player  $A$  have a strictly dominant strategy?
- (c) Is there any value of  $x$  such that player  $B$  has a strictly dominant strategy?
- (d) Is there any value of  $x$  such that player  $B$  has a strictly dominated strategy?
- (e) Is there any value of  $x$  such that the above game has a (pure-strategy) Nash Equilibrium?
- (f) Derive the range of the values of  $x$  such that the strategy  $(B, C)$  is Pareto-optimal.
7. Consider a game where there are two players,  $A$  and  $B$ . Player  $A$  can choose among three actions,  $TOP(T)$ ,  $MIDDLE(M)$ ,  $BOTTOM(B)$ , and player  $B$  can choose among two actions  $LEFT(L)$  and  $RIGHT(R)$ . The payoffs that players get as a result of the two players' actions are as summarized in the following matrix.

A \ B	L	R
T	0,3	6,2
M	3,10	2,12
B	4,4	3,5

- (a) Is there any outcome that is not Pareto optimal?
- (b) Is any such outcome a Nash equilibrium?



## Chapter 3

# Basic tools of political economics

1. A fraternity decides to use the Borda Count voting rule to choose its new president. There are 3 candidates: a biology major (Joe), an economics major (Ray) and a math major (Pat).

There are four groups in the fraternity: group  $A$  has 10 members, group  $B$  has 7 members, group  $C$  has 5 members and group  $D$  has 8 members. Preferences are as follows:

	$A$	$B$	$C$	$D$
1st choice	Joe	Pat	Pat	Ray
2nd choice	Pat	Joe	Ray	Joe
3rd choice	Ray	Ray	Joe	Pat

- (a) Does a Condorcet winner exist? If yes, then who is the Condorcet winner? If not, then show that there does not exist one.
  - (b) Does a Condorcet loser exist? If yes, then who is the Condorcet loser? If not, then show that there does not exist one.
  - (c) Who is elected president?
  - (d) Does the Borda Count Voting rule satisfy the condition of Independence of Irrelevant Alternatives or not? Justify your answer.
2. Suppose that there are five people in a community who rank projects  $A$ ,  $B$ ,  $C$ , and  $D$  (which correspond to different levels of provision of a certain policy and are therefore ordered with project  $A$  entailing the lowest level of provision and project  $D$  the highest) as follows:

	1	2	3	4	5
1st choice	<i>A</i>	<i>A</i>	<i>D</i>	<i>C</i>	<i>B</i>
2nd choice	<i>D</i>	<i>C</i>	<i>B</i>	<i>B</i>	<i>C</i>
3rd choice	<i>C</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>D</i>
4th choice	<i>B</i>	<i>D</i>	<i>A</i>	<i>A</i>	<i>A</i>

- (a) Graph the individual preferences of each individual.
- (b) Does a Condorcet winner exist? If yes, what is the Condorcet winner? If not, what is the problem?
3. Suppose that there are three candidates running for election: a liberal, a moderate, and a conservative. The voting rule is plurality voting with runoff: if no candidate wins a clear majority in the first round, the top two vote-getters are paired off in a runoff majority election. There are three groups of voters: group *A* has 33 members, group *B* has 30 members, and group *C* has 37 members. Preferences are as follows:

	Group <i>A</i>	Group <i>B</i>	Group <i>C</i>
1st choice	conservative	moderate	liberal
2nd choice	moderate	liberal	moderate
3rd choice	liberal	conservative	conservative

- (a) Who wins the election? Is he a Condorcet winner?

Now, suppose that there is an additional candidate, who is a libertarian. The voters' preferences over the four candidates are as follows (Group A-1 has 29 members, group A-2 has 4 members, group B has 30 members, and group C has 37 members):

	Group <i>A</i> – 1	Group <i>A</i> – 2	Group <i>B</i>	Group <i>C</i>
1st choice	conservative	libertarian	moderate	liberal
2nd choice	moderate	conservative	liberal	moderate
3rd choice	liberal	moderate	conservative	conservative
4th choice	libertarian	liberal	libertarian	libertarian

- (b) Who wins the election?
- (c) Does the plurality rule with runoff satisfy the condition of Independence of Irrelevant Alternatives? Justify your answer.

4. 300 people living in a community are considering how much to spend in waste recycling. There are four groups of people in the community who differ with respect to their preferences for expenditures in waste recycling,  $R$ . The first group contains 75 individuals whose preferences are described by the following function

$$V_1(R) = 40R - 2R^2$$

The second group contains 50 individuals whose preferences are described by the following function

$$V_2(R) = 30R - 3R^2$$

The third group contains 110 individuals whose preferences are described by the following function

$$V_3(R) = 80R - R^2$$

The fourth group contains 65 individuals whose preferences are described by the following function

$$V_4(R) = 80R - 2R^2$$

Suppose that there are only 4 alternative levels of spending: 5, 10, 20, and 40.

- (a) What is the level of spending that would be chosen under sequential majority rule?
  - (b) What is the level of spending that would be chosen under Borda Count?
  - (c) What is the level of spending that would be chosen under plurality rule?
  - (d) What is the level of spending that would be chosen under plurality rule with runoff?
5. Suppose that there are four different candidates from two parties competing in an election: Conservative-1 ( $C1$ ), Conservative-2 ( $C2$ ), Liberal-1 ( $L1$ ), and Liberal-2 ( $L2$ ). There are four groups of voters who have the following preferences:

	A	B	C	D
1st choice	C1	L1	L2	C2
2nd choice	C2	L2	L1	C1
3rd choice	L1	C1	C1	L1
4th choice	L2	C2	C2	L2

Group A contains 35% of the voters, group B 30%, group C 25%, and group D the remaining 10%. The winner is determined by plurality rule.

- (a) Who wins the election? Is the winner a Condorcet winner?  
Now, suppose that candidate L2 voluntarily withdraws from the race.
  - (b) Who wins the election? Is the winner a Condorcet winner?
  - (c) Briefly explain why it may make sense for candidate L2 to withdraw from the electoral competition.
  - (d) Which condition in the Arrow's impossibility theorem does the plurality rule violate?
6. Consider two towns trying to select one of the three alternatives  $\{A, B, C\}$  which would be implemented in both towns. There are two stages in the selection process. In the first one, each town decides the alternative to support in the second stage. The first town has 2 electoral votes and the second town has 1. The alternative with the largest electoral support is selected. There are three groups in each town. The preferences of the first town are given by:

	X	Y	Z
1st Choice	A	C	B
2nd Choice	C	B	C
3rd Choice	B	A	A

where there are 8 in group X, 6 in Y and 4 in Z. On the other hand, the preferences in the second town are given by:

	M	N	K
1st Choice	A	B	A
2nd Choice	B	A	C
3rd Choice	C	C	B

where group  $M$  has 5 people,  $N$  has 4 and  $K$  has 3.

- (a) What is a Condorcet winner in this example? Does one exist?
  - (b) What is a Condorcet loser in this example? Does one exist?
  - (c) Suppose the first town decides using a Borda Count and the second town uses majority rule to decide the alternative to support. What is the selected alternative?
  - (d) Suppose now that instead of two separate stages, towns jointly decide in a single plurality election. Is there a Condorcet winner? A loser?
  - (e) Who wins the election? Is it the Condorcet winner?
7. Consider three towns trying to select one of the three alternatives  $\{A, B, C\}$  which would be implemented in all three towns. There are two stages in the selection process. In the first one, each town decides the alternative to support in the second stage. The first town has 2 electoral votes, the second town has 1 and the third has 2. The alternative with the largest electoral support is selected. There are three groups in each town. The preferences of the first town are given by:

	$X$	$Y$	$Z$
1st Choice	$A$	$A$	$B$
2nd Choice	$C$	$B$	$A$
3rd Choice	$B$	$C$	$C$

where there are 8 in group  $X$ , 6 in  $Y$  and 4 in  $Z$ . The preferences in the second town are given by:

	$M$	$N$	$K$
1st Choice	$A$	$B$	$A$
2nd Choice	$B$	$A$	$C$
3rd Choice	$C$	$C$	$B$

where group  $M$  has 5 people,  $N$  has 4 and  $K$  has 3.

Lastly, the preferences in the third town are given by:

	$T$	$J$	$L$
1st Choice	$C$	$B$	$C$
2nd Choice	$B$	$C$	$A$
3rd Choice	$A$	$A$	$B$

where group  $T$  has 5 people,  $J$  has 10 and  $L$  has 7.

- (a) What is a Condorcet winner in this example? Does one exist?
- (b) What is a Condorcet loser in this example? Does one exist?
- (c) Suppose every town decides which proposal to support with the plurality rule. What is the selected alternative?
- (d) Suppose now that instead of two separate stages, towns jointly decide in a single plurality election. Is there a Condorcet winner? A loser?
- (e) Who wins the election? Is it a Condorcet winner. Why?

# Chapter 4

## Voters

1. Answer the following true or false questions and briefly explain your answer:
  - (a) According to the pivotal voter model, a voter in the United States' national elections is more likely to vote than a voter in Luxembourg.
  - (b) Group rule-utilitarian individuals vote in order to maximize their own utility exclusively.
  - (c) According to the ethical voter model, if in expectation there are more supporters of  $a$  than  $b$  then the turnout of group  $a$  is going to be lower than the one of  $b$ .
  - (d) In the context of the uncertain voter model, newspaper readers are less likely to vote.
  - (e) According to the uncertain voter model, if multiple elections are held simultaneously, voters in the voting booth may not vote in one of the elections because of the mere physical cost of voting.
2. Consider a pivotal voter model with  $n = 3$  voters, each of them supporting either alternative  $a$  or alternative  $b$ . The probability that a voter supports one of the alternatives is  $1/2$ , which is common knowledge. Each voter decides whether to vote or not, and if she does, she votes for the alternative she supports. The alternative to be implemented is decided through majority rule and in case of a tie a fair coin is flipped to select an alternative. The cost of voting is  $C_i \in \{0.2, 0.8\}$  for each voter  $i \in \{1, 2, 3\}$  and its distribution is  $P(0.2) = P(0.8) = 1/2$ .

- (a) Suppose the realization of the random variables is that voters 1 and 2 support  $a$  while voter 3 supports  $b$ , and that  $C_1 = C_2 = 0.2$  and  $C_3 = 0.8$ . Assume that these realizations are all common knowledge among voters. Moreover, each voter gets utility equal to 1 if the alternative she supports is implemented and  $-1$  otherwise. Is there a Nash equilibrium in this game? And if yes, is it unique?
- (b) Following the setting in (a), suppose now that the utility values are  $1/2$  and  $-1/2$  instead of 1 and  $-1$ . Is there a Nash equilibrium in this game? And if yes, is it unique?
- (c) Consider now the situation in which preferences over alternatives and voting costs have not realized yet. Furthermore, assume that their joint distribution is such that we always have  $C_i = 0.2$  for supporters of  $a$  and  $C_i = 0.8$  for supporters of  $b$ . The utility from the implemented alternative is still 1 for the supported one and  $-1$  otherwise.

Consider the strategy profile of voting if  $C_i = 0.2$  and not voting otherwise. Which type of player has the highest probability of being pivotal? Is this a BNE? What is the expected turnout?

- (d) Following the same setting as in (c) assume now that there are  $n = 5$  voters. How do the probability of being a pivotal voter and expected turnout change with respect to part (c)? Is the strategy profile in part (c) still an equilibrium? Explain your results.
3. Consider the ethical voter model in this chapter. Showing each step of your calculations, derive the following
    - (a) The expected voting cost  $E[C_i^e]$ ;
    - (b) The aggregate expected utility of the group of voters supporting  $j = a, b$ ,  $\pi_j(C_a, C_b)$ ;
    - (c) The unique Nash equilibrium strategy profile  $C^*$ .
  4. Consider an election where a society has to decide between alternatives  $a$  and  $b$ . The voting population is a continuum of measure one which is divided between supporters of the two alternatives. Let  $\mu$  be the fraction of voters supporting alternative  $a$  whose density function is given by  $f(\mu) = 3\mu^2$  over the support  $[0, 1]$ . Each voter  $i$  bears a cost of voting  $C_i$ , which is assumed to be uniformly distributed on the support  $[0, 1]$ , and benefits from a utility equal to 1 if the alternative she supports is implemented and  $-1$  otherwise.

- (a) What is the probability that a voter is pivotal?
- (b) Suppose now that voters are rule utilitarians:
- i. State the condition for alternative  $a$  to be implemented.
  - ii. What are the payoff functions for the two groups of voters?
  - iii. For each group  $j$ , find the equation characterizing the best response function (no need to find a closed form solution).
  - iv. Explain how you would find the Nash equilibrium strategies  $C_a^*$  and  $C_b^*$ .
5. In the context of the uncertain voter model, consider two alternatives  $a$  and  $b$  with joint support over  $\{-2, -1\} \times \{1, 2\}$ . The joint distribution of  $(a, b)$  is such that  $\Pr(-2, 2) = \frac{1}{8}$ ,  $\Pr(-2, 1) = \Pr(-1, 2) = \frac{2}{8}$  and  $\Pr(-1, 1) = \frac{3}{8}$ . The distribution of preferred ideologies is uniform over  $[-2, 2]$ . Voters evaluate alternative ideologies according to  $V_i(x) = -(x_i - x)^2$ .
- Each voter  $i$  derives a direct benefit,  $D_i^e$  from voting which is equal to 0 and  $\frac{3}{8}$  with equal probability. Moreover, let's assume when the cost of voting is equal to the direct benefit of voting, individuals choose to vote. What is the expected turnout in this election?
6. As in the uncertain voter model there is a continuum of voters with ideological preferences that are distributed uniformly over the  $[-1, 1]$ . Voters can choose to cast ballots in two elections with two competing policy platforms in each election. The policies in the first election are denoted by  $a_1$  and distributed according to a distribution with the density

$$f_{a_1}(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{8(x+1)}{3} & \text{for } -1 \leq x \leq -1/2 \\ \frac{4(1/2-x)}{3} & \text{for } -1/2 < x \leq 1/2 \\ 0 & \text{for } 1/2 < x \end{cases}$$

and the one of the second election is denoted by  $a_2$  and distributed according

$$f_{a_2}(x) = \begin{cases} 0 & \text{for } x < -1/2 \\ \frac{4(x+1/2)}{3} & \text{for } -1/2 \leq x \leq 1/2 \\ \frac{8(1-x)}{3} & \text{for } 1/2 < x \leq 1 \\ 0 & \text{for } 1 < x \end{cases}.$$

The policies in the second election are denoted by  $b_1$  and  $b_2$  and distributed according to

$$f_{b_1}(x) = \begin{cases} 0 & \text{for } x < -1 \\ 2(x+1) & \text{for } -1 \leq x \leq -1/2 \\ \frac{2(1-x)}{3} & \text{for } -1/2 < x \leq 1 \\ 0 & \text{for } 1 < x \end{cases}$$

and,

$$f_{b_2}(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{2(x+1)}{3} & \text{for } -1 \leq x \leq 1/2 \\ 2(1-x) & \text{for } 1/2 < x \leq 1 \\ 0 & \text{for } 1 < x \end{cases},$$

Suppose that voter  $i$  evaluates a policy according to  $V_i(x) = -(x_i - x)^2$ , where  $x_i$  is the preferred ideology of the voter  $i$  and derives benefit  $D \in \mathbb{R}$  by participating in either election.

- a. How do you compare the two elections in terms of voters' uncertainty?
  - b. Which election do you expect to have a higher turnout?
7. Three individuals 1, 2, and 3 are participating in 2 simultaneous elections. The direct benefit of voting for each voter is large enough to convince everyone to participate in both elections, and this is known to everyone. The first election is between two alternatives  $a_1$  and  $a_2$ , while the second election is a choice between  $b_1$  and  $b_2$ . Each individual has preferences over the joint outcome of both elections as follows:

$$\begin{aligned} (a_1, b_1) &\succ_1 (a_1, b_2) \succ_1 (a_2, b_1) \succ_1 (a_2, b_2) \\ (a_2, b_2) &\succ_2 (a_2, b_1) \succ_2 (a_1, b_2) \succ_2 (a_1, b_1) \\ (a_1, b_2) &\succ_3 (a_2, b_1) \succ_3 (a_2, b_2) \succ_3 (a_1, b_1), \end{aligned}$$

where  $X \succ_i Y$  means individual  $i$  strictly prefers  $X$  over  $Y$ .

- a. What is the outcome of sincere voting behavior?
- b. Is the outcome in part (a) an equilibrium if the voters were behaving strategically?
- c. Characterize one equilibrium profile under the strategic voting behavior.

## Chapter 5

# Electoral competition

1. Consider a Downsian electoral competition over the line  $[-1, 1]$  where voters are uniformly distributed. Suppose there are two candidates,  $\{D, R\}$  and the ties are broken by a coin toss.
  - (a) What is the Nash Equilibrium of this game?
  - (b) Now suppose that there is an ideological candidate  $C$  that positions himself at  $-1$  whatever the other candidates do. Is there a Nash Equilibrium of the game? If so, what is it?
  - (c) Suppose now that there is another ideological candidate at  $1$ . Is there an equilibrium and what is it?
2. Consider the Downsian model of electoral competition but suppose that now we have a two dimensional policy space. There are only four discrete policies from which politicians can choose:

$$x_i \in \{(1, 1), (-1, 1), (1, -1), (-1, -1)\}.$$

As before, each voter has preferences that decrease with the Euclidean distance between her bliss point and the promised policy. There are 25 voters at each of these points where the candidates can place themselves.

- (a) Is it an equilibrium for both candidates to place themselves at  $(1, 1)$ ?
- (b) Suppose one of the candidates are placed at  $(-1, 1)$  and the other at  $(1, 1)$ . Is this an equilibrium?

- (c) Can you find any set of positions for the candidates that does not result in an equilibrium?
3. Consider the Downsian model of electoral competition but suppose that now we have a two dimensional policy space. There are only discrete policies that the politicians can choose:

$$x_i \in \{(0, 0), (1, 1), (-1, 1), (1, -1), (-1, -1)\}$$

As before, each voter has preferences that decrease with the Euclidean distance between her bliss point and the promised policy. There are 20 voters at each of these points where the candidates can place themselves.

- (a) Is it an equilibrium for both candidates to place themselves at  $(1, 1)$ ?
- (b) Suppose one of the candidates are placed at  $(-1, 1)$  and the other at  $(1, 1)$ . Is this an equilibrium?
- (c) Is it an equilibrium for both candidates to place themselves at  $(0, 0)$ ?
- (d) Is there a unique equilibrium?
- (e) Does the Median Voter Theorem 1 hold in this example?
4. A town will elect an officer and in the town there are only 400 people all of which share the same ideology. Citizens of the town declare candidacy, everyone votes and the ties are decided with a coin flip. Normalize utility to 0 if a citizen does not run and their most preferred policy is enacted (which will occur as long as any person runs as everyone share to same policy position).
- (a) Find conditions on  $B$  in terms of  $C$  such that there is only one citizen running for the office.
- (b) Find conditions on  $B$  in terms of  $C$  such that there are only two citizens running for the office.
- (c) Find conditions on  $B$  in terms of  $C$  such that there are  $K$  citizens running for the office.
5. A town will elect an officer and in the town there are two groups of 400 people with the same ideology. Citizens of the town declare candidacy, everyone votes and the ties are decided with a coin flip. A voter gets utility -1 if the elected candidate is of the other group of ideology and 0 otherwise.

- (a) Is there any equilibrium with a single candidate? If there is, what are the conditions one has to impose on  $B$  and  $C$  to get it, and is it unique? (Assume that every citizen gets a utility of  $-\infty$  if no citizen becomes a candidate.)
  - (b) Is there an equilibrium with two candidates? If there is, what are the conditions one has to impose on  $B$  and  $C$  to get it, and is it unique?
  - (c) Are there parameters where two citizens from the same ideological camp run for the election in the equilibrium?
  - (d) Can you find parameters such that every citizen running for the office is an equilibrium?
6. Consider the citizen-candidate model of electoral competition. Suppose that the population of citizens contains 1,000 individuals. Of these, 450 people have the most preferred policy equal to  $-1$ , 450 people have the most preferred policy equal to  $1$  and the remaining 100 people prefer  $0$ . How many citizens care about the policies is captured by parameter  $\alpha \geq 0$ . That is, when a policy  $y$  is implemented, a citizen with most preferred policy  $x_i$  receives the utility  $-\alpha|y - x_i|$  from policy,  $y$ . Moreover, any citizen can run for office. The cost of running is  $C$  and the benefit of the office is  $B$ . When no candidate runs for office, each citizen gets utility  $-100$ .
- (a) Suppose  $\alpha = 0$ . What are the conditions on  $B$  in terms of  $C$  to have no citizen run for the office? How does this depend on  $\alpha$ ?
  - (b) Suppose  $B = 3$  and  $C = 2$ . Can you find an  $\alpha$  such that there is a one-candidate equilibrium and the candidate has the ideal position of  $-1$ ?
  - (c) Is there any combination of  $B$ ,  $C$  and  $\alpha$  such that there is a one-candidate equilibrium where the candidate has the ideal position of  $-1$ ?
  - (d) Now suppose  $B = 3$  and  $C = 2$ . What is the unique one candidate equilibrium? Why is it unique? Does this change with  $\alpha$ ?
  - (e) What are the conditions on  $B$  in terms of  $C$  and  $\alpha$  that guarantees a two-candidate equilibrium where candidates have ideal points  $-1$  and  $1$ ? What happens as  $\alpha$  increases, why?
  - (f) Suppose  $B = 0$  and  $C = 1$ . Is there some  $\alpha$  that we will still have a two-candidate equilibrium?

- (g) Suppose  $B = 4$  and  $C = 2$ . Can you find  $\alpha$  such that there is a two-candidate equilibrium where both candidates have 0 as their ideal positions.
- (h) Suppose  $B > C > 0$ . Is there any combination of  $B$ ,  $C$  and  $\alpha$  such that there is an equilibrium with three candidates, all with different ideal policies?
7. Consider the citizen-candidate model of electoral competition. Suppose that the population of citizens contains  $N$  individuals. Of these,  $N_1$  people have the most preferred policy equal to  $-1$ ,  $N_{-1}$  people have the most preferred policy equal to  $1$  and the remaining  $N_0$  people prefer  $0$ . Moreover, citizens can run for the office. The cost of running is  $C$  and the benefit of the office is  $B$ . When no candidate runs for the office, each citizen gets utility  $-100$ .
- (a) Suppose  $B = 3$  and  $C = 2$ . What are the conditions on the population parameters  $N_i$  such that there is a one-candidate equilibrium with a candidate with ideal position  $1$  running?
- (b) Suppose  $B = 4$  and  $C = 2$ . What are the conditions on the population parameters  $N_i$  such that there is a two-candidate equilibrium with candidates with ideal points  $-1$  and  $1$  running?
- (c) Suppose  $B = 6$  and  $C = 2$ . What are the conditions on the population parameters  $N_i$  such that there is a three-candidate equilibrium with candidates with distinct ideal points?
8. Consider the citizen-candidate model of electoral competition. Suppose that the population of citizens contains  $1,000$  individuals. Of these,  $450$  people have the most preferred policy equal to  $-1$ ,  $450$  people have the most preferred policy equal to  $1$  and the remaining  $100$  people prefer  $0$ . How much citizen  $i$  cares about policy is captured by parameter  $\alpha_i \geq 0$ . That is, when a policy  $y$  is implemented, a citizen with most preferred policy  $x_i$  receives the utility  $-\alpha_i|y - x_i|$  from policy. Moreover, citizens can run for the office. The cost of running is  $C$  and the benefit of the office is  $B$ . When no candidate runs for the office, each citizen gets utility  $-100$ .
- (a) Suppose  $B = 3$  and  $C = 2$ . Are there any conditions on  $\alpha_1$ ,  $\alpha_0$  and  $\alpha_{-1}$  such that there is a one-candidate equilibrium where the candidate has the ideal point  $1$ ?

- (b) Suppose  $B = 3$  and  $C = 2$ . Are there any conditions on  $\alpha_1$ ,  $\alpha_0$  and  $\alpha_{-1}$  such that there is a one-candidate equilibrium where the candidate has the ideal point 0?
- (c) Suppose  $B = 4$  and  $C = 2$ . Are there any conditions on  $\alpha_1$ ,  $\alpha_0$  and  $\alpha_{-1}$  such that there is a one-candidate equilibrium where the candidates has ideal points  $-1$  and  $1$ ?



## Chapter 6

# Parties

1. Answer whether each of the following statements is true or false. Briefly explain each of your answers.
  - (a) Consider the voter mobilization model with two elections,  $e_1$  and  $e_2$ . The primitives of the model are the same for both elections, but with two differences: (i) supporters of  $a$  care only about voting for  $e_1$  and not  $e_2$ ; (ii) supporters of  $b$  care about voting in both elections exactly as much as supporters of  $a$  care about  $e_1$ . As a result, in equilibrium parties invest more for mobilization in  $e_2$  than  $e_1$  in order to convince more supporters of  $a$  to vote.
  - (b) In the voter mobilization model, parties can invest different amounts in equilibrium.
  - (c) Consider a policy platforms model with group  $a$ ,  $b$ , and  $c$  representing the 60%, 20%, and 20% of the population respectively. If politicians  $a$  and  $b$  are in the same party, then  $a$  decides the policy platform since she has the majority of the votes.
  - (d) In the policy platforms model, a party configuration is stable if neither politicians have the incentive to leave and form a smaller party.
  - (e) In the candidate recruitment model there is no equilibrium where a party selects a high-skilled candidate while the other chooses a mediocre one.
  - (f) In a candidate recruitment model, increasing the party's benefit from winning an election will not avoid a mediocracy.
  - (g) Aristocracy is more likely to occur in a majoritarian system.

2. In the voter mobilization model, it was assumed that both parties obtain equal benefits if their preferred policy is implemented. Let us assume that party  $j$  obtains benefit  $B_j$  with  $B_a > B_b$ . Show that in equilibrium the optimal level of investment for party  $a$  is larger than the one for party  $b$ .
3. Consider the voter mobilization model in section 6.1. Now suppose that the party supporting alternative  $a$  has a constraint on the level of investment to encourage its supporters to vote. Suppose that the the highest investment possible for  $P_a$  is  $\bar{I} < I_a^*$ , where  $I_a^*$  denotes the optimal investment level of the party with no budget constraint. Suppose that  $D_a^e = 1, D_b^e = 0.25, B = 1$ , and the investment constraint is given by  $\bar{I} = 0.15$ . Explain whether there exists a Nash equilibrium in which party  $a$  invests the highest possible level.
4. Consider the policy platforms model described in section 6.2 with  $x^a = (-1, -1), x^b = (1, 1)$  and  $x^c = (0, 1)$ . Answer the following questions:
  - (a) Consider the party configuration  $(\{b\}, \{a, c\})$ . Is the policy  $x = (0.8, 0.2)$  in the Pareto set  $P^{\{a, c\}}$ ? If not, show graphically the area of points that Pareto dominate  $x$ . Write down the equation that characterizes the policy platforms in  $P^{\{a, c\}}$ .
  - (b) Given the party configuration  $(\{a\}, \{b\}, \{c\})$ , compute the indirect utility of each politician for each possible election outcome. Why is politician in party  $\{c\}$  running unopposed the only equilibrium?
  - (c) Suppose that 20% of the population is in group  $b$ , 25% is in group  $c$ , and 55% is in group  $a$ . How does it change the equilibrium policy platforms? What about the equilibrium party configurations?
5. In the last few decades, the Italian political system has been characterized by a very large number of political parties (between thirty and forty). However, the party configuration of the 2006 Italian general elections involved two main coalitions: the center-right wing coalition (led by Silvio Berlusconi and including 19 parties) and the center-left coalition (led by Romano Prodi and including 18 parties). How can you interpret this equilibrium party configuration using the policy platform model? What can we infer about the preferred policies  $x_j$ 's of the parties in each coalition?
6. Consider the policy platforms model with a two-dimensional policy space  $X = X_1 \times X_2, X_1 = X_2 = [-1, 1]$  and a continuum of voters

divided in three groups (each containing  $1/3$  of the population) denoted by  $j \in \{a, b, c\}$ . The indirect utility for each member of group  $j$  has the quadratic form  $V_j(x) = -(x_1^j - x_1)^2 - (x_2^j - x_2)^2$  with most preferred policies  $x^j$ 's given by

$$x^a = (-1, -1), \quad x^b = (1, 1), \quad \text{and } x^c = (-1, 0.5).$$

As in section 6.2, one voter of each group is a politician and has the same preferences as each member of her group.

- (a) Draw the most preferred policies of each group and at least two indifference curves in the policy space.
  - (b) What is the set of potential party configurations running for election?
  - (c) Characterize and draw the Pareto set of each party. How do they differ compared to the case where  $x^c = (-1, 1)$  (as in section 6.2)?
  - (d) Find the equilibrium policy platform for each possible party configuration.
  - (e) Given the results in (d), find the equilibrium party configurations. Compare your findings with the  $x^c = (-1, 1)$  case. What is your interpretation of such differences?
7. Consider one of the past presidential elections in the U.S., for instance the election in 2016. Using the model described in section 6.3 try to explain how the primaries and the general election can fit into narrative of the model about the two phases of the election. Based on your personal judgment about the primary contestants and candidates in the general election, answer the following questions.
- (a) Explain whether each party chose mediocracy or aristocracy as its strategy.
  - (b) Based on your answer to the previous part and the candidate recruitment model in this chapter, can you explain why each party chose mediocre or the aristocratic candidate and under what conditions they would have changed their choice of candidate.
  - (c) Argue whether there was a discouragement effect in the organizational phase of each party.



## Chapter 7

# Lobbies

1. Consider the model of ex-ante lobbying where one lobby may offer campaign contributions to one of two candidates running in an election.
  - (a) According to this model, will the candidate choose an equilibrium policy that is equal to the policy most preferred by the lobby?
  - (b) According to this model, will the candidate receive any transfers from the lobbyist?

Consider now the model of ex-post lobbying where one lobby may offer transfers to an elected politician.

- (c) According to this model, will the politician choose an equilibrium policy that is equal to the policy most preferred by the median voter?
- (d) According to this model, will the politician receive any transfers from the lobbyist?

Suppose now that the lobby may choose whether to offer campaign contributions to one of the two candidates running in an election, or to offer transfers to the politician who wins the election.

- (e) Which one would the lobby choose?
2. Answer whether each of the following statements is true or false. Briefly explain each of your answers.
    - (a) In the model of ex ante lobbying where one lobby may offer campaign contributions to one of the two candidates running in an

- election, both candidates choose an equilibrium policy that is equal to the average of the policy most preferred by the lobby and the policy most preferred by the median voter.
- (b) In the model of ex ante lobbying where two lobbies may offer campaign contributions to the two candidates running in an election, both candidates choose an equilibrium policy that is equal to the average of the policies most preferred by the two lobbies.
  - (c) In the model of ex post lobbying where one lobby may offer transfers to an elected politician, the politician chooses an equilibrium policy that is equal to the policy most preferred by the lobby.
  - (d) In the model of ex post lobbying where two lobbies may offer transfers to an elected politician, the politician chooses an equilibrium policy that is equal to the average of the policies most preferred by the two lobbies.
  - (e) In the models of ex ante and ex post lobbying with only one lobby, it is always the case that ex ante lobbying biases policy away from the outcome that would result if lobbying were not allowed more than ex post lobbying.
3. Consider a Downsian model of electoral competition where two candidates,  $L$  and  $R$ , are competing for elections. The policy space  $X$  is unidimensional and the policy most preferred by the median voter is  $X_M = 0$ . The candidates compete by choosing policies they would implement if elected. Candidate  $L$  announces policy  $X_L$  and candidate  $R$  announces policy  $X_R$ . The objective of each candidate is to maximize the probability of winning the election. There is one right-wing lobby that may offer candidate  $R$  campaign contributions  $C_R$  to enhance his electoral chances. The probability candidate  $R$  wins the election is

$$\pi_R = \frac{1}{2} + C_R.$$

The utility function of the lobby is

$$U^R(X, C_R) = -(1 - X)^2 - C_R^2.$$

- (a) Which policies will candidate  $L$  and candidate  $R$  choose in equilibrium?
- (b) In equilibrium, how much money will the lobby contribute to the electoral campaign of candidate  $R$ ?

4. Consider a Downsian model of electoral competition where two candidates,  $L$  and  $R$ , are competing for elections. The policy space  $X$  is uni-dimensional and the policy most preferred by the median voter is  $X_M = 0$ . The candidates compete by choosing policies they would implement if elected. Candidate  $L$  announces policy  $X_L$  and candidate  $R$  announces policy  $X_R$ . The objective of each candidate is to maximize the probability of winning the election.

There are two lobbies,  $A$  and  $B$ . Lobby  $A$  may offer candidate  $R$  campaign contributions  $C_A$  to enhance his electoral chances and lobby  $B$  may offer candidate  $L$  campaign contributions  $C_B$  to enhance her electoral chances. The probability candidate  $R$  wins the election is

$$\pi_R = \frac{1}{2} + C_A - \lambda C_B$$

The utility function of lobby  $A$  is

$$U^A(X, C_A) = -(\alpha - X)^2 - \frac{1}{2}C_A^2,$$

and the utility function of lobby  $B$  is

$$U^B(X, C_B) = -(\beta - X)^2 - \frac{1}{2}C_B^2.$$

- (a) Derive the campaign contributions of lobbies  $A$  and  $B$ ,  $C_A$  and  $C_B$ , as a function of the policies of candidates  $L$  and  $R$ ,  $X_L$  and  $X_R$ .
- (b) Derive the policy choices of the two candidates in the unique Nash equilibrium of the electoral competition game.
- (c) In equilibrium, how much money will each lobby contribute to the electoral campaign of the candidates and what is the probability candidate  $R$  wins the election?
- (d) Consider four possible cases. Case 1:  $\alpha = 1$ ,  $\beta = -1$  and  $\lambda = 1$ . Case 2:  $\alpha = 1$ ,  $\beta = -1$  and  $\lambda = 0.5$ . Case 3:  $\alpha = 1$ ,  $\beta = -2$  and  $\lambda = 1$ . Case 4:  $\alpha = 1$ ,  $\beta = -2$  and  $\lambda = 0.5$ . For each case, calculate the policies that candidates  $L$  and  $R$  choose in equilibrium.
- (e) For each of the four cases considered in part (d), evaluate the extent to which lobbying biases policy away from the outcome that would occur if lobbying were not allowed.

5. Consider a situation where an elected politician  $P$  has to choose a policy  $X$  and there is a lobby  $R$  that may offer the politician a monetary transfer  $T_R$  in order to attempt to influence his policy choice. The utility function of the politician is

$$U^P(X, T_R) = -X^2 + T_R$$

and the utility function of the lobby is

$$U^R(X, T_R) = -1(1 - X)^2 - T_R.$$

- (a) Which policy will the politician choose in equilibrium?  
 (b) In equilibrium, how much money will the lobby transfer to the politician?
6. Consider a situation where an elected politician  $P$  has to choose some policy  $X$  and there are two lobbies (that do not communicate)  $A$  and  $B$  that may offer the politician monetary transfers  $T_A$  and  $T_B$  in order to attempt to influence her policy choice. The utility function of the politician is

$$U^P(X, T_A, T_B) = -X^2 + \lambda(T_A + T_B),$$

the utility function of the lobby  $A$  is

$$U^A(X, T_A) = -(P_A - X)^2 - \alpha_A T_A,$$

and the utility function of lobby  $B$  is

$$U^B(X, T_B) = -(P_B - X)^2 - \alpha_B T_B.$$

Suppose that  $P_A > 0$  and  $P_B > 0$ .

- (a) Solve for the willingness to pay of the lobbyists for some policy  $X > 0$ .  
 (b) Solve for the equilibrium level policy chosen by the politician.

Now suppose that  $P_A > 0$ ,  $P = P_A = -P_B$ . Moreover, now lobby  $A$  transfers money only if  $X > 0$  and  $B$  only if  $X < 0$ .

- (c) Solve for the willingness of pay for the lobbyists for any policy.  
 (d) Let  $\alpha_A = \alpha_B = 1$  and  $P = 1$ . Determine the equilibrium policy chosen by the politician for each  $\lambda$ .

- (e) Now suppose that  $\alpha_A = 2$ ,  $\alpha_B = 1$  and  $P = 1$ . What is the equilibrium policy chosen?
7. Answer whether each of the following statements on the endogenous lobbies model is true or false and carefully explain each of your answers.
- (a) The stages of the game consist only of voting and then lobbying.
  - (b) Lobbies make contributions to the candidates before the elections take place.
  - (c) In both models of exogenous and endogenous lobbies the policy outcome is a compromise between the politician's and the lobby's preferences.
  - (d) There can be an equilibrium where all existing lobbies participate in the policy-making process.
  - (e) Lobbies, instead of voters, may be crucial in bringing the policy outcome to the center of the policy spectrum.
  - (f) It is never optimal for the politician to bargain with lobbies having completely opposite policy preferences.
8. Answer the following questions.
- (a) In order to maintain fairness in political elections, some European countries set limits on the amount of money candidates can spend for their electoral campaigns. As these limits surely benefit smaller parties lacking financial resources (by limiting larger parties' expenses during electoral campaign), they also greatly reduce the role of lobbies.  
Based on the lobbying models you saw in this chapter, how should the policy outcomes in the European countries differ compared to the ones in the USA (where there is no limit)? What could be the rationale of setting such limits?
  - (b) In 2003 the US Congress passed a law called "Medicare Part D", aimed at helping seniors buy prescription drugs. However, one of the most criticized points of this reform was that the federal government is not allowed to bargain over medicine prices, as this activity was fully outsourced to insurance companies delivering the subsidized coverage determined by the new law. It is known that the main reason behind the government giving away this bargaining right is due to the pharmaceutical companies funnelling

about 2 billion dollars to congressmen through lobbies in the past fifteen years.

Using the implications of the lobbying models you studied in this chapter, explain why such a law is still in force despite the majority of US citizens reportedly prefer a stronger role of the government in negotiating prescription drugs prices.

## Chapter 8

# Political careers

1. Answer whether each of these statements is true or false. Briefly explain your answer
  - (a) A politician who leaves politics before retirement is a career politician.
  - (b) In the BNE of the political careers model, individuals with  $p \in \left[ p^*, \frac{1+p^*}{2} \right]$  run for office in the second period.
  - (c) In equilibrium, individuals having voluntary political careers do not run for office in the second period.
  - (d) In equilibrium, people with  $p = 0$  may consider a political career.
2. Consider two countries,  $A$  and  $B$ , with exactly the same characteristics in terms of market and political sector wages, skills distributions, and public benefits generated by politicians. However, workers' productivity in the market sector of country  $A$  is tracked much better than in country  $B$ , therefore allowing more workers to find out their actual market skill after one period of work.

If both  $A$  and  $B$  were to increase the per-period politician salary  $s$  by the same amount, what would be the effect on the average quality of career politicians in the two countries?

3. In the past decade there has been a heated debate in Italy about the retirement annuity of politicians (so called *vitalizio*). In fact, retired members of the Parlamento and Regional Councils who accumulated enough tenure in office are entitled to receive a pension for life. Some parties, like the brand new Five Star Movement, are strongly against

such *vitalizio* claiming that it is waste of public money and an unfair benefit to politicians, and proposed to completely abolish it once in power.

Based on the political careers model, how would such a policy affect the Italian political sector?

4. In the model of politicians' careers, suppose the specification of the probability of having higher market ability, i.e.  $\pi(p)$  is given by

$$\pi(p) = \alpha + \lambda * p^2$$

- (a) What is the fraction of high market ability individuals among those with political ability? Compare this with the original specification for a given level of  $\lambda$ .
- (b) Now suppose that political abilities are distributed uniformly over  $[0, 2]$ , how would the fraction differ between the two specifications?
5. Consider two countries,  $A$  and  $B$ , with different politicians' turnover in office. In particular,  $A$  has a 50% turnover rate, while  $B$  has only 35%. Based on the political career model, what are the possible sources of such different turnover rates in the two countries?
6. Consider an individual in the first year of her working life with no political ability ( $p = 0$ ). Let  $\bar{w}$  denote the posted wage in the market for those with unknown abilities, in the first period.
- (a) Suppose that this individual decides to work in the market sector in both periods. Write down an expression for the expected payoff of her decision.
- (b) Write an expression for the expected payoff of joining the political sector in the first period and switching to the market sector in the second period.
7. Consider the first period of an economy, only populated by individuals in their first year of working life deciding between the market and political sectors. Explain how many different wages can be observed in the competitive market sector, how are these wages determined? Now, suppose in the first period some individuals join the market sector and some join the political sector. How many different wage

levels are posted in the market sector in the second period, and briefly explain what are these wages based on.



## Chapter 9

# Public goods

1. Answer whether each of the following statements is true or false in regards to the Political Economy of Public Goods, specifically the selection of the level of funding for some public good,  $G$ .

- (a) Let income be homogeneous across agents, preferences be single-peaked and heterogeneous, and the variable differentiating agents' preference be  $\alpha_i$ . If  $\alpha_i$  is distributed symmetrically, *i.e.*, its mean is also its median, then the democratically selected level of public good provision will be efficient.
- (b) Under public provision, efficiency requires

$$MRT = \prod_{i=1}^N MRS_i$$

- (c) If a model uses proportional taxes to raise funds for public expenditure, the government budget constraint requires revenue raised in taxes to equal the expenditure on public goods.
  - (d) When considering provision of public goods, efficiency conditions depend on the specific form of the Social Welfare Function.
  - (e) When considering provision of public goods, efficiency conditions are dependent upon whether a tax is raised proportionally or as a lump sum.
2. A community of 10 people is considering building a public swimming pool of size  $G$ .

Individuals derive utility from a homogeneous private good  $X$  (the numeraire) and a swimming pool of size  $G$ . Their utility function is

$$U(X, G) = \ln X + \alpha \ln G$$

where  $\alpha = 1$  for four individuals,  $\alpha = 3$  for three individuals,  $\alpha = 4$  for two individuals, and  $\alpha = 9$  for one individual. Each individual has an income of 200. The cost of a unit of swimming pool is 500.

The community decides to build the pool and chooses the size by majority voting.

- (a) If costs are shared equally among all individuals, what pool size will be chosen?
  - (b) Is the outcome of majority voting efficient?
3. The 250 people living in a community are considering how much to spend on national defense.

Income is distributed in the community as: 100 people have income of \$100, 50 people have income of \$200, 100 people have income of \$300. People's preferences for defense expenditures ( $E$ ) are described by the following functions

$$V_i(E) = y_i E - E^2$$

where  $y_i$  is the income of individual  $i$ .

Suppose that there are 5 alternative levels of spending (0, 50, 100, 150, and 200) and people vote on them in ascending order according to sequential majority voting.

- (a) What is the level of expenditure that will be chosen?
  - (b) Is the outcome efficient?
  - (c) Does the outcome depend on the order in which the votes are taken? Explain.
4. A community of  $N = 10$  people is considering building a public library of size  $G$ . Individuals derive utility from consuming a homogeneous private good  $X$  (the numeraire) and a public library of size  $G$ . Their utility function is

$$U(X_i, G) = \ln X_i + \alpha_i \ln G$$

where 4 individuals have  $\alpha_i = \frac{1}{8}$  and income  $I_i = 800$ , 2 individuals have  $\alpha_i = \frac{1}{4}$  and income  $I_i = 600$ , and 4 individuals have  $\alpha_i = \frac{1}{2}$  and income  $I_i = 400$ . The cost of a unit of public library is  $c = 10$ .

The community decides to build the public library and chooses its size by voting. It is also decided that the cost associated with building the public library will be shared in proportion to an individual's income: that is, each individual will pay a tax as a share of their income equal to

$$T = \frac{cG}{N\bar{I}}.$$

- (a) Derive the indirect utility functions of the individuals in the community.
  - (b) Show that the indirect utility functions are single-peaked.
  - (c) Solve for the size of the public library that will be chosen by referendum.
  - (d) Solve for the efficient size of the public library.
  - (e) Suppose that the community is constrained to choose only between two alternative sizes of the public library,  $G_H = 1000$  or  $G_L = 100$ . What size of the public library will be chosen under majority rule? Note: preferences are not necessarily symmetric. You cannot, therefore assume that the level closest to an individual's most preferred size  $G_i^*$  is the level he'd most prefer.
5. Two individuals,  $A$  and  $B$ , have to decide whether or not to contribute to a public good. The income individual  $A$  is \$1000 and the income of individual  $B$  is \$900. The benefit to individual  $A$  if only one individual contributes to the public good is equivalent to \$50 and if both individuals contribute is equivalent to \$100. The benefit to individual  $B$  if only one individual contributes to the public good is equivalent \$100 and if both individuals contribute is equivalent to \$300. The individual cost of contributing to the public good is \$100 for both individuals.
- (a) If the two individuals choose simultaneously whether or not to contribute, what are the Nash Equilibria of the voluntary contribution game? Are the Nash Equilibria Pareto Optimal?
  - (b) Suppose now that individual  $B$  has the option of paying \$150 to make contributions compulsory for both herself and individual  $A$ , instead of letting each individual decide whether or not to contribute. Would  $B$  pay that amount?

6. Consider a society inhabited by 20 individuals ( $N=20$ ). The society is divided into three groups (types)  $j = A, B, C$ ; 11 individuals belong to group  $A$ , 5 individuals belong to group  $B$ , and 4 individuals belong to group  $C$ . Suppose that type  $j$ 's preferences over a publicly provided good  $G$  and a privately provided good (the numeraire)  $X$  are

$$U^j(X_j, G) = \ln X_j + \alpha_j \ln G$$

where  $\alpha_j$  represents group  $j$ 's taste for the public good, and let  $\alpha_A = 1/4$ ,  $\alpha_B = 1/2$ , and  $\alpha_C = 3/4$ . Assume, in addition, that all individuals have identical wealth  $Y = 1400$ . Finally, the cost of producing one unit of the public good is  $c = 20$ .

- (a) Compute the amount of public good provided by the community through voluntary contributions assuming that all individuals in the same group make the same contribution and one generic individual in each group decides the level of contribution of all the individuals in the group.
  - (b) Suppose now that in order to finance the provision of the public good the government taxes all individuals equally (that is, each individual pays  $cG/N$ ). Compute the amount of public good provided by the community under majority rule.
  - (c) Compute the socially optimal amount of public good in this economy. How does it compare to the quantity you found in parts (a) and (b)?
7. (a) Suppose a town is deciding how much money to spend on a public works project,  $x$ . Let's say the town has agreed to spend no more than \$10,000 on the project, so  $x$  is within  $[0, 10000]$ . There are 100 voters in the town and voter  $i$ ,  $i = 1, \dots, 100$ , evaluates alternative expenditures  $x$  according to the indirect utility function  $V^i(x) = |a_i + x|$ , where  $a_i$  is an individual specific parameter. Is it possible to use Median Voter Theorem I to establish the existence of a Condorcet winner? Does the answer depend on the values of the parameters  $a_1, \dots, a_N$ ?
- (b) Suppose that those same townspeople had different indirect utility functions, while everything else stayed the same. So voter  $i$  evaluates alternative expenditures  $x$  according to the indirect utility function  $V^i(x) = (I_i/10 - x)^2$ , where  $I_i$  is voter  $i$ 's income. Furthermore, assume that 10 voters have incomes of \$100,000; 20

voters have incomes of \$30,000; 30 voters have incomes of \$20,000; and the remaining voters have incomes of \$10,000. Is it possible to use one of the two Median Voter Theorems to establish the existence of and find a Condorcet winner?



## Chapter 10

# Public schools

1. Consider the model of the provision of public education discussed in this chapter.
  - (a) According to this model, when will the superintendent hold a referendum to change the current (status quo) level of expenditure in public education?
  - (b) According to the model, if the superintendent does hold a referendum and the referendum is successful (that is, the community votes in favor of the proposal by the School Board), what is the relationship between the outcome of the referendum (that is, the new level of expenditure in public education implemented after the referendum) and the status quo level?
  - (c) What are the three testable implications of this model if the superintendent is a bureaucrat a la Niskanen?
  - (d) What are the three testable implications of this model if the superintendent is a Downsian politician?
2. The 100 residents of a community are deciding the level of public education spending in their community. There are two groups of people, 60 young resident who have a school-age child and 40 seniors living on their own. Although the senior residents value the education of the children in their community, their valuation is different from the young parents. To reflect this, we modify primitives of the model discussed in this chapter. Let the preferences of the young parents be given by  $U_i^y(c_i, E) = \ln c_i + 2 \ln E$  and those of the senior residents by  $U_i^o(c_i, E) = \ln c_i + \ln E$ . Suppose that the current level of expenditure is \$100, and every individual has an income of \$200.

- (a) What is the optimal level of spending on public education for an individual in each group?
  - (b) If the superintendent is a Downsian politician, characterize the political-economic equilibrium.
  - (c) If the superintendent is a bureaucrat, characterize the political-economic equilibrium.
3. Consider again the model of provision of the public education in this chapter in a community populated by 10 individuals with similar preferences over the numeraire good and public education spending with  $\alpha = \frac{1}{2}$ . However, individuals do not enjoy the same income level. 2 individuals earn \$40, 6 individuals earn \$100, and the remaining 2 earn \$200 of income. Suppose that the current level of spending on public education is \$50. What is the optimal level of spending on public education for an individual in each income group? How does the political-economic equilibrium differ from the one discussed in the chapter,
- (a) If the superintendent is a Downsian politician?
  - (b) If the superintendent is a bureaucrat?
4. In this chapter, we assumed that the goal of an appointed superintendent is to maximize the size of the budget. However, the body in charge of this appointment may be elected by the public. The appointed superintendent may have a fixed term and the decision of being reappointed can reflect the satisfaction of the electorate with the appointed official. Consider the role of the state superintendent; Find one state where the superintendent is elected and one that is appointed directly by an elected official. How often an elected superintendent is reelected and how often an appointed superintendent is reappointed after his or her fixed term comes to an end?

Now let's consider the model of the political provision of primary education with some slight modifications. We leave the primitives of the political economy intact except for the goal of the superintendent, who is an appointed bureaucrat. Let the status-quo level of education on public education be denoted by  $E_{SQ}$ , the political-economic equilibrium level of spending on public schools when the superintendent is elected by  $E_{PESE}^*$ . Now, let's suppose that the appointed superintendent's utility from successfully implementing  $E$  is given by

$$E - (E - E_{PESE}^*)^2.$$

What is the political-economic equilibrium in this modified model? How does this compare to the case where the bureaucrat is just maximizing the budget?

5. People living in a community are considering how much to spend in public education. Their preferences for education expenditure ( $E$ ) are described by the following indirect utility function, which is the same for all individuals in the community:

$$V(E) = 100E - E^2.$$

- (a) Suppose that the community chooses how much to spend in public education by voting over all possible levels of  $E$ , but you do not know which voting rule the community is using. What is the level of expenditure that will be chosen?
- (b) Suppose instead that the community chooses how much to spend in public education by holding a referendum. Suppose also that there is a School Board who has power over the agenda, and can face the voters with a choice between an expenditure level of its choice and the current (status-quo) level  $E_0$ . Assuming that the School Board's objective is to maximize the amount of education expenditures, characterize the level of expenditure that will be chosen as a function of  $E_0$ .
6. The 100 people living in a community are considering how much to spend in public education. Individuals derive utility from a homogeneous private good  $X$  (the numeraire) and per-capita education expenditure  $E$ . Their utility function is:

$$U(X, E) = \ln X + \ln E.$$

Each individual has income of 100 and education expenditures are financed with a proportional income tax. Let  $t$  denote the tax rate.

- (a) Suppose that the community chooses how much to spend in public education by majority rule. What is the level of expenditure that will be chosen?
- (b) Suppose instead that the community chooses how much to spend in public education by holding a referendum. Suppose also that there is a School Board who has power over the agenda, and can face the voters with a choice between an expenditure level of its

choice and the current (status-quo) level  $E_0 = 10$ . Assuming that the School Board's objective is to maximize the amount of education expenditures, what is the level of expenditure that will be chosen?

7. Answer whether each of the following statements is true or false. Briefly explain each of your answers.
- (a) If the superintendent behaves as a bureaucrat, it will call a referendum whenever the status quo level of expenditure in public education is higher than the level most preferred by the residents of its school district.
  - (b) If the superintendent behaves as an elected Downsian politician, it will call a referendum whenever the status quo level of expenditure in public education is lower than the level most preferred by the residents of its school district.
  - (c) Whenever referenda occur in school districts managed by appointed bureaucrats, we should observe a positive correlation between the alternative level of expenditure in public education proposed by the superintendent and the status quo level.
  - (d) Whenever referenda occur in school districts managed by elected politicians, we should observe a negative correlation between the alternative level of expenditure in public education proposed by the superintendent and the status quo level.
  - (e) If we ever observe a referendum where a superintendent proposes to decrease the level of spending in public education, then we can conclude that particular School Board does not behave as a bureaucrat a la Niskanen.

## Chapter 11

# Higher education

1. Answer whether each of the following statements is true or false. Briefly explain each of your answers.
  - (a) The creation of student loans for higher education represents a Pareto improvement (with respect to a system where student loans do not exist).
  - (b) The creation of private universities represents a Pareto improvement (with respect to a system where all universities are public).
  - (c) A mixed higher education system with a private and a public component is always efficient.
2. Each of the 100 individuals living in a community has a certain amount of wealth,  $W$ , which she can either invest in education,  $E$ , or in a physical capital good,  $K$ , yielding a return  $R = 2$ . Furthermore, she is endowed with an ability level,  $A$ , that positively affects the return of her investment in education  $AE^\beta$ , where  $\beta = 0.5$ . The individual derives utility from her consumption,  $C$ , and the utility function is

$$U(C) = C.$$

People differ with respect to their wealth,  $W$ , as well as their ability,  $A$ . 20 people have  $W = 5$  and  $A = 4$ , 20 people have  $W = 10$  and  $A = 8$ , 40 people have  $W = 15$  and  $A = 16$ , and 20 people have  $W = 25$  and  $A = 20$ .

- (a) If education is only provided through a private system, how much education will each individual choose to invest in? Is the resulting allocation efficient?

- (b) If education is only provided through a public system financed by a proportional tax on wealth where the tax rate  $t$  is chosen by majority voting, how much education will be provided? Is the level of provision efficient? Is the level of provision constrained efficient?
3. Answer whether each of the following statements is true or false. Briefly explain each of your answers.
- (a) If higher education is only provided through a private system, where homogeneous individuals with respect to wealth and ability can choose different levels of education, then the resulting allocation is always efficient.
- (b) If higher education is only provided through a public system financed by a proportional tax on wealth, where everybody receives the same level of education and heterogeneous individuals with respect to wealth and ability choose the tax rate by majority voting, then the resulting level of education is always constrained efficient, but is also always inefficient.
- (c) A mixed higher education system, where all individuals receive the same basic level of education through the public system and each individual can then invest in additional education provided through a private system, always represents a Pareto Improvement over both a pure private and a pure public system.
4. Each of the 90 individuals living in a community has a certain amount of wealth,  $W$ , which she can either invest in education,  $E$ , or in a physical capital good,  $K$ , yielding return  $R = 3$ . Furthermore she is endowed with an ability level,  $A$ , that positively affects the return of her investment in education which is equal to  $A\sqrt{E}$ . The individual derives utility from her consumption,  $C$ , and the utility function is  $U(C) = C$ .
- People differ with respect to their wealth and ability. 30 people have  $A = 6$  and  $W = 30$  and another 30 people have  $A = 12$  and  $W = 36$ . The remaining 30 people have  $A = 18$  and  $W = 54$ .
- (a) If education is only provided through a private system, how much education will each individual choose to invest in? (Show your calculations.)
- (b) Is the resulting allocation efficient? Briefly explain your answer.

- (c) If education is only provided through a public system financed by a proportional tax on wealth where the tax rate  $t$  is chosen by majority voting, how much education will be provided? (Show your calculations.)
- (d) Is the level of provision constrained efficient? Briefly explain your answer.
- (e) Suppose that the education system in the community is a pure public system. A politician calls for a referendum to introduce a private system on top of the public one. That is, he proposes a transition from a pure public system to a mixed public/private system. Will the referendum pass? Provide the intuition for the result.
5. Each of the 100 individuals living in a community has a certain amount of wealth,  $W$ , which she can either invest in education,  $E$ , or in a physical capital good,  $K$ , yielding return  $R = 2$ . Furthermore she is endowed with an ability level,  $A$ , that positively affects the return of her investment in education which is equal to  $A\sqrt{E}$ . The individual derives utility from her consumption,  $C$ , and the utility function is  $U(C) = C$ .
- People differ with respect to their wealth and ability. 30 people have  $A_1 = 10$  and  $W_1 = 20$ . The remaining 70 people have  $A_2 = 5$  and  $W_2 = 30$ .
- (a) What would be the education level chosen by each individual under the pure private system?
- (b) What would be the education level chosen by each individual under the pure public system? What is the implied tax rate?
- (c) Now suppose that the education system in the community will be decided with a majority vote. That is, every individual in the community will either vote for the public education or private education. Explain why Arrow's impossibility theorem does not apply and why we can use the Median Voter Theorem.
- (d) Which education system will be victorious in education?
- (e) Which education system is more efficient? And does the election pick that system? Why or why not?
6. Consider a finite population of  $n$  individuals with  $W_i = W$  for all  $i = 1, \dots, n$  but different ability levels  $A_i$ . Show that the Condorcet winner

of the voter problem with respect to the level of public education in a mixed system,  $E_{mix}^{PUB}$ , is equal to the Condorcet winner of a pure public higher education system.

7. Consider a pure private higher education system with the addition of a loan market. In particular, assume that each individual  $i \in \{1, 2, \dots, n\}$  chooses to allocate their initial wealth,  $W_i$ , to either education,  $E_i$ , or to the loan market which gives a return of  $R$ . As in the standard model, education gives returns equal to  $A_i E_i^\beta$  where  $\beta = 0.5$  and the utility is linear in consumption,  $U(C_i) = C_i$ .

- (a) Find the optimal amount of education for each individual.

Now assume that individuals whose education choice is constrained by a low  $W_i$  can borrow money from the loan market to be invested in education. Let  $b_i$  denote the amount borrowed by individual  $i$ . The amount to be repaid is then  $b_i R$ .

- (b) What is the optimal amount  $b_i^*$  each individual is willing to borrow?

Assume the loan market is managed in a centralized way by the government, whose only objective is to collect money from those investing in the loan market and lend money to borrowers, with its only constraint being

$$\bar{L} \equiv \sum_i l_i \geq \sum_i b_i \equiv \bar{B}$$

where  $l_i$  is the amount invested in the loan market by individual  $i$ .

Suppose now the case where, in order to satisfy the loan market constraint, the government can give out individual loans for a maximum amount of  $\bar{b} = \bar{L}/N_b$ , where  $N_b$  is the number of borrowers (i.e. financially constrained individuals). That is,  $b_i \leq \bar{b}$ .

- (c) Derive the social welfare improvement (from a utilitarian point of view) of the introduction of such policy.
- (d) If  $\bar{L} \geq \sum_i b_i^* \equiv \bar{B}^*$ , what would be a better loan policy in terms of social welfare?

Now assume that  $\bar{L} < \bar{B}^*$  and that the government decides to run a policy which prioritizes some of the borrowers with respect to others.

- (e) Which borrowers should have priority in getting a loan in order to maximize the welfare gain? Would a merit-based loan (i.e. based only on  $A_i$ ) be enough of a priority criterion?

Now assume that all borrowers have same wealth  $W$  and consider a policy in which the government sets a loan eligibility cutoff  $\underline{A}$  such that only individuals with  $A_i \geq \underline{A}$  get a loan. The amount of each loan is constant and equal to  $\tilde{b}$ .

- (f) Explain in words when such a policy can bring about a social welfare gain.



## Chapter 12

# Redistribution

1. Answer whether each of the following statements is true or false. Briefly explain each of your answers.
  - (a) A lump-sum transfer financed by proportional taxation of labor income is an efficient redistribution policy.
  - (b) A lump-sum transfer financed by proportional taxation of fixed endowments is an efficient redistribution policy.
  - (c) A transfer proportional to income financed by lump-sum taxation is an efficient redistribution policy.
2. The 100 individuals living in a community derive utility from consumption,  $C$ , and leisure,  $\ell$  and are endowed with one unit of time they can allocate between work and leisure. Their utility function is

$$U(C, \ell) = C + \ln \ell$$

People differ with respect to their wage rate,  $W$ . Of the 100 townspeople, 60 individuals have  $W = 1$ , and the remaining 40 individuals have  $W = 4$ . If an individual works an amount of time  $L$ , her earned income is equal to  $WL$ .

The community imposes a linear income tax: that is, a proportional tax on earned income to finance lump-sum redistribution to all individuals. The tax rate,  $t$ , and the per-capita amount of redistribution,  $S$ , are chosen by majority voting. The budget for the community has to balance.

- (a) Derive the individuals' labor supply.

- (b) Characterize the community's income distribution.
  - (c) Derive the indirect utility functions for the individuals in the community and show they satisfy the "single-crossing" property.
  - (d) Characterize the community's budget constraint.
  - (e) What is the tax rate that will be chosen?
3. The individuals living in a community derive utility from consumption,  $C$ , and leisure,  $\ell$ . Their utility function is

$$U(C, \ell) = \frac{(1 - \ell)^2}{2}$$

People differ with respect to how productive they are with their use of time. An individual with productivity  $X$  has  $1 + X$  (effective) units of time she can allocate between work and leisure. The average productivity of the population is  $\bar{X} = 0.5$  and the median is  $X_M = 0.2$ . The wage rate is the same for everybody and is  $W = 1$ . Hence, if an individual works an amount of time  $L$ , her earned income is equal to  $WL = L$ .

The community imposes a linear income tax: that is, a proportional tax on earned income to finance lump-sum redistribution to all individuals. The tax rate,  $t$ , and the per-capita amount of redistribution,  $S$ , are chosen by majority voting. The budget for the community has to balance.

- (a) Derive the individuals' labor supply.
  - (b) Calculate the mean and median of the community's income distribution.
  - (c) Derive the indirect utility functions for the individuals in the community and show they satisfy the "single-crossing" property.
  - (d) Characterize the community's budget constraint.
  - (e) What is the tax rate that will be chosen?
4. The 150 individuals living in a community derive utility from consumption,  $C$ , and leisure,  $\ell$ . Their utility function is

$$U(C, \ell) = C - \alpha(1 - \ell)^2.$$

People differ with respect to their preferences for leisure,  $\alpha$ . Specifically, 100 people have  $\alpha = 1$ , 25 people have  $\alpha = 2$ , and the remaining

25 people have  $\alpha = 3$ . Each individual is endowed with one unit of time to allocate between work and leisure. The wage rate,  $W$ , is the same for everybody and is equal to 1. Hence, if an individual works an amount of time  $L$ , her leisure time is  $1 - L$  and her earned income is equal to  $L$ .

The community imposes a linear income tax: that is, a proportional tax on earned income to finance lump-sum redistribution to all individuals. The tax rate,  $t$ , and the per-capita amount of redistribution,  $S$ , are chosen by majority voting. The budget for the community has to balance.

- (a) Derive the individuals' labor supply.
  - (b) Calculate the mean and median of the community's income distribution.
  - (c) Derive the indirect utility functions for the individuals in the community and show they satisfy the single-crossing property.
  - (d) Characterize the community's budget constraint.
  - (e) What are the values of  $t$  and  $S$  that will be chosen by majority voting?
5. Consider the model of income redistribution where individuals differ with respect to their potential wage rate and their utility function is additively separable and linear in consumption.
- (a) What is the marginal effect of an increase in the tax rate on average income?
  - (b) What is the marginal effect of an increase in the level of subsidy on average income?
  - (c) What is the equilibrium relationship between the tax rate and the level of subsidy that characterizes the government's budget constraint?
  - (d) What is the effect of an extension of the franchise (that is, a transition from a system where only rich people can vote to a system with universal suffrage), on the amount of redistribution?
  - (e) Suppose we observe the redistribution policies of two countries,  $A$  and  $B$ , and country  $A$ 's policy is characterized by a much higher tax rate and subsidy level than the policy of country  $B$ . Provide two possible explanations for this observation.

6. Consider all possible income redistribution policies.
  - (a) Provide an example of an efficient redistribution policy and briefly explain why it is efficient.
  - (b) Provide an example of an inefficient redistribution policy and briefly explain why it is inefficient.
7. The 100 individuals living in a community derive utility from consumption,  $C$ , and leisure,  $\ell$ . Their utility function is

$$U(C, \ell) = C + 2\ell - \ell^2.$$

People differ with respect to how productive they are with their use of time. An individual with productivity  $X$  has  $1 + X$  (effective) units of time she can allocate between work and leisure. Productivity is distributed as follows: 10 people have  $X = 0.1$ , 20 people have  $X = 0.2$ , 30 people have  $X = 0.3$ , 20 people have  $X = 0.8$  and 20 people have  $X = 1$ .

The wage rate is the same for everybody and is equal to  $W = 2$ . Hence, if an individual works an amount of time  $L$ , her earned income is equal to  $WL = 2L$ .

The community imposes a linear income tax: that is, a proportional tax on earned income to finance lump-sum redistribution to all individuals. The tax rate,  $t$ , and the per-capita amount of redistribution,  $S$ , are chosen by majority voting. The budget for the community has to balance.

- (a) Derive the individuals' labor supply.
- (b) Calculate the mean and median of the community's income distribution.
- (c) Derive the indirect utility functions for the individuals in the community and show they satisfy the single-crossing property.
- (d) Characterize the community's budget constraint.
- (e) What are the values of  $t$  and  $S$  that will be chosen by majority voting?

# Chapter 13

## Health care

1. Answer whether each of these statements is true or false. Briefly explain your answer
  - (a) If a private insurance company can perfectly distinguish between people in different groups and pool their risks, then the optimal fee is the same as the optimal savings amount in the absence of the company.
  - (b) In the model described in section 13.1.1, the middle class would prefer the rich class policy.
  - (c) If insurance companies cannot distinguish people of different types, the private market is inefficient.
  - (d) Implementing universal health care coverage is the Constrained Pareto Optimal outcome of a Social Planner.
2. Suppose that the individuals making decision in the first period, discount the value of their consumption in period 2 by a factor  $\beta \in (0, 1)$ . Thus, the preferences of the individual  $i$  can be represented by

$$U^i(c_1^i, c_2^i) = (1 - \exp(-2c_1^i)) + \beta(1 - \exp(-2c_2^i)).$$

Again, let's denote individual  $i$ 's wage by  $w_i$  and her probability of getting sick and being unable to work by  $\rho_i$ .

- (a) Derive an expression for individual  $i$ 's optimal saving,  $s_i$ , in period one.
- (b) Let  $w = 1$ ,  $\beta = 0.9$ ,  $\rho = 0.1$  what is the optimal level of savings?

3. Consider a competitive private health insurance sector. The insurance company is risk neutral and collects premiums to cover expected claims and policy maintenance costs. The fixed maintenance cost is known and denoted by  $c > 0$ . There is a population of 100 individuals, where the top 10 percent earn wage  $w_{rich} = 10$  and have a probability of getting sick  $\rho_{rich} = 1/5$ . The bottom 10 percent of the population's wage is  $w_{poor} = 1$  and face  $\rho_{poor} = 1/2$ . The rest of the population earn  $w_{middle} = 4$  and have  $\rho_{middle} = 2/5$ .
  - (a) Assuming that each group's policy needs to be financially self-sufficient and that the insurer can distinguish among different groups, what is the optimal fee for each group?
  - (b) Assume that the insurer cannot distinguish among the different groups. Find the optimal fees for each policy such that each policy only attracts its intended target group.
4. Now suppose the same setup as in the previous problem. Assume that the insurer is able to distinguish among different groups but is not restricted by self-sufficiency of each group's policy. Can there be any Pareto improvement over the outcomes under self-sufficiency?
5. In 2010, President Obama signed into law a major reform of the US health care system called the Affordable Care Act (ACA). Under ACA, insurers can only consider one of the following 4 personal factors in their premium setting: age, geographical location, tobacco use, and family structure. This means that the insurers cannot use factors such as income, pre-existing health conditions, or chronic illnesses to set the plans' premiums. Discuss how this provision of ACA may affect the efficiency of the health care market? Suppose that, as in the model in this chapter, there is a correlation between income and health risk of individuals.
6. In the public health care system described in section 13.1.2, explain why the tax rate in the political-economic equilibrium is  $\tau_{PE}^* = 0.25$  while the constrained Pareto optimal rate is  $\tau_{CPO}^* = 0$ ?
7. Consider the electoral competition and health care model in section 13.2, but suppose that candidate  $\mathcal{B}$  maximizes the welfare function

given by the weighted sum below.

$$\sum_{i=1}^{100} \lambda_i [\log(w_i(1 - \tau)) + \rho_i \log(b) + (1 - \rho_i) \log(w_i)]$$

$$s.t. \quad b = 10\tau$$

where

$$\lambda_i = \begin{cases} 1/10 & \text{if } \textit{poor} \\ 2/10 & \text{if } \textit{middle} \\ 1/10 & \text{if } \textit{rich} \end{cases}$$

- (a) Find the policy that would be enacted by  $\mathcal{B}$  if elected and compute the utility of the members of each group under this policy.
- (b) For which candidate will the middle class vote? Who wins the election?



## Chapter 14

# Mobility

1. In the context of the local public goods model, show that the political-economic equilibrium is a locally efficient outcome for both communities.
2. Answer whether each of these statements is true or false. Briefly explain your answer
  - (a) In a political-economic equilibrium of the local public goods model, an individual living in community  $j$  may prefer the public good allocation of the other community,  $G_k^{PE}$  (with  $k \neq j$ ), given the same level of consumption  $c_j^{PE}$ .
  - (b) In the local redistribution model, richer individuals will need a higher increase in  $S_j$  compared to the poor in order to keep the same level of indirect utility after a given increase in taxes.
  - (c) In a two-community redistribution equilibrium, each community-specific median voter maximizes her indirect utility with respect to both tax rates  $\tau_1$  and  $\tau_2$ .
3. Consider the local public goods model in section 14.1, but assume now that  $\alpha_i$  is uniformly distributed over  $[0, 2]$  and that individuals' utility is given by

$$U^i(c_i, G_j) = \ln c_i + \frac{\alpha}{3} \ln G_j.$$

- (a) Find the individual's optimal level of public good  $G_i^*$  and the level of provision of local public good in community  $j$ ,  $G_j^*$ .
- (b) Characterize the sorting equilibrium. (Hint: base your analysis on the  $\alpha_i \sim U[0, 1]$  case.)

- (c) Find the political-economic equilibrium.
4. Consider the local redistribution model with two communities of section 14.2, but suppose now that each individual's utility is given by

$$U(c, h) = \ln(c) + \lambda h$$

The rest of the model is unchanged.

- (a) Find the demand for housing as a function of taxes, subsidies, and income.
- (b) Determine whether preferences satisfy the single-crossing property. Can we apply one of the MVTs?
- (c) Characterize the redistribution equilibrium
5. Consider the federalism model with two local communities,  $j = 1, 2$ , with production technologies given by

$$F_j(n_j) = \frac{n_j^{\frac{1}{1+j}}}{64},$$

where  $n_j$  denotes the number of workers in the community  $j$ . The total number of poor people in both communities, adds up to 1000. Given the assumption of competitive labor markets and free mobility between the two communities, what is the rent of the rich landowners and the wage of the poor workers in each community? Also, compare the two communities in terms of their production technology.

Calculate the labor demand in each community under the following restrictions on the redistribution policies.

- (a)  $S_1 = S_2 = 0$
- (b)  $S_1 = S_2$
6. Explain why the local levels of transfers with  $S_1 \neq S_2$  are not efficient in the federalism model.
7. In the context of the federalism model, consider two communities with similar access to the production technologies described in this chapter. Show that in the presence of a central government that can establish a system of matching grants  $(T_1, T_2, z_1, z_2)$ , the equilibrium level of the local transfers  $(S_1, S_2)$  satisfies  $S_1 = S_2$ . Also, discuss how the redistribution policy of the central government affect the efficiency of the equilibrium outcome.

# Chapter 15

## Answer key

### Answer key: Chapter 2

- (a) Efficiency requires the following conditions to be satisfied:

$$\begin{aligned}x_2^A &= 10 + x_1^A \\x_1^B &= 10 + x_2^B \\x_1^A + x_1^B + x_2^A + x_2^B &= 40\end{aligned}$$

Hence, all allocations  $(x_1^A + x_1^B + x_2^A + x_2^B)$  that satisfy these conditions are efficient. Equity requires the following conditions to be satisfied

$$\begin{aligned}x_2^A &= x_1^B \\x_2^A &= 10 + x_1^A \\x_1^B &= 10 + x_2^B \\x_1^A + x_1^B + x_2^A + x_2^B &= 40\end{aligned}$$

Hence, the only allocation that satisfies the equity conditions is

$$x_2^A = 15, x_1^B = 15, x_1^A = 5, x_2^B = 5$$

- (b) The allocations that satisfy the efficiency conditions are the same as above. For the equity conditions, we add another equation which is

$$0.6x_2^A = 0.4x_1^B.$$

We get this condition in the following way. First, as in the general setup, the allocations that satisfy equity conditions should satisfy

$$W_{UA}U_{x_1^A}^A = W_{UB}U_{x_1^B}^B.$$

In this question,

$$\begin{aligned} W_{UA}U_{x_1^A}^A &= 0.4U_{x_1^A}^A = 0.4(40 - 2x_1^A) \\ W_{UB}U_{x_1^B}^B &= 0.6U_{x_1^B}^B = 0.6(60 - 2x_1^B). \end{aligned}$$

Therefore,  $W_{UA}U_{x_1^A}^A = W_{UB}U_{x_1^B}^B$  yields

$$0.6(30 - x_1^B) = 0.4(20 - x_1^A). \quad (15.1)$$

Next, we can rearrange the terms in the efficiency conditions and get  $30 - x_1^B = x_2^A$  and  $20 - x_1^A = x_1^B$  in the following way. Recall that we had the following three conditions for efficiency

$$\begin{aligned} x_2^A &= 10 + x_1^A \\ x_1^B &= 10 + x_2^B \\ x_1^A + x_1^B + x_2^A + x_2^B &= 40 \end{aligned}$$

Plugging the first and the second equation into the third equation, we get  $x_2^A + x_1^B = 30$ . Therefore,

$$30 - x_1^B = x_2^A \quad (15.2)$$

$$20 - x_1^A = 20 - (x_2^A - 10) = 30 - x_2^A = x_1^B \quad (15.3)$$

Finally, by plugging (15.2) and (15.3) into (15.1), we get

$$0.6x_2^A = 0.4x_1^B$$

In summary, equity requires the following four conditions to be satisfied

$$\begin{aligned} 0.6x_2^A &= 0.4x_1^B \\ x_2^A &= 10 + x_1^A \\ x_1^B &= 10 + x_2^B \\ x_1^A + x_1^B + x_2^A + x_2^B &= 40. \end{aligned}$$

Hence, the only allocation that satisfies the equity conditions is

$$x_2^A = 12, x_1^B = 18, x_1^A = 2, x_2^B = 8$$

Alternatively, we can obtain the same solution by solving the Social Planner's problem is

$$\max 0.4[40x_1^A + 60x_2^A - (x_1^A)^2 - (x_2^A)^2] + 0.6[60x_1^B + 40x_2^B - (x_1^B)^2 - (x_2^B)^2]$$

$$\text{subject to } x_1^A + x_1^B + x_2^A + x_2^B = 40,$$

which can be rewritten as

$$\begin{aligned} \max L = & 0.4[40x_1^A + 60x_2^A - (x_1^A)^2 \\ & - (x_2^A)^2] + 0.6[60x_1^B + 40x_2^B - (x_1^B)^2 - (x_2^B)^2] \\ & + \lambda[40 - x_1^A - x_1^B - x_2^A - x_2^B] \end{aligned}$$

which yields the following first-order conditions.

$$\begin{aligned} 0.4(40 - 2x_1^A) - \lambda &= 0 & [x_1^A] \\ 0.4(60 - 2x_2^A) - \lambda &= 0 & [x_2^A] \\ 0.6(60 - 2x_1^B) - \lambda &= 0 & [x_1^B] \\ 0.6(40 - 2x_2^B) - \lambda &= 0 & [x_2^B] \\ x_1^A + x_1^B + x_2^A + x_2^B - 40 &= 0 & [\lambda] \end{aligned}$$

From the first and the second equation, we get

$$x_2^A = x_1^A + 10,$$

which was the first condition we got for efficiency. From the third and the fourth equation, we get

$$x_1^B = x_2^B + 10,$$

which was the second condition for efficiency. And, the last equation is the resource constraint. Now, from the first and the third condition, we get

$$x_1^B = \frac{50 + 2x_1^A}{3}$$

By solving the four equations, we get exactly the same allocation

$$x_2^A = 12, x_1^B = 18, x_1^A = 2, x_2^B = 8$$

2. (a) Efficiency requires the following conditions to be satisfied:

$$\begin{aligned}x_1^A &= x_2^A + 1, \\x_1^B &= x_2^B + 1, \text{ and} \\x_1^A + x_1^B + x_2^A + x_2^B &= 30.\end{aligned}$$

Hence, all allocations  $(x_1^A + x_1^B + x_2^A + x_2^B)$  that satisfy these conditions are efficient. Equity requires the following conditions to be satisfied:

$$\begin{aligned}x_1^A &= x_1^B, \\x_1^A &= x_2^A + 1, \\x_1^B &= x_2^B + 1, \text{ and} \\x_1^A + x_1^B + x_2^A + x_2^B &= 40.\end{aligned}$$

Hence, the only allocation that satisfies the equity conditions is

$$x_2^A = 7, x_1^B = 8, x_1^A = 8, x_2^B = 7.$$

- (b) The allocations that satisfy the efficiency conditions are the same as above. Equity requires the following conditions to be satisfied:

$$\begin{aligned}0.25x_1^A &= 0.75x_1^B, \\x_1^A &= x_2^A + 1, \\x_1^B &= x_2^B + 1, \text{ and} \\x_1^A + x_1^B + x_2^A + x_2^B &= 30.\end{aligned}$$

Hence, the only allocation that satisfies the equity conditions is

$$x_2^A = 11, x_1^B = 4, x_1^A = 12, x_2^B = 3.$$

3. (a) Yes, it violates the condition that all markets are complete.  
 (b) Since agents cannot trade good  $x_2$ , there will not be any exchange of good  $x_1$  either, because neither agent is willing to give away their units of good  $x_1$ . Therefore, the only equilibrium allocation is that each agent consumes their endowment:

$$(x_1^{*A}, x_2^{*A}, x_1^{*B}, x_2^{*B}) = (\bar{x}_1^A, \bar{x}_2^A, \bar{x}_1^B, \bar{x}_2^B) = (1, 2, 2, 1).$$

- (c) The equilibrium allocation is not Pareto optimal. It does not equalize the agent's marginal rates of substitution.
- (d) Efficiency requires

$$MRS_{x_1x_2}^A = \frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B} = MRS_{x_1x_2}^B.$$

Plugging in the conditions

$$\begin{aligned}\bar{x}_1^A + \bar{x}_1^B &= 3 \\ \bar{x}_2^A + \bar{x}_2^B &= 3,\end{aligned}$$

we get

$$\bar{x}_1^A = \bar{x}_2^A, \quad \bar{x}_1^B = \bar{x}_2^B.$$

Hence, if the initial endowment  $(\bar{x}_1^A, \bar{x}_2^A, \bar{x}_1^B, \bar{x}_2^B)$  satisfies  $\bar{x}_1^A = \bar{x}_2^A$  and  $\bar{x}_1^B = \bar{x}_2^B$ , it is efficient.

- (e) Let us denote the relative price of good  $x_2$  by  $\bar{p}$ , i.e.,  $\bar{p} = \frac{p_2}{p_1}$ . By solving the agent  $A$ 's utility-maximization problem

$$\begin{aligned}\max_{x_1^A, x_2^A} \quad & \frac{1}{2} \ln x_1^A + \frac{1}{2} \ln x_2^A \\ \text{s.t.} \quad & x_1^A + \bar{p}x_2^A = \bar{x}_1^A + \bar{p}\bar{x}_2^A = 1 + 2\bar{p},\end{aligned}$$

we get

$$x_1^{*A} = \frac{1 + 2\bar{p}}{2}, \quad x_2^{*A} = \frac{1 + 2\bar{p}}{2\bar{p}}$$

Similarly, we can also get

$$x_1^{*B} = \frac{2 + \bar{p}}{2}, \quad x_2^{*B} = \frac{2 + \bar{p}}{2\bar{p}}$$

Using the market clearing condition for good  $x_1$ , we get

$$\frac{1 + 2\bar{p}}{2} + \frac{2 + \bar{p}}{2} = 3 \Rightarrow \bar{p} = 1$$

Hence, we get the equilibrium allocation for the economy

$$x_1^{*A} = x_2^{*A} = x_1^{*B} = x_2^{*B} = \frac{3}{2}.$$

And, any pair of the prices  $(p_1, p_2)$  that satisfy  $p_1 = p_2$  is an equilibrium price.

4. (a) There are two Nash equilibria. In one equilibrium, player  $A$  plays  $L$  and player  $B$  plays  $Y$ . In the other equilibrium, player  $A$  plays  $R$  and player  $B$  plays  $N$ . Both Nash equilibria are Pareto optimal.
- (b) In the subgame perfect equilibrium of the game player  $A$  plays  $L$  and player  $B$  plays  $Y$ . The equilibrium is Pareto optimal.
- (c) In the subgame perfect equilibrium of the game player  $B$  plays  $N$  and player  $A$  plays  $R$ . The equilibrium is Pareto optimal.
5. (a) The strategy pair  $(a^*, b^*)$  is a Nash equilibrium if player  $A$  solves the maximization problem

$$\max_a [100 - 2(a + b^*)]a$$

and player  $B$  solves the following maximization problem:

$$\max_b [100 - 2(a^* + b)]b.$$

Hence,  $(a^*, b^*)$  must satisfy:

$$\begin{aligned} a^* &= 25 - \frac{1}{2}b^* \\ b^* &= 25 - \frac{1}{2}a^* \end{aligned}$$

Solving this pair of equations we obtain that the Nash Equilibrium of the game is

$$a^* = b^* = \frac{50}{3}.$$

- (b) When player  $B$  moves second, he faces the following problem, given the action  $a$  previously chosen by player  $A$ :

$$\max_b [100 - 2(a + b)]b.$$

Hence, player  $B$ 's reaction (or best response) function to player  $A$ 's action is

$$b^*(a) = 25 - \frac{1}{2}a.$$

When player  $A$  moves, she anticipates player  $B$ 's reaction to each action  $a$  she may take. Hence, player  $A$ 's problem is:

$$\max_a [100 - 2(a + b^*(a))]a$$

or

$$\max_a \left[ 100 - 2 \left( a + \left( 25 - \frac{1}{2}a \right) \right) \right] a.$$

Hence, player A's optimal action is

$$a^* = 25$$

and player B's optimal reaction is

$$b^* = 12.5$$

which characterizes the subgame perfect equilibrium of the game.

6. (a) Player A has a strictly dominated strategy which is  $B$ . To see that, let us compare A's action  $T$  and action  $B$ . When player B chooses action  $L$ , the action  $T$  yields payoff 4 and action  $B$  yields payoff 1 for player A. When player B chooses action  $C$ , action  $T$  yields the payoff 1 and action  $B$  yields 0 for player A. When player B chooses action  $R$ , the action  $T$  yields 5 and the action  $B$  yields 3. That is, regardless of player B's action, the action  $T$  yields a strictly higher payoff than the action  $B$  for player A. Therefore, strategy  $B$  is strictly dominated by strategy  $T$ .
- (b) Player A does not have any strictly dominant strategies. Since player A's strategy  $B$  is a strictly dominated strategy, we only need to compare strategy  $T$  and  $M$ . When player B chooses the strategy  $L$ , strategy  $M$  yields a higher payoff than strategy  $T$  for player A. When player B chooses the strategy  $C$ ,  $T$  yields higher payoff than the strategy  $M$ . Therefore, player does not have any strictly dominant strategies.
- (c) There does not exist any value of  $x$  such that player B has a strictly dominant strategy. First, let us compare player B's strategies  $L$  and  $R$ .  $L$  yields a higher payoff than  $R$  for player B when A chooses  $T$ , and  $R$  yields a higher payoff than  $L$  when A chooses  $B$ . Moreover, the strategy  $C$  yields a lower payoff than  $L$  and  $R$  when A chooses  $T$  or  $M$ . Therefore, player B does not have any strategies that yield strictly higher payoffs than the other actions for all the strategies of player A.
- (d) Yes. If  $x < 3$ , the strategy  $C$  yields strictly lower payoff than the strategy  $L$ , regardless of the strategy chosen by player A. Therefore, when  $x < 3$ , player B has a strictly dominated strategy  $C$ .

- (e) There is no value of  $x$  such that the game has a pure-strategy Nash Equilibrium. To check it, we proceed as follows. First, we fix the strategy of player  $B$  and get the best response of player  $A$ . Once we get the best response of  $A$ , then we check whether the strategy of  $B$  we fixed is a best response to  $A$ 's strategy.

First, suppose that player  $B$  plays  $L$ . Then,  $A$ 's best response is  $M$ . Given  $A$ 's choice of  $M$ , player  $B$ 's strategy  $L$  is not a best response, since he can be better-off by deviating to  $R$ . Hence, there is no equilibrium where player  $B$  chooses action  $L$ . Second, suppose that player  $B$  plays  $C$ . Then,  $A$ 's best response is  $T$ , and given  $A$ 's choice of  $T$ , player  $B$  will be better-off by deviating to  $L$ . Hence, there is no equilibrium where player  $B$ 's action is  $C$ . Lastly, suppose that player  $B$  plays  $R$ . Then,  $A$ 's best response is  $T$ . And, given  $A$ 's choice of  $T$ , player  $B$  will be better-off by deviating to  $L$ . Hence, regardless of the value of  $x$ , there is no (pure-strategy) equilibrium in the game.

Also note that the fact that there is no (pure-strategy) equilibrium regardless of the value of  $x$  can be shown in a more compact way, if we use the method of *Iterated Deletion of Strictly Dominated Strategies* and the fact that  $B$  is a strictly dominated strategy for the player  $A$ , as follows.

Since  $B$  is a strictly dominated strategy for player  $A$  player  $A$  will never use that strategy in the equilibrium. Hence, we erase  $B$ , and the game is reduced to the following form.

A\B	L	C	R
T	4,2	1,-1	5,0
M	6,2	0,0	2,3

Now, in the reduced game,  $C$  is a strictly dominated strategy for player  $B$ . Hence, we delete  $C$  for player  $B$  and the game is reduced as follows.

A\B	L	R
T	4,2	5,0
M	6,2	2,3

Now, when  $B$  plays  $L$ ,  $A$ 's best response is  $M$ , which will give player  $B$  an incentive to deviate to  $R$ . When  $B$  plays  $R$ ,  $A$ 's best response is  $T$ , which will give player  $B$  an incentive to deviate to  $L$ . Hence, we can see that there is no equilibrium regardless of the value of  $x$ .

- (f) The pair  $(B, C)$  is Pareto optimal if and only if  $x > 3$ . If  $x \leq 3$ , we can see from the table that there are other pairs such as  $(B, L)$  and  $(M, R)$  that Pareto dominate  $(B, C)$ . On the other hand, since the maximum payoff for player  $B$  from other pairs is 3,  $(B, C)$  will not be Pareto dominated by any other pairs if  $x > 3$ .
7. (a) There are two pairs of strategies that yield outcomes that are not Pareto optimal. One is (T,L) which yields the payoff (0,3), and the other one is (B,R) which yields the payoff (3,5).
- (b) Neither outcome is a Nash equilibrium.

### Answer key: Chapter 3

1. (a) There exists a Condorcet winner, and it is Joe. To check the existence of a Condorcet winner and a Condorcet loser, we do the following pairwise comparison.
- If we compare Joe with Pat, then Joe gets 18 votes while Pat gets 12 votes. Hence, Joe wins.
  - If we compare Joe with Ray, then Joe gets 17 votes while Ray gets 13 votes. Hence, Joe wins.
  - If we compare Pat with Ray, then Pat gets 22 votes while Ray gets 8 votes. Hence, Pat wins.
- (b) From the above comparison, we see that there also exists a Condorcet loser, and it is Ray.
- (c) Joe receives 65 points, Pat 64 points and Ray 51 points. So, Joe is elected president. He is the Condorcet winner.
- (d) Now, Pat gets 64 points, and both Joe and Ray get 58 points. Pat is elected president.
- (e) The Borda Count Voting rule does not satisfy the *Independence of Irrelevant Alternatives*(IIA). The only change in the preferences was in the rank between Joe and Ray for group  $B$ , which changed the winner of the election from Joe to Pat. Since the rank between Joe and Pat was changed by the rank of Ray, who was an irrelevant alternative, this voting rule does not satisfy IIA.
2. (a) See Figure 15.1.
- (b) Project  $C$  is a Condorcet winner. Note that project  $C$  is a Condorcet winner despite the fact that some individuals have preferences that are not single-peaked.

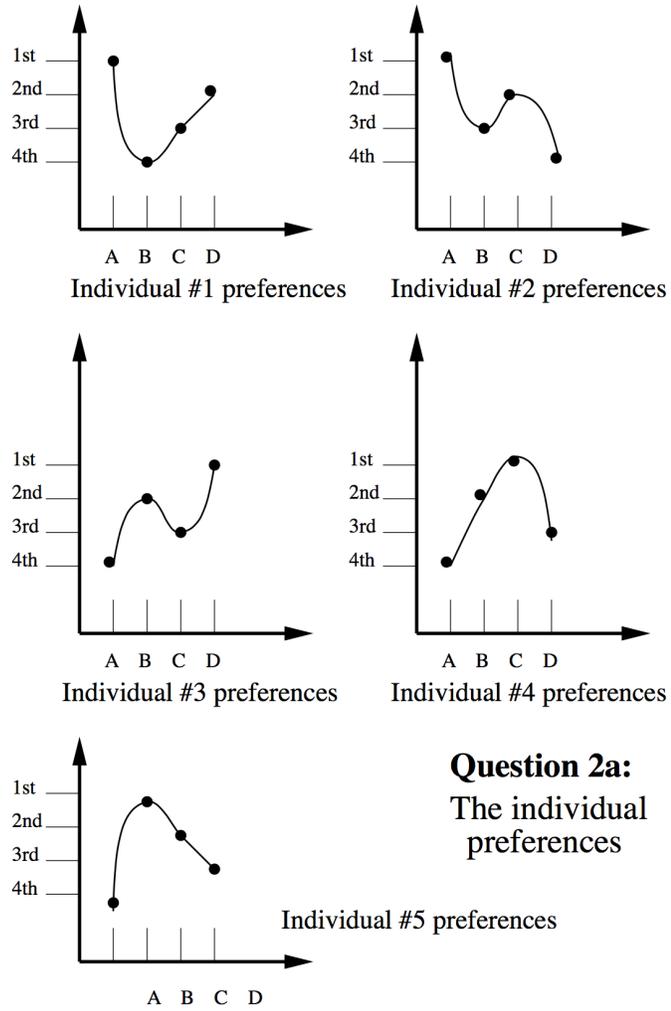


Figure 15.1: Graphs of individual preferences.

3. (a) In the first round, the liberal candidate receives 37 votes, the moderate candidate 30 votes and the conservative candidate 33 votes. Since no candidate wins a majority, the two candidates with the most votes, that is the liberal and the conservative, move to the runoff round where the liberal candidate receives 67 votes and wins the election.  
And, the liberal candidate is not a Condorcet winner. If we do pairwise comparison of the liberal and the moderate, then the liberal gets 37 votes and the moderate gets 63 votes. Thus, the liberal is defeated by the moderate in the pairwise comparison.
- (b) The moderate wins the election.
- (c) The plurality rule with runoff does not satisfy the condition of IIA. The rank between the conservative and the moderate and the rank between the liberal and the moderate were both changed by an irrelevant alternative, the libertarian.
4. (a) 20.  
(b) 10.  
(c) 40.  
(d) 10.
5. (a) C1 gets 35 % of the votes, L1 gets 30%, L2 gets 25%, and C2 gets the remaining 10% of the votes. Therefore, C1 wins the election. Clearly, C1 is not a Condorcet winner. If we do pairwise comparison of C1 and L1, then C1 gets 45% of the votes, and L1 gets 55% of the votes.
- (b) After the withdrawal of L2, L1 wins the election. L1 is a Condorcet winner. To verify it, we do the following pairwise comparison.
- If we compare L1 and L2, L1 gets 75% of the votes and L2 gets the rest. Hence, L1 wins.
  - If we compare L1 and C1, L1 gets 55% of the votes and C1 gets the rest. Hence, L1 wins.
  - If we compare L1 and C2, L1 gets 55% of the votes and C2 gets the rest. Hence, L1 wins.
- (c) In the point of view of L2, the electoral outcome (L1) when she withdraws is preferable to the electoral outcome (C1) when she participates in the race. Therefore, the candidate L2 may want to withdraw from the race.

- (d) The plurality rule also violates the IIA condition, since the rank between C1 and L1 is changed by the existence of L2 in the set of alternatives.
6. (a) First note that any decision that is made by the first town will be accepted in the second stage. Since  $C$  is a Condorcet winner there, it also is a Condorcet winner of the entire election.
    - (b) By the same logic as above, it is  $A$ .
    - (c) The important thing is that the first town has 2 electoral votes which is more than the other town's 1. Hence, it suffices to look at the choice of the first town. And we can easily see that they are choosing  $C$ .
    - (d)  $A$  is the Condorcet winner and  $B$  is the loser.
    - (e)  $A$  wins the election.
  7. (a)  $A$  is the Condorcet winner.
    - (b)  $C$  is the Condorcet loser.
    - (c)  $A$  is the selected alternative.
    - (d)  $A$  is the Condorcet winner and  $C$  is the loser.
    - (e)  $A$  wins the election.

### Answer key: Chapter 4

1. (a) False. The United States have a voting population much larger than Luxembourg, which implies that the probability of being pivotal is much lower in the USA. Hence, turnout should be higher in Luxembourg.
  - (b) False. By definition group rule-utilitarians follow an equilibrium strategy that, if followed by all the group members, maximizes the aggregate utility of the group.
  - (c) True. This is based on the example where  $\mu$  is distributed according to the density function  $2\mu$ . In fact, minorities are actually expected to have a higher probability of voting than non-minorities.
  - (d) False. Newspaper readers are generally more informed, and therefore less prone to make voting mistakes. The uncertain voter model implies that individuals with a lower probability of making mistakes when voting are more likely to turn out.

(e) False. The uncertain voter model can explain why some individual abstain for some elections even though they are already at the voting station, but the reason is not related to the physical cost of voting. It is instead related by the adversity of individuals in making voting mistakes.

2. (a) There is no equilibrium in this game. To see that, define the dummy variable  $V_i$  taking value 1 if voter  $i$  votes and 0 otherwise, the vector  $V = (V_1, V_2, V_3)$ , voter  $i$ 's utility  $U_i$  under the current strategy profile  $V$ , and voter  $i$ 's utility when she unilaterally deviates from strategy profile  $V$ ,  $U'_i$ . Let's now analyze each possible strategy profile:

i.  $V_1 = V_2 = V_3 = 1 \implies U_1 = 0.8, U_2 = 0.8, U_3 = -1.8$ .  
All of them have an incentive to unilaterally deviate since  $U'_1 = 1, U'_2 = 1, U'_3 = -1$ .

ii.  $V_1 = 1, V_2 = 1, V_3 = 0 \implies U_1 = 0.8, U_2 = 0.8, U_3 = -1$ .  
Both voters 1 and 2 have an incentive to unilaterally deviate since  $U'_1 = 1, U'_2 = 1$ .

iii.  $V_1 = 1, V_2 = 0, V_3 = 0 \implies U_1 = 0.8, U_2 = 1, U_3 = -1$ . Voter 3 has an incentive to unilaterally deviate since  $U'_3 = -0.8$  (by deviating the expected benefit is 0 since it will be a tie).

iv.  $V_1 = V_2 = V_3 = 0 \implies U_1 = 0, U_2 = 0, U_3 = 0$ . All of them have an incentive to unilaterally deviate since  $U'_1 = 0.8, U'_2 = 0.8, U'_3 = 0.2$ .

v.  $V_1 = 0, V_2 = 1, V_3 = 0 \implies U_1 = 1, U_2 = 0.8, U_3 = -1$ . Voter 3 has an incentive to unilaterally deviate since  $U'_3 = -0.8$ .

vi.  $V_1 = 0, V_2 = 1, V_3 = 1 \implies U_1 = 0, U_2 = -0.2, U_3 = -0.8$ .  
Voter 1 has an incentive to unilaterally deviate since  $U'_1 = 0.8$ .

vii.  $V_1 = 1, V_2 = 0, V_3 = 1 \implies U_1 = -0.2, U_2 = 0, U_3 = -0.8$ .  
Voter 2 has an incentive to unilaterally deviate since  $U'_2 = 0.8$ .

viii.  $V_1 = 0, V_2 = 0, V_3 = 1 \implies U_1 = -1, U_2 = -1, U_3 = 0.2$ .  
Voters 2 and 3 have an incentive to unilaterally deviate since  $U'_1 = -0.2, U'_2 = -0.2$ .

(b) With the utility levels being 0.5 if the preferred alternative wins and  $-0.5$  if the other one does, we have two equilibria: one where only voter 1 votes and one where only voter 2 does. To see that we can simply consider all the strategy profiles as in part (a)

above and notice that for profiles  $V = (1, 0, 0)$  and  $V = (0, 1, 0)$  voter 3 has no incentive to deviate since the cost of voting is too high compared to the utility loss from alternative  $a$  winning.

- (c) In order to determine whether this strategy profile is a BNE we first need to determine the probability of being a pivotal voter. Given the strategy profile, we know that the probability of another player  $i$  voting is

$$p \equiv \Pr(V_i = 1) = \Pr(C_i = 0.2) = 0.5$$

where  $V_i$  is a dummy variable taking value 1 if  $i$  votes and 0 otherwise.

We can now construct the probability distribution of  $\sigma$ , which denotes the number of voters other than  $i$  that vote:

$$\Pr(\sigma = 0) = (1 - p)(1 - p) = 0.25$$

$$\Pr(\sigma = 1) = 2(1 - p)p = 0.5$$

$$\Pr(\sigma = 2) = p^2 = 0.25$$

The probability of being pivotal conditional on  $\sigma$  and preferred alternative  $j \in \{a, b\}$ , denoted  $\pi(\sigma, j)$ , is instead

- For  $j = a$

$$\pi(0, a) = 1, \quad \pi(1, a) = 0, \quad \pi(2, a) = 0$$

- For  $j = b$

$$\pi(0, b) = 1, \quad \pi(1, b) = 1, \quad \pi(2, b) = 0$$

Since the change in utility given by voting when being pivotal is equal to 1, we get the following expected benefit from voting given the preferred alternative  $j$  and that the other voters play according to the strategy profile

$$EB_i(a) = \pi(0, a) \Pr(\sigma = 0) = 0.25$$

$$EB_i(b) = \pi(0, b) \Pr(\sigma = 0) + \pi(1, b) \Pr(\sigma = 1) = 0.75.$$

As we can see,  $EB_i(a) > 0.2$  and  $EB_i(b) < 0.8$ . That is, voters preferring  $a$  will vote (since the expected benefit of voting is higher than the cost) while those preferring  $b$  will not (since the cost of voting is higher than its expected benefit), which is

exactly the profile strategy we are considering. As a result, this is an equilibrium.

Now we can compute the expected turnout as

$$\begin{aligned} E(\text{turnout}) &= 1 * p^3 + (2/3) * 3 * p^2(1-p) + (1/3) * 3 * p(1-p)^2 \\ &= 0.125 + 2 * 0.125 + 0.125 \\ &= 0.5. \end{aligned}$$

(d) Let's go over the same steps as in (c) but with  $n = 5$ .

As above, given the strategy profile the probability of player  $i$  voting is

$$p \equiv \Pr(V_i = 1) = \Pr(C_i = 0.2) = 0.5.$$

The probability distribution of  $\sigma$  is instead:

$$\begin{aligned} \Pr(\sigma = 0) &= (1-p)^4 = 0.0625 \\ \Pr(\sigma = 1) &= 4(1-p)^3p = 0.25 \\ \Pr(\sigma = 2) &= 6(1-p)^2p^2 = 0.375 \\ \Pr(\sigma = 3) &= 4(1-p)p^3 = 0.25 \\ \Pr(\sigma = 4) &= p^4 = 0.0625 \end{aligned}$$

The probability of being pivotal conditional on  $\sigma$  and preferred alternative  $j \in \{a, b\}$  is exactly as before.

- For  $j = a$

$$\begin{aligned} \pi(0, a) &= 1, & \pi(1, a) &= 0, & \pi(2, a) &= 0 \\ \pi(3, a) &= 0, & \pi(4, a) &= 0 \end{aligned}$$

- For  $j = b$

$$\begin{aligned} \pi(0, b) &= 1 & \pi(1, b) &= 1 & \pi(2, b) &= 0 \\ \pi(3, a) &= 0 & \pi(4, a) &= 0 \end{aligned}$$

The expected benefit from voting is then

$$\begin{aligned} EB_i(a) &= \pi(0, a) \Pr(\sigma = 0) = 0.0625 \\ EB_i(b) &= \pi(0, b) \Pr(\sigma = 0) + \pi(1, b) \Pr(\sigma = 1) = 0.3125 \end{aligned}$$

which implies  $EB_i(a) < 0.2$  and  $EB_i(b) < 0.8$ . As a result, voters supporting  $a$  have the incentive to deviate and this strategy profile is not an equilibrium.

3. (a) Individuals vote only if  $C_i^e < C_j$  and abstain otherwise. Therefore, they will bear the voting cost if and only if  $C_i^e < C_j$ . We have then

$$\begin{aligned}
 E[C_i] &= \int_0^1 \mathbf{1}\{C < C_j\} C dC \\
 &= \int_0^{C_j} C dC \\
 &= \frac{C^2}{2} \Big|_0^{C_j} \\
 &= \frac{C_j^2}{2} - \frac{0}{2} = \frac{C_j^2}{2}
 \end{aligned}$$

where  $\mathbf{1}\{A\}$  is the indicator function taking value 1 if the statement  $A$  is true and 0 otherwise.

- (b) Given that alternative  $a$  is implemented if

$$\mu > \frac{C_b}{C_a + C_b} \equiv \bar{\mu}$$

while alternative  $b$  is implemented otherwise, we can write the aggregate expected utility of group  $a$  as follows

$$\begin{aligned}
 \pi_a(C_a, C_b) &= \int_0^1 \mu [\mathbf{1}\{\mu \leq \bar{\mu}\} (-1 - E[C_i]) + \mathbf{1}\{\mu > \bar{\mu}\} (1 - E[C_i])] d\mu \\
 &= \int_0^{\bar{\mu}} \mu \left[ -1 - \frac{C_a^2}{2} \right] d\mu + \int_{\bar{\mu}}^1 \mu \left[ 1 - \frac{C_a^2}{2} \right] d\mu \\
 &= \int_{\bar{\mu}}^1 \mu d\mu - \int_0^{\bar{\mu}} \mu d\mu - \frac{C_a^2}{2} \int_0^1 \mu d\mu \\
 &= \frac{\mu^2}{2} \Big|_{\bar{\mu}}^1 - \frac{\mu^2}{2} \Big|_0^{\bar{\mu}} - \frac{C_a^2}{2} \frac{\mu}{2} \Big|_0^1 \\
 &= \frac{1}{2} - \frac{\bar{\mu}^2}{2} - \frac{\bar{\mu}^2}{2} + \frac{0}{2} - \frac{C_a^2}{2} \left( \frac{1}{2} - \frac{0}{2} \right) \\
 &= \frac{1}{2} - \bar{\mu}^2 - \frac{C_a^2}{4} \\
 &= \frac{1}{2} - \left( \frac{C_b}{C_a + C_b} \right)^2 - \frac{C_a^2}{4}
 \end{aligned}$$

Likewise, the aggregate expected utility of group  $b$  is given by

$$\begin{aligned}
\pi_b(C_a, C_b) &= \int_0^1 (1 - \mu) [\mathbf{1}\{\mu > \bar{\mu}\} (-1 - E[C_i]) + \mathbf{1}\{\mu \leq \bar{\mu}\} (1 - E[C_i])] d\mu \\
&= \int_0^{\bar{\mu}} (1 - \mu) \left[1 - \frac{C_b^2}{2}\right] d\mu + \int_{\bar{\mu}}^1 (1 - \mu) \left[-1 - \frac{C_b^2}{2}\right] d\mu \\
&= \int_0^{\bar{\mu}} (1 - \mu) d\mu - \int_{\bar{\mu}}^1 (1 - \mu) d\mu - \frac{C_b^2}{2} \int_0^1 \mu d\mu \\
&= \mu \Big|_0^{\bar{\mu}} - \frac{\mu^2}{2} \Big|_0^{\bar{\mu}} - \mu \Big|_{\bar{\mu}}^1 + \frac{\mu^2}{2} \Big|_{\bar{\mu}}^1 - \frac{C_b^2}{2} \frac{\mu}{2} \Big|_0^1 \\
&= \bar{\mu} - 0 - \frac{\bar{\mu}^2}{2} + 0 - 1 + \bar{\mu} + \frac{1}{2} - \frac{\bar{\mu}^2}{2} - \frac{C_b^2}{2} \left(\frac{1}{2} - \frac{0}{2}\right) \\
&= 2\bar{\mu} - \frac{1}{2} - \bar{\mu}^2 - \frac{C_b^2}{4} \\
&= 2 \left(\frac{C_b}{C_a + C_b}\right) - \frac{1}{2} - \left(\frac{C_b}{C_a + C_b}\right)^2 - \frac{C_b^2}{4}
\end{aligned}$$

- (c) To find the equilibrium strategy profile we first need to find the best response functions  $C_a^*(C_b)$  and  $C_b^*(C_a)$ , then put them in a system of equations and finally solve for  $C_a^*$  and  $C_b^*$ .

The best response function for group  $j$  is given by the F.O.C. of the aggregate expected utility with respect to  $C_j$ , taking as given the cutoff cost value of the other group.

$$\begin{aligned}
\frac{\partial \pi_a(C_a, C_b)}{\partial C_a} &= 2 \frac{C_b^2}{(C_a + C_b)^3} - \frac{C_a}{2} = 0 \\
\frac{\partial \pi_b(C_a, C_b)}{\partial C_b} &= 2 \frac{C_a}{(C_a + C_b)^2} - 2 \frac{C_a C_b}{(C_a + C_b)^3} - \frac{C_b}{2} = 0
\end{aligned}$$

Finally, solving this system of equations we get

$$C_a^* = C_b^* = \frac{\sqrt{2}}{2}$$

4. (a) The probability is zero. This is because set of voters is a continuum which implies that a single voter has zero measure. As a result, her vote alone cannot influence the outcome and therefore can never be pivotal.

- (b) i. With the assumption of group rule-utilitarians voters, we have that each group  $j \in \{a, b\}$  will choose a cost cutoff  $C_j$  such that only individuals with  $C < C_j$  vote. Given the assumption of uniformly distributed voting cost over  $[0, 1]$ ,  $C_j$  also represents the fraction of group  $j$  members that turn out to vote. As a result, the condition for alternative  $a$  to be implemented can be expressed as

$$\mu C_a > (1 - \mu)C_b \implies \mu > \frac{C_b}{C_a + C_b}$$

- ii. Denoting by  $\mathbf{1}\{\}$  the indicator function (taking value 1 if the statement in brackets is true and 0 otherwise) and by  $D \equiv C_b/(C_a + C_b)$ , the aggregate payoff function for group  $a$  is computed as follows

$$\begin{aligned} \pi_a(C_a, C_b) &= \\ \int_0^1 \left[ \mathbf{1}\{\mu > D\} \mu \left(1 - \frac{C_a^2}{2}\right) - \mathbf{1}\{\mu \leq D\} \mu \left(1 + \frac{C_a^2}{2}\right) \right] 3\mu^2 d\mu \end{aligned}$$

which becomes

$$\begin{aligned} \pi_a(C_a, C_b) &= \int_D^1 3\mu^3 d\mu - \int_0^D 3\mu^3 d\mu - \frac{C_a^2}{2} \int_0^1 3\mu^3 d\mu \\ &= \frac{3}{4} - \frac{3D^4}{4} - \frac{3D^4}{4} - \frac{3C_a^2}{8} \\ &= \frac{3}{4} - \frac{3}{2} \left( \frac{C_b}{C_a + C_b} \right)^4 - \frac{3C_a^2}{8} \end{aligned}$$

For group  $b$  we have instead

$$\begin{aligned} \pi_b(C_a, C_b) &= \\ \int_0^1 \left[ \mathbf{1}\{\mu \leq D\} (1 - \mu) \left(1 - \frac{C_b^2}{2}\right) - \mathbf{1}\{\mu > D\} (1 - \mu) \left(1 + \frac{C_b^2}{2}\right) \right] 3\mu^2 d\mu \end{aligned}$$

which becomes

$$\begin{aligned}\pi_b(C_a, C_b) &= \int_0^D 3(\mu^2 - \mu^3)d\mu - \int_D^1 3(\mu^2 - \mu^3)d\mu \\ &\quad - \frac{C_b^2}{2} \int_0^1 3(\mu^2 - \mu^3)d\mu \\ &= D^3 - \frac{3D^4}{4} - 1 + D^3 + \frac{3}{4} - \frac{3D^4}{4} - \frac{C_b^2}{2} \left(1 - \frac{3}{4}\right) \\ &= 2 \left(\frac{C_b}{C_a + C_b}\right)^3 - \frac{1}{4} - \frac{3}{2} \left(\frac{C_b}{C_a + C_b}\right)^4 - \frac{3C_b^2}{8}\end{aligned}$$

- iii. The equation that characterize the best response function for group  $a$  given  $C_b$  is given by F.O.C.

$$\frac{\partial \pi_a(C_a, C_b)}{\partial C_a} = 6 \left(\frac{C_b}{C_a + C_b}\right)^3 - \frac{3}{4}C_a = 0$$

which can be rearranged as

$$\left(\sqrt[3]{C_a}\right)^4 + C_b \sqrt[3]{C_a} - 2C_b = 0$$

As for group  $b$  we have instead

$$\frac{\partial \pi_b(C_a, C_b)}{\partial C_b} = -6 \left(\frac{C_b}{C_a + C_b}\right)^2 + 6 \left(\frac{C_b}{C_a + C_b}\right)^3 - \frac{3}{4}C_b = 0$$

which can be rearranged as

$$\left(\frac{C_b}{C_a + C_b}\right)^3 - \left(\frac{C_b}{C_a + C_b}\right)^2 - \frac{C_b}{8} = 0$$

- iv. The Nash equilibrium will be given by those voting cost cut-offs  $C_a^*$  and  $C_b^*$  that solve the system of equations composed by the two best response functions above.

5. Given the joint probability distribution of the two alternatives we get

$$C_i(b) - C_i(a) = -x_i \frac{9}{2}.$$

The optimal voting decision in the second stage of the game is to vote for  $a$  if  $x_i < 0$  and vote for  $b$  if  $x_i > 0$ . Now given the optimal voting rule if one were to vote,

- If  $(x_a, x_b) = (-2, 2)$  or  $(x_a, x_b) = (-1, 1)$  no individual would make a mistake.
- If  $(x_a, x_b) = (-2, 1)$ , those with  $x_i \in (-\frac{1}{2}, 0)$  would vote for  $a$  but should vote for  $b$ . The cost is  $3 + 6x_i$ .
- If  $(x_a, x_b) = (-1, 2)$ , those with  $x_i \in (0, \frac{1}{2})$  would vote for  $b$  but should vote for  $a$ . The cost is  $3 - 6x_i$ .
- If  $(x_a, x_b) = (-2, 1)$  no one would make a mistake.

Thus, the expected voting cost for an individual with preferred policy  $x_i$  is

$$C_i^e(x_i) = \begin{cases} 0 & \text{if } x_i \in [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] \\ \frac{3}{4} + \frac{3}{2}x_i & \text{if } x_i \in (-\frac{1}{2}, 0) \\ \frac{3}{4} - \frac{3}{2}x_i & \text{if } x_i \in (0, \frac{1}{2}) \end{cases}$$

For any realization of  $D_i$ , every individual with  $x_i \in [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$  votes. For any realization of  $D_i$  every individual with  $x_i \in (-\frac{1}{4}, \frac{1}{4})$  would never vote since the cost of voting is greater than the two possible realization of the direct benefits.

When  $D_i = \frac{3}{8}$ , which occurs with probability  $\frac{1}{2}$ , individuals with  $x_i \in (-1/2, -1/4] \cup [1/4, 1/2)$  will participate in the election. When  $D_i = 0$  these individuals abstain. Thus, the turnout in the election is  $3.25/4 = 0.81\%$ .

6. (a) The two distributions have different means. However, the variance of both triangular distributions in the second election is larger than the first one. The variance of the distributions in the second election is  $13/72$  and for the first one is  $7/72$ . Since the ideal ideologies are distributed uniformly, more uncertainty is implied by the second election in terms of variance.
- (b) We expect that the cost of voting for each alternative is higher for every individual in the second election. Thus, the optimal value of the second stage is lower in the second election. Since the voters have the same distribution of preferred alternatives in both elections, we expect a higher turnout in the first election
7. (a) Sincere voting behavior implies that each voter in each election votes for her most preferred alternative. The most preferred alternative for 1 and 3 is  $a_1$  and thus it will be elected. In the second election,  $b_2$  is the most preferred alternative for 2 and

- 3, so it will be elected. Thus, the outcome of sincere voting is  $(a_1, b_2)$  which happens to be the most preferred outcome for 3.
- (b) The outcome of the election cannot be any better for the third individual than the one under sincere voting. So, there is no unilateral profitable deviation for 3. Let's check whether the second individual can change her voting decision in the election unilaterally to improve the joint outcome for herself. By changing her voting profile from  $(a_2, b_2)$  to  $(a_2, b_1)$  the outcome of the election changes to  $(a_2, b_1)$  and she will be strictly better off. Thus, the sincere voting strategy profile is not an equilibrium.
- (c) The following voting profile is one of the possible equilibria: 1 votes for  $(a_1, b_1)$ , 2 votes for  $(a_2, b_1)$ , and 3 votes for  $(a_2, b_1)$ .

## Answer key: Chapter 5

1. (a) The Nash equilibrium is both candidates positioning themselves at the median, 0.
  - (b) Suppose both candidates position themselves at  $\frac{1}{3}$ . Then, each candidate gets  $\frac{1}{3}$  of the total vote. If a Downsian candidate deviates to a position to the right of  $\frac{1}{3}$ , she gets a vote share strictly less than  $\frac{1}{3}$  and what she loses is absorbed by the other Downsian candidate. Hence, she loses the election. This is also true if she moves to the other direction. Hence, this is an equilibrium.
  - (c) Now positioning at 0 is an equilibrium with  $\frac{1}{4}$  of a chance to win the election.
2. (a) Note that, when placed at the same policy, both candidates win the election with half probability. Let's consider any deviation. If a candidate deviates to  $(-1, -1)$ , he receives all 25 votes from that point but shares any other votes (except obviously with those in  $(1, 1)$ ) and hence cannot improve his winning probability. Similar calculations also hold for other deviations and hence this is an equilibrium.
  - (b) It is. To see this, consider the deviation of the candidate who is placed at  $(1, 1)$  to  $(1, -1)$ . Now he earns half the votes from  $(-1, -1)$  but loses half the votes in  $(1, 1)$ . Hence, he gets no benefit. All the rest of the possible deviations give the same result.

- (c) The above results are enough, because we can use symmetry to argue that there are no positions that do not yield an equilibrium.
3. (a) No, a deviation to  $(0, 0)$  wins the election.  
 (b) No, a deviation to  $(0, 0)$  wins the election.  
 (c) It is, any deviation loses votes. Deviation to any policy by any candidate would give the candidate 20 votes instead of 50.  
 (d) Yes. In a situation where one of the candidates is not positioned at  $(0, 0)$ , that candidate will deviate to  $(0, 0)$ .  
 (e) The preferences are not defined on a unidimensional space and hence the median voter theorem as we defined it in the class does not hold.
4. (a) Since everyone in the town has the same ideology, normalize the utility from ideology to 0. Hence, if a candidate does not run, he receives 0. When a candidate runs alone, he gets the utility  $B - C$ . When there are two candidates, both get  $B/2 - C$ . Hence, only one candidate running for the office is an equilibrium if and only if  $B - C > 0$  and  $B/2 - C < 0$ . This implies  $B \in [C, 2C]$ .  
 (b) When there are two candidates running for the office, they each get  $B/2 - C$ . If one deviates not to run, he gets 0. Hence, we need to have  $B > 2C$ . For a third candidate not to run, it must be true that  $B/3 - C < 0$ , which implies that  $B \in [2C, 3C]$ .  
 (c) By the same logic above, the answer will be  $B \in [KC, (K + 1)C]$ .
5. (a) Assume there is one candidate running. There are three deviations we need to check. First is obviously that of the candidate, but as long as  $B - C > -\infty$ , he will run. Now consider a citizen from the same ideological group as the candidate. If he runs he will get  $B/2 - C$ . Hence, we need the condition  $B < 2C$ .  
 Now let's check the incentives of the citizens who belong to the other ideological group. If a member of that group does not run, he gets -1. On the other hand, if he runs, he will get  $B/2 - C - 1/2$  as he will win the election with half probability. Then, there is no profitable deviation if  $B < 2C - 1$ .  
 Since  $2C - 1 < 2C$ , the only condition we need is  $B < 2C - 1$ . Then essentially, the benefit of the office does not fully justify the cost and only a single candidate runs for the office.

Note that this equilibrium cannot be unique since a candidate from either group could run and these are different outcomes with a single candidate running.

- (b) There can be two kinds of equilibrium where two candidates are running for the office. It is either the case that two candidates with the same position are running or different positions. Let's take the first case.

Let's first show that two candidates running from the same position cannot happen in an equilibrium. In this case, both candidates get the utility  $B/2 - C$ . A deviation to not running gives the utility 0. Hence, no deviation would require  $B/2 - C > 0$  which implies  $B > 2C$ . On the other hand, if a citizen from the other ideological group runs, he gets  $B - C$ , since he will win the election for sure. But that requires  $B < C - 1$ . Hence, we need:  $2C < C - 1$  which implies  $C < -1$ . But that does not make sense as  $C > 0$ .

Now consider the case where one candidate per group is running for the office. Each candidate has  $1/2$  chance of winning the election. Hence, they get  $B/2 - C - 1/2$ . A deviation to not running would give  $-1$ . Hence, we need  $B > 2C - 1$ . Notice now that the non-candidate citizens would never run for the office, since if they do, their ideology would lose the election for sure and would still pay the cost of running.

Obviously, the equilibrium we found is unique.

- (c) First note that there must be an equal number of candidates running for the office from both camps, otherwise the side with more candidates loses for sure. Hence, let's consider a case where four candidates are running for the office.

In that case, a candidate's utility is  $B/4 - C - 1/2$ . If he deviates to not running he'd get 0, since they'd win the election. This would imply that we need  $B/4 - C > 1/2$  or  $B > 4C + 2$ .

Now consider a generic citizen. If he runs for the election, they lose the race for sure, and hence he would never get the benefit and still would pay the cost. Hence, there is no reason to check for such a deviation.

So yes, there is an equilibrium where two candidates from the same camp run for the office. It requires the benefits from the office to be quite large.

- (d) If everyone runs for the office, they have an equal chance of winning and the enacted policy is of their group with half probability. Hence, their utility is  $B/400 - C - 1/2$ . However, if an individual does not run, he will vote for someone within his group and since everyone else will receive a single vote, his ideology will be enacted. This gives him a utility of 0. Hence, the no deviation requirement is  $B > 200 + 400C$ .

Therefore, it is possible that every citizen will run for the office.

6. (a) When there is no citizen running for the office, each citizen gets  $-100$ . If a citizen deviates to run, he will get  $B - C$ . Hence, for no candidate to be an equilibrium, we need  $B < C - 100$ . The answer does not depend on  $\alpha$ , since when considering the deviation of a citizen to run, he would definitely win the race and hence get 0 utility from ideology.
- (b) There cannot be such an equilibrium for any  $\alpha$ . To see this, consider a deviation by a citizen with the ideal position 0. If there is only one candidate with ideal position  $-1$  running for the office, she'd get  $-\alpha \leq 0$ . If she instead runs for the office, she will win the election for sure, resulting in a payoff of 1. Hence, she has a profitable deviation. There cannot be any such equilibrium.
- (c) There is. In the suggested equilibrium, the candidate gets  $B - C$  and he will not deviate as long as  $B > C - 100$ . The citizens at  $-1$  get 0. If one of them decides to run, she'd get  $\frac{B}{2} - C$ . Hence, to avoid deviation, we need  $\frac{B}{2} \leq C$ . A citizen at 0 gets utility  $-\alpha$ . If he deviates instead, he will win the election for sure and get  $B - C$ . Hence, we need  $-\alpha \geq B - C$ . Lastly, a citizen at  $-1$  will get  $-2\alpha$ . Instead if she runs, she will get  $\frac{B}{2} - C$ . Hence, we have only one condition that provides the requested equilibrium  $\alpha = 0$  and  $B = C$ . Notice that this comes from the condition on the citizen with the ideal point at 0.
- (d) The unique one candidate equilibrium is where the candidate has the ideal position of 0. The citizens on the either end ( $-1$  or  $1$ ) will never deviate to run, since they will lose if they do. The non-candidate citizens at 0, on the other hand, will get  $-1/2$  if any one of them run since they will win with probability 0.5. Hence, they won't have an incentive to run.

This equilibrium is unique (within one candidate equilibria) given the parameters, since we know that any other equilibria is not possible by the previous part of the question. The result does not depend on  $\alpha$ .

- (e) First consider the candidates. Each candidate will win the election with half probability. Each candidate's payoff at the equilibrium is  $-\alpha + B/2 - C$ . Note that the first component ( $\alpha$ ) comes from the fact that with probability 0.5, she gets to set public policy equal to her favorite and with probability 0.5, the other gets to do so and she gets  $-2\alpha$ . If a candidate deviates to not running, the other candidate wins for sure, and hence gets  $-2\alpha$ . Hence, we get the condition  $B \geq 2(C - \alpha)$ .

Now consider citizens. A citizen at 1 or  $-1$  will not deviate to run. By doing so, she will not only lose for sure, but also will ensure that the other candidate running with the same ideology will lose. On the other hand, a citizen from 0 will lose with probability 1 as well, implying that she will not run.

Hence, the condition necessary is  $B \geq 2(C - \alpha)$ . As  $\alpha$  increases, the cost of not running increases, in the sense that ideology becomes more important and citizens need lower benefit from the office to run.

- (f) Remember that the condition we previously found was  $B \geq 2(C - \alpha)$ . If  $B = 0$  and  $C = 1$ , we need  $\alpha > 1$  to have the previous equilibrium. That is even if the candidates do get benefits from the office, their ideological preference may induce them to run for office.
- (g) No we can't. Note that when a citizen from  $-1$  or  $1$  decide to run, she'd win since she'd carry 450 votes compared to 550/2 of the other two. If she does not run, she gets  $-\alpha$  and if she runs, she gets  $B - C$ . Hence, we need  $-\alpha \geq 2$ , which is impossible.
- (h) No. The candidate at 0 will not run knowing that she won't win the election.
7. (a) Given  $B = 3$  and  $C = 2$ , no citizen with ideal point 1 will challenge the actual candidate. Also given  $B = 3$  and  $C = 2$ , a citizen at 0 or  $-1$  will deviate to run for the office if winning is certain. If a candidate from 0 runs, she will get  $N_0 + N_{-1}$  votes. If a candidate from  $-1$  runs, she will get  $N_0/2 + N_{-1}$  votes. Hence, we need  $N_1 > N_0 + N_{-1}$ .

- (b) First of all, note that candidates should be winning the election with positive probability since no candidate will run if she will lose for sure. This implies  $N_1 = N = -1$ . Moreover, we need to have no profitable deviation for a citizen at 0. This happens when she knows that she will lose the election if she were to run and hence requires  $N_0 < N_1 = N_{-1}$ .
- (c) By the same logic as above, we need  $N_0 = N_1 = N_{-1}$ .
8. (a) No, there are no such conditions. Since  $B > C$  and the citizen at 0 will win for sure if she decides to run, she will deviate regardless of  $\alpha_0$ .
- (b) This will hold for any  $\alpha_i \geq 0$ . Reason is simple, citizens at  $-1$  or  $1$  cannot unilaterally change the outcome of the election by running and citizens at 0 do not find it optimal to run because of  $B$  and  $C$  and does not change the policy if they run.
- (c) Regardless of  $\alpha_i$ , a citizen at 0 will not deviate to run for the office since she will lose for sure. This is also true for a citizen at  $-1$  or  $1$ , since deviating to run ensures that the candidate from the other end of the spectrum will win.
- Now consider a candidate at  $i$ ,  $i \in \{-1, 1\}$ . Under the proposed equilibrium, he gets  $-\alpha_i$ . If she deviates, she'd get  $-2\alpha_i$ . This implies the condition  $-2\alpha_i > -\alpha_i$ , which is satisfied when  $\alpha_i > 0$ .

### Answer key: Chapter 6

1. (a) False. In equilibrium the amount invested in mobilization by both parties,  $I^*$ , is decreasing in the distance between the direct benefits of voting of the two type of supporters. Therefore, parties will invest more in  $e_1$  since both supporters receive exactly the same benefit from voting.
- (b) False. In equilibrium both parties invest exactly the same amount  $I^*$ .
- (c) False. The policy platform inside each party is chosen under unanimity rule. Hence, a politician representing a minority group has the veto power over a policy proposed by another member of the party.
- (d) True. A party where its politicians would rather leave and form a smaller one cannot exist in equilibrium.

- (e) True. The only two equilibria of the model are mediocracy (both parties selecting low ability candidates) and aristocracy (both parties selecting high ability candidates).
- (f) False. A higher  $\gamma$  implies that the party gets a relatively higher payoff from the effort exerted during the recruitment phase. Hence, the party will benefit from decreasing the discouragement effect at this stage and will push for a mediocre selection, which will lead to mediocracy.
- (g) True. The majoritarian is a win-it-all system where there is no positive outcome from losing. As a result, parties are incentivized to select the best candidates in order to increase the likelihood to win the election.
2. By modifying the objective function of the two parties to reflect the difference in their benefit from the implementation of their positive policy and setting the two F.O.C.s equal to each other we get:

$$\begin{aligned} & \left( \frac{\exp(I_a^* + D_a^e + I_b^* + D_b^e)}{(\exp(I_a^* + D_a^e) + \exp(I_b^* + D_b^e))^2} \right) B_a - I_a^* \\ & = \left( \frac{\exp(I_a^* + D_a^e + I_b^* + D_b^e)}{(\exp(I_a^* + D_a^e) + \exp(I_b^* + D_b^e))^2} \right) B_b - I_b^*. \end{aligned}$$

Thus,

$$I_a^* - I_b^* = (B_a - B_b) \frac{\exp(I_a^* + D_a^e + I_b^* + D_b^e)}{(\exp(I_a^* + D_a^e) + \exp(I_b^* + D_b^e))^2},$$

and since  $B_a > B_b$  we get

$$\frac{\exp(I_a^* + D_a^e + I_b^* + D_b^e)}{(\exp(I_a^* + D_a^e) + \exp(I_b^* + D_b^e))^2} > 0$$

As a result, the optimal investment of the party  $a$  is higher than the one of party  $b$ .

3. It can be shown that for a given investment of the party  $b$  the objective function of party  $a$  is single-peaked. To verify whether such equilibrium exists, we plug in  $I_a = 0.15$  in the first order condition of the party  $b$ . By numerically solving the first order condition of the party  $b$ , the best response of  $b$  to  $I_a = \bar{I} = 0.15$  is  $I_b = 0.223$ . Now, it remains to show that  $I_a = 0.15$  is the best response of  $a$  to

$I_b = 0.223$ . We plug in for  $I_b = 0.223$  in the first order condition of  $a$  to find the unconstrained best response of  $a$ . By numerically solving the equation, we get that the unconstrained best response of  $a$  is  $I_a = 0.218$ . One can show that the objective function of party  $a$  has a single peak in  $a$ 's investment for any given investment of  $b$ . Thus, the best response of  $a$  will be to invest the maximum investment level. Thus,  $(I_a^*, I_b) = (0.15, 0.223)$  is a Nash equilibrium of the constrained problem.

4. (a) The policy platform  $x = (0, 8, 0, 2)$  is not in the Pareto set  $P^{\{a,c\}}$ , and therefore cannot be implemented in equilibrium given this party configuration. To see that, suppose that  $\{a, c\}$  actually chooses this policy. Politicians in this configuration will get indirect utility equal to

$$\begin{aligned} V_a(x) &= -(0.8 + 1)^2 - (0.2 + 1)^2 = -4.68 \\ V_c(x) &= -(0.8)^2 - (0.2 - 1)^2 = -1.28 \end{aligned}$$

Now consider, for example, policy platform  $x' = (0, 0)$ . We have that

$$\begin{aligned} V_a(x') &= -(1)^2 - (1)^2 = -2 \\ V_c(x') &= -(0)^2 - (-1)^2 = -1 \end{aligned}$$

and therefore both politicians are strictly better off choosing  $x'$  instead of  $x$ . As a result,  $x$  cannot be in the Pareto set since there is at least one policy platform that Pareto dominates it.

The shaded area in Figure 15.2 includes all the policy platforms that Pareto dominate  $x$ . In fact, all these points lie on indifference curves closer to the most preferred points of both  $a$  and  $c$ .

Graphically, the Pareto set is given by the line connecting  $x^a$  to  $x^c$ . The equation characterizing this Pareto set is then

$$x_2 = 2x_1, \quad x_1 \in [-1, 0].$$

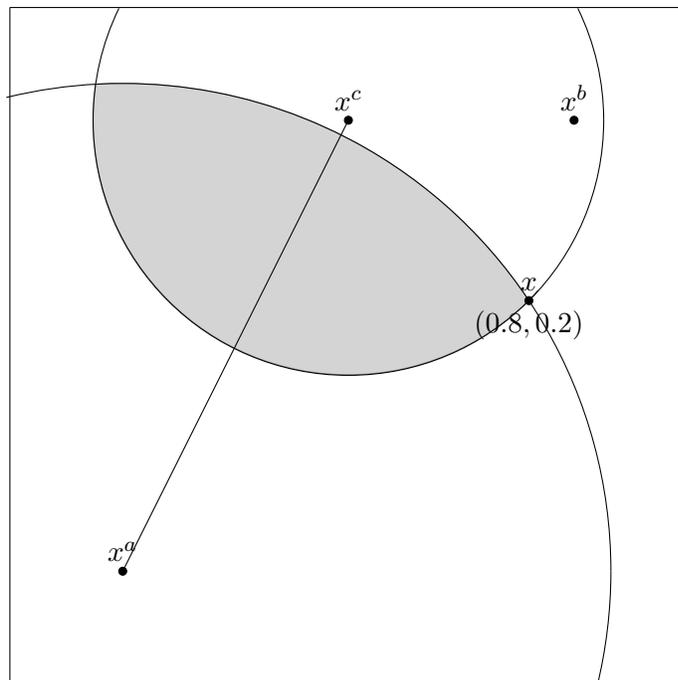


Figure 15.2: Policy platforms that Pareto dominate  $x$ .

(b) With party configuration  $(\{a\}, \{b\}, \{c\})$  we have three possible outcomes:

i. If  $a$  wins, she will offer policy platform  $x^a = (-1, -1)$ . The indirect utilities are then

$$\begin{aligned} V_a(x^a) &= 0 \\ V_b(x^a) &= -(1+1)^2 - (1+1)^2 = -8 \\ V_c(x^a) &= -(1)^2 - (1+1)^2 = -5. \end{aligned}$$

ii. If  $b$  wins, she will offer policy platform  $x^b = (1, 1)$ . The indirect utilities are then

$$\begin{aligned} V_a(x^b) &= -(-1-1)^2 - (-1-1)^2 = -8 \\ V_b(x^b) &= 0 \\ V_c(x^b) &= -(-1)^2 - (1-1)^2 = -1. \end{aligned}$$

iii. If  $c$  wins, she will offer policy platform  $x^c = (0, 1)$ . The indirect utilities are then

$$\begin{aligned} V_a(x^c) &= -(-1)^2 - (-1-1)^2 = -5 \\ V_b(x^c) &= -(1)^2 - (1-1)^2 = -1 \\ V_c(x^c) &= 0. \end{aligned}$$

Now given these values we can determine the equilibrium platforms. Specifically, an equilibrium platform is a situation in which no other party in the configuration has the incentive to run for the elections since it would lose. We can restrict our analysis to the cases where parties run unopposed. We have then

i.  $\{a\}$  runs unopposed:

A. If also  $b$  runs,  $c$  would vote for  $b$  since

$$V_c(x^b) = -1 > -5 = V_c(x^a).$$

Hence,  $b$  would win.

B. If also  $c$  runs,  $b$  would vote for  $c$  since

$$V_b(x^c) = -1 > -8 = V_b(x^a).$$

Hence,  $c$  would win.

ii.  $\{b\}$  runs unopposed:

A. If also  $a$  runs,  $c$  would still vote for  $b$  since

$$V_c(x^b) = -1 > -5 = V_c(x^a).$$

Hence,  $b$  would still win.

B. If also  $c$  runs,  $a$  would vote for  $c$  since

$$V_a(x^c) = -5 > -8 = V_a(x^b).$$

Hence,  $c$  would win.

iii.  $\{c\}$  runs unopposed:

A. If also  $a$  runs,  $b$  would still vote for  $c$  since

$$V_b(x^c) = -1 > -8 = V_b(x^a).$$

Hence,  $c$  still would win.

B. If also  $b$  runs,  $a$  would still vote for  $c$  since

$$V_a(x^c) = -5 > -8 = V_a(x^b).$$

Hence,  $c$  would still win.

As a result, the only equilibrium platform in this party configuration is  $\{c\}$  running unopposed.

- (c) By having 55% of the population supporting his policy platform, and given that voters vote sincerely, the politician of group  $a$  will always win the elections by running unopposed. Therefore, unless there are some party configurations offering  $x^a$  as equilibrium platform, the only stable equilibrium party configurations will be those where  $\{a\}$  runs unopposed.

The question now is which parties other than  $\{a\}$  can offer an equilibrium platform equal to  $x^a$ . Notice that policy platforms in a party are chosen under unanimity rule, which implies that also minority groups have veto power over the decision of running for the elections. This implies that changes in group shares of the population have no effect on equilibrium platforms. In this specific case, if either politician  $b$  or  $c$  are in the same party with  $a$  they always have the power to vote against participating in the elections and let the other party (if there is one) run unopposed. First, we can exclude parties  $\{b\}$ ,  $\{c\}$  and  $\{b, c\}$  since  $x^a$  is not even included in their Pareto sets. So let's now consider the remaining three parties:

- i. Party  $\{a, b\}$  does not have  $x^a$  as equilibrium platform since  $b$  strictly prefers  $x^c = (0, 1)$  to  $x^a$  and will use its right to veto in order to let  $\{c\}$  run unopposed.
- ii. By the same token, party  $\{a, c\}$  does not have  $x^a$  as equilibrium platform since  $c$  strictly prefers  $x^b = (1, 1)$  to  $x^a$ .
- iii. Party  $\{a, b, c\}$  has  $x^a$  as equilibrium platform since, being the only party, every policy platform in  $P^{\{a,b,c\}}$  will be implemented in equilibrium.

As a result, the equilibrium party configurations are  $(\{a\}, \{b\}, \{c\})$ ,  $(\{a\}, \{b, c\})$ , and  $(\{a, b, c\})$  offering policy platform  $x^a = (-1, -1)$ .

5. First notice that, since this configuration is an equilibrium, each smaller party is better off staying in the current coalition rather than leaving and form a smaller party. Moreover, the equilibrium policy platform must be the outcome of a mediation between all smaller parties included in the coalition such that: (i) none of them would rather prefer the outcome induced by the other coalition; (ii) none of them would be better off forming a smaller party with other members of the coalition.

The fact that smaller parties split almost perfectly between the two coalitions indicates that the  $x^j$ 's might be quite polarized in two main focal policies: the center-right and the center-left one. In fact, if the  $x^j$ 's were more spread out over the policy spectrum we could have, for example, another coalition positioned in the center (since parties close to the center might be better off forming their own coalition and offer a policy closer to their preferred ones).

However, this last point is based on the assumptions that all small parties have more or less the same size in the population, which is actually not true in this Italian case. Both coalitions were centered around two main parties (the ones of the respective leaders), namely *L'Unione* for the center-left and *La casa delle libert * for the center-right, and almost all the other parties represented a much lower fraction of the population. As a result, even if a subset of parties in a coalition had preferences closer to each other than with the main party, they cannot form their own coalition because they would lose the elections, as well as suffer from a worse equilibrium policy platform given by the new winning coalition that does not include them.

6. (a) The most preferred policies and indifference curves of the three groups are displayed in Figure 15.3.

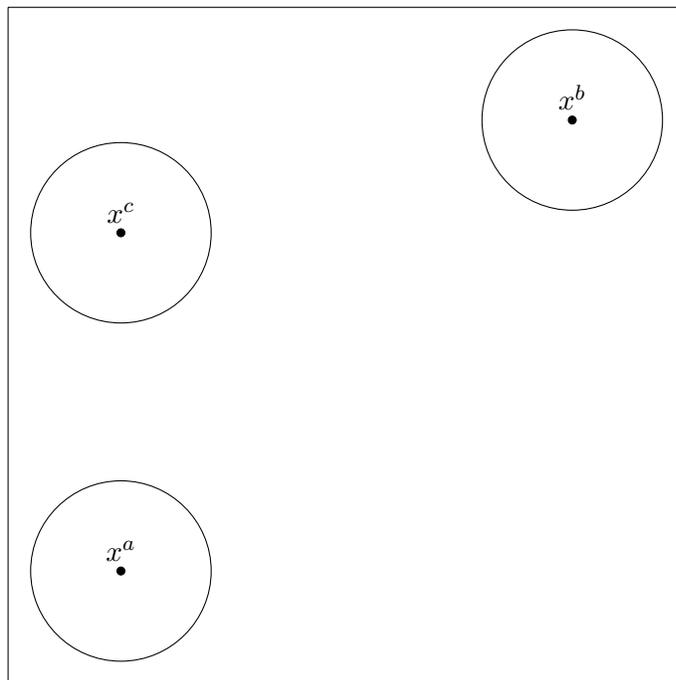


Figure 15.3: Most preferred policies and indifference curves.

- (b) This set includes all possible party configurations (also those which are not played in equilibrium), that is:

$$\{(\{a\}, \{b\}, \{c\}), (\{a, b\}, \{c\}), (\{a\}, \{b, c\}), (\{a, c\}, \{b\}), (\{a, b, c\})\}$$

- (c) The Pareto sets are

$$P^{\{a\}} = \{(-1, -1)\}$$

$$P^{\{b\}} = \{(1, 1)\}$$

$$P^{\{c\}} = \{(-1, 0.5)\}$$

$$P^{\{a,b\}} = \{(x_1, x_2) : x_2 = x_1 \in [-1, 1]\}$$

$$P^{\{b,c\}} = \{(x_1, x_2) : x_2 = 0.75 + 0.25x_1, x_1 \in [-1, 1]\}$$

$$P^{\{a,c\}} = \{(-1, x_2) : x_2 \in [-1, 0.5]\}$$

$$P^{\{a,b,c\}} = \{(x_1, x_2) : x_1, x_2 \in [-1, 1], x_1 \leq x_2 \leq 0.75 + 0.25x_1\}$$

Pareto sets  $P^{\{a\}}$ ,  $P^{\{b\}}$ , and  $P^{\{a,b\}}$  are the same as in section 6.2.  $P^{\{b,c\}}$  contains all points lying on the upward sloping line  $x_2 = 0.75 + 0.25x_1$ , while previously it consisted on a constant flat line with  $x_2 = 1$ . Therefore, if in section 6.2  $b$  and  $c$  perfectly agreed on policy dimension  $x_2$ , now there is a disagreement and getting close to one of the parties preferred policy platform means that the other party has to sacrifice in both policy dimensions.

Both  $a$  and  $c$  still perfectly agree on policy dimension  $x_1$ , but now their position is closer in terms of the first dimension as well. Indeed,  $P^{\{b,c\}}$  is now smaller since it does no more contain policies involving  $x_2 > 0.5$ . Finally, the area of points included in  $P^{\{a,b,c\}}$  indicates how close the positions of each group are. In section 6.2 we had that the (Euclidean) distance between  $x^a$  and  $x^c$  and between  $x^b$  and  $x^c$  was equal to 2, and the area of points in the set was equal to 2 as well. Now the distance between  $x^a$  and  $x^c$  has decreased to 1.5 while the one between  $x^b$  and  $x^c$  has increased to 2.06156. The area of points in  $P^{\{a,b,c\}}$  is now 1.5, implying that overall the groups have closer policy positions.

- (d) First notice that if a party includes more than one group and decides to run (implying that both internal groups will end up vote for the party), the party will win the elections since it will get at least 2/3 of the votes. The following are the equilibrium policy platform for each party configuration:

- $(\{a\}, \{b\}, \{c\})$ : Each party  $j$  that chooses to run will offer its preferred policy  $x^j$ . Notice that both group  $a$  and group  $b$  would rather have  $x^c$  than the policy offered by the other party ( $x^b$  and  $x^a$ , respectively) and therefore party  $\{c\}$  will run unopposed in equilibrium.
- $(\{a, b\}, \{c\})$ : We have that group  $a$  strictly prefers  $x^c$  to policies  $(x, x) \in P^{\{a, b\}}$  such that  $x \in (\sqrt{1.125} - 1, 1]$ , while group  $b$  strictly prefers  $x^c$  to policies  $(x, x) \in P^{\{a, b\}}$  such that  $x \in [-1, 1 - \sqrt{2.125})$ . Therefore, the equilibrium platform is going to be either politician  $a$  or  $b$  running unopposed and offering policy  $(x, x)$  such that  $x \in (1 - \sqrt{2.125}, \sqrt{1.125} - 1)$ .
- $(\{a, c\}, \{b\})$ : Both groups  $a$  and  $c$  strictly prefer each policy platform in the Pareto set  $P^{\{a, c\}}$  to  $x^b$ . Hence, in equilibrium either politician  $a$  or politician  $c$  runs unopposed offering a policy platform  $x \in P^{\{a, c\}}$ .
- $(\{b, c\}, \{a\})$ : While group  $b$  strictly prefers each policy platform in the Pareto set  $P^{\{b, c\}}$  to  $x^a$ , group  $c$  strictly prefers  $x^a$  to policies  $(x_1, x_2) \in P^{\{b, c\}}$  such that  $x_1 \in [1 - \sqrt{2.11765}, 1]$ . Hence, in equilibrium either politician  $b$  or politician  $c$  runs unopposed offering a policy platform

$$(x_1, x_2) \in \{(x_1, x_2) : x_1 \in [-1, 1 - \sqrt{2.11765}), x_2 = 0.75 + 0.25x_1\}$$

- $(\{a, b, c\})$ : In this case any policy platform in  $P^{\{a, b, c\}}$  can be implemented in equilibrium.
- (e) In order to determine whether a party configuration is stable we need to look at the incentive of each politician or group of politicians to leave the party and form a smaller one. Let's consider each party configuration separately:
- i.  $(\{a\}, \{b\}, \{c\})$ : This configuration is stable by definition since each politician has his own party.
  - ii.  $(\{a, b\}, \{c\})$ : This is a stable configuration since if either  $a$  or  $b$  leaves the party they'll have to compete against  $\{c\}$ , which will win (since both groups  $a$  and  $b$  prefers  $x^c$  to the policy offered by the other party) and implement  $(-1, 0.5)$ , and both of them will be worse off compared to the equilibrium policy platform.
  - iii.  $(\{a, c\}, \{b\})$ : This configuration is stable only if party  $\{a, c\}$  offers  $x^c = (-1, 0.5)$ . If another equilibrium platform is of-

ferred,  $\{c\}$  has the incentive to form his own party and offer  $x^c$ , which would eventually be the equilibrium outcome.

- iv.  $(\{b, c\}, \{a\})$ : Exactly as in configuration  $(\{a, c\}, \{b\})$ , stability is ensured only by  $\{b, c\}$  offering  $x^c = (-1, 0.5)$ . Otherwise,  $\{c\}$  has the incentive to leave and induce equilibrium outcome  $x^c$ .
- v  $(\{a, b, c\})$ : This configuration is stable only if the offered policy platform is  $(-0.25, -0.25)$ , which is the only configuration preventing  $c$  to leave the party and  $a$  and  $b$  to form their own separate one.

From these results we can see that there are no qualitative differences in terms of stability compared to the case where  $x_c = (-1, 1)$ . The only thing that changes are the values of the policy platform that ensure stability of the configurations.

7. The solution to this problem depends on your subjective understanding of the election under consideration. We focus on the 2016 presidential election in here, your answers may differ from the one here.
  - (a) In the recruitment phase, the Democratic party ran with lower number of serious candidates than the Republican party. On the Democratic side, Hillary Clinton and Bernie Sanders were competing for the nomination, while on the Republican side more than 17 major candidates filed as candidates. On the Democratic side, many party officials believed Hillary Clinton is one of the most qualified and experienced politicians running for the party's nominee. However, on the Republican side there was no clear choice to be qualified as the most qualified and the most experienced candidate in years. One could argue, abstracting away from many details of the reality, that the Democratic party chose aristocracy while the Republican party decided to go with mediocracy.
  - (b) It would be difficult to explain the outcome in part (a) under this model. Since in equilibrium both parties should choose either two mediocre or two aristocratic candidates.
  - (c) One can argue Clinton's qualification's suppressed a heated competition in the Democratic party's primary. However, due to the mediocracy in the Republican party, even those with minimal political experience or revealed political ability continued to cam-

paigned vigorously throughout the primaries stage and energized the party's base.

## Answer key: Chapter 7

1.
  - (a) Yes. The equilibrium policy chosen by both candidates is the policy most preferred by the lobby. Otherwise, one candidate would have an advantage over the other and win the election.
  - (b) No. The lobbyist will not need to transfer resources to the candidate to induce her to choose its most preferred policy because competition between the two candidates induces them to choose the same policy.
  - (c) No. The equilibrium policy chosen by the politician is an average of the policy most preferred by the lobby and the one most preferred by the median voter.
  - (d) Yes. Since without any transfer the politician would implement the policy most preferred by the median voter, in order to obtain a policy that is more favorable, the lobby has to make positive transfers.
  - (e) The lobby would choose to offer campaign contributions to one of the two candidates running in an election for two reasons. First, the lobby gets its preferred policy, and second it does not have to pay for it.
2.
  - (a) False. Both candidates choose the most preferred policy of the lobby.
  - (b) True. Because of competition between the two lobbies neither lobby gets their most preferred policy.
  - (c) False. The politician chooses an equilibrium policy that is equal to the average of the policy most preferred by the lobby and the policy most preferred by the median voter.
  - (d) False. The politician chooses an equilibrium policy that is equal to the average of the policies most preferred by the two lobbies and the policy most preferred by the median voter.
  - (e) True. The policy outcome with ex ante lobbying is the policy most preferred by the lobby which is always farther away from the policy most preferred by the median voter than the average of the policy most preferred by the lobby and the policy most

preferred by the median voter, which is the policy outcome with ex post lobbying.

3. (a) We first need to characterize the campaign contribution of lobby  $R$  as a function of the policy choices of the two candidates. The lobby chooses  $C_R$  to maximize its expected utility:

$$\begin{aligned} EU_R(X_L, X_R, C_R) &= \pi_L [-(1 - X_L)^2 - C_R^2] + \pi_R [-(1 - X_R)^2 - C_R^2] \\ &= \left(\frac{1}{2} - C_R\right) [-(1 - X_L)^2 - C_R^2] + \left(\frac{1}{2} + C_R\right) [-(1 - X_R)^2 - C_R^2] \\ &= \left(\frac{1}{2} - C_R\right) [-(1 - X_L)^2] + \left(\frac{1}{2} + C_R\right) [-(1 - X_R)^2] - C_R^2 \end{aligned}$$

Hence,

$$C_R = \frac{1}{2} [(1 - X_L)^2 - (1 - X_R)^2]$$

We can now characterize the equilibrium policy choices of the two candidates. Candidate  $L$  chooses  $X_L$  to maximize

$$\pi_L = \frac{1}{2} - \frac{1}{2} [(1 - X_L)^2 - (1 - X_R)^2]$$

from which it follows that

$$X_L^* = 1$$

Candidate  $R$  chooses  $X_R$  to maximize

$$\pi_R = \frac{1}{2} + \frac{1}{2} [(1 - X_L)^2 - (1 - X_R)^2]$$

from which it follows that

$$X_R^* = 1$$

Hence, the policies candidates  $L$  and  $R$  will choose in equilibrium are

$$X_L^* = X_R^* = 1$$

- (b)

$$C_R^* = \frac{1}{2} [(1 - X_L^*)^2 - (1 - X_R^*)^2]$$

Hence, in equilibrium the lobby does not contribute any money to the electoral campaign of candidate  $R$ .

4. (a) We first need to characterize the campaign contributions of lobby  $A$  and lobby  $B$  as a function of the policy choices of the two candidates. Lobby  $A$  chooses  $C_A$  to maximize its expected utility:

$$\begin{aligned}
 EU_A(X_L, X_R, C_A) &= \pi_L [-(\alpha - X_L)^2 - 0.5C_A^2] + \pi_R [-(\alpha - X_R)^2 - 0.5C_A^2] \\
 &= \left(\frac{1}{2} + \lambda C_B - C_A\right) [-(\alpha - X_L)^2 - 0.5C_A^2] + \\
 &\quad \left(\frac{1}{2} - \lambda C_B + C_A\right) [-(\alpha - X_R)^2 - 0.5C_A^2] \\
 &= \left(\frac{1}{2} + \lambda C_B - C_A\right) [-(\alpha - X_L)^2] + \\
 &\quad \left(\frac{1}{2} - \lambda C_B + C_A\right) [-(\alpha - X_R)^2] - 0.5C_A^2
 \end{aligned}$$

Hence,

$$C_A = [(\alpha - X_L)^2 - (\alpha - X_R)^2]$$

Lobby  $B$  chooses  $C_B$  to maximize its expected utility:

$$\begin{aligned}
 EU_B(X_L, X_R, C_B) &= \pi_L [-(\beta - X_L)^2 - 0.5C_B^2] + \pi_R [-(\beta - X_R)^2 - 0.5C_B^2] \\
 &= \left(\frac{1}{2} + \lambda C_B - C_A\right) [-(\beta - X_L)^2 - 0.5C_B^2] + \\
 &\quad \left(\frac{1}{2} - \lambda C_B + C_A\right) [-(\beta - X_R)^2 - 0.5C_B^2] \\
 &= \left(\frac{1}{2} + \lambda C_B - C_A\right) [-(\beta - X_L)^2] + \\
 &\quad \left(\frac{1}{2} - \lambda C_B + C_A\right) [-(\beta - X_R)^2] - 0.5C_B^2
 \end{aligned}$$

Hence,

$$C_B = \lambda [(\beta - X_R)^2 - (\beta - X_L)^2]$$

- (b) We now characterize the equilibrium policy choices of the two candidates. Candidate  $L$  chooses  $X_L$  to maximize

$$\pi_L = \frac{1}{2} [(\alpha - X_L)^2 - (\alpha - X_R)^2] + \lambda^2 [(\beta - X_R)^2 - (\beta - X_L)^2]$$

from which it follows that

$$X_L^* = \frac{\alpha + \lambda^2 \beta}{1 + \lambda^2}$$

Candidate  $R$  chooses  $X_R$  to maximize

$$\pi_R = \frac{1}{2} + [(\alpha - X_L)^2 - (\alpha - X_R)^2] - \lambda^2 [(\beta - X_R)^2 - (\beta - X_L)^2]$$

from which it follows that

$$X_R^* = \frac{\alpha + \lambda^2 \beta}{1 + \lambda^2}$$

- (c) In equilibrium, since  $X_L^* = X_R^*$ ,  $C_L^* = C_R^* = 0$ . Hence,  $\pi_R = \pi_L = 1/2$ .
- (d) If  $\alpha = 1$ ,  $\beta = -1$  and  $\lambda = 1$ ,  $X_L^* = X_R^* = 0$ . If  $\alpha = 1$ ,  $\beta = -1$  and  $\lambda = 0.5$ ,  $X_L^* = X_R^* = 0.6$ . If  $\alpha = 1$ ,  $\beta = -2$  and  $\lambda = 1$ ,  $X_L^* = X_R^* = -0.5$ . If  $\alpha = 1$ ,  $\beta = -2$  and  $\lambda = 0.5$ ,  $X_L^* = X_R^* = 0.4$ .
- (e) If lobbying were not allowed the outcome of electoral competition would be the policy most preferred by the median voter.

$$X_L^* = X_R^* = 0$$

Hence, the bias is the largest in case 2, followed by case 3 and then 4. In case 1 there is no bias.

5. (a) We first need to characterize the willingness to pay of lobby  $R$  for any policy  $X$  the elected politician  $P$  may implement other than his most preferred policy 0. This is

$$T_R = -(1 - X)^2 + 1 = -X^2 + 2X.$$

We can now characterize the equilibrium policy choice of the politician, who chooses  $X$  to maximize

$$-X^2 + (-X^2 + 2X) = -2X^2 + 2X.$$

From which it follows that

$$X^* = \frac{1}{2}$$

- (b)

$$T_R^* = -X^{*2} + 2X^* = \frac{3}{4}$$

6. (a) For lobby  $i$ , the willingness of pay can be found using the equation

$$-P_i^2 = -(P_i - X)^2 - \alpha_i T_i$$

which implies

$$T_i^* = \frac{-X^2 + 2P_i X}{\alpha_i}$$

- (b) Given part (a), one can rewrite the problem of the politician as:

$$\max -X^2 + \lambda \left[ \frac{-X^2 + 2P_A X}{\alpha_A} + \frac{-X^2 + 2P_B X}{\alpha_B} \right]$$

Taking the first order condition and equating it to zero gives

$$X^* = \frac{\frac{\lambda P_A}{\alpha_A} + \frac{\lambda P_B}{\alpha_B}}{1 + \lambda/\alpha_A + \lambda/\alpha_B}$$

- (c) Now lobby  $A$  will only transfer money if the policy chosen  $X$  is positive. However, notice that the previous problem changes only slightly and we have

$$T_A^* = \begin{cases} \frac{-X^2 + 2PX}{\alpha_A} & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we have

$$T_B^* = \begin{cases} \frac{-X^2 - 2PX}{\alpha_B} & \text{if } X < 0 \\ 0 & \text{otherwise} \end{cases}$$

- (d) First note that given the parameter specifications, we have

$$T_A^* = \begin{cases} -X^2 + 2X & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_B^* = \begin{cases} -X^2 - 2X & \text{if } X < 0 \\ 0 & \text{otherwise} \end{cases}$$

The optimal policy larger than 0 is the maximizer of  $-X^2 + \lambda(-X^2 + 2X)$ . Hence, if the politician were to chose a positive policy, it would be  $X_{>0}^* = \frac{\lambda}{1+\lambda}$ . Similarly,  $X_{<0}^* = -\frac{\lambda}{1+\lambda}$ .

Notice that both policies provide the politician with the utility  $\frac{\lambda^2}{1+\lambda} > 0$ . Hence, politician is indifferent between choosing  $\frac{\lambda}{1+\lambda}$  or  $-\frac{\lambda}{1+\lambda}$  and both are equilibria.

(e) Notice that now we have

$$T_A^* = \begin{cases} \frac{-X^2 + 2X}{2} & \text{if } X > 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_B^* = \begin{cases} -X^2 - 2X & \text{if } X < 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that the politician is indifferent in choosing  $X$  or  $-X$  since his ideological bliss point is 0. Hence, he will always select to a negative policy since she will get paid more by lobby  $B$  for the same amount of deviation. Hence, now the unique equilibrium is

$$X^* = -\frac{\lambda}{1+\lambda}.$$

7. (a) False. In the first stage voters decide whether to participate in an election as candidates, and before lobbying the elected candidate chooses the set of lobbies she will bargain with.
- (b) False. The model of Ferri and Merlo is about ex-post lobbying where lobbies try to influence the candidate in exchange of transfers after she is elected.
- (c) True. In both models of exogenous and endogenous lobbying the policy outcome is the result of a bargaining process between the candidate (which represent the preferences of the voters) and the lobbies trying to influence the policy.
- (d) False. One of the main findings of Ferri and Merlo is that in equilibrium no elected candidate chooses to bargain with more than one lobby.

- (e) True. By including the citizen-candidate framework in the model, elections may end up having candidates with extreme preferences who, once elected, will engage in bargaining with a lobby having opposite preferences in order to maximize the amount of transfers received.
  - (f) False. Elected candidates will engage in bargaining with a lobby having opposite preferences in order to maximize the amount of transfers.
8. (a) Since we are dealing with electoral campaign contributions, the right model to consider is the ex-ante exogenous lobbies one. For each lobby  $k$ , a binding limit on the contributions amount does not allow to (potentially) spend the optimal amount  $c_k^*(x_D, x_R)$ . Hence, its influence on policy-making is weaker (even though, in equilibrium, the amount spent is 0). As a result, we would expect European policies to be closer to the median voter preferences. The rationale behind setting these limits is to reduce the policy bias induced by lobbying, which is particularly strong if carried out before elections take place.
- (b) Pharmaceutical companies make enormous amount of revenues by not having the government bargaining over drugs prices, which makes lobbying in favor of the law an extremely profitable investment. Since they spent a lot of money in lobbying, we can disregard the ex-ante lobbying model (which entails zero contributions in equilibrium) and focus on the other two models in order to interpret these facts.

Notice first that, since a large majority of the voters is against this law (at least for the bargaining part), we can assume that elected politicians represent the same preferences in Congress. Moreover, we can also assume that pharmaceutical companies lobbies have essentially low or no competition. Pharmaceutical companies are too big, and there seems to be no other big competitor in the market trying to compete against them.

As a result, both (ex-post) exogenous and endogenous lobbying models are able to explain these facts in a very similar fashion. With no competition in the lobbying market, the exogenous lobbying model implies that lobbies will strongly bias the policy outcome. Furthermore, the large amount of transfers paid implies that the preferred policy by the lobby,  $x_L$ , is very distant from

the median voter preference (assumed to be zero in the model). In a similar way, we know that from the endogenous lobbying model the elected representative will choose to bargain with only one lobby and that the more distant are the preferences between them, the higher will be the transfer the politician is able to ask for.

### Answer key: Chapter 8

1. (a) False. A career politician is a politician who works in the political sector until retirement. A politician who leaves politics before retirement in order to work in the private sector is instead one with *political careers*.
  - (b) True. In fact, these politicians have involuntary political careers since they want to run for re-election but they know they will be defeated. The fact that these individuals run in spite of not being elected is required in order to sustain the voters beliefs that only people with  $p > p^*$  will run for election but, given the average political skill of the challenger equal to  $\frac{1+p^*}{2}$ , only those incumbents with  $p > \frac{1+p^*}{2}$  will actually win the elections.
  - (c) True. As the definition of political careers itself states, individuals with such careers leave office after the second period. Specifically, those having VPC find optimal to use the first period in politics to reveal their skill  $p$  to the market and then go to the private sector.
  - (d) False. Individuals with  $p = 0$  may consider a political career in general (it is in their choice set) but they do not in equilibrium, where only people with  $p > p^*$  run for the elections.
2. We can characterize the only difference between the two countries as  $\theta_A > \theta_B$ , i.e. workers in the private sector have a higher probability of revealing their skills after one period of work. From the comparative statics of the model we know that, in equilibrium, an increase in  $s$  by the same amount in both  $A$  and  $B$  will increase the quality of politicians more in country  $B$  (because of the lower  $\theta$ ) than in country  $A$ .
  3. In the context of this model, we can interpret this retirement annuity as a second period payoff,  $r > 0$ . The Five Star Movement agenda of

removing such *vitalizio* implies setting  $r = 0$ . From the comparative statics of the model we know that in equilibrium such a policy would: (i) not change the average quality of politicians, (ii) increase turnover in office, (i) decrease the average quality of political careers. That is, such a policy will mainly increase the share of higher skills politicians who leave office after just one period and keep in Parlamento lower quality career politicians.

4. (a) For a given level of  $\lambda$ , integrating the probability specification over  $[0, 1]$  with respect to  $p$  gives  $\alpha + \lambda/3$ . Thus, at every level of  $\lambda$ , the fraction of high ability individuals among those with political ability is higher in the original specification.
- (b) The fraction of high ability individuals under the first specification is  $2\alpha + 2\lambda$  and under the second specification is  $2\alpha + 8/3\lambda$ . Thus, for every value of  $\lambda$  the fraction of high ability individuals among those with political ability is higher in the second specification.
5. Based on the comparative statics of the model we can interpret these different turnover rates in the following ways:
  - Politicians' per-period salary  $s$  is lower in  $A$  than in  $B$ .
  - High-skilled in country  $A$  are paid more than in country  $B$ .
  - Country  $B$  gives career politicians higher second period benefits  $r$  compared to country  $A$ .
6. (a) Recall that individuals only know their political ability and are unaware of their market ability. Let  $V^0(M, M)$  denote the expected value of working in market sector in the initial period. Then,

$$V^0(M, M) = \bar{w} + \theta\alpha w_h + \theta(1 - \alpha)w_l + (1 - \theta)\bar{w}.$$

(b)

$$V^0(P, M) = s + \bar{w}.$$

7. In the first period, no one's market ability is revealed yet, thus, the only offered wage level is  $\bar{w}$  that is offered to anyone who decides to join the market sector. This wage is based on the belief of firms about the distribution of high and low ability individuals among those who decide to join the private sector in equilibrium.

In the second period, the following types of individuals take part in the market sector. Those who were employed in the market sector in period one and their market ability is revealed to be high or low. Those who were employed in the market sector in period one and their market ability was not revealed. Those who joined the political sector in period one and now with their revealed political ability have decided to join the market sector. Those individuals joining the economy for the first time in the second period with their unknown market or political ability.

New entrants to the economy and those with employment in the market sector in the first period with unrevealed types will be pooled together and offered the same wage, the mass of individuals who are offered this wage level are of a positive measure.

Individuals with employment in the market sector in the first period with revealed types, are offered two distinct wage levels  $w_h$  and  $w_l$  for those with the high type and the low type respectively. Both wage levels are offered to a positive measure of individuals.

Lastly, a distinct wage  $w_p$  is offered to every politician of the revealed type  $p$  who decides to join the market sector for the second period. as  $p > 0$  is continuously distributed, the wage offered for every type is of measure zero.

## Answer key: Chapter 9

1. (a) True. CW will be the option most preferred by the agent with median  $\alpha$ . The efficient alternative will be the one most preferred by the agent with mean  $\alpha$ .
- (b) False.  $MRT = \sum_{i=1}^N MRS_i$ .
- (c) True. Regardless of the nature of the tax.
- (d) False. The efficiency conditions are indifferent to the SWF.
- (e) False. The efficiency condition will still be  $MRT = \sum_i MRS_i$  though the specified form this takes may change.
2. (a) The indirect utility function of a generic individual is

$$V(G) = \ln \left( I - \frac{cG}{n} \right) + \alpha \ln G$$

where  $I$  denotes income,  $c$  the cost of one unit of the public good and  $n$  the number of individuals in the community. Since  $I = 200$ ,  $c = 500$  and  $n = 10$ , this function reduces to

$$V(G) = \ln(200 - 50G) + \alpha \ln G$$

To find the “peak” of the indirect utility function we calculate the derivative with respect to  $G$  and put it equal to zero:

$$\frac{dV}{dG} = \frac{-50}{200 - 50G} + \frac{\alpha}{G} = 0.$$

Solving for  $G$  we obtain:

$$\hat{G} = \frac{\alpha}{1 + \alpha} 4.$$

Using the Median Voter Theorem, we know that the outcome of majority voting is the level of  $G$  most preferred by the median voter. Since the median of the distribution of  $\alpha$  in the community is equal to 3, the outcome of majority voting is

$$\hat{G}^M = \frac{3}{1 + 3} 4 = 3.$$

- (b) The outcome of majority voting is efficient if and only if the median of the distribution of  $\alpha$  in the community is equal to the mean. In this case, the median is equal to 3 and the mean is also equal to 3. Hence, the outcome of majority voting is efficient.
3. (a) The median voter theorem applies. This means that the outcome of majority voting is the level of expenditure most preferred by the individual with median income (that is with income equal to \$200). The indirect utility function of the median voter is

$$V(E) = 200E - E^2$$

and the peak of this function is at

$$E^* = 100.$$

Hence, the level of expenditure that will be chosen by majority voting is 100.

- (b) In this case mean income and median income coincide (they are both equal to \$200). Hence, the outcome of majority voting is efficient.
- (c) The outcome of majority voting is independent of the order in which the votes are taken since it is a Condorcet winner.
4. (a) The indirect utility function is given by:

$$V(G) = \ln\left(I_i - \frac{cG}{N}\right) + \alpha_i \ln(G)$$

$$V(G) = \ln(I_i - G) + \alpha_i \ln(G)$$

- (b) Note that

$$\frac{dV(G)}{dG} = \frac{-1}{I_i - G} + \frac{\alpha_i}{G}$$

which implies that the unique value of  $G$  such that  $\frac{dV(G)}{dG} = 0$  is given by

$$G^* = \frac{\alpha_i I_i}{1 + \alpha_i}.$$

To verify that  $V$  is single-peaked, we have to show that  $G^*$  is a maximum and not a minimum, which is true if the second derivative is always negative:

$$\frac{d^2V}{dG^2} = \frac{-1}{(I_i - G)^2} - \frac{\alpha_i}{G^2} < 0$$

- (c) Median voter theorem applies. The peak for the first group is 50, the peak for second group is 75, the peak for the third group is 60. So the median peak is 60, which implies that  $G^M = 60$  is the outcome of majority voting.
- (d) Efficiency requires that

$$\sum MRS = MRT$$

where

$$MRS_i = \frac{\alpha_i (I_i - G)}{G}$$

and

$$MRT = 10.$$

Hence,

$$4 \left( \frac{100 - G}{G} \right) + 2 \left( \frac{300 - G}{3G} \right) + 4 \left( \frac{600 - G}{9G} \right) = 10 \implies G^E = 57.35.$$

- (e) First note that  $G = 50$  is the peak of the indirect utility function of the first group and  $G = 75$  is the peak of the indirect utility function for the second group. Hence, the third group will be decisive. The utility they derive from a public library of size 50 is  $\ln(550) + \frac{1}{9} \ln(50)$  and from 75 is  $\ln(525) + \frac{1}{9} \ln(75)$ . To be able to compare these two equations, multiply both sides by 9. Then, you can see that the utility from size 50 is larger than the one from size 75 since:

$$\begin{aligned} 9 \ln(550) + \ln(50) &> 9 \ln(525) + \ln(75) \\ 550^9 \cdot 50 &> 525^9 \cdot 75 \end{aligned}$$

Hence,  $G = 50$  is chosen.

5. (a) The normal form representation of the game is given by:

	B	
	Contribute	Don't
A		
Contribute	1000,1100	950,1000
Don't	1050,900	1000,900

It is trivial to see that the equilibria of this game are (Don't, Contribute) and (Don't, Don't). The efficient allocations are given by (Contribute, Contribute) and (Don't, Contribute). Hence, only one equilibrium is an optimal allocation.

- b. Notice that in the equilibrium studied above, individual  $B$  gets \$900. He would, on the other hand, get \$1100 if contribution was compulsory. Since the difference is larger than \$150, he would make this payment and contribution would be compulsory.
6. (a) Let  $G_j^e$  be the expected amount of public good provided by individuals belonging to group  $j$ . By assumption, all individuals of the same type (group) will provide the same amount of public good in equilibrium. Individuals belonging to group  $A$  expect that the total provision of the public good will be

$$G = 11G_A + 5G_B^e + 4G_C^e.$$

Therefore, the maximization problem of a generic individual of type  $A$  who has to decide the level of contribution of all the

individuals in group  $A$  is

$$\begin{aligned} \max_{x_A, G_A} \ln x_A + \frac{1}{4} \ln (11G_A + 5G_B^e + 4G_C^e) \\ \text{s.t. } x_A + 20G_A = 1400. \end{aligned}$$

The Lagrangian is

$$\mathcal{L} = \ln x_A + \frac{1}{4} \ln (11G_A + 5G_B^e + 4G_C^e) + \lambda (1400 - x_A - 20G_A).$$

At the optimal solution:

$$\begin{aligned} MRS^A = 20 &\implies \frac{11}{4} \frac{x_A}{G} = 20 \\ \text{and } x_A + 20G_A &= 1400 \end{aligned}$$

that is

$$\begin{cases} x_A = \frac{80}{11} (11G_A + 5G_B^e + 4G_C^e) \\ x_A + 20G_A = 1400. \end{cases}$$

Hence:

$$\frac{4}{11} (11G_A + 5G_B^e + 4G_C^e) + G_A = 70.$$

Therefore,  $A$ 's reaction function is

$$G_A = \frac{770 - 20G_B^e + 16G_C^e}{55}.$$

Similarly, we can derive the  $B$ 's reaction function:

$$G_B = \frac{350 - 22G_A^e + 8G_C^e}{15}$$

and the  $C$ 's reaction function is

$$G_C = \frac{210 - 11G_A^e + 5G_B^e}{7}.$$

In the Nash equilibrium we have that  $G_j^N = G_j^e$  for all  $j = A, B, C$ :

$$\begin{cases} G_A^N = \frac{770 - 20G_B^e + 16G_C^e}{55} \\ G_B^N = \frac{350 - 22G_A^e + 8G_C^e}{15} \\ G_C^N = \frac{210 - 11G_A^e + 5G_B^e}{7}. \end{cases}$$

By solving this system we get  $G_A^N = \frac{98}{11}$ ,  $G_B^N = \frac{14}{5}$ , and  $G_C^N = 14$ . The total amount of public good provided by the community through voluntary contributions is given by  $G^N = 11 \frac{98}{11} + 5 \frac{14}{5} + 4 \times 14 = 168$ .

- (b) The indirect utility function of an individual of type  $j$  as a function of the public spending is given by:

$$V^j(G) = \ln(1400 - G) + \alpha_j \ln G.$$

Each individual votes for the level of  $G$  that maximizes his utility:

$$G_j^* = \operatorname{argmax}_G \ln(1400 - G) + \alpha_j \ln G \quad (15.4)$$

The F.O.C. gives:

$$\begin{aligned} \frac{V^j(G)}{dG} : -\frac{1}{1400 - G} + \frac{\alpha_j}{G} &= 0 \\ \implies -G + \alpha_j(1400 - G) &= 0 \\ \implies G_j^* + \frac{\alpha_j}{1 + \alpha_j} 1400 & \end{aligned}$$

Remark:  $G_j^*$  is increasing in  $\alpha_j$ . Since voters' preferences are single peaked, the median voter theorem holds. Therefore, the outcome of the political process is:

$$G^p + \frac{\alpha_M}{1 + \alpha_M} 1400 = 280$$

where  $\alpha_M = \alpha_A$ .

- (c) Pareto Optimality requires that  $\sum_j MRS^j = c$  and  $MRS^j = \alpha_j \frac{x_j}{G}$ . Hence:

$$\sum_{j \in \{A, B, C\}} N^j \alpha_j \frac{1400 - G}{G} = 20$$

where  $\sum_j N^j \alpha_j = \frac{33}{4}$ . Hence

$$\begin{aligned} 1400 - G &= \frac{80}{33} G. \\ G^E &= 408.85 \end{aligned}$$

Notice that the provision of  $G$  that emerges in a majority voting democracy coincides with the Pareto optimal provision if and only if  $\alpha^M = \bar{\alpha}$ . If  $\bar{\alpha} < \alpha_M \implies$  over-provision of the public good, and if  $\bar{\alpha} > \alpha_M \implies$  under-provision of the public good.

Here  $\bar{\alpha} = \sum_j \frac{N^j \alpha_j}{N} = \frac{33}{80}$  and  $\alpha^M = \frac{1}{4} \implies$  under-provision of the public good.

Finally, the private provision of the public good is also not efficient and entails under-provision. In summary, we have that

$$G^N < G^P < G^E.$$

7. (a) Notice that  $V^i(x) = |a_i + x|$  is not generally single-peaked in  $[0, 10000]$  and hence Median Voter Theorem I cannot guarantee the existence of a Condorcet winner. This does, however, depend on  $a_i$ . Notice that  $a_i$  determines the alternative that provides the lowest value of  $V^i$ , specifically  $-a_i$  will be the lowest point. Hence, if there exist no  $a_i \in (-10000, 0)$ , then the preferences are monotonic (a.k.a. single peaked on the extremes) and we can guarantee the existence of a Condorcet winner by the MVT I. It will be the extreme most-preferred by the median voter (ordered by their “peaks”).
- (b) Notice that  $V^i$  is not single-peaked in  $[0, 10000]$  for all our voters (it is for those with income of \$100,000, but not for the others) and hence Median Voter Theorem I cannot guarantee the existence of a Condorcet winner. This does, however, depend on  $I_i$ . Notice that  $I_i/10$  determines the alternative that provides the lowest value of  $V^i$ . This falls in  $(0, 10000)$  for voters with income of \$10,000, \$20,000 and \$30,000. So we cannot guarantee the existence of a Condorcet winner by the MVT I. We do, however, observe that those with more income prefer less spending. So we can employ MVT II which tells us that a Condorcet winner exists and is the alternative preferred by the voter with median income.

## Answer key: Chapter 10

1. (a) The superintendent will hold a referendum when the status quo level is smaller than the level most preferred by the median voter (or the level most preferred by all citizens). In this case the superintendent can induce the citizens to vote for an increase in the level of expenditure in public education.
- (b) There is a negative relationship between the status quo level and the outcome of the referendum. The smaller the status quo level the higher the level of expenditure in public education citizens are willing to support.

- (c) First, the probability of success of a referendum should be higher if  $E^{SQ} < E^{MV}$ . Second, there should be a negative correlation between  $E^{SB}$  and  $E^{SQ}$ . Third, there should be zero correlation between  $E^{SB}$  and  $E^{MV}$ .
- (d) First, the probability of success of a referendum should be independent of the relationship between  $E^{SQ}$  and  $E^{MV}$ . Second, there should be zero correlation between  $E^{SB}$  and  $E^{SQ}$ . Third, there should be a positive correlation between  $E^{SB}$  and  $E^{MV}$ .
2. (a) First, note that since everyone earns the same level of income, the income tax rate charged to every individual is simply the chose level of per-capita education spending divided by the average income. Thus, the budget constraint for each individual  $i$  is given by

$$200(1 - \tau) = 200\left(1 - \frac{E}{200}\right) = c_i(E).$$

Plugging the expression above for the equilibrium consumption of every individual into the utility function of young people gives

$$\begin{aligned} V_i^y(E) &= \ln\left(200\left(1 - \frac{E}{200}\right)\right) + 2 \ln E \\ &= \ln 200 + \ln\left(1 - \frac{E}{200}\right) + 2 \ln E. \end{aligned}$$

And, plugging it for the senior residents gives

$$\begin{aligned} V_i^o(E) &= \ln\left(200\left(1 - \frac{E}{200}\right)\right) + \ln E \\ &= \ln 200 + \ln\left(1 - \frac{E}{200}\right) + \ln E. \end{aligned}$$

The first order conditions with respect to the expenditure on education,  $E$  for the young residents gives:

$$E^{*y} = \frac{2}{3}200 = \frac{400}{3} = 133.33$$

and doing the same for the senior citizens gives the optimal expenditure level by

$$E^{*o} = \frac{1}{2}200 = 100.$$

Thus, the optimal spending level for the young people is 33% higher than the one preferred by the senior citizens.

- (b) Preferences for all individuals are single-peaked and the space of possible spending levels is uni-dimensional. The median voter which in here is an individual from the young community decides has the preferred spending level of 133.33. Thus, the superintendent holds a referendum over this spending level and the referendum is passed.
- (c) The superintendent would like to increase the spending as high as the median voter does not become worse off compared to the status quo. In here, the status quo is the preferred spending level for the seniors in the community. However, the median voter, who is a young resident, clearly prefers any spending level higher than the status quo, up to her preferred level which is 133.33. However, the appointed superintendent will go higher than this and chooses  $E_S^*$  such that  $V_i^y(E_{SQ}) = V_i^y(E_S^*)$ . Thus,

$$\begin{aligned} \ln 200 + \ln\left(1 - \frac{100}{200}\right) + 2 \ln 100 &= \ln 200 + \ln\left(1 - \frac{E_S^*}{200}\right) + 2 \ln E_S^* \\ \implies \ln\left((E_S^*)^2 - (E_S^*)^3/200\right) &= 8.52 \\ \implies (E_S^*)^2 - (E_S^*)^3/200 &= 5014.05 \end{aligned}$$

Thus,  $E_S^* = 161.6$ .

3. (a) Note that a middle class resident will be the median voter in this setup. The mean income is given by  $\bar{I} = 108$ . Since individuals only vary with respect to their income and are homogeneous with respect to their preferences (single-peaked), the optimal per-capita expenditure level is the same for every individual, regardless of their level of income. Thus, using the first order condition from the indirect utility of any individual we get  $E^* = 108/3 = 36$ . Which in this case is the level of expenditure that the superintendent proposes in a referendum and it passes.
- (b) In the previous part we calculated the optimal level of expenditure to be \$36 per-capita. The status-quo of \$50 is to the right side of the optimal level of spending. The indirect utility function is single-peaked with the peak located at the optimal expenditure level. Thus, any level higher than the status-quo proposed by the superintendent makes every individual worse-off and will be rejected unanimously. Thus, in equilibrium the superintendent either will not call for a referendum or calls for a referendum and it will be rejected unanimously and the status-quo level prevails.

4. The equilibrium level proposed by an elected superintendent is given by  $E_{PESE}^* = \frac{\alpha}{1+\alpha}\bar{I}$ . Plugging it in the utility of the appointed superintendent in here gives

$$E - (E - \frac{\alpha}{1+\alpha}\bar{I})^2.$$

Taking the first order conditions with respect to  $E$  we get

$$1 - 2(E - \frac{\alpha}{1+\alpha}\bar{I}) = 0.$$

Thus, the optimal level preferred by the appointed superintendent is

$$E_{PESA}^* = \frac{1 + \alpha + \alpha\bar{I}}{2(1 + \alpha)}.$$

Or  $E_{PESA}^* = \frac{1}{2}E_{PESE}^*$ . Thus, under this specification of the superintendent's objective function, the preferred level is 1/2 to the right of the one preferred by the individuals. Now, if the status quo is to the right of the individual's optimal level and to the right of the appointed superintendent, then the superintendent calls for a referendum proposing  $E_{PESA}^*$  and it will be passed unanimously as every individual is better off. However, this is still above the efficient level of expenditure.

If the status quo is to the right of  $E_{PESE}^*$  and to the left of  $E_{PESA}^*$ , then there will be no referendum or one where the superintendent proposes  $E_{PESA}^*$  and rejected unanimously.

Finally, if the status quo is to the left of  $E_{PESE}^*$  (obviously to the left of  $E_{PESA}^*$  too), let's define  $E_S$  as the largest possible  $E_S$  such that  $V_i(E_S) = V_i(E_{SQ})$ . Then, the superintendent proposes

$$\min(E_S, E_{PESA}^*)$$

and it passes unanimously, thus still above the efficient level  $E_{PESE}^*$ .

5. (a) The expenditure level that will be chosen is  $E = 50$ . It will be chosen by unanimity independent on the voting rule which is used.
- (b) If  $E_0 < 50$ , the expenditure level that will be chosen as a function of  $E_0$  is  $E(E_0) = 100 - E_0$ . If  $E \geq 50$  the expenditure level that will be chosen as a function of  $E_0$  is  $E(E_0) = E_0$ .

6. (a) Under majority rule, the Condorcet winner will be selected. We can use MVT I to find that CW. Note the budget constraint for the government under a proportional taxation rate is

$$\sum_{i=1}^N tI_i = N \times E^{pub},$$

where  $N = 100$ ,  $I_i = 100$  for all people and  $E^{pub}$  is per capital public education expenditure. Simple substitution will tell us that  $E^{pub} = t\bar{I} = t \times 100$ . Hence, the indirect utility function over the tax rate  $t$  is

$$V(t) = \ln((1-t)I_i) + \ln(t\bar{I}).$$

Following MVT we find the peak of this indirect utility function for the median voter (They are all median voters as everyone has the same income  $I_i = \bar{I} = 100$ .), which is the most preferred tax rate of the median voter and therefore the Condorcet winner. Let's take the derivative of the indirect utility function and set it equal to zero:

$$\frac{dV(t)}{dt} : \frac{-1}{(1-t)I_i} + \frac{1}{t\bar{I}} = 0$$

$$(1-t)I_i = t\bar{I} \implies (1-t)100 = t100 \implies t^* = .5$$

Recall  $E^{pub} = t^*\bar{I}$ . The expenditure level that will be chosen is  $E^{pub} = 50$ .

- (b) The school board will act as bureaucrats a la Niskanen, meaning they will pick the highest level of expenditure such that at least half of the community will support the referendum. Hence, we need to ask, at what level of expenditure will the community be indifferent between the proposed level and the status quo. Said in math terms, let's find  $t$  such that

$$V(t) = V(.10).$$

This is not very hard:

$$\ln((1-t)I_i) + \ln(t\bar{I}) = \ln((1-.1)I_i) + \ln(.1\bar{I})$$

$$\ln((1-t)100) + \ln(t100) = \ln(90) + \ln(10)$$

$$t^* = .1 \text{ or } .9$$

The school board will pick the higher  $t$  hence expenditure will be  $E^{s.b.} = t^*100 = 90$ .

7. (a) False. A bureaucrat only calls a referendum when the status quo level is lower than the level most preferred by the residents.
- (b) True. A bureaucrat only calls a referendum when the status quo level is different than the level most preferred by the residents.
- (c) False, there should be a negative correlation.
- (d) False, there should be a no correlation.
- (e) True. A bureaucrat a la Niskanen would not lower the level of spending on education.

### Answer key: Chapter 11

1. (a) True. Availability of loans will alleviate some of the wealth constraint and allow individuals with lower wealth to afford a higher level of education, if they see fit.
- (b) True. In the absence of general equilibrium effects, this will be a Pareto improvement since individuals have a larger set to choose from.
- (c) False. It may not be a Pareto Improvement over a pure private system since it may make low-ability individuals worse off by over-educating them.
2. (a) False. This holds only if the wealth constraints are not binding for anyone.
- (b) False. It is true that the resulting allocation will be inefficient. However, one cannot guarantee constrained efficiency unless the mean and the median of the relevant statistics are the same.
- (c) False. A mixed system always represents a Pareto Improvement over a pure public system, but it is not Pareto Rankable with respect to a pure private system when the wealth constraints are binding for some individuals under a private system.
3. (a) If education is provided only through a private system, each individual will choose  $E$  to maximize

$$C = (W - E)R + AE^\beta = (W - E)2 + A\sqrt{E}$$

subject to the constraint that  $E \leq W$ . Hence,

$$E = \min \left\{ W, \left( \frac{A}{4} \right)^2 \right\}$$

which implies that each of the 20 individuals with  $W = 5$  and  $A = 4$  will choose  $E = 1$ , each of the 20 individuals with  $W = 10$  and  $A = 8$  will choose  $E = 4$ , each of the 40 individuals with  $W = 15$  and  $A = 16$  will choose  $E = 15$  (they would like to choose  $E = 16$ , but the wealth constraint is binding for them), and each of the 20 individuals with  $W = 25$  and  $A = 20$  will choose  $E = 25$ . The resulting allocation is not efficient because 40 individuals are wealth constrained and cannot invest in the optimal level of education.

- (b) If education is provided only through a public system, then  $E = t\bar{W} = 14t$  where the tax rate  $t$  is chosen by majority voting. The indirect utility function is

$$V(t) = (1 - t)WR + AE^\beta = (1 - t)W2 + A\sqrt{14t}$$

which is has a single peaked at

$$t = 14 \left( \frac{A}{4W} \right)^2.$$

Hence, for each of the 20 individuals with  $W = 5$  and  $A = 4$  the most preferred tax rate is  $t = \frac{14}{25}$ , for each of the 20 individuals with  $W = 10$  and  $A = 8$  the most preferred tax rate is  $t = \frac{14}{25}$ , for each of the 40 individuals with  $W = 15$  and  $A = 16$  the most preferred tax rate is  $t = \frac{224}{225}$ , and for each of the 20 individuals with  $W = 25$  and  $A = 20$  the most preferred tax rate is  $t = \frac{14}{25}$ . Using the Median Voter Theorem, the tax rate that will be chosen by majority rule is  $t = \frac{14}{25}$  and hence the level of education that will be provided through the public system is  $E = 7.84$ . This level is not efficient. It is also not constrained efficient, since the constrained efficient level of provision of education through a public system is

$$E = \left( \frac{\bar{A}}{4} \right)^2 = \left( \frac{12.8}{4} \right)^2 = 10.24$$

4. (a)  $E_i^{PR} = \min \left\{ W_i, \left( \frac{A_i}{4} \right)^2 \right\}$ . This implies that  $E_1 = \min \{30, 1\} = 1$ ,  $E_2 = \min \{36, 4\} = 4$  and  $E_3 = \min \{54, 9\} = 9$ .
- (b) Since the wealth constraint is not binding for any of the individuals, the allocation is efficient.

- (c)  $E_i^{PU} = \left[ \frac{\bar{W}}{6} \left( \frac{A}{\bar{W}} \right)_{MED} \right]^2$ . Since  $\bar{W} = 40$  and  $\text{med} \left( \frac{A}{\bar{W}} \right) = 1/3$ , we get  $E^{PU} = (20/9)^2$ . The implied tax rate is  $t = 1/40(20/9)^2 = 10/81$ .
- (d) You can immediately see that the public provision is not constrained efficient since  $\bar{W} \neq W_{MED}$  but  $\bar{A} = A_{MED}$ , that is the distributions of  $W$  and  $A$  are not symmetric.
- (e) Notice that transitioning from a pure public system to the mixed private/public system will not change the incentives of the first group since they have the lowest level of ability and are already receiving more than what they ideally would. The remaining groups, however, would vote in favor of the transition since they would ideally purchase more education.
5. (a)  $E_i^{PR} = \min \left\{ W_i, \left( \frac{A_i}{4} \right)^2 \right\}$ . This implies that  $E_1 = \min \{20, 25/4\} = 25/4$  and  $E_2 = \min \{30, 25/16\} = 25/16$ .
- (b)  $E_i^{PU} = \left[ \frac{\bar{W}}{4} \left( \frac{A}{\bar{W}} \right)_{MED} \right]^2$ . Since  $\bar{W} = 24$  and  $\left( \frac{A}{\bar{W}} \right)_{MED} = 2$ , we get  $E = E_1 = E_2 = 9$ . The implied tax rate is  $9/24$ .
- (c) Arrow's impossibility theorem does not apply to this situation since there are only two alternatives and the possible utilities from education are limited. Hence, the unrestricted domain assumption is violated. We can use the Median Voter Theorem. This is fundamentally because the single-peakedness is trivially satisfied when there are only two choices. Then, by construction, there can only be one peak of the indirect utility function.
- (d) It suffices to look at the incentives of group 1 since the median voter will be one of them. Under the private school system, they get a utility of  $2(20\frac{25}{4}) + 25 = \frac{105}{2}$ . On the other hand, under the public system, they get  $40(\frac{15}{24}) + 30 = 55$ . Hence, the public education system is implemented.
- (e) The efficient system is the private system since none of the credit constraints are binding. However, the public education system is voted victorious. The reason is that the relatively poorer individuals, where the median voter belongs, are more able and can therefore get a cheaper education for a better outcome under the public education.

6. As a first step, we need to characterize the indirect utility of each individual  $i$  as a function of the tax rate  $\tau$ . In order to do that, we have to substitute the optimal amount of topped up private education,  $E_{mix}^{PR}$ , in the utility function. To simplify notation, define the following

$$E_i^* \equiv \left[ \frac{\beta A_i}{R} \right]^{\frac{1}{1-\beta}}$$

Now let's express  $E_{mix}^{PR}$  as a function of  $\tau$  by substituting for  $E_{mix}^{PUB} = \tau W$ :

$$E_{mix}^{PR} = \begin{cases} \min\{E_i^*, W\} - \tau W & , \text{ if } \tau < \frac{E_i^*}{W} \\ 0 & , \text{ if } \tau \geq \frac{E_i^*}{W} \end{cases}$$

where we used the fact that  $\min\{y - x, z - x\} = \min\{y, z\} - x$ .

Notice that if  $\min\{E_i^*, W\} = E_i^*$  we have that  $E_i^*/W \geq 1$ , while if  $\min\{E_i^*, W\} = W$  we get  $0 \leq E_i^*/W < 1$  (assuming both  $A_i$  and  $W$  are positive). Since it must be that  $\tau \in [0, 1]$ , we can distinguish two cases.

First, if  $E_i^* \geq W \implies E_i^*/W \geq 1$  and therefore we will always get  $E_{mix}^{PR} = (1 - \tau)W$ . That is, the individual will spend all her after-tax wealth in private education. The indirect utility is then given by

$$V_i(\tau) = A_i W^\beta$$

which is constant. Hence, there is no unique  $\tau^*$  which maximizes the indirect utility. Every value  $\tau \in [0, 1]$  yields the same utility.

Second, if  $E_i^* < W \implies E_i^*/W \in [0, 1)$  we get

$$E_{mix}^{PR} = \begin{cases} E_i^* - \tau W & , \text{ if } \tau < \frac{E_i^*}{W} \\ 0 & , \text{ if } \tau \geq \frac{E_i^*}{W} \end{cases}$$

Hence

$$V_i(\tau) = \begin{cases} (W - E_i^*) R + A_i (E_i^*)^\beta & , \text{ if } \tau < \frac{E_i^*}{W} \\ (1 - \tau) W R + A_i (\tau W)^\beta & , \text{ if } \tau \geq \frac{E_i^*}{W} \end{cases}$$

which shows that taxes affect indirect utility only if they are at least as high as  $E_i^*/W$ .

If we take the F.O.C. just considering the interval  $\tau \in [E_i^*/W, 1]$  we get

$$-WR + \beta W A_i (\tau W)^{\beta-1} = 0$$

then solving for  $\tau$  we get

$$\begin{aligned}\tau_i^* &= \frac{1}{W} \left( \frac{\beta A_i}{R} \right)^{\frac{1}{1-\beta}} \\ &= \frac{E_i^*}{W}\end{aligned}$$

The S.O.C. is

$$\beta(\beta - 1)W^2 A_i (\tau W)^{\beta-2} < 0$$

which follows from  $\beta \in (0, 1)$ . The value  $\tau_i^*$  is then the global maximum in this interval and  $V(\tau)$  is decreasing for  $\tau > \tau_i^*$ .

The maximum indirect utility for  $\tau \in [E^*/W, 1]$  is then

$$V_i(\tau_i^*) = (W - E_i^*) R + A_i (E_i^*)^\beta$$

which is exactly the same as the indirect utility for  $\tau < \frac{E_i^*}{W}$ . It follows that for individuals with  $E_i^* < W$  each value  $\tau \in \left[0, \frac{E_i^*}{W}\right]$  is optimal, and therefore single-peakedness is not satisfied.

However, notice that

$$\frac{\partial V'(\tau; A_i)}{\partial A_i} = \begin{cases} 0 & \text{if } E^* \geq W \\ 0 & \text{if } E^* < W \text{ and } \tau < \frac{E_i^*}{W} \\ \beta W (\tau W)^{\beta-1} & \text{if } E^* < W \text{ and } \tau \geq \frac{E_i^*}{W} \end{cases}$$

where  $\beta W (\tau W)^{\beta-1} > 0$ . Therefore

$$\frac{\partial V'(\tau; A_i)}{\partial A_i} \geq 0$$

for all  $A_i$  and we can apply MVT II which implies that the CW tax rate is the one preferred by the median ability individual  $A_M$ .

Finally, we know that  $\tau_i^*$  is a global maximizer (even if not unique) given each  $A_i$ , and therefore

$$\begin{aligned}E_{mixed}^{PUB} &= \tau_M^* W \\ &= \left( \frac{\beta A_M}{R} \right)^{\frac{1}{1-\beta}}\end{aligned}$$

which is equal to the level of education provided in a pure public system with  $W_i = W$ :

$$\begin{aligned} E^{PUB} &= \left[ \frac{\beta \bar{W}}{R} \left( \frac{A_i}{W_i} \right)_M \right]^{\frac{1}{1-\beta}} \\ &= \left[ \frac{\beta W}{R} \left( \frac{A_M}{W} \right) \right]^{\frac{1}{1-\beta}} \\ &= \left( \frac{\beta A_M}{R} \right)^{\frac{1}{1-\beta}} \end{aligned}$$

7. (a) This is exactly as the pure private system. Individuals solve

$$\begin{aligned} \max_{E_i} (W - E_i)R + A_i E_i^\beta \\ \text{s.t. } E_i \leq W_i \end{aligned}$$

which yields

$$E_i^{PR} = \min \{E^*, W_i\}$$

where

$$E_i^* = \left( \frac{\beta A_i}{R} \right)^{\frac{1}{1-\beta}}$$

- (b) The borrowers are defined as those with  $E_i^{PR} = W_i < E_i^*$ . After choosing their level of education  $E_i^{PR}$  they decide the optimal level of borrowings  $b_i^*$  by solving the following

$$\begin{aligned} \max_{b_i} -b_i R + A_i (W_i + b_i)^\beta \\ \text{s.t. } b_i \geq 0 \end{aligned}$$

The solution ignoring the non-negativity constraint is given by

$$\begin{aligned} b_i^* &= \left( \frac{\beta A_i}{R} \right)^{\frac{1}{1-\beta}} - W_i \\ &= E_i^* - W_i \end{aligned}$$

which satisfies the constraint since  $E_i^* - W_i > 0$ . Therefore, individuals optimally borrow just the amount needed to reach the optimal educational level  $E_i^*$ .

- (c) The utilitarian social welfare gain is the sum of the individuals' utility gains given by the implementation of the policy. Since only the borrowers' utility is affected by the policy, we can just focus on the social welfare gain for this group.

Let  $\mathcal{B}$  be the set of borrowers. Each  $i \in \mathcal{B}$  will optimally borrow  $b'_i = \min\{\bar{b}, b_i^*\}$ . Using the definition of  $b_i^*$ , the utility gain  $\Delta_i$  for individual  $i \in \mathcal{B}$  is given by

$$\Delta_i = \begin{cases} -(E_i^* - W_i)R + A_i(E_i^*)^\beta & \text{if } E_i^* - W_i \leq \bar{b} \\ -\bar{b}R + A_i(W_i + \bar{b})^\beta & \text{if } E_i^* - W_i > \bar{b} \end{cases}$$

and the utilitarian social welfare gain is then

$$\begin{aligned} \sum_{i \in \mathcal{B}} \Delta_i &= -R \sum_{i \in \mathcal{B}} b'_i + \sum_{i \in \mathcal{B}} A_i(W_i + b'_i)^\beta \\ &= \sum_{b_i^* \leq \bar{b}} \left[ -R(E_i^* - W_i) + A_i(E_i^* - W_i)^\beta \right] - N_b R \bar{b} + \sum_{b_i^* > \bar{b}} A_i \bar{b}^\beta \end{aligned}$$

- (d) If  $\bar{L} \geq \sum_i b_i^* \equiv \bar{B}^*$ , the government is able to give the optimal loan amount to each borrower. As a result, giving exactly  $b_i^*$  to each borrower  $i \in \mathcal{B}$  is a better policy than setting the credit limit  $\bar{b}$ . This is simply because those individuals with  $b_i^* > \bar{b}$  will be able to get the additional financial resources needed to attain the optimal level of education  $E_i^*$ .
- (e) In order to determine which individuals should have priority in getting a loan to maximize the social welfare gain we need to look at the marginal utility of the borrowers at  $E_i = E_i^{PR} = W_i$ . In particular, the priority should be given to those who have higher marginal utility, since a dollar lent to this borrowers will yield a higher Social Welfare gain.

Let  $\tilde{U}_i(E_i)$  be the utility of individual  $i$  as a function of education purchased. We have

$$\left. \frac{\partial \tilde{U}_i}{\partial E_i} \right|_{E_i=W_i} = -RW_i + \beta \frac{A_i}{W_i^{1-\beta}} > 0$$

where the inequality comes from the fact that, by definition, the level of education  $E_i = W_i$  is strictly less than the optimal amount  $E_i^*$ , and therefore an increase in education leads automatically to an increase in utility.

The equation above shows that marginal utility at  $E_i = W_i$  is increasing in  $A_i$  and decreasing in  $W_i$ . As a result, conditional on  $A_i$  the government should give priority to poorer people, while conditional on  $W_i$  the priority should be given to high ability individuals.

A merit-based priority criterion is not enough to optimally allocate loans in this case because it does not take into account wealth.

- (f) Following from point (e), if wealth,  $W$ , is constant among borrowers the only criterion we need to prioritize is ability,  $A_i$ . By setting a minimum ability level in order to give the loan, there will always be a social welfare gain as long as at least one of the borrowers has  $A_i \geq \underline{A}$ . However, this is clearly not the best policy unless all borrowers have same ability  $A \geq \underline{A}$ .

In fact, it turns out that the best policy would be to give loans as an increasing function of ability for two main reasons:

- i. If multiple individuals are eligible to get a loan and they have different ability levels, it would be more efficient to lend more money to those with higher ability and less to low-ability individuals instead of a constant amount  $\tilde{b}$ .
- ii. Some eligible borrowers may not need the entire loan  $\tilde{b}$  in order to attain optimal education. Therefore, we can increase efficiency by reallocating this excess loan amount to other individuals (eligible or not eligible).

Obviously, a policy which perfectly discriminates individuals by their ability can be very hard and costly to implement in practice (e.g. high administrative costs) and therefore the minimum ability requirement policy might be a better choice from a cost-benefit analysis point of view.

## Answer key: Chapter 12

1. (a) False. Proportional taxation of labor income will induce inefficiency since it will distort the labor supply decisions.
- (b) True. Proportional taxation of fixed endowments will not distort decisions.
- (c) False. Proportional transfers will distort the labor supply decisions.

2. (a) To derive the labor supply for an individual we first plug the budget constraints into the utility function so that it only depends on  $L$  as in

$$U(L) = WL(1 - t) + S + \ln(1 - L)$$

and then take the derivative with respect to  $L$  and put it equal to zero.

$$\frac{\partial U}{\partial L} = W(1 - t) - \frac{1}{1 - L} = 0.$$

Solving for  $L$  we obtain that

$$L^* = 1 - \frac{1}{W(1 - t)}.$$

Remember that  $L^*$  must be positive, which implies that

$$W > W^*$$

where

$$W^* = \frac{1}{1 - t}$$

is the critical wage rate below which individuals do not work.

Note that as long as  $t$  is positive, each of the 60 individuals with wage rate equal to 1 is never going to work, while each of the 40 individuals with wage rate equal to 4 is going to work as long as  $t < 0.75$ . (Since when  $t \geq 0.75$  nobody works and so income is zero, we can restrict attention to redistribution policies such that  $t < 0.75$ .) Hence,

$$L^* = \begin{cases} 0 & \text{if } W = 1 \\ 1 - \frac{1}{4(1-t)} & \text{if } W = 4. \end{cases}$$

- (b) Since  $y = WL$ , we have that

$$y = \begin{cases} 0 & \text{if } W = 1 \\ 4 - \frac{1}{(1-t)} & \text{if } W = 4. \end{cases}$$

Hence, average income is equal to

$$\bar{y} = \frac{40 \left( 4 - \frac{1}{1-t} \right)}{100}$$

and median income is

$$y_M = 0.$$

- (c) The indirect utility functions of the individuals in the community are

$$V(t, S) = \begin{cases} S & \text{if } W = 1 \\ 4(1-t) - 1 + S - \ln(4(1-t)) & \text{if } W = 4 \end{cases}$$

and the slopes of their indifference curves are

$$\frac{dS}{dt} = \begin{cases} 0 & \text{if } W = 1 \\ 4 - \frac{1}{(1-t)} & \text{if } W = 4. \end{cases}$$

Hence, preferences satisfy the single-crossing property.

- (d) The community's budget constraint is

$$S = t\bar{y} = \frac{40t \left(4 - \frac{1}{1-t}\right)}{100}.$$

- (e) Since the median voter has wage rate  $W = 1$ , the median voter does not work and chooses the tax rate that maximizes the amount of subsidy he receives (note that after substituting  $S$  with  $t\bar{y}$  using the community's budget constraint, the indirect utility function of the median voter is  $V(t) = t\bar{y}$ ).

Hence, the median voter solves the following maximization problem:

$$\max_t \frac{40t \left(4 - \frac{1}{1-t}\right)}{100}.$$

The first order condition for this maximization problem is

$$.4 \left(4 - \frac{1}{(1-t)^2}\right) = 0$$

and the only solution between 0 and 1 is  $t = 0.5$ . Hence, the tax rate that will be chosen by majority voting is  $t = 0.5$ .

3. (a) To derive the labor supply for an individual we first plug the budget constraints into the utility function so that it only depends on  $L$

$$U(L) = L(1-t) + S - 0.5(L-X)^2$$

and then take the derivative with respect to  $L$  and put it equal to zero

$$\frac{\partial U}{\partial L} = (1-t) - L + X = 0.$$

Solving for  $L$  we obtain that

$$L^* = X + (1 - t).$$

Note that  $L^*$  is always positive (so everybody works).

- (b) Since  $y = L$ , we have that average income is equal to

$$\bar{y} = \bar{X} + (1 - t) = 0.5 + (1 - t)$$

and median income is

$$y_M = X_M + (1 - t) = 0.2 + (1 - t).$$

- (c) The indirect utility functions of the individuals in the community are

$$V(t, S) = (X + (1 - t))(1 - t) + S - 0.5(1 - t)^2 = X(1 - t) + S + 0.5(1 - t)^2$$

and the slopes of their indifference curves are

$$\frac{dS}{dt} = X + (1 - t).$$

Hence, preferences satisfy the single-crossing property.

- (d) The community's budget constraint is

$$S = t\bar{y} = 0.5t + t(1 - t).$$

- (e) The median voter solves the following maximization problem:

$$\max_t V_M(t) = 0.2(1 - t) + 0.5t + t(1 - t) + 0.5(1 - t)^2.$$

The first order condition for this maximization problem is

$$-0.2 + 0.5 + 1 - 2t - 1 + t = 0$$

and the solution is  $t = 0.3$ . Hence, the tax rate that will be chosen by majority voting is  $t = 0.3$ .

4. (a) The problem of the individual is:

$$\begin{aligned} \max \quad & U(L) = L(1 - t) + S - \alpha(L)^2 \\ \text{s.t.} \quad & L \in [0, 1] \end{aligned}$$

The first order condition delivers:

$$L = \frac{1-t}{2\alpha}$$

Note that we have  $L \in [0, 1]$  for all  $\alpha \in \{1, 2, 3\}$  and  $t \in [0, 1]$ . Hence, everyone works and the labor supply function is given by:

$$L^* = \frac{1-t}{2\alpha}$$

(b) Mean income is given by

$$\bar{y} = (1-t) \frac{1}{6} \left( 2 + \frac{1}{4} + \frac{1}{6} \right) = \frac{29}{72}(1-t)$$

and median income is given by

$$y_M = \frac{1-t}{2}$$

(c) The indirect utility function is equal to:

$$\begin{aligned} V(t, S) &= \frac{(1-t)^2}{2\alpha} + S - \alpha \frac{(1-t)^2}{(2\alpha)^2} \\ &= \frac{(1-t)^2}{4\alpha} + S \end{aligned}$$

and hence

$$\frac{dS}{dt} = L^* = \frac{(1-t)}{2\alpha}.$$

It follows that  $\frac{dS}{dt}$  is always monotonically decreasing in  $\alpha$ . Hence, the single-crossing property is satisfied.

(d) The community's budget constraint is characterized by the following equation:

$$S = t\bar{y} = \frac{29}{72}t(1-t)$$

(e) The median voter belongs to the taste group with  $\alpha = 1$ . Since this group values leisure the least, they work the most and hence earn the most. Hence, they will prefer no taxes or subsidies.

5. (a) Average income decreases due to the fact that fewer people work and those who work work less.
- (b) There is no effect. Since utility is additively separable and linear in consumption there is no income effect.
- (c) The equilibrium relationship is described by the Laffer Curve. The subsidy first increases with the tax rate but then it decreases.
- (d) The likely effect is an increase in redistribution. If the poor are the majority but only the rich vote, then the median voter dislikes redistribution. An extension of the franchise changes the identity of the median voter from the rich to the poor, who likes redistribution.
- (e) One possible explanation is that in country A the median voter is relatively less productive than the median voter in country B, or country A is more unequal than country B. Another possible explanation is that the citizens in the two countries have different preferences.
6. (a) The only efficient income redistribution policy consists of lump-sum taxes (or proportional taxes on fixed endowments) and lump-sum transfers, because lump-sum taxes and transfers do not distort individual behavior (Second Fundamental Welfare Theorem)
- (b) Any other income redistribution policy is inefficient because it distorts individual behavior.
7. (a) To derive the labor supply for an individual we first plug the budget and time constraints into the utility function so that it only depends on  $L$ ,

$$U(L) = 2L(1 - t) + S + 2(1 + X - L) - (1 + X - L)^2$$

and then take the derivative with respect to  $L$  and put it equal to zero.

$$\frac{\partial U}{\partial L} = 2(1 - t) - 2 + 2(1 + X - L) = 0$$

Solving for  $L$  we obtain that

$$L^* = 1 + X - t.$$

Note that  $L^*$  is always positive (so everybody works).

- (b) Since  $y = 2L$ , we have that average income is equal to

$$\bar{y} = 2(1 + \bar{X} - t) = 3 - 2t$$

and median income is

$$y_M = 2(1 + X_M - t) = 2.6 - 2t.$$

- (c) The indirect utility functions of the individuals in the community are

$$V(t, S) = 2(1 + X - t)(1 - t) + S + 2t - t^2 = 2 + 2X - 2t - 2tX + S + t^2$$

and the slopes of their indifference curves are

$$\frac{dS}{dt} = 2(1 + X - t) = y.$$

Since the derivative of the slope of the indifference curves with respect to  $X$  is positive (in fact, it is equal to 2), we conclude that preferences satisfy the “single-crossing” property.

- (d) The community’s budget constraint is

$$S = t\bar{y} = t(3 - 2t).$$

- (e) The median voter solves the following maximization problem:

$$\max_t V_M(t) = 2 + 2X_M - 2t - 2tX_M + t(3 - 2t) + t^2 = 2 + 0.6 - 2t - 0.6t + t(3 - 2t) + t^2$$

The first order condition for this maximization problem is

$$-2 - 0.6 + 3 - 4t + 2t = 0$$

and the solution is

$$t = \frac{-2 - 0.6 + 3}{2} = 0.2$$

Hence, the tax rate that will be chosen by majority voting is  $t = 0.2$  and the per-capita level of redistribution is  $S = 0.52$ .

**Answer key: Chapter 13**

1. (a) True. This happens because, given the assumptions of the model, under perfect information the company chooses group-specific fee and benefit that maximize the utility of the same group under the financial self-sufficiency constraint. It follows that the optimal fee  $f_g^*$  for group  $g$  is exactly equal as the amount any individual in this group would have saved in absence of an insurance market.
  - (b) True, the middle class would be better off with the rich class policy compared to their. In order to prevent that, the insurance can put a constraint to the rich class maximization problem in order to avoid members of middle class pretending to be rich. However, the introduction of this constraint causes a utility loss for the rich class.
  - (c) True. By solving the SP problem we get that the optimal fee under the private market with perfect information,  $f_g^*$ , is efficient. However, imperfect information causes utility losses to different class member because of the constraints the insurance company has to put on the maximization problem in order to avoid lower classes pretending to be of higher classes and getting a better contract.
  - (d) False. The CSPP gives an optimal tax rate of  $\tau_{CPO}^* = 0$ , which directly implies no universal health care coverage provided by the government. On the other hand, universal coverage is the outcome of the political-economic equilibrium of the model.
2. (a) Let  $w_i$  and  $s_i$  denote the wage and saving level of individual  $i$ . Then, the expected utility function can be written in terms of  $s_i$  as

$$E[U^i(s_i)] = (1 - \exp(-2(w_i - s_i))) + \beta\rho(1 - \exp(-2s_i)) + \beta(1 - \rho)(1 - \exp(-2(w_i + s_i))).$$

The first order condition with respect to  $s_i$  gives

$$-2\exp(-2(w_i - s_i)) + \beta\rho\exp(-2s_i) + \beta(1 - \rho)\exp(-2(w_i + s_i)).$$

This a non-linear function in  $s_i$  and generally there is no closed form solution to it. The optimal level of saving  $s_i^*$  is the solution to the above equation.

- (b) Solving for the optimal level of saving in the F.O.C. gives  $s^* = 0.097$ . In the absence of discounting the optimal saving will be higher at 0.12.
3. (a) The insurance company will provide the policy that maximizes each groups indirect utility function. Following the F.O.C. in section 13.1.1.,  $f_{poor}^* = 1/4$ ,  $f_{middle}^* = 4/5$ ,  $f_{rich}^* = 1$ .
- (b) We first set  $f_{rich}$  so that the poor cannot benefit from it. That is

$$\begin{aligned} & \max_{f_{rich}} V_{rich}(f_{rich}) \\ \text{s.t. } & V_{poor}(f_{rich}) \leq V_{poor}(f_{poor}) \end{aligned}$$

we need that,  $2f - 5/2f^2 \leq 1/16$ , i.e.  $f_{rich} \geq 0.77$  or  $f_{rich} \leq 0.03$ .  $f_{rich}^* = 1$  satisfies the first inequality and will not attract the poor. Then, we do the same to ensure that the middle class individuals do not prefer the plan intended for the rich individuals to their own.  $V_{middle}(f_{middle}) = 10.24$ . We need for  $f_{rich}$  to satisfy  $28/5f - 2f^2 \leq 19.84$ . Which is trivially satisfied for all values.

Now it remains to check whether the poor would prefer the middle class plan. For this we need  $f_{middle}$  to satisfy  $3/4f - 5/4f^2 \leq 1/16$ , i.e.  $f_{middle} \geq 1/2$  or  $f_{middle} \leq 1/10$ .

We see that the first best outcome which is the collection of the optimal policies in the previous part is still achievable even if the insurer cannot distinguish between the different groups.

4. The financial sufficiency condition requires that

$$2b_{rich} + 5b_{poor} + 32b_{middle} = 10f_{poor} + 10f_{rich} + 80f_{middle}$$

Thus, any collection of benefits and fees that satisfy the above is financially sufficient. In particular, the collection of benefits and fees under the self-sufficiency condition satisfy the unconstrained sufficiency condition above. Note that any change in the fees and benefits for one group, requires to be balanced off by a change in the fees and benefits of another group, thus making them worse off. Therefore, there is no Pareto improvement over the outcome in the previous problem.

5. This provision of the ACA will reduce the insurer's ability to distinguish between different groups of individuals. However, the insurer can still use other characteristics to price different groups of people.

For instance, one can argue that the geographical location is highly correlated with the income level of individuals and thus we expect a significant correlation between the health risk of an individual and her geographical location, while controlling for age and other personal characteristics permissible under the ACA. An evidence would be higher premiums in the Exchange Markets in low-income counties in the U.S. . In any case, restricting insurer's ability to discriminate based on the observable characteristics' of the potential customers would disturb the efficiency of the health insurance market. However, in here we are not taking the other objectives of the policy maker into account.

6. One reason is the characteristics of the population. Let's consider a population where instead, 10 are rich, 40 are middle class, and 20 are poor with the same wage and illness rate. Then,  $\tau_{CPO}^* = 0.18$ . Or, when 10 are rich, 20 are middle class, and 40, then  $\tau_{CPO}^* = 0.22$ .
7. (a) Candidate  $\mathcal{B}$  maximization problem can be rewritten as

$$\begin{aligned} & \max_{\tau} \sum_{i=1}^{100} \lambda_i [\log(w_i(1-\tau)) + \rho_i \log(10\tau) + (1-\rho_i) \log(w_i)] \\ & = \max_{\tau} 4 \left[ \log(2(1-\tau)) + \frac{1}{4} \log(10\tau) + \frac{3}{4} \log(2) \right] \\ & \quad + 4 \left[ \log(3(1-\tau)) + \frac{1}{2} \log(10\tau) + \frac{1}{2} \log(3) \right] \\ & \quad + 4 \left[ \log(4(1-\tau)) + \frac{1}{4} \log(10\tau) + \frac{3}{4} \log(4) \right] \end{aligned}$$

The F.O.C. with respect to  $\tau$  is

$$-\frac{1}{2(1-\tau)} + \frac{1}{40\tau} - \frac{1}{3(1-\tau)} + \frac{1}{20\tau} - \frac{1}{4(1-\tau)} + \frac{1}{40\tau} = 0$$

which yields

$$\tau^* = \frac{6}{59}$$

and that corresponds to a benefit of  $b^* = \frac{60}{59}$ .

Substituting  $\tau^*$  and  $b^*$  back in the utility function of each group member we get the utility levels if  $\mathcal{B}$  wins election

$$\begin{aligned} U_{poor} &= 0.841857 \\ U_{middle} &= 1.179462 \\ U_{rich} &= 1.534098 \end{aligned}$$

- (b) The utility of the middle class if  $\mathcal{B}$  wins the election is lower than the one they would get if  $\mathcal{R}$  wins, where the latter is equal to 1.84. Therefore, the middle class will vote for  $\mathcal{R}$ . The same holds for the other classes, whose utility under  $\mathcal{B}$  is lower compared to  $\mathcal{R}$ . As a result, the whole population votes for  $\mathcal{R}$  who eventually wins the election.

### Answer key: Chapter 14

1. In order to determine whether the political-economic equilibrium (PE) is locally efficient we have to solve the Social Planner (SP) problem for each community  $j \in \{1, 2\}$ . Since efficiency does not depend on the functional form of the Social Welfare Function, we will use the utilitarian SWF for convenience. The SP problem in community  $j$  is given by

$$\begin{aligned} \max_{\{c_i\}_i^{n_j}, G_j} & \sum_{i=1}^{n_j} (\ln c_i + \alpha_i \ln G_j) \\ \text{s.t.} & \sum_{i=1}^{n_j} c_i + G_j = n_j \end{aligned}$$

which leads to the Lagrangian

$$\max_{\{c_i\}_i^{n_j}, G_j} \sum_{i=1}^{n_j} (\ln c_i + \alpha_i \ln G_j) - \lambda \left( n_j - \sum_{i=1}^{n_j} c_i + G_j \right).$$

The F.O.C.s are then

$$\begin{aligned} \frac{1}{c_i} &= \lambda, \forall i \\ \sum_{i=1}^{n_j} \frac{\alpha_i}{G_j} &= \lambda, \forall i \\ \sum_{i=1}^{n_j} c_i + G_j &= n_j \end{aligned}$$

Substituting we get

$$c_i = \frac{G_j}{\sum_{i=1}^{n_j} \alpha_i}, \forall i$$

$$G_j = n_j - \sum_{i=1}^{n_j} c_i,$$

and the level of public good  $G_j$  is then

$$G_j = n_j - \sum_{i=1}^{n_j} \frac{G_j}{\sum_{i=1}^{n_j} \alpha_i}$$

$$\implies G_j \left( 1 + \frac{n_j}{\sum_{i=1}^{n_j} \alpha_i} \right) = n_j$$

$$\implies G_j = \left( \frac{1}{1 + \frac{1}{\bar{\alpha}}} \right) n_j$$

where  $\bar{\alpha}$  is the mean  $\alpha$  among the individuals living in community  $j$ . The last equation can be re-written as

$$G_j^{SP} = \left( \frac{\bar{\alpha}}{1 + \bar{\alpha}} \right) n_j$$

which is the same level of local public good in the political-economic equilibrium, since (in expectation) mean and median  $\alpha$  are the same given the uniform distribution assumption.

Substituting for  $G_j^{SP}$  in the consumption equation above we get, for each individual  $i$ ,

$$c_i = c^{SP} = \frac{1}{1 + \bar{\alpha}}.$$

Finally, substituting the mean values of  $\alpha$  in each community  $j$ , namely  $\bar{\alpha}_1 = 1/4$  and  $\bar{\alpha}_2 = 3/4$ , we will end up getting exactly the same values as the political-economic equilibrium of section 14.1. Therefore, the political-economic equilibrium outcome is efficient.

2. (a) True. Suppose for example that in community  $j$  there is a lower provision of public good compared to community  $k$ , which implies that each individual in  $j$  consumes more private good (recall that

income is equal to 1 for all individuals). Since the utility is increasing in  $G$ , the individual may prefer the provision of public good in the other community as long as  $G_k^{PE} > G_j^{PE}$  and consumption is fixed at  $c_j^{PE}$ . Notice that the question specifically uses only the term "prefers" and does not mention whether the budget constraint must be satisfied.

- (b) True. In fact we have that

$$\frac{d}{dy} \left( \frac{dS_j}{d\tau_j} \right) > 0$$

which implies that indifference curves of richer people are steeper. Therefore, given an increase in  $\tau_j$  richer individuals will need a higher increase in subsidies compared to low income people in order to make up for the utility loss generated by higher taxes.

- (c) False. The median voter of community  $j$  maximizes her indirect utility over  $\tau_j$  *taking as given* the value  $\tau_k$ ,  $k \neq j$ . In fact, we can look at the median voter indirect utility as a best response function with respect of the action chosen by the median voter in the other community.
3. (a) From the balanced-budget requirement for local community  $j$  we know that

$$\tau_j = \frac{G_j}{n_j}$$

and the individual's budget constraint will then be

$$(1 - \tau_j) = 1 - \frac{G_j}{n_j} = c(G_j)$$

where, in equilibrium, consumption  $c(G_j)$  is going to be constant across individual in the same community  $j$ .

Substituting for consumption in the utility function we get then

$$V_i(G_j) = \ln \left( 1 - \frac{G_j}{n_j} \right) + \frac{\alpha}{3} \ln G_j$$

and from the F.O.C.s with respect to  $G_j$  we get individual  $i$ 's most preferred public good level

$$G_i^* = \left( \frac{\alpha_i}{3 + \alpha_i} \right) n_j.$$

Finally, given the single-peakedness of the indirect utility  $V_i()$  we can apply MVT I and the level of public good provided in community  $j$  is

$$G_j^* = \left( \frac{\alpha_{M_j}}{3 + \alpha_{M_j}} \right) n_j,$$

where  $\alpha_{M_j}$  is the median  $\alpha$  in the community.

- (b) The sorting equilibrium consists in a strategy profile where each individual has no incentive to move to the other community. That is, if individual  $i$  lives in  $j = 1$  we will have  $V_i(G_1^*) \geq V_i(G_2^*)$ , and viceversa for those living in  $j = 2$ .

Based on the case of  $\alpha_i$  uniformly distributed on  $[0, 1]$  in section 14.1, we can characterize the following sorting equilibrium

- Individuals with  $\alpha_i \in [0, 1]$  will live in community 1.
- Individuals with  $\alpha_i \in [1, 2]$  will live in community 1.

with those with  $\alpha_i = 1$  being indifferent between the two communities and therefore living in either one or the other.

- (c) Given the sorting equilibrium we have that the median voters in the two communities will have  $\alpha_{M_1} = \frac{1}{2}$  and  $\alpha_{M_2} = \frac{3}{2}$ . The political-equilibrium values of public good and consumption in each community is then:

$$c_1^{PE} = \frac{6}{7}, c_2^{PE} = \frac{2}{3}, G_1^* = \frac{1}{7} \left( \frac{n}{2} \right), G_2^* = \frac{1}{3} \left( \frac{n}{2} \right)$$

4. (a) Given the equilibrium prices  $\tilde{p}_1 = (1 + \tau_1)\theta$  and  $\tilde{p}_1 = (1 + \tau_1)\theta$ , individual  $y$  problem is given by

$$\begin{aligned} & \max_{c,h} \ln(c) + \lambda h \\ \text{s.t. } & y + S_j = h(1 + \tau_j)\theta + c \end{aligned}$$

Substituting for  $c$  from the budget constraint we can reduce the problem to the choice of  $h$

$$\max_{c,h} \ln(y + S_j - h(1 + \tau_j)\theta) + \lambda h$$

From the F.O.C. we get

$$(1 + \tau_j)\theta + \lambda(y + S_j) - \lambda(1 + \tau_j)\theta h = 0$$

which gives the following optimal housing choice as a function of  $S_j$ , and  $\tau_j$ .

$$h^*(\tau_j, S_j; y) = \frac{y + S_j}{(1 + \tau_j)\theta} - \frac{1}{\lambda}$$

- (b) Substituting  $h^*(\tau_j, S_j; y)$  in the utility function we get the following indirect utility

$$\begin{aligned} V(\tau_j, S_j; y) &= \ln\left(\frac{(1 + \tau_j)\theta}{\lambda}\right) + \frac{y + S_j}{(1 + \tau_j)\theta} - \frac{1}{\lambda} \\ &= \ln(1 + \tau_j) + \ln(\theta) - \ln(\lambda) + \frac{y + S_j}{(1 + \tau_j)\theta} - \frac{1}{\lambda} \end{aligned}$$

In order to determine whether preferences over  $\tau_j$  and  $S_j$  satisfy the single crossing property, we first need to compute the the slope of their indifference curves:

$$\begin{aligned} dV &= \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \tau} d\tau \\ &= \frac{1}{(1 + \tau)\theta} dS + \frac{(1 + \tau)\theta - (y + S)}{(1 + \tau)^2\theta} d\tau \end{aligned}$$

where subscript  $j$  is removed for notational convenience.

Setting  $dV = 0$  we get

$$\frac{dS}{d\tau} = \frac{y + S}{1 + \tau} - \theta$$

and

$$\frac{d}{dy} \left( \frac{dS}{d\tau} \right) = \frac{1}{1 + \tau} > 0$$

Hence, the single-crossing property is satisfied and we can apply MVT III.

- (c) From part (b) above we can see that individuals with higher income display steeper indifference curves. This implies that wealthier people are more adverse to higher taxation since, compared to individuals with lower income, they will need a higher increase in subsidy  $S_j$  in order to make up for the utility loss of a given increase in the tax rate  $\tau_j$ . Therefore, the richer the individual, the lower is her preferred level of redistribution.

As a result, individuals will sort themselves in the two communities based in their income  $y$ . In particular, we get the community partitions  $C_1 = [0, y_m]$  and  $C_2 = [y_m, 1]$ , where  $y_m$  represents an individual who is indifferent between the two communities.

We can now plug this sorting result into the government budget constraints of each community:

$$\begin{aligned}\int_0^{y_m} S_1 dy &= \int_0^{y_m} \tau_1 \theta h^*(\tau_1, S_1; y) dy \\ \implies S_1 y_m &= \tau_1 \theta \left( \frac{y_m^2 + S_1 y_m}{(1 + \tau_1) \theta} - \frac{y_m}{\lambda} \right) \\ \implies S_1 &= \frac{\tau_1 [\lambda y_m - \theta(1 + \tau_1)]}{\lambda}\end{aligned}$$

and

$$\begin{aligned}\int_{y_m}^1 S_2 dy &= \int_{y_m}^1 \tau_2 \theta h^*(\tau_2, S_2; y) dy \\ \implies S_2(1 - y_m) &= \tau_2 \theta \left( \frac{(1 - y_m^2) + S_2(1 - y_m)}{(1 + \tau_2) \theta} - \frac{1 - y_m}{\lambda} \right) \\ \implies S_2 &= \frac{\tau_2 [(1 + y_m) - \theta(1 + \tau_2)]}{\lambda}\end{aligned}$$

Then plugging  $S_1$  and  $S_2$  in the indirect utility function we get

$$V(\tau_1; y) = \ln(1 + \tau_1) + \ln(\theta) - \ln(\lambda) + \frac{y}{(1 + \tau_1)\theta} + \frac{y_m}{\theta} - \frac{1 + \tau_1}{\lambda}$$

and

$$V(\tau_2; y) = \ln(1 + \tau_2) + \ln(\theta) - \ln(\lambda) + \frac{y}{(1 + \tau_2)\theta} + \frac{y_m + \tau_2}{\theta} - \frac{1 + \tau_2}{\lambda}$$

In order to find the value  $y_m$  we need to equate  $V(\tau_1; y) = V(\tau_2; y)$ . Unfortunately, we can see that there are infinite values of  $y_m$  that satisfy this equation ( $y_m$  cancels out from both sides).

However, we can still compute  $\tau_1$  and  $\tau_2$  that maximize the individual's utility by taking the F.O.C.s and rearranging

$$\begin{aligned}\tau_1^*(y) &= \frac{\lambda}{2} \left( 1 \pm \sqrt{1 - \frac{4y}{\theta\lambda}} \right) - 1 \\ \tau_2^*(y) &= \frac{\theta\lambda}{2(\theta - \lambda)} \left( 1 \pm \sqrt{1 - \frac{4y(\theta - \lambda)}{\theta^2\lambda}} \right) - 1\end{aligned}$$

which for certain values of  $\theta$  and  $\lambda$  can be unique and inside the unit interval  $[0, 1]$ .

Finally, the median income in community 1 is going to be  $y_m/2$ , while in community 2 we have  $(1 + y_m)/2$ . As a result, the level of taxation in a political-economic equilibrium is given by

$$\tau_1^* = \frac{\lambda}{2} \left( 1 \pm \sqrt{1 - \frac{2y_m}{\theta\lambda}} \right) - 1$$

$$\tau_2^* = \frac{\theta\lambda}{2(\theta - \lambda)} \left( 1 \pm \sqrt{1 - \frac{2(1 + y_m)(\theta - \lambda)}{\theta^2\lambda}} \right) - 1$$

where the value of  $y_m$  is the equilibrium income cutoff level such that

- (a) Individuals with  $y \in [0, y_m]$  will live in community 1.
- (b) Individuals with  $y \in [y_m, 1]$  will live in community 2.
- (c) No individual has incentive to move to the other community.

5. The production functions are given by

$$F_1(n_1) = \left( \frac{n_1}{5} \right)^{\frac{1}{2}}$$

$$F_2(n_2) = \left( \frac{n_2}{5} \right)^{\frac{1}{3}}$$

Again, the competitive labor markets in each community implies that the workers receive a wage equal to their marginal productivity. Thus,

$$w_1 = F_1'(n_1) = \frac{1}{2\sqrt{5}} n_1^{-\frac{1}{2}}$$

$$w_2 = F_2'(n_2) = \frac{1}{3\sqrt[3]{5}} n_2^{-\frac{2}{3}}$$

The rich land owners receive the remaining rents

$$r_1 = \left( \frac{n_1}{64} \right)^{\frac{1}{2}} - \frac{n_1}{16} n_1^{-\frac{1}{2}}$$

$$r_2 = \left( \frac{n_2}{64} \right)^{\frac{1}{3}} - \frac{n_2}{12} n_2^{-\frac{2}{3}}.$$

And,  $n_1 + n_2 = 1000$ .

- (a) The marginal product of labor in both communities would be equalized. Thus,  $n_1$  that solves

$$\frac{1}{16}n_1^{-\frac{1}{2}} = \frac{1}{12}(1000 - n_1)^{-\frac{2}{3}}.$$

is given by  $n_1 = 774$  and  $n_2 = 226$ .

- (b) The free mobility of labor between the two communities implies that the wages earned by the poor working class is equalized in two communities. In here,

$$\frac{1}{16}n_1^{-\frac{1}{2}} + S = \frac{1}{12}(1000 - n_1)^{-\frac{2}{3}} + S$$

It's clear to see the population of the two communities would not change in presence of this equal transfer thus no distortion of the optimal allocation of labor happens in here.

6. In this setting the cost-less and free mobility of labor ensures that the wages paid in the two communities are equal. If the only deciding factor in payment to workers is the competitive labor market in each community, then the wage equality implied by the free mobility of labor is equivalent to equal marginal productivity of labor. As we know, this condition implies the optimal allocation of labor between the two production technologies.

Now suppose that  $S_1 < S_2$ . The competitive markets still pay the workers a wage equal to the marginal productivity of labor in that community, but the community adds the additional transfer on top of their wage to their payment. Now the free mobility implies that  $MPL_1(n_1) = MPL(n_2) + (S_2 - S_1)$ , thus,  $MPL_1(n_1) > MPL(n_2)$ . This means that the second community is hosting more poor residents than the efficient level. Although the second community is producing more, the total production of the economy is lower than what it could be under the no transfer regime.

7. Let's characterize the maximization problem of different players in the economy. The poor workers seek to maximize their consumption, which is the sum of the competitive labor market wage and the transfers. Thus, workers maximize  $\frac{\beta_j}{n_j} + S_j$  by choosing their community  $j$ . Free and cost-less mobility implies  $\frac{\beta_1}{n_1} + S_1 = \frac{\beta_2}{n_2} + S_2$ .

The rich land-owners in community  $j$  try to maximize their net income given the lump-sum tax and the contribution rate of the central

government, by choosing the transfer to the workers in their community. Therefore, they maximize:  $\beta_j(\ln(n_j) - 1) - (1 - z_j)S_j n_j - T_j$  by choosing  $S_j$ . Note that the land-owners do realize that  $n_j$  is a function of the subsidy level  $S_j$  that they choose. The labor demand in community  $j = 1, 2$  can be written as  $n_j = n \left( \frac{\beta_j}{\beta_1} \right)$ , which implies

$$n_1 = \left( \frac{\beta_1}{\beta_1 + \beta_2 + n(S_2 - S_1)/2} \right) n$$

$$n_2 = \left( \frac{\beta_2}{\beta_1 + \beta_2 + n(S_1 - S_2)/2} \right) n$$

So the land-owner 1 chooses  $S_1$  to maximize

$$\beta_1 \left[ \ln \left( \frac{\beta_1}{\beta_1 + \beta_2 + n(S_2 - S_1)/2} \right) + \ln(n) - 1 \right]$$

$$- (1 - z_1)S_1 \left( \frac{\beta_1}{\beta_1 + \beta_2 + n(S_2 - S_1)/2} \right) n - T_j$$

The first order condition with respect to  $S_1$  gives

$$\frac{\beta_1 n (2(\beta_1 + \beta_2) + n(S_2 - S_1)) + 4\beta_1 S_1 (1 - z_1)(\beta_1 + \beta_2)}{(2(\beta_1 + \beta_2) + n(S_2 - S_1))^2} = 0$$

The central government's objective is to  $(z_1, z_2, T_1, T_2)$  to give incentive to the landowners to choose equal level of subsidies, i.e.  $S_1 = S_2$ .

We will not try to explicitly solve for the solution to the first order condition above in here. We are interested in how the land-owners' subsidy best response changes as the central government increase  $z_1$ , i.e. its contribution to the redistribution in community 1. Note that as  $z_1$  increases, the term  $4\beta_1 S_1 (1 - z_1)(\beta_1 + \beta_2)$  in the numerator of the F.O.C. decreases, and that for  $n$  large enough, a decrease in  $S_1$  increases  $-\beta_1 n^2 S_1$  faster than it decreases  $4\beta_1 S_1 (1 - z_1)(\beta_1 + \beta_2)$ . Thus, the subsidy decision of community 1 is decreasing in the government's contribution. Similarly, for the second community. We can conclude there are rates of contribution  $(z_1^*, z_2^*)$  such that equilibrium subsidies chose by the land-owners,  $(S_1^*, S_2^*)$ , satisfy  $S_1^* = S_2^*$ . It's important to note that the lump-sum tax  $T_j$  does not appear in the decision making of the land-owners. Thus, the central government chooses  $(z_1^*, z_2^*)$  so that the subsidies are equalized and then chooses the lump-sum taxes just to balance the budget. To conclude, we just showed that in presence of a central government, by imposing lump-sum taxes and

contributing to the local redistribution effort, efficiency can be established.