

Main formulae for Part 3 Advanced circuit theory and technology

Complex numbers

$$z = a + jb = r(\cos\theta + j\sin\theta) = r\angle\theta,$$

where $j^2 = -1$ Modulus, $r = |z| = \sqrt{a^2 + b^2}$

$$\text{Argument, } \theta = \arg z = \tan^{-1} \frac{b}{a}$$

$$\text{Addition: } (a + jb) + (c + jd) = (a + c) + j(b + d)$$

$$\text{Subtraction: } (a + jb) - (c + jd) = (a - c) + j(b - d)$$

Complex equations: If $a + jb = c + jd$, then $a = c$ and $b = d$

If $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$ then

$$\text{Multiplication: } z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2)$$

$$\text{and Division: } \frac{z_1}{z_2} = \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$

De Moivre's theorem:

$$[r\angle\theta]^n = r^n \angle n\theta = r^n (\cos n\theta + j \sin n\theta)$$

General

$$Z = \frac{V}{I} = R + j(X_L - X_C) = |Z|\angle\phi$$

where $|Z| = \sqrt{[R^2 + (X_L - X_C)^2]}$ and

$$\phi = \tan^{-1} \frac{X_L - X_C}{R}$$

$$X_L = 2\pi fL \quad X_C = \frac{1}{2\pi fC} \quad Y = \frac{I}{V} = \frac{1}{Z} = G + jB$$

Series: $Z_T = Z_1 + Z_2 + Z_3 + \dots$

$$\text{Parallel: } \frac{1}{Z_T} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \dots$$

$$P = VI \cos\phi \text{ or } P = I_R^2 R \quad S = VI \quad Q = VI \sin\phi$$

$$\text{Power factor} = \cos\phi = \frac{R}{Z}$$

If $V = a + jb$ and $I = c + jd$ then $P = ac + bd$

$$Q = bc - ad \quad S = VI^* = P + jQ$$

R-L-C series circuit

$$f_r = \frac{1}{2\pi\sqrt{LC}} \quad Q = \frac{\omega_r L}{R} = \frac{1}{\omega_r CR} = \frac{1}{R} \sqrt{\frac{L}{C}}$$

$$= \frac{V_L}{V} = \frac{V_C}{V} = \frac{f_r}{f_2 - f_1}$$

$$f_r = \sqrt{f_1 f_2}$$

LR-C network

$$f_r = \frac{1}{2\pi} \sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \quad R_D = \frac{L}{CR} \quad Q = \frac{I_C}{I_r} = \frac{\omega_r L}{R}$$

LR-CR network

$$f_r = \frac{1}{2\pi\sqrt{LC}} \sqrt{\left(\frac{R_L^2 - L/C}{R_C^2 - L/C}\right)}$$

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = a \begin{vmatrix} e & f \\ h & j \end{vmatrix} - b \begin{vmatrix} d & f \\ g & j \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Delta-star

$$Z_1 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C}, \text{ etc.}$$

Star-delta

$$Z_A = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}, \text{ etc.}$$

Impedance matching

$$|z| = \left(\frac{N_1}{N_2}\right)^2 |Z_L|$$

Complex waveforms

$$I = \sqrt{\left(I_0^2 + \frac{I_{1m}^2 + I_{2m}^2 + \dots}{2}\right)}$$

$$i_{AV} = \frac{1}{\pi} \int_0^\pi i d(\omega t) \quad \text{form factor} = \frac{\text{r.m.s.}}{\text{mean}}$$

$$P = V_0 I_0 + V_1 I_1 \cos\phi_1 + V_2 I_2 \cos\phi_2 + \dots \text{ or } P = I^2 R$$

$$\text{power factor} = \frac{P}{VI}$$

$$\text{Harmonic resonance: } n\omega L = \frac{1}{n\omega C}$$

Fourier series

If $f(x)$ is a periodic function of period 2π then its Fourier series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, for the range $-\pi$ to $+\pi$:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \quad (n = 1, 2, 3, \dots)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \quad (n = 1, 2, 3, \dots)$$

If $f(x)$ is a periodic function of period L then its Fourier series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos \left(\frac{2\pi nx}{L} \right) + b_n \sin \left(\frac{2\pi nx}{L} \right) \right\}$$

where for the range $-\frac{L}{2}$ to $+\frac{L}{2}$:

$$a_0 = \frac{1}{L} \int_{-L/2}^{L/2} f(x) dx$$

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos \left(\frac{2\pi nx}{L} \right) dx \quad (n = 1, 2, 3, \dots)$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin \left(\frac{2\pi nx}{L} \right) dx \quad (n = 1, 2, 3, \dots)$$

Harmonic analysis

$$a_0 \approx \frac{1}{p} \sum_{k=1}^p y_k \quad a_n \approx \frac{2}{p} \sum_{k=1}^p y_k \cos nx_k$$

$$b_n \approx \frac{2}{p} \sum_{k=1}^p y_k \sin nx_k$$

Hysteresis and eddy current

Hysteresis loss/cycle = $A\alpha\beta$ J/m³

or hysteresis loss = $k_h v f (B_m)^n$ W

Eddy current loss/cycle = $k_e (B_m)^2 f^2 t^3$ W

Dielectric loss

Series representation: $\tan \delta = R_S \omega C_S = 1/Q$

Parallel representation: $\tan \delta = \frac{1}{R_P \omega C_P}$

Loss angle $\delta = (90^\circ - \phi)$

Power factor = $\cos \phi \approx \tan \delta$

Dielectric power loss = $V^2 \omega C \tan \delta$

Field theory

Coaxial cable: $C = \frac{2\pi \epsilon_0 \epsilon_r}{\ln \frac{b}{a}}$ F/m $E = \frac{V}{r \ln \frac{b}{a}}$ V/m

$$L = \frac{\mu_0 \mu_r}{2\pi} \left(\frac{1}{4} + \ln \frac{b}{a} \right) \text{ H/m}$$

Twin line: $C = \frac{\pi \epsilon_0 \epsilon_r}{\ln \frac{D}{a}}$ F/m

$$L = \frac{\mu_0 \mu_r}{\pi} \left(\frac{1}{4} + \ln \frac{D}{a} \right) \text{ H/m}$$

Energy stored: in a capacitor, $W = \frac{1}{2} CV^2$ J;

in an inductor $W = \frac{1}{2} LI^2$ J

in electric field per unit volume,

$$w_f = \frac{1}{2} DE = \frac{1}{2} \epsilon_0 \epsilon_r E^2 = \frac{D^2}{2\epsilon_0 \epsilon_r} \text{ J/m}^3$$

in a non-magnetic medium,

$$w_f = \frac{1}{2} BH = \frac{1}{2} \mu_0 H^2 = \frac{B^2}{2\mu_0} \text{ J/m}^3$$

Attenuators

Logarithmic ratios:

$$\text{in decibels} = 10 \lg \frac{P_2}{P_1} = 20 \lg \frac{V_2}{V_1} = 20 \lg \frac{I_2}{I_1}$$

$$\text{in nepers} = \frac{1}{2} \ln \frac{P_2}{P_1} = \ln \frac{V_2}{V_1} = \ln \frac{I_2}{I_1}$$

Symmetrical T -attenuator:

$$R_0 = \sqrt{(R_1^2 + 2R_1 R_2)} = \sqrt{(R_{OC} R_{SC})}$$

$$R_1 = R_0 \left(\frac{N-1}{N+1} \right) \quad R_2 = R_0 \left(\frac{2N}{N^2-1} \right)$$

Symmetrical π -attenuator:

$$R_0 = \sqrt{\left(\frac{R_1 R_2^2}{R_1 + 2R_2} \right)} = \sqrt{(R_{OC} R_{SC})}$$

$$R_1 = R_0 \left(\frac{N^2-1}{2N} \right) \quad R_2 = R_0 \left(\frac{N+1}{N-1} \right)$$

L-section attenuator: $R_1 = \sqrt{[R_{OA}(R_{OA} - R_{OB})]}$

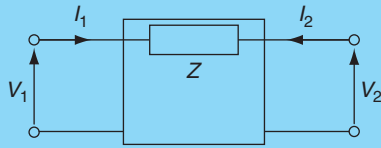
$$R_2 = \sqrt{\left(\frac{R_{OA} R_{OB}^2}{R_{OA} - R_{OB}} \right)}$$

ABCD parameters

Network

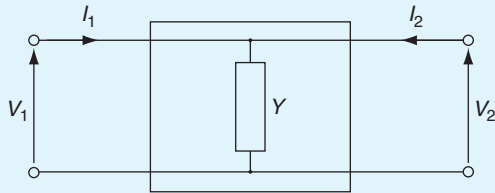
ABCD transmission matrix

Series impedance



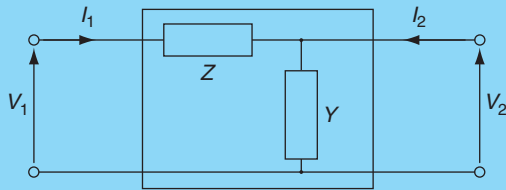
$$\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$$

Shunt admittance



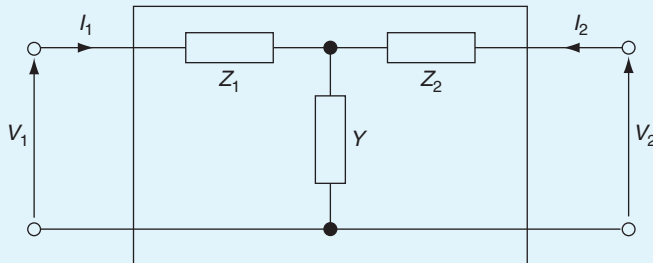
$$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}$$

L-network



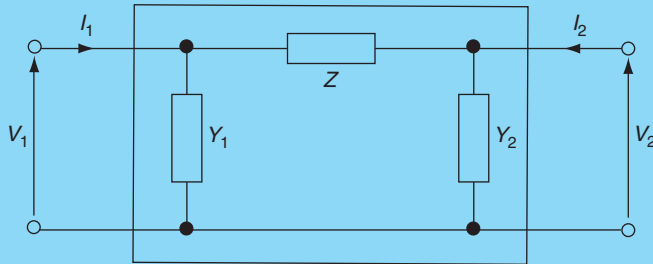
$$\begin{pmatrix} (1+YZ) & Z \\ Y & 1 \end{pmatrix}$$

T-network



$$\begin{pmatrix} (1+YZ_1) & (Z_1+Z_2+YZ_1Z_2) \\ Y & (1+YZ_2) \end{pmatrix}$$

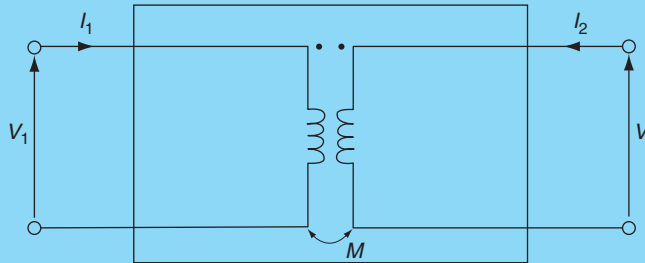
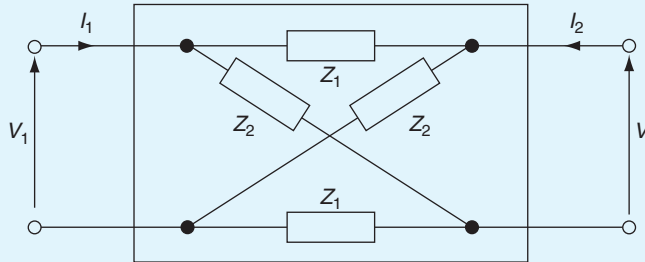
π -network



$$\begin{pmatrix} (1+Y_2Z) & Z \\ (Y_1+Y_2+Y_1Y_2Z) & (1+Y_1Z) \end{pmatrix}$$

(Continued)

ABCD parameters (Continued)

Network	ABCD transmission matrix
<p>Pure mutual inductance</p> 	$\begin{pmatrix} 0 & j\omega M \\ \frac{1}{j\omega M} & 0 \end{pmatrix}$
<p>Symmetrical lattice</p> 	$\begin{pmatrix} \left(\frac{Z_1 + Z_2}{Z_2 - Z_1}\right) & \left(\frac{2Z_1 Z_2}{Z_2 - Z_1}\right) \\ \left(\frac{2}{Z_2 - Z_1}\right) & \left(\frac{Z_1 + Z_2}{Z_2 - Z_1}\right) \end{pmatrix}$

Characteristic impedance, $Z_0 = \sqrt{\frac{B}{C}}$

Filter networks

Low-pass T or π : $f_c = \frac{1}{\pi \sqrt{LC}}$ $R_0 = \sqrt{\frac{L}{C}}$

$$C = \frac{1}{\pi R_0 f_c} \quad L = \frac{R_0}{\pi f_c}$$

$$Z_{0T} = R_0 \sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}$$

$$Z_{0\pi} = \frac{R_0}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}}$$

High-pass T or π : $f_c = \frac{1}{4\pi \sqrt{LC}}$ $R_0 = \sqrt{\frac{L}{C}}$

$$C = \frac{1}{4\pi R_0 f_c} \quad L = \frac{R_0}{4\pi f_c}$$

$$Z_{0T} = R_0 \sqrt{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]}$$

$$Z_{0\pi} = \frac{R_0}{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]}$$

Low- and high-pass:

$$Z_{0T} Z_{0\pi} = Z_1 Z_2 = R_0^2$$

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = e^\gamma = e^{\alpha + j\beta} = e^\alpha \angle \beta$$

Phase angle $\beta = \omega \sqrt{LC}$

time delay $= \sqrt{LC}$

m -derived filter sections:

$$\text{Low-pass } m = \sqrt{\left[1 - \left(\frac{f_c}{f_\infty}\right)^2\right]}$$

$$\text{High-pass } m = \sqrt{\left[1 - \left(\frac{f_\infty}{f_c}\right)^2\right]}$$

Magnetically coupled circuits

$$E_2 = -M \frac{dI_1}{dt} = \pm j\omega M I_1$$

$$M = N_2 \frac{d\phi_2}{dI_1} = N_1 \frac{d\phi_1}{dI_2} = k \sqrt{L_1 L_2} = \frac{L_A - L_B}{4}$$

Transmission lines

$$\text{Phase delay } \beta = \omega\sqrt{LC} \quad \text{wavelength } \lambda = \frac{2\pi}{\beta}$$

$$\text{velocity of propagation } u = f\lambda = \frac{\omega}{\beta}$$

$$I_R = I_S e^{-n\gamma} = I_S e^{-n\alpha} \angle -n\beta$$

$$V_R = V_S e^{-n\gamma} = V_S e^{-n\alpha} \angle -n\beta$$

$$Z_0 = \sqrt{(Z_{OC} Z_{SC})} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

$$\gamma = \sqrt{[(R + j\omega L)(G + j\omega C)]}$$

$$\text{Reflection coefficient, } \rho = \frac{I_r}{I_i} = \frac{Z_O - Z_R}{Z_O + Z_R} = -\frac{V_r}{V_i}$$

$$\text{Standing-wave ratio, } s = \frac{I_{\max}}{I_{\min}} = \frac{I_i + I_r}{I_i - I_r} = \frac{1 + |\rho|}{1 - |\rho|}$$

$$\frac{P_r}{P_i} = \left(\frac{s-1}{s+1}\right)^2$$

Transients

C-R circuit $\tau = CR$

Charging: $v_C = V(1 - e^{-(t/CR)})$ $v_r = Ve^{-(t/CR)}$

$$i = Ie^{-(t/CR)}$$

Discharging: $v_c = v_R = Ve^{-(t/CR)}$ $i = Ie^{-(t/CR)}$

L-R circuit $\tau = \frac{L}{R}$

Current growth: $v_L = Ve^{-(Rt/L)}$

$$v_R = V(1 - e^{-(Rt/L)})$$

$$i = I(1 - e^{-(Rt/L)})$$

Current decay: $v_L = v_R = Ve^{-(Rt/L)}$ $i = Ie^{-(Rt/L)}$

These formulae are available for download at the website:

www.routledge.com/cw/bird

