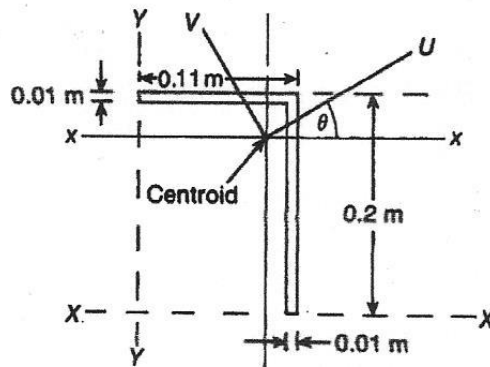


CHAPTER 15 UNSYMMETRICAL BENDING OF BEAMS

EXERCISE 58, Page 337

1. Determine the direction and magnitudes of the principal second moments of area of the angle bar shown.



Section	a	y	ay	ay^2	i_H
1	1×10^{-3}	0.195	1.95×10^{-4}	3.803×10^{-5}	8.33×10^{-9}
2	2×10^{-3}	0.1	2×10^{-4}	2×10^{-5}	6.667×10^{-6}
Σ	3×10^{-3}	—	3.95×10^{-4}	5.803×10^{-5}	6.675×10^{-6}

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{3.95 \times 10^{-4}}{3 \times 10^{-3}} = 0.1317 \text{ m}$$

$$I_{xx} = \sum ay^2 + \sum i_H = 5.803 \times 10^{-5} + 6.675 \times 10^{-6} = 6.471 \times 10^{-5} \text{ m}^4$$

$$I_{xx} = I_{xx} - (\bar{y})^2 \sum a = 6.471 \times 10^{-5} - (0.1317)^2 \times 3 \times 10^{-3} = \mathbf{1.268 \times 10^{-5} \text{ m}^4}$$

Section	a	x	ax	ax^2	i_V
1	1×10^{-3}	0.05	5×10^{-5}	2.5×10^{-6}	8.333×10^{-7}
2	2×10^{-3}	0.105	2.1×10^{-4}	2.205×10^{-5}	1.667×10^{-8}
Σ	3×10^{-3}	—	2.6×10^{-4}	2.455×10^{-5}	8.5×10^{-8}

$$\bar{x} = \frac{\sum ax}{\sum a} = \frac{2.6 \times 10^{-4}}{3 \times 10^{-3}} = 0.0867 \text{ m}$$

$$I_{YY} = \sum ax^2 + \sum i_v = 2.455 \times 10^{-5} + 8.5 \times 10^{-8} = 2.54 \times 10^{-5} \text{ m}^4$$

$$I_{yy} = I_{YY} - (\bar{x})^2 \sum a = 2.54 \times 10^{-5} - (0.0867)^2 \times 3 \times 10^{-3} = \mathbf{2.849 \times 10^{-6} \text{ m}^4}$$

$$\begin{aligned} I_{xy} &= \sum Ah\bar{k} = 1 \times 10^{-3} \times (0.05 - 0.0867) \times (0.195 - 0.1317) \\ &\quad + 2 \times 10^{-3} \times (0.105 - 0.0867) \times (0.1 - 0.1317) \\ &= -2.323 \times 10^{-6} - 1.16 \times 10^{-6} \end{aligned}$$

i.e. $I_{xy} = \mathbf{-3.483 \times 10^{-6} \text{ m}^4}$

$$\begin{aligned} \tan 2\theta &= \frac{2I_{xy}}{I_y - I_x} = \frac{2 \times -3.483 \times 10^{-6}}{2.849 \times 10^{-6} - 1.27 \times 10^{-5}} \\ &= 0.707 \end{aligned}$$

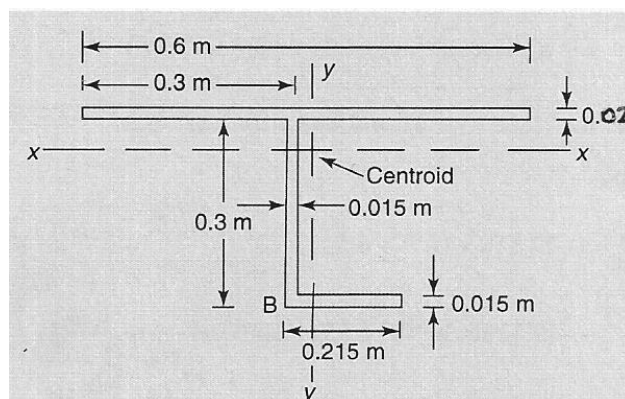
from which, $\theta = \tan^{-1} 0.707 = \mathbf{17.64^\circ}$

$$\begin{aligned} I_{UU} &= \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y) \sec 2\theta \\ &= 7.775 \times 10^{-6} + 4.9255 \times 10^{-6} \times 1.225 \end{aligned}$$

i.e. $I_{UU} = \mathbf{1.381 \times 10^{-5} \text{ m}^4}$

Similarly, $I_{VV} = \mathbf{1.741 \times 10^{-6} \text{ m}^4}$

2. Determine the direction and magnitudes of the principal second moments of area of the section shown.



Section	a	y	ay	ay^2	i_H
1	0.012	0.31	3.72×10^{-3}	1.153×10^{-3}	4×10^{-7}
2	4.5×10^{-3}	0.15	6.75×10^{-4}	1.0125×10^{-4}	3.375×10^{-5}
3	3×10^{-3}	7.5×10^{-3}	2.25×10^{-5}	1.6875×10^{-7}	5.625×10^{-8}
Σ	0.0195	–	4.418×10^{-3}	1.255×10^{-3}	3.42×10^{-5}

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{4.418 \times 10^{-3}}{0.0195} = 0.227 \text{ m}$$

$$I_{xx} = \sum ay^2 + \sum i_H = 1.255 \times 10^{-3} + 3.42 \times 10^{-5} = 1.2892 \times 10^{-3} \text{ m}^4$$

$$I_{xx} = I_{xx} - (\bar{y})^2 \sum a = 1.2892 \times 10^{-3} - (0.227)^2 \times 0.0195 = \mathbf{2.844 \times 10^{-4} \text{ m}^4}$$

Section	a	x	ax	ax^2	i_V
1	0.012	0.3	3.6×10^{-3}	1.08×10^{-3}	3.6×10^{-4}
2	4.5×10^{-3}	0.3	1.35×10^{-3}	4.05×10^{-4}	8.44×10^{-8}
3	3×10^{-3}	0.408	1.224×10^{-3}	4.99×10^{-4}	1×10^{-5}
Σ	0.0195	–	6.174×10^{-3}	1.984×10^{-3}	3.701×10^{-4}

$$\bar{x} = \frac{\sum ax}{\sum a} = \frac{6.174 \times 10^{-3}}{0.0195} = 0.3166 \text{ m}$$

$$I_{yy} = \sum ax^2 + \sum i_V = 1.984 \times 10^{-3} + 3.701 \times 10^{-4} = 2.354 \times 10^{-3} \text{ m}^4$$

$$I_{yy} = I_{yy} - (\bar{x})^2 \sum a = 2.354 \times 10^{-3} - (0.3166)^2 \times 0.0195 = \mathbf{3.994 \times 10^{-4} \text{ m}^4}$$

$$\begin{aligned}
I_{xy} &= \sum Ah\bar{k} = 0.012 \times (0.3 - 0.3166) \times (0.31 - 0.227) \\
&\quad + 4.5 \times 10^{-3} \times (0.3 - 0.3166) \times (0.15 - 0.227) \\
&\quad + 3 \times 10^{-3} \times (0.408 - 0.3166) \times (7.5 \times 10^{-3} - 0.227) \\
&= -1.654 \times 10^{-5} + 5.752 \times 10^{-6} - 6.02 \times 10^{-8}
\end{aligned}$$

i.e. $I_{xy} = \mathbf{-7.098 \times 10^{-5} \text{ m}^4}$

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{-1.4194 \times 10^{-4}}{1.15 \times 10^{-4}}$$

$$= -1.2340$$

from which, $2\theta = \tan^{-1}(-1.2340) = -50.99^\circ$

and $\theta = -25.49^\circ$

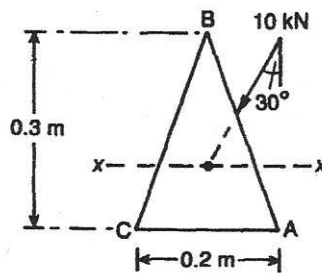
$$I_{UU} = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\sec(-50.99^\circ)$$

$$= 3.419 \times 10^{-4} - 1.15 \times 10^{-4} \times 1.589$$

i.e. $I_{UU} = 1.592 \times 10^{-4} \text{ m}^4$

Similarly, $I_{VV} = 5.246 \times 10^{-4} \text{ m}^4$

3. Determine the stresses at the corners and the maximum deflection of a cantilever of length 3 m, loaded at its free end with a concentrated load of 10 kN, as shown.



$$I_{yy} = \frac{BH^3}{36} = \frac{0.2 \times 0.3^3}{36} = 1.5 \times 10^{-4} \text{ m}^4$$

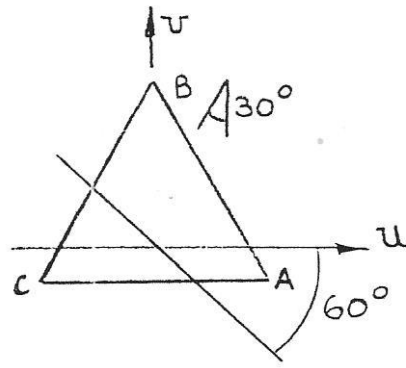
$$I_{vv} = \frac{2 \times 0.3 \times 0.1^3}{12} = 5 \times 10^{-5} \text{ m}^4$$

$$M_{max} = 10 \times 3 = 30 \text{ kN m}$$

$$\theta = 0 \text{ as } I_{xy} = 0$$

$$\beta = \tan^{-1}(-I_{uu} \times \tan \alpha / I_{vv})$$

$$\beta = -60^\circ$$



θB

$$u_B = 0 \quad v_B = 0.2 \text{ m}$$

At C, $u_C = -0.1 \quad v_C = -0.1$

$$\sigma_B = \frac{(M \cos \alpha) v}{I_{UU}} + \frac{(M \sin \alpha) u}{I_{VV}}$$

$$= + 30 \times 10^3 (1154.7 + 0) = + \mathbf{34.64 \text{ MPa}}$$

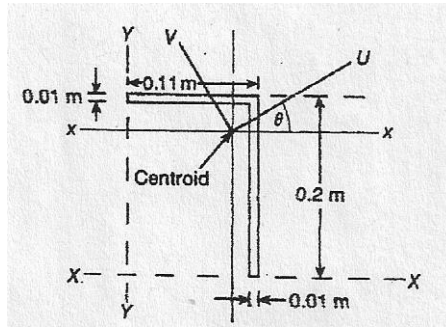
$$\sigma_c = + 30 \times 10^3 (-577.35 - 1000) = - \mathbf{47.32 \text{ MPa}}$$

$$u_A = 0.1 \quad v_A = -0.1$$

$$\sigma_A = 30 \times 10^3 (-577.35 + 1000) = \mathbf{12.68 \text{ MPa}}$$

EXERCISE 59, Page 341

1. If a cantilever of length 3 m, and with a cross-section as shown below, is subjected to a vertically applied downward load at its free end, of magnitude 4 kN, determine the position and value of the maximum bending stress.



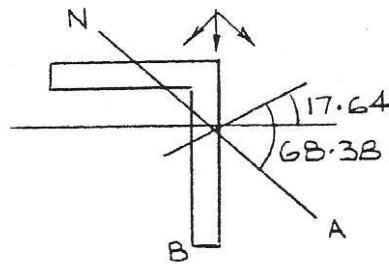
To find NA

From Problem 1, Exercise 58,

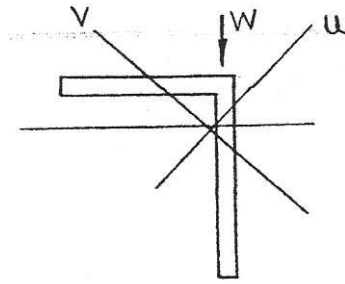
$$\begin{aligned}\beta &= \tan^{-1} (-I_{uu} \times \tan \theta / I_{vv}) \\ &= \tan^{-1} (-1.381 \times 10^{-5} \times \tan (17.64) / 1.741 \times 10^{-6})\end{aligned}$$

i.e.

$$\beta = -68.38^\circ$$



$$\begin{aligned}u_B &= x_B \cos \theta + y_B \sin \theta \\ &= (0.1 - 0.0867) \cos 17.64^\circ - 0.1317 \sin 17.64^\circ \\ &= 0.0127 - 0.03991 = -0.0272 \text{ m} \\ v_B &= y_B \cos \theta - x_B \sin \theta \\ &= -0.1255 - 4.03 \times 10^{-3} \\ &= -0.1295 \text{ m}\end{aligned}$$



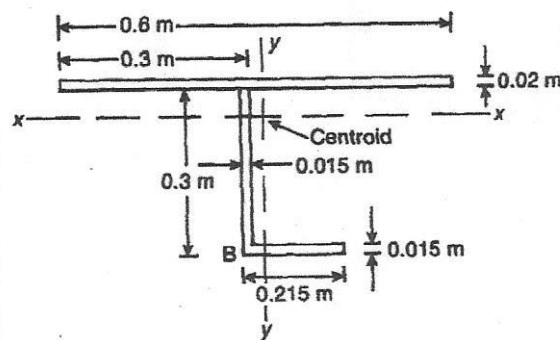
$$M = 4 \times 3 = 12 \text{ kN m}$$

$$\begin{aligned}\sigma &= \frac{(12 \cos \theta) v_B}{I_{UU}} + \frac{(12 \sin \theta) u_B}{I_{VV}} \\ &= \frac{12 \times 10^3 \times 0.953 \times (-0.1295)}{1.381 \times 10^{-5}} + \frac{12 \times 10^3 \times 0.303 \times (-0.027)}{1.741 \times 10^{-6}} \\ &= -107.24 - 56.39\end{aligned}$$

i.e.

$$\sigma_B = -163.63 \text{ MPa}$$

2. A simply supported beam of length 4 m, and with a cross-section as shown below, is subjected to a centrally placed concentrated load of 20 kN, acting perpendicularly to the xx axis, and through the centroid of the beam. Determine the stress at mid-span at the point B in the cross-section of the beam.



From Problem 2, Exercise 58,

$$\begin{aligned}\beta &= \tan^{-1} (-I_{uu} \tan \theta / I_{vv}) \\ &= \tan^{-1} (-1.592 \times 10^{-4} \tan (-25.49) / 5.246 \times 10^{-4})\end{aligned}$$

i.e.

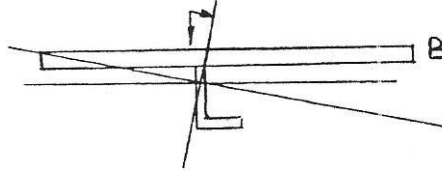
$$\beta = \tan^{-1} (0.1447) = 8.23^\circ$$

$$u_B = (0.6 - 0.3166) \cos (-25.49) + (0.32 - 0.227) \sin (-25.49)$$

$$= 0.2558 - 0.04 = \mathbf{0.2158 \text{ m}}$$

$$v_B = (0.32 - 0.227) \cos(-25.49) - (0.6 - 0.3166) \sin(-25.49)$$

$$= 0.084 + 0.122 = \mathbf{0.206 \text{ m}}$$



$$\sigma_B = - \left[\frac{20 \times 10^3 \times \cos(-25.49) \times 0.206}{1.592 \times 10^{-4}} + \frac{20 \times 10^3 \times \sin(-25.49) \times 0.2158}{5.246 \times 10^{-4}} \right],$$

a minus sign because the beam is sagging in mid-span

$$= - [23.36 - 3.54]$$

i.e.

$$\sigma_B = \mathbf{-19.82 \text{ MPa}}$$

3. Determine the components of deflection under the load in the directions of the principal axes of bending for the beam of Problem 1 above, given that $E = 2 \times 10^{11} \text{ N/m}^2$.

$$\delta_u = \frac{-W \sin \theta l^3}{3EI_v} = \frac{-4 \times 10^3 \times \sin(17.64) \times 3^3}{3 \times 2 \times 10^{11} \times 1.741 \times 10^{-6}}$$

i.e.

$$\delta_u = \mathbf{-0.031 \text{ m}}$$

$$\delta_v = \frac{-W \cos \theta l^3}{3EI_u} = \frac{-4 \times 10^3 \times \cos(17.64) \times 3^3}{3 \times 2 \times 10^{11} \times 1.381 \times 10^{-5}}$$

i.e.

$$\delta_v = \mathbf{-0.0124 \text{ m}}$$

4. Determine the components of the central deflection in the directions of the principal axes of bending for the beam of Problem 2 above, given that $E = 1 \times 10^{11} \text{ N/m}^2$.

$$\delta_u = \frac{-W \sin \theta l^3}{3EI_v} = \frac{-20 \times 10^3 \times \sin(-25.49) \times 4^3}{48 \times 2 \times 10^{11} \times 5.246 \times 10^{-4}}$$

i.e. $\delta_u = 2.19 \times 10^{-4} \text{ m}$

$$\delta_v = \frac{-W \cos \theta l^3}{3EI_u} = \frac{-20 \times 10^3 \times \sin(-25.49) \times 4^3}{48 \times 2 \times 10^{11} \times 1.592 \times 10^{-4}}$$

i.e. $\delta_v = -1.51 \times 10^{-3} \text{ m}$
