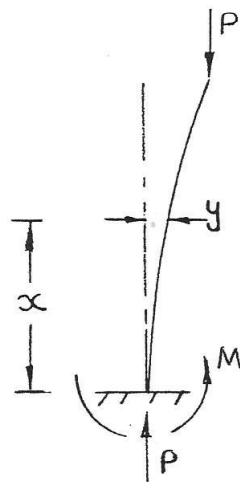
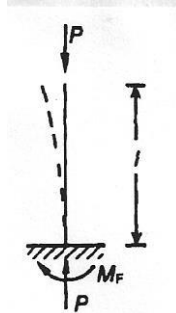


# CHAPTER 14 THE BUCKLING OF STRUTS

## EXERCISE 56, Page 316

1. Determine the Euler buckling load for the axially loaded strut shown.



$$EI \frac{d^2 y}{dx^2} = -Py + M$$

Let  $\alpha^2 = P/EI$

Then 
$$\frac{d^2 y}{dx^2} + \alpha^2 y = \frac{M}{EI}$$

The complete solution is

$$y = A \cos \alpha x + B \sin \alpha x + M/(EI\alpha^2)$$

$$\frac{dy}{dx} = -\alpha A \sin \alpha x + \alpha B \cos \alpha x$$

$$\frac{d^2 y}{dx^2} = -\alpha^2 A \cos \alpha x - \alpha^2 B \sin \alpha x$$

At  $x = 0, y = 0$

Hence,  $A = -M/(EI\alpha^2)$

At  $x = 0, \frac{dy}{dx} = 0$  hence,  $B = 0$

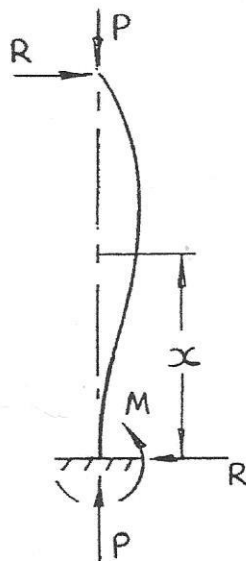
At  $x = l, \frac{d^2y}{dx^2} = 0$

Hence,  $-\alpha^2 A \cos \alpha l = 0$  or  $\cos \alpha l = 0$

i.e.  $\alpha l = \pi/2$

Hence,  $P_{cr} = \frac{\pi^2 EI}{4l^2}$

2. Determine the Euler buckling load for an initially straight axially loaded strut which is pinned at one end and fixed at the other.



$$EI \frac{d^2y}{dx^2} = -Py + R(l - x)$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \frac{R}{EI}(l - x)$$

The complete solution is  $y = A \cos \alpha x + B \sin \alpha x + \frac{R}{P}(l - x)$

At  $x = 0$ ,  $y = 0$

Therefore,  $A = -R/P$

At  $x = 0$ ,  $\frac{dy}{dx} = 0$ , therefore,  $B = R/\alpha P$

Hence, 
$$y = \frac{R}{P} \left[ -l \cos \alpha x + \frac{\sin \alpha l}{\alpha} + (l - x) \right]$$

At  $x = l$ ,  $y = 0$

Therefore, 
$$0 = \frac{R}{P} \left[ -l \cos \alpha l + \frac{\sin \alpha l}{\alpha} \right]$$

or  $\tan \alpha l = \alpha l$

Hence,  $\alpha l = 4.5 \text{ rads}$

so that 
$$P_{cr} = \frac{20.2EI}{l^2}$$

**3.** Find the Euler crushing load for a hollow cylindrical cast-iron column of 0.15 m external diameter and 20 mm thick, if it is 6 m long and hinged at both ends. Assume that  $E = 75 \times 10^9 \text{ N/m}^2$ . Compare this load with that given by the Rankine formula using constants of 540 MN/m<sup>2</sup> and 1/1600. For what length of column would these two formulae give the same crushing load?

$$I = \frac{\pi(0.15^4 - 0.11^4)}{64} = 1.766 \times 10^{-5} \text{ m}^4$$

$$A = 8.168 \times 10^{-3} \text{ m}^2$$

$$k^2 = 2.162 \times 10^{-3} \text{ and } k = 0.046 \text{ m}$$

and  $P_e = 363.1 \text{ kN}$

$$\begin{aligned} P_r &= \frac{\sigma_c A}{\left[ 1 + a \left( \frac{l}{k} \right)^2 \right]} = \frac{540 \times 10^3 \times 8.16 \times 10^{-3}}{\left[ 1 + \frac{1}{1600} \left( \frac{36}{2.162 \times 10^{-3}} \right)^2 \right]} \\ &= \frac{4410.7}{1 + 10.4} = 386.7 \text{ kN} \end{aligned}$$

$$\frac{\pi^2 EI}{l^2} = \frac{\sigma_c A}{\left[1 + \left(\frac{l}{k}\right)^2\right]}$$

$$\text{i.e.} \quad \frac{1307230}{l^2} = \frac{540 \times 10^6 \times 8.168 \times 10^{-3}}{\left[1 + \frac{1}{1600} \left(\frac{l^2}{2.162 \times 10^{-3}}\right)\right]}$$

$$\text{or} \quad 2.964 (1 + 0.289 l^2) = l^2$$

$$\text{i.e.} \quad 2.964 + 0.857 l^2 = l^2$$

$$\text{from which,} \quad l = \mathbf{4.55 \text{ m}}$$

**4.** A short steel tube of 0.1 m outside diameter, when tested in compression, was found to fail under an axial load of 800 kN. A 15 m length of the same tube when tested as a pin-jointed strut failed under a load of 30 kN. Assuming that the Euler and Rankine-Gordon formulae apply to the strut, calculate (a) the tube inner diameter, and (b) the denominator constant in the Rankine-Gordon formula. Assume that  $E = 196.5 \text{ GN/m}^2$ .

$$(a) \quad 30 \times 10^3 = \frac{\pi^2 EI}{l^2}$$

$$I = \frac{30 \times 10^3 \times 15^2}{\pi \times 196.5 \times 10^9} = \mathbf{3.48 \times 10^{-6} \text{ m}^4}$$

$$3.48 \times 10^{-6} = \frac{\pi^2 (0.1^4 - d^4)}{64}$$

$$7.09 \times 10^{-5} - 0.1^4 = -d^4$$

$$\text{i.e. the tube inner diameter, } d = \sqrt[4]{2.91 \times 10^{-5}} = \mathbf{0.0734 \text{ m}}$$

$$(b) A = 3.617 \times 10^{-3} \text{ m}^2 \quad \text{and} \quad k^2 = 9.62 \times 10^{-4}$$

$$P_r = \frac{800 \times 10^3}{\left[ 1 + a \left( \frac{l}{k} \right)^2 \right]}$$

$$l + a \frac{15^2}{9.62 \times 10^{-4}} = 26.67$$

and  $a = 25.67 \times \frac{9.62 \times 10^{-4}}{152} = 1.097 \times 10^{-4}$

**The denominator constant in the Rankine-Gordon formula is:**

$$\frac{1}{a} = \frac{1}{1.097 \times 10^{-4}} = \mathbf{9117}$$

**5.** A steel pipe of 36 mm diameter, 6 mm thick and 1 m long is supported so that the ends are hinged, but all expansion is prevented. The pipe is unstressed at 0 °C. Calculate the temperature at which buckling will occur. Assume the following:  $\sigma_c = 325 \text{ MN/m}^2$ ,  $a = 1/7500$ ,  $E = 200 \text{ GN/m}^2$  and  $\alpha = 11.1 \times 10^{-6} / ^\circ\text{C}$ .

$$I = \frac{\pi}{64} \left[ (36 \times 10^{-3})^4 - (24 \times 10^{-3})^4 \right]$$

i.e.  $I = 6.61 \times 10^{-8} \text{ m}^4$

$$A = 5.655 \times 10^{-4} \text{ m}^2 \quad \text{and} \quad k^2 = 1.17 \times 10^{-4}$$

$$\sigma = \alpha l T E$$

$$P_R = \alpha l T E A = \mathbf{1255.4 \text{ T}} \tag{1}$$

$$P_R = \frac{325 \times 10^6 \times 5.655 \times 10^{-4}}{\left( 1 + \frac{1}{7500} \left( \frac{1^2}{1.17 \times 10^{-4}} \right) \right)} = \frac{183.79 \times 10^{-3}}{2.1397}$$

$$= \mathbf{85696 \text{ N}} \tag{2}$$

Equating equation (1) and equation (2) gives:

i.e. **the temperature at which buckling will occur,  $T = 85696/1255.4 = 68.4^\circ\text{C}$**

6. The table below shows the results of a series of buckling tests carried out on a steel tube of external diameter 35 mm and internal diameter 25 mm. Assuming the Rankine-Gordon formula to apply, determine the numerator and denominator constants for this tube.

$l$ (mm)	600	1000	1400	1800
$P_R$ (kN)	150	125	110	88

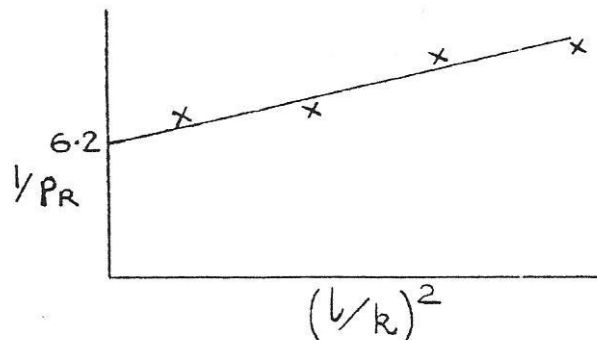
$$I = \frac{\pi \left( (35 \times 10^3)^4 - (25 \times 10^3)^4 \right)}{64} = 5.449 \times 10^{-8} \text{ m}^4$$

$$A = 4.712 \times 10^{-4} \text{ m}^2$$

$$k^2 = 1.156 \times 10^{-4} \text{ m}^2 = 115.6 \text{ mm}^2$$

$l$ (mm)	600	1000	1400	1800
$(l/k)^2$	3113	8650	16950	28020
$1/P_R$	6.667	8	9.09	11.36

Plotting  $1/P_R$  against  $(l/k)^2$  gives the following graph:



From the graph,

$$\text{intercept} = \frac{1}{\sigma_c A} = 6.2 \quad \text{and} \quad \frac{1}{P_R} = 6.2 \quad \text{hence,} \quad P_R = 0.162$$

from which,  $\sigma_c = \frac{0.162}{A} = 344 \text{ MN/m}^2$  = the numerator constant of the Rankine-Gordon formula

Also from graph,

$$1 + a \left( \frac{l}{k} \right)^2 = \frac{\sigma_c A}{P_R}$$

From the graph,

$$\left( \frac{l}{R} \right)^2 = 30000$$

$$1/P_R = 11.25$$

and

$$P_R = 0.0889$$

$$a \times 30000 = \left( \frac{345 \times 4.712 \times 10^{-4}}{0.0889} \right) - 1$$

$$= 0.829$$

and  $a = 0.829/30000 = \frac{1}{36200}$  = the denominator constant of the Rankine-Gordon formula

**7.** The result of two tests on steel struts with pinned ends were found to be:

Test number	1	2
Slenderness ratio	50	80
Average stress at failure (MN/m <sup>2</sup> )	266.7	194.4
$A = 1 \text{ m}^2$		

(a) Assuming that the Rankine-Gordon formula applies to both struts, determine the numerator and denominator constants of the Rankine-Gordon formula.

(b) If a steel bar of rectangular section  $0.06 \text{ m} \times 0.019 \text{ m}$  and of length  $0.4 \text{ m}$  is used as a strut with both ends clamped, determine the safe load using the constants derived in (a) and employing a safety factor of 4.

(a) 
$$P_R = \frac{\sigma_c A}{\left[ 1 + a \left( \frac{l}{k} \right)^2 \right]}$$

or 
$$\frac{P_R}{A} = \frac{\sigma_c}{\left[ 1 + a \left( \frac{l}{k} \right)^2 \right]}$$

Let  $\sigma_c = 350 \text{ MN/m}^2$

For  $\left(\frac{l}{k}\right) = 50$  and  $a = 1/8000$

$$(b) \quad \frac{P_R}{A} = \frac{350}{\left[1 + \frac{1}{8000}(50)^2\right]} = 266.7$$

For  $\left(\frac{l}{k}\right) = 80$

$$\frac{P_R}{A} = \frac{350}{\left[1 + \frac{1}{8000}(80)^2\right]} = 194.4$$

$$A = 0.06 \times 0.019 = 1.14 \times 10^{-3} \text{ m}^2$$

$$I = 3.4295 \times 10^{-8}$$

$$k = 5.485 \times 10^{-3}$$

$$\frac{l}{k} = 72.93$$

$$P_R = \frac{350 \times 1.14 \times 10^{-3}}{\left[1 + \frac{1}{4} \times \frac{1}{8000} \times 5318.6\right]}$$

i.e.  $P_R = 342 \text{ kN}$

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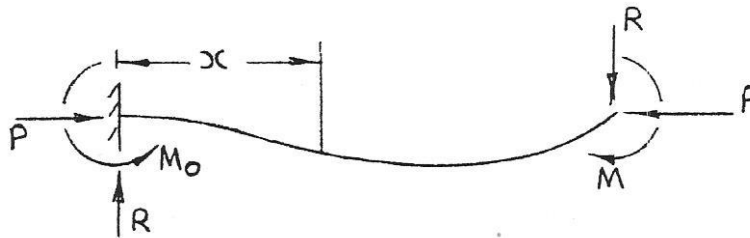


### EXERCISE 57, Page 324

1. A long slender strut of length  $L$  is encastred at one end and pin jointed at the other. At its pinned end, it carries an axial load  $P$ , together with a couple  $M$ . Show that the magnitude of the couple at the clamped end is given by the expression

$$M \left( \frac{\alpha L - \sin \alpha L}{\alpha L \cos \alpha L - \sin \alpha L} \right)$$

Determine the value of this couple if  $P$  is one quarter of the Euler buckling load for this class of strut.



$$EI \frac{d^2 y}{dx^2} = -Py + M_o - Rx$$

$$y = A \cos \alpha x + B \sin \alpha x + \frac{M_o}{\alpha^2 EI} - \frac{Rx}{\alpha^2 EI}$$

$$\frac{dy}{dx} = -\alpha A \sin \alpha x + \alpha B \cos \alpha x - \frac{R}{\alpha^2 EI}$$

$$\frac{d^2 y}{dx^2} = -\alpha^2 A \cos \alpha x - \alpha^2 B \sin \alpha x$$

At  $x = 0$ ,  $y = 0$ , hence,  $A = -M_o/(\alpha^2 EI)$

At  $x = 0$ ,  $\frac{dy}{dx} = 0$ , hence,  $\alpha B = R/(\alpha^2 EI)$

or  $B = R/(\alpha^3 EI)$

At  $x = l$ ,  $y = 0$ , hence,  $0 = A \cos \alpha l + B \sin \alpha l + M_o/(\alpha^2 EI) - Rl/(\alpha^2 EI)$

i.e.  $0 = -M_o/(\alpha^2 EI) \cos \alpha l + R/(\alpha^3 EI) \sin \alpha l + M_o/(\alpha^2 EI) - Rl/(\alpha^2 EI)$

i.e.  $0 = (M_o/\alpha^2 EI) (1 - \cos \alpha l) + R/(\alpha^2 EI) \left( \frac{\sin \alpha l}{\alpha} - l \right)$

from which, 
$$R = \frac{-M_o (1 - \cos \alpha l)}{\left( \frac{\sin \alpha l}{\alpha} - l \right)} \quad (1)$$

Taking moments about the left end gives:

$$M + Rl = M_o$$

or 
$$R = \frac{M_o - M}{l} \quad (2)$$

Equating equations (1) and (2) gives:

$$\frac{M_o - M}{l} = \frac{-M_o (1 - \cos \alpha l)}{\left( \frac{\sin \alpha l}{\alpha} - l \right)}$$

$$M_o - M = \frac{-M_o l (1 - \cos \alpha l)}{\left( \frac{\sin \alpha l}{\alpha} - l \right)}$$

$$M = \frac{M_o l (1 - \cos \alpha l)}{\left( \frac{\sin \alpha l}{\alpha} - l \right)} + M_o$$

$$= \frac{M_o \left[ l (1 - \cos \alpha l) + \left( \frac{\sin \alpha l}{\alpha} - l \right) \right]}{\left( \frac{\sin \alpha l}{\alpha} - l \right)}$$

$$= \frac{M_o \left( -l \cos \alpha l + \frac{\sin \alpha l}{\alpha} \right)}{\left( \frac{\sin \alpha l}{\alpha} - l \right)} \times \frac{\alpha}{\alpha}$$

$$= \frac{M_o (-\alpha l \cos \alpha l + \sin \alpha l)}{(\sin \alpha l - \alpha l)}$$

or 
$$M_o = \frac{M (\alpha l - \sin \alpha l)}{(\alpha l \cos \alpha l - \sin \alpha l)}$$

$$P_{cr} = \frac{20.25EI}{l^2}$$

Therefore,

$$P = \frac{5.063EI}{l^2}$$

$$\alpha^2 = \frac{P}{EI} = \frac{5.063}{l^2}$$

and

$$\alpha = \frac{2.25}{l}$$

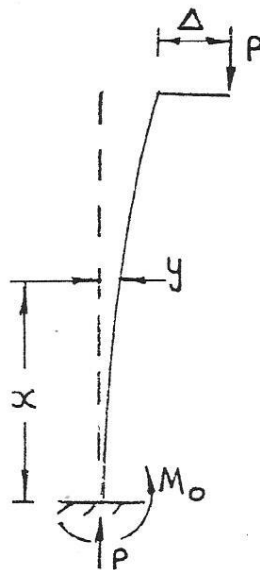
$$M_o = \frac{M(2.25 - 0.778)}{2.25 \times (-0.628) - 0.778}$$

$$= \frac{-1.472M}{2.191}$$

i.e.

$$M_o = -0.672m$$

**2.** A long strut, initially straight, securely fixed at one end and free at the other, is loaded at the free end with an eccentric load whose line of action is parallel to the original axis. Deduce an expression for the deviation of the free end from its original position.



$$EI \frac{d^2 y}{dx^2} = -Py + M_0$$

Let  $\alpha^2 = P/EI$

Then 
$$\frac{d^2 y}{dx^2} + \alpha^2 y = \alpha^2 \frac{M_0}{P}$$

The complete solution is

$$y = A \cos \alpha x + B \sin \alpha x + \frac{M_0}{P}$$

$$\frac{dy}{dx} = -\alpha A \sin \alpha x + \alpha B \cos \alpha x$$

$$\frac{d^2 y}{dx^2} = -\alpha^2 A \cos \alpha x - \alpha^2 B \sin \alpha x$$

At  $x = 0, y = 0$

hence, 
$$0 = A + \frac{M_0}{P}$$

i.e. 
$$A = \frac{-M_0}{P}$$

At  $x = 0, \frac{dy}{dx} = 0$

hence, 
$$0 = \alpha B \quad \text{i.e. } B = 0$$

At  $x = l, M = P\Delta = EI \left( \frac{d^2 y}{dx^2} \right)_{x=l}$

Hence, 
$$\frac{P\Delta}{EI} = -\alpha^2 A \cos \alpha l$$

Therefore, 
$$A = -\frac{P\Delta}{EI} \times \frac{EI}{P} \sec \alpha l = -\Delta \sec \alpha l$$

and 
$$M = -P \times -\Delta \sec \alpha l$$

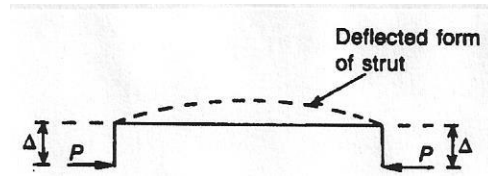
and 
$$M_0 = P\Delta \sec \alpha l$$

Hence, 
$$y = -\Delta \sec \alpha l \times \cos \alpha x + \Delta \sec \alpha l$$

**Deflection at the free end**  $= -\Delta \sec \alpha l \times \cos \alpha l + \Delta \sec \alpha l$

$$= \Delta(\sec \alpha l - 1) \quad \text{since } x = 0$$

3. A tubular steel strut of 70 mm external diameter and 50 mm internal diameter is 3.25 m long. The line of action of the compressive forces is parallel to, but eccentric from, the axis of the tube, as shown below.



Find the maximum allowable eccentricity of these forces if the maximum permissible deflection (total) is not greater than 15 mm. Assume that:  $E = 2 \times 10^{11} \text{ N/m}^2$  and  $P = 114.7 \text{ kN}$

Let  $\Delta$  = eccentricity

$$EI \frac{d^2 y}{dx^2} = -P(y + \Delta)$$

Then

$$\frac{d^2 y}{dx^2} + \frac{P}{EI} y = -\frac{P}{EI} \Delta$$

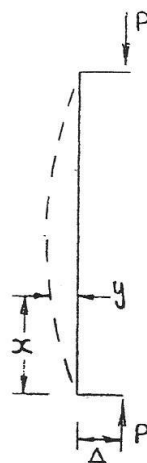
Let  $\alpha^2 = P/EI$

then

$$\frac{d^2 y}{dx^2} + \alpha^2 y = -\alpha^2 \Delta$$

The complete solution is

$$y = A \cos \alpha x + B \sin \alpha x - \Delta$$



At  $x = 0, y = 0$

hence,  $0 = A - \Delta$  from which,  $A = \Delta$

At  $x = l$ ,  $y = 0$

i.e.  $0 = \Delta \cos \alpha l + B \sin \alpha l - \Delta$

from which, 
$$B = \frac{\Delta(1 - \cos \alpha l)}{\sin \alpha l} = \frac{\Delta 2 \sin^2 \frac{\alpha l}{2}}{2 \sin \frac{\alpha l}{2} \cos \frac{\alpha l}{2}} = \Delta \tan \frac{\alpha l}{2} \quad \text{from double angles}$$

i.e. 
$$y = \Delta \left[ \cos \alpha x + \tan \frac{\alpha l}{2} \sin (\alpha x) - 1 \right]$$

The maximum deflection  $\delta$  occurs at  $x = l/2$ ,

i.e. 
$$\delta = \Delta \left( \cos \frac{\alpha l}{2} + \tan \frac{\alpha l}{2} \sin \frac{\alpha l}{2} - 1 \right)$$

$$= \Delta \cos \frac{\alpha l}{2} \left( 1 + \tan^2 \frac{\alpha l}{2} - \frac{1}{\cos \frac{\alpha l}{2}} \right)$$

$$= \Delta \cos \frac{\alpha l}{2} \left( \sec^2 \frac{\alpha l}{2} - \frac{1}{\cos \frac{\alpha l}{2}} \right)$$

i.e. 
$$\delta = \Delta \left( \sec \frac{\alpha l}{2} - 1 \right)$$

The maximum bending moment =  $M_{max}$

where  $M_{max} = P(\delta + \Delta)$

or 
$$M_{max} = P \Delta \sec \left( \sqrt{\frac{P}{EI}} \frac{l}{2} \right)$$

Second moment of area,  $I = \frac{\pi(70^4 - 50^4)}{64} = 871790 \text{ mm}^4$

$EI = 1.744 \times 10^{11}$

and  $\alpha = \sqrt{\frac{114.7 \times 10^3}{1.744 \times 10^{11}}} = 8.111 \times 10^{-4}$

Also,  $\frac{\alpha l}{2} = \frac{8.111 \times 10^{-4} \times 3.25}{2} = 1.318 \times 10^{-3}$   
 $\cos \frac{\alpha l}{2} = 0.25$

and  $\sec \frac{\alpha l}{2} = 1/0.25 = 4$

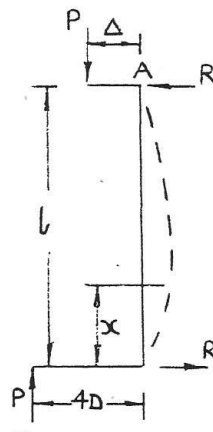
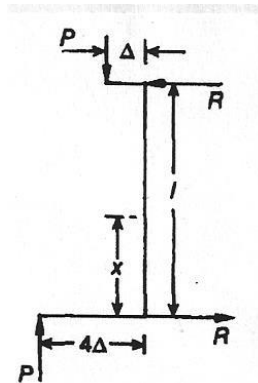
Now,  $\delta = \Delta \left( \sec \frac{\alpha l}{2} - 1 \right)$

hence,  $15 = \Delta(4 - 1)$

from which, **maximum eccentricity,  $\Delta = 5 \text{ mm}$**

4. The eccentrically loaded strut shown is subjected to a compressive load  $P$ . If  $EI = 20\,000 \text{ Nm}^2$ , determine the position and value of the maximum deflection assuming the following data apply:

$P = 5000 \text{ N}$ ,  $l = 3 \text{ m}$  and  $\Delta = 0.01 \text{ m}$



Taking moments about A gives:

$$P\Delta + Rl = P \times 4\Delta$$

from which,

$$R = 3 P\Delta/l$$

$$\begin{aligned} EI \frac{d^2 y}{dx^2} &= -P(4\Delta + y) + Rx \\ &= -P(4\Delta + y) + (3P\Delta/l)x \\ &= -P\Delta \left( 4 - \frac{3x}{l} \right) - Py \end{aligned}$$

Let  $\alpha^2 = P/EI$

then 
$$\frac{d^2 y}{dx^2} + \alpha^2 y = -\alpha^2 \Delta \left( 4 - \frac{3x}{l} \right)$$

i.e. 
$$y = A \cos \alpha x + B \sin \alpha x - \Delta \left( 4 - \frac{3x}{l} \right) \quad (1)$$

and 
$$\frac{dy}{dx} = -\alpha A \sin \alpha x + \alpha B \cos \alpha x + \frac{3\Delta}{l}$$

At  $x = 0$ ,  $y = 0$

hence, 
$$0 = A \cos 0 + B \sin 0 - \Delta(4 - 0)$$

from which, 
$$A = 4\Delta$$

At  $x = l$ ,  $y = 0$

hence, 
$$0 = A \cos \alpha l + B \sin \alpha l - \Delta \left( 4 - \frac{3l}{l} \right)$$

i.e. 
$$0 = A \cos \alpha l + B \sin \alpha l - \Delta$$

from which, 
$$\begin{aligned} B &= \frac{\Delta - A \cos \alpha l}{\sin \alpha l} \\ &= \frac{\Delta - 4\Delta \cos \alpha l}{\sin \alpha l} = \frac{\Delta(1 - 4 \cos \alpha l)}{\sin \alpha l} \end{aligned}$$

Hence, from equation (1),

$$y = 4\Delta \cos \alpha x + \Delta (1 - 4 \cos \alpha l) \sin \alpha x / \sin \alpha l - \Delta (4 - 3x/l)$$

and 
$$\frac{dy}{dx} = -4\alpha\Delta \sin \alpha x + \alpha\Delta (1 - 4 \cos \alpha l) \cos \alpha x / \sin \alpha l + 3\Delta/l$$



For maximum  $y$ ,  $\frac{dy}{dx} = 0$

Hence,  $0 = -4\alpha\Delta \sin \alpha x + \alpha\Delta (1 - 4 \cos \alpha l) \cos \alpha x / \sin \alpha l + 3\Delta l$

Now,  $\alpha = \sqrt{\frac{P}{EI}} = \sqrt{\frac{5000}{20000}}$

or  $\alpha = 0.5$

To calculate  $x$ :

Try  $x = 1.5 \text{ m}$ , therefore,  $\alpha x = 42.97^\circ$  and  $\alpha l = 85.94^\circ$

Substituting these values gives:

$$0 = -4 \times 0.5 \times 0.682 + 0.5 (1 - 0.2829) \times 0.732 / 0.997 + 1$$

or  $0 = -1.364 + 0.263 + 1 = -0.101$  which is incorrect

Try  $x = 1.4 \text{ m}$ , therefore,  $\alpha x = 40.11^\circ$

or  $0 = -1.288 + 0.3586 \times 0.764 / 0.997 + 1 = -0.0129$

Try  $x = 1.35 \text{ m}$ , therefore,  $\alpha x = 38.67^\circ$

or  $0 = -1.2498 + 0.3586 \times 0.78 / 0.997 + 1 = +0.003$

Try  $x = 1.38 \text{ m}$ , therefore,  $\alpha x = 39.53^\circ$

or  $0 = -1.273 + 0.3596 \times 0.771 + 1 = 4.356 \times 10^{-3}$

i.e.  $x = 1.38 \text{ m}$

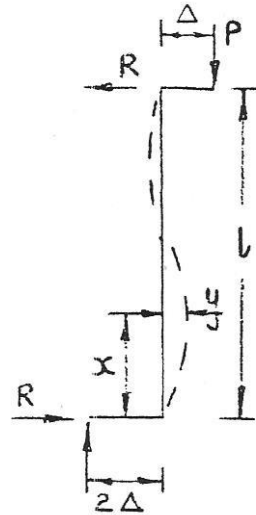
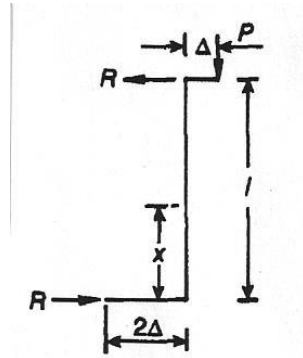
and  $\delta = 4 \times 0.01 \times 0.771 + 0.01 \times 0.719 \times 0.636 - 0.01 (4 - 1.38)$

$$= 0.03084 + 4.573 \times 10^{-3} - 0.0262$$

i.e.  $\delta = 9.21 \times 10^{-3} \text{ m}$

5. Show that for the eccentrically loaded strut shown the bending moment at any distance  $x$  is given

by: 
$$M = P\Delta \left( -2 \cos \alpha x + \frac{(1 + 2 \cos \alpha l)}{\sin \alpha l} \sin \alpha x \right)$$



$$Rl = 3P\Delta$$

i.e. 
$$R = \frac{3P\Delta}{l}$$

At distance  $x$ ,

$$EI \frac{d^2 y}{dx^2} = -P(y + 2\Delta) + Rx$$

It may be shown that the complete solution is:

$$y = A \cos \alpha x + B \sin \alpha x + \frac{Rx}{P} - 2\Delta \quad (1)$$

At  $x = 0, y = 0$  i.e.  $A = 2\Delta$

At  $x = l, y = 0$

i.e. 
$$0 = 2\Delta \cos \alpha l + B \sin \alpha l + \frac{3P\Delta l}{P} - 2\Delta$$

i.e. 
$$0 = 2\Delta \cos \alpha l + B \sin \alpha l + \Delta$$

from which,  $B = \frac{-\Delta(1+2\cos \alpha l)}{\sin \alpha l}$

Hence, from equation (1),

$$y = 2\Delta \cos \alpha x - \frac{\Delta(1+2\cos \alpha l)}{\sin \alpha l} \sin \alpha x + \frac{\left(\frac{3P\Delta}{l}\right)x}{P} - 2\Delta$$

i.e.  $y = 2\Delta \cos \alpha x - \frac{\Delta(1+2\cos \alpha l)}{\sin \alpha l} \sin \alpha x + \frac{3\Delta x}{l} - 2\Delta$

Then  $\frac{dy}{dx} = \Delta \left[ -2\alpha \sin \alpha x - \alpha \frac{(1+2\cos \alpha l)}{\sin \alpha l} \cos \alpha x + \frac{3}{l} \right]$

and  $\frac{d^2y}{dx^2} = \Delta \alpha^2 \left[ -2\cos \alpha x + \frac{(1+2\cos \alpha l)}{\sin \alpha l} \sin \alpha x \right]$

$$M = EI \frac{d^2y}{dx^2} = P\Delta \left[ -2\cos \alpha x + \frac{(1+2\cos \alpha l)}{\sin \alpha l} \sin \alpha x \right]$$

**6.** An initially curved strut, whose initial deflected form is small and parabolic, is symmetrical about its mid-point. If the strut is subjected to a compressive axial load  $P$  at its pinned ends, show that the maximum compressive stress is given by:

$$\frac{P}{A} \left[ 1 + \frac{\Delta y}{k^2} \frac{8EI}{Pl^2} \left( \sec \frac{\alpha l}{2} - 1 \right) \right]$$

where  $\Delta$  = initial central deflection and  $k$  = least radius of gyration

Determine  $\Delta$  for such a strut, assuming the geometrical and material properties of worked problem 6 apply.

Let  $y_0 = \frac{4\Delta x(l-x)}{l^2} = \frac{4\Delta x}{l^2} - \frac{4x^2\Delta}{l^2}$

then  $\frac{dy_0}{dx} = \frac{4\Delta}{l^2} - \frac{8x\Delta}{l^2}$

and 
$$\frac{d^2 y_0}{dx^2} = -\frac{8\Delta}{l^2}$$

Now, 
$$\frac{d^2 y}{dx^2} = +\frac{d^2 y_0}{dx^2} - \alpha^2 y$$

The complete solution to this is

$$y = A \cos \alpha x + B \sin \alpha x - \frac{8\Delta}{\alpha^2 l^2} \quad (1)$$

and 
$$\frac{dy}{dx} = -\alpha A \sin \alpha x + \alpha B \cos \alpha x$$

At  $x = 0$ ,  $y = 0$ , i.e.  $0 = A - \frac{8\Delta}{\alpha^2 l^2}$

i.e. 
$$A = \frac{8\Delta}{\alpha^2 l^2}$$

At  $x = l/2$ ,  $\frac{dy}{dx} = 0$ ,

hence, 
$$0 = -\alpha \left( \frac{8\Delta}{\alpha^2 l^2} \right) \sin \frac{\alpha l}{2} + \alpha B \cos \frac{\alpha l}{2}$$

from which, 
$$B = \frac{\left( \frac{8\Delta}{\alpha^2 l^2} \right) \sin \frac{\alpha l}{2}}{\cos \frac{\alpha l}{2}} = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \tan \frac{\alpha l}{2}$$

From equation (1),

$$y = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \cos \alpha x + \left( \frac{8\Delta}{\alpha^2 l^2} \right) \tan \frac{\alpha l}{2} \sin \alpha x - \frac{8\Delta}{\alpha^2 l^2}$$

The maximum deflection occurs at  $x = l/2$ ,

i.e. 
$$\delta = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \left[ \cos \frac{\alpha l}{2} + \frac{\sin^2 \frac{\alpha l}{2}}{\cos \frac{\alpha l}{2}} - 1 \right]$$

$$\begin{aligned}
&= \left( \frac{8\Delta}{\alpha^2 l^2} \right) \left[ \cos \frac{\alpha l}{2} + \frac{\left( 1 - \cos^2 \frac{\alpha l}{2} \right)}{\cos \frac{\alpha l}{2}} - 1 \right] \\
&= \left( \frac{8\Delta}{\alpha^2 l^2} \right) \left[ \cos \frac{\alpha l}{2} + \frac{1}{\cos \frac{\alpha l}{2}} - \cos \frac{\alpha l}{2} - 1 \right]
\end{aligned}$$

i.e.  $\delta = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \left[ \sec \frac{\alpha l}{2} - 1 \right] \quad (2)$

$$M_{\max} = P\delta$$

Therefore,  $\sigma_b = \frac{P\delta \bar{y}}{Ak^2}$

$$\sigma_{(direct)} = \frac{P}{A}$$

Hence,  $\sigma_{\max} = \frac{P}{A} \left( 1 + \frac{\delta \bar{y}}{k^2} \right)$

$$= \frac{P}{A} \left( 1 + \left( \frac{8\Delta \bar{y}}{\alpha^2 l^2 k^2} \right) \left[ \sec \frac{\alpha l}{2} - 1 \right] \right) \quad \text{from equation (2)}$$

i.e.  $\sigma_{\max} = \frac{P}{A} \left( 1 + \frac{\Delta \bar{y}}{k^2} \left( \frac{8EI}{Pl^2} \right) \left[ \sec \frac{\alpha l}{2} - 1 \right] \right) \quad \text{since } \alpha^2 = P/EI$

Now,  $P_e = \pi^2 EI/l^2$

Therefore,  $\sigma_{\max} = \frac{P}{A} \left( 1 + \frac{\Delta \bar{y}}{k^2} \left( \frac{8P_e}{\pi^2 P} \right) \left[ \sec \frac{\alpha l}{2} - 1 \right] \right)$

and as  $\alpha^2 = P/EI$ ,

$$\frac{\alpha l}{2} = \frac{1}{2} \sqrt{\frac{Pl^2}{EI}} = \frac{1}{2} \sqrt{\frac{P\pi^2}{P_e}} = \frac{\pi}{2} \sqrt{\frac{P}{P_e}}$$

Therefore,  $\sigma_{\max} = \frac{P}{A} \left( 1 + \frac{\Delta \bar{y}}{k^2} \left( \frac{8P_e}{\pi^2 P} \right) \left[ \sec \left( \frac{\pi}{2} \sqrt{\frac{P}{P_e}} \right) - 1 \right] \right)$

or 
$$\sigma = \frac{P}{A} \left( 1 + \frac{\Delta \bar{y}}{k^2} \left( \frac{8EI}{Pl^2} \right) \left[ \sec \frac{\alpha l}{2} - 1 \right] \right) \quad \text{where } P_e = \pi^2 EI / l^2$$

$EI = 2 \times 10^{11} \times 6.594 \times 10^{-5} = 1.318 \times 10^7, \quad l = 10 \text{ m}, \quad \alpha l = 69.88^\circ, \quad \sigma = -75 \text{ MPa},$

$P = 0.196 \text{ MN}, \quad A = 9.327 \times 10^{-3} \text{ m}^2, \quad \bar{y} = 12.5 \text{ cm}$

Hence, 
$$\sigma = -21 \times 10^6 - \frac{0.196 \times 10^6 \times 12.5 \times 10^{-2} \times 8 \times 2 \times 10^{11} \Delta}{0.196 \times 10^6 \times 100} \left[ \sec \frac{\alpha l}{2} - 1 \right]$$

or 
$$-75 \times 10^6 = -21 \times 10^6 - 2000 \times 10^6 \cdot \Delta \times 0.2199$$

i.e. 
$$-54 \times 10^6 = -439.8 \times 10^6 \Delta$$

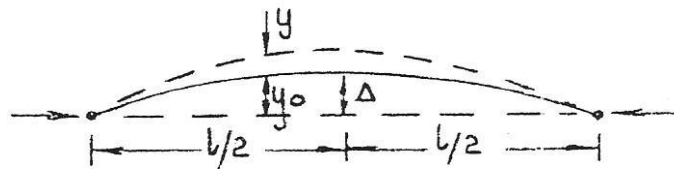
from which,  $\Delta = 0.123 \text{ m} = \mathbf{12.3 \text{ cm}}$

**7.** An initially curved strut, whose initial deflected form is small and circular, is subjected to a compressive axial load  $P$  at its pinned ends. Show that the total deflection  $y$  at any distance  $x$  is given by:

$$y = \frac{8\Delta}{\alpha^2 l^2} \left[ \cos \alpha x + \tan \frac{\alpha l}{2} \sin \alpha x - 1 \right]$$

where  $\Delta$  is the initial central deflection.

Determine  $\Delta$  for such a strut, assuming the geometrical and material properties of worked problem 6 apply.



$$\Delta(2R_o - \Delta) = \frac{l^2}{4}$$

i.e. 
$$2R_o \Delta - \Delta^2 = \frac{l^2}{4}$$

but 
$$\Delta^2 \rightarrow 0$$

Therefore,  $R_o = \frac{l^2}{8\Delta}$

However,  $\frac{1}{R_o} = \frac{d^2 y}{dx^2}$

Therefore,  $EI \left( \frac{d^2 y}{dx^2} - \frac{d^2 y_o}{dx^2} \right) = -Py$

becomes:  $EI \left( \frac{d^2 y}{dx^2} - \frac{8\Delta}{l^2} \right) = -Py$

or  $\frac{d^2 y}{dx^2} + \alpha^2 y = \frac{8\Delta}{l^2}$

The solution is:  $y = A \cos \alpha x + B \sin \alpha x + \frac{8\Delta}{\alpha^2 l^2}$

At  $x = 0, y = 0$ , therefore,  $A = \frac{8\Delta}{\alpha^2 l^2}$

$$\frac{dy}{dx} = -\alpha A \sin \alpha x + \alpha B \cos \alpha x$$

At  $x = l/2, \frac{dy}{dx} = 0$

Hence,  $0 = -\alpha A \sin \frac{\alpha l}{2} + \alpha B \cos \frac{\alpha l}{2}$

i.e.  $0 = -\alpha \left( \frac{8\Delta}{\alpha^2 l^2} \right) \sin \frac{\alpha l}{2} + \alpha B \cos \frac{\alpha l}{2}$

from which,  $\left( \frac{8\Delta}{\alpha^2 l^2} \right) \sin \frac{\alpha l}{2} = B \cos \frac{\alpha l}{2}$

and  $B = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \frac{\sin \frac{\alpha l}{2}}{\cos \frac{\alpha l}{2}} = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \tan \frac{\alpha l}{2}$

Hence,  $y = \left( \frac{8\Delta}{\alpha^2 l^2} \right) \left[ \cos \alpha x + \tan \frac{\alpha l}{2} \sin \alpha x - 1 \right]$

Now,  $\frac{d^2 y}{dx^2} = -\alpha^2 A \cos \alpha x - \alpha^2 B \sin \alpha x$

$$\text{i.e.} \quad \frac{d^2 y}{dx^2} = -\alpha^2 \left( \frac{8\Delta}{\alpha^2 l^2} \right) \cos \alpha x - \alpha^2 \left( \frac{8\Delta}{\alpha^2 l^2} \tan \frac{\alpha l}{2} \right) \sin \alpha x$$

$$\text{i.e.} \quad \frac{d^2 y}{dx^2} = - \left( \frac{8\Delta}{l^2} \right) \left[ \cos \alpha x + \tan \frac{\alpha l}{2} \sin \alpha x \right]$$

$M_{max}$  occurs at  $x = l/2$

$$\text{i.e.} \quad M_{max} = EI \left( \frac{d^2 y}{dx^2} - \frac{1}{R_o} \right)$$

$$EI \frac{d^2 y}{dx^2} = \frac{2 \times 10^{11} \times 6.594 \times 10^5 \times 8\Delta}{100} [0.8197 + 0.6986 + 0.5727]$$

$$= \mathbf{1286925\Delta}$$

$$\frac{EI}{R_o} = 2 \times 10^{11} \times 6.594 \times 10^{-5} \times \frac{8\Delta}{100} = \mathbf{1055040 \Delta}$$

$$\text{Hence,} \quad M_{max} = EI \left( \frac{d^2 y}{dx^2} - \frac{1}{R_o} \right) = (1286925 + 1055040)\Delta$$

$$= \mathbf{231885\Delta}$$

$$\text{However,} \quad M_{max} = \frac{53.96 \times 10^6 \times 6.594 \times 10^{-5}}{12.5 \times 10^{-2}} = \mathbf{28465 \text{ Nm}}$$

$$\text{Hence,} \quad 231885\Delta = 28465$$

$$\text{from which,} \quad \Delta = 28465/231885 = 0.123 \text{ m} = \mathbf{12.3 \text{ cm}}$$


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