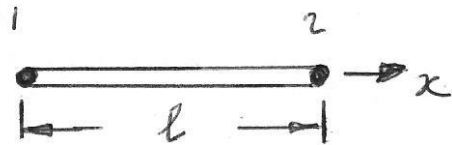


## CHAPTER 19 THE FINITE ELEMENT METHOD

### EXERCISE 68, Page 407

1. Using the finite element method, determine the elemental stiffness matrix for a uniform section rod, of length  $l$  and with end nodes.



2 node rod element:

$$u = \alpha_1 + \alpha_2 x$$

At  $x = 0$ ,  $u = u_1$

i.e. 
$$u_1 = \alpha_1$$

At  $x = l$ ,  $u = u_2$

i.e. 
$$u_2 = \alpha_1 + \alpha_2 l$$

or 
$$\alpha_2 l = u_2 - \alpha_1 = u_2 - u_1$$

and 
$$\alpha_2 = \frac{u_2 - u_1}{l}$$

Therefore, 
$$u = u_1 + (u_2 - u_1) \frac{x}{l}$$

Let 
$$\xi = \frac{x}{l}$$

then 
$$\begin{aligned} u &= u_1 + (u_2 - u_1) \xi \\ &= u_1 (1 - \xi) + u_2 \xi \end{aligned}$$

or 
$$u = \begin{pmatrix} (1 - \xi) & \xi \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = (N)(u_i) \quad (1)$$

or

$$(N) = \begin{pmatrix} (1-\xi) & \xi \end{pmatrix} \quad (2)$$

$$(B) = \frac{d(N)}{dx}$$

or

$$(B) = \frac{d(N)}{l d\xi} = \begin{pmatrix} -\frac{1}{l} & \frac{1}{l} \end{pmatrix} = \frac{d(N)}{dx}$$

$$(D) = (E)$$

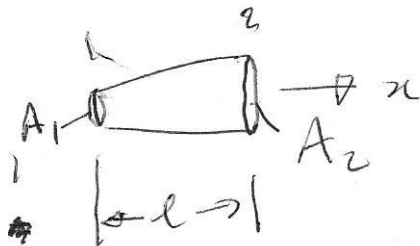
$$(k) = \int (B)^T (D) (B) dx \times A$$

$$= \int_0^l \frac{AE}{l^2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} l d\xi$$

i.e.

$$(k) = \frac{AE}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

2. Determine the stiffness matrix for a rod element whose cross-sectional area varies uniformly from  $A_1$ , at node 1 to  $A_2$  at node 2.



From Problem 1,

$$(N) = \begin{pmatrix} (1-\xi) & \xi \end{pmatrix}$$

Similarly, the cross-sectional area at  $\xi$ ,

$$(A) = \begin{pmatrix} A_1(1-\xi) & A_2 \end{pmatrix}$$

$$(B) = \frac{d(N)}{dx}$$

or

$$(B) = \frac{d(N)}{l d\xi} = \frac{1}{l} \begin{pmatrix} -1 & 1 \end{pmatrix}$$

$$(D) = (E)$$

$$\begin{aligned}
(k) &= \int (B)^T (D)(B) A l d\xi \\
&= \int \frac{E}{l} \begin{pmatrix} -1 & 1 \end{pmatrix} \frac{1}{l} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \begin{pmatrix} A_1(1-\xi) & A_2 \end{pmatrix} l d\xi \\
&= \frac{E}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int_0^1 \begin{pmatrix} A_1(1-\xi) & A_2 \end{pmatrix} d\xi \\
&= \frac{E}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \left( \xi - \frac{\xi^2}{2} \right) & A_2 \frac{\xi^2}{2} \end{pmatrix}_0^1 \\
&= \frac{E}{l} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{A_1}{2} & \frac{A_2}{2} \end{pmatrix}
\end{aligned}$$

i.e. 
$$(k) = \frac{(A_1 + A_2)E}{2l} \begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ -\mathbf{1} & \mathbf{1} \end{pmatrix}$$

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