

## CHAPTER 9 COMPLEX STRESS AND STRAIN

### EXERCISE 40, Page 208

1. In a two dimensional system of stress,  $\sigma_x = 100$  MPa,  $\sigma_y = 50$  MPa and  $\tau_{xy} = 40$  MPa.

Determine the principal stresses in this stress system, together with their directions, and the maximum shearing stresses.

$$\sigma_x = 100 \text{ MPa}, \quad \sigma_y = 50 \text{ MPa} \quad \text{and} \quad \tau_{xy} = 40 \text{ MPa}$$

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \frac{1}{2}(150) \pm \frac{1}{2}\sqrt{(50)^2 + 4(40)^2} \\ &= 75 \pm 47.17 = 75 + 47.17\end{aligned}$$

i.e.  $\sigma_1 = 122.17 \text{ MPa}$

and  $\sigma_2 = 75 - 47.17 = 27.83 \text{ MPa}$

$$2\theta = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \tan^{-1}\left(\frac{2 \times 40}{100 - 50}\right)$$

i.e.  $2\theta = \tan^{-1}(1.6) = 57.99^\circ$

from which,  $\theta = 29^\circ$

$$\hat{\tau} = \frac{\sigma_1 - \sigma_2}{2} = \frac{122.17 - 27.83}{2}$$

i.e.  $\tau = 47.17 \text{ MPa}$

2. In a two dimensional system of stress,  $\sigma_x = 60$  MPa,  $\sigma_y = -120$  MPa and  $\tau_{xy} = 50$  MPa.

Determine the principal stresses in this stress system, together with their directions, and the maximum shearing stresses.

$$\sigma_x = 60 \text{ MPa}, \quad \sigma_y = -120 \text{ MPa} \quad \text{and} \quad \tau_{xy} = 50 \text{ MPa}$$

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \frac{1}{2}(60 - 120) \pm \frac{1}{2}\sqrt{(60 - (-120))^2 + 4(50)^2} \\ &= -30 + 102.96\end{aligned}$$

i.e.  $\sigma_1 = \mathbf{72.96 \text{ MPa}}$

and  $\sigma_2 = -30 - 102.96 = -\mathbf{132.96 \text{ MPa}}$

$$2\theta = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \tan^{-1}\left(\frac{2 \times 50}{60 - (-120)}\right)$$

i.e.  $2\theta = \tan^{-1}(0.5555) = 29.05^\circ$

from which,  $\theta = \mathbf{14.53^\circ}$

$$\hat{\tau} = \frac{\sigma_1 - \sigma_2}{2} = \frac{72.96 - (-132.96)}{2}$$

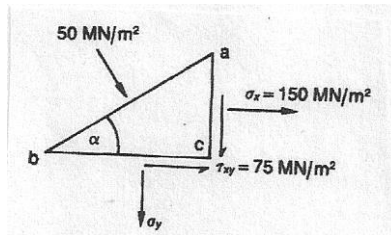
i.e.  $\tau = \mathbf{102.96 \text{ MPa}}$

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## EXERCISE 41, Page 210

1. At a point in a two-dimensional stress system, the known stresses are as shown below.

Determine the principal stresses  $\sigma_y$ ,  $\alpha$  and  $\hat{\tau}$



NB From the diagram, it can be seen that the stress on the inclined plane is a principal stress, because it is not accompanied by a shear stress. Moreover, as the other two normal stresses  $\sigma_x$  and  $\sigma_y$  are positive, the **50 MN/m<sup>2</sup> stress is  $\sigma_2$  and negative.**

Resolving horizontally gives:

$$50 \sin \alpha \, ab = 150 \, ac + 75 \, bc = 0$$

$$50 \sin \alpha + 150 \frac{ac}{ab} + 75 \frac{bc}{ab} = 0$$

$$50 \sin \alpha + 150 \sin \alpha + 75 \cos \alpha = 0$$

i.e.  $200 \sin \alpha = -75 \cos \alpha$

from which,  $\frac{\sin \alpha}{\cos \alpha} = \frac{-75}{200} \quad \text{or} \quad \tan \alpha = \frac{-75}{200}$

and  $\alpha = \tan^{-1}\left(\frac{-75}{200}\right) = -20.56^\circ$

Resolving vertically gives:

$$50 \cos \alpha \, ab + 75 \, ac + \sigma_y \, bc = 0$$

i.e.  $46.82 - 26.34 + 0.936\sigma_y = 0$

and  $\sigma_y = -21.88 \text{ MN/m}^2$

$$\begin{aligned} \sigma_1 &= \frac{1}{2} (150 - 21.88) + \frac{1}{2} \sqrt{(150 + 21.88)^2 + 4 \times 75^2} \\ &= 64.06 + 114.06 \end{aligned}$$

i.e.  $\sigma_1 = 178.1 \text{ MN/m}^2$

*Alternative Method*

$$\sigma_2 = -50 = \frac{1}{2}(150 + \sigma_y) - \frac{1}{2}\sqrt{(150 - \sigma_y)^2 + 4 \times 75^2}$$

$$-50 = 75 + \sigma_y/2 - \frac{1}{2}\sqrt{(150 - \sigma_y)^2 + 4 \times 75^2}$$

$$-125 \times 2 = \sigma_y - \frac{1}{2}\sqrt{(150 - \sigma_y)^2 + 4 \times 75^2}$$

$$(\sigma_y + 250)^2 = [(150 - \sigma_y)^2 + 4 \times 75^2]$$

$$\sigma_y^2 + 500 \sigma_y + 62\,500 = 22\,500 - 300 \sigma_y + \sigma_y^2 + 22\,500$$

$$800 \sigma_y = -17\,500$$

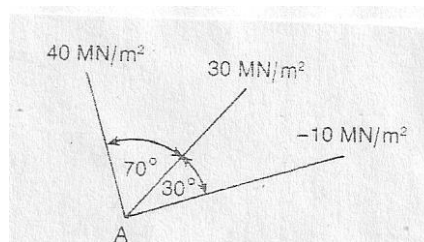
and  $\sigma_y = -21.88 \text{ MN/m}^2$  (as before)

$$\sigma_1 = \frac{1}{2}(150 - 21.88) + \frac{1}{2}\sqrt{150 + 21.88^2 + 4 \times 75^2}$$

$$= 64 + \frac{1}{2} \times 228.1$$

i.e.  $\sigma_1 = 178.1 \text{ MN/m}^2$

2. At a certain point A in a piece of material, the magnitudes of the direct stresses are  $-10 \text{ MN/m}^2$ ,  $30 \text{ MN/m}^2$  and  $40 \text{ MN/m}^2$ , as shown below. Determine the magnitude and direction of the principal stresses and the maximum shear stress.



$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau_{xy} \sin 2\theta$$

Let  $\sigma_x = -10 \text{ MN/m}^2$

For  $\theta = 30^\circ$ ,

Therefore,  $\sigma_{30} = 30 = -\frac{1}{2}(-10 + \sigma_y) + \frac{1}{2}(-10 - \sigma_y) \cos 60 + \tau_{xy} \sin 60$

i.e.  $30 = -5 + \frac{\sigma_y}{2} - \left(5 + \frac{\sigma_y}{2}\right) 0.5 + 0.866 \tau_{xy}$

i.e.  $30 = -5 + 0.5\sigma_y - 0.25\sigma_y - 2.5 + 0.866 \tau_{xy}$

i.e.  $37.5 = 0.25 \sigma_y + 0.866 \tau_{xy}$  (1)

For  $\theta = 100^\circ$ ,

$$40 = \frac{1}{2}(-10 + \sigma_y) + \frac{1}{2}(-10 - \sigma_y) \cos 200 + \tau_{xy} \sin 200$$

$$40 = -5 + 0.5\sigma_y - (5 + \sigma_y/2)(-0.9397) - 0.342 \tau_{xy}$$

$$40 = -0.3015 + 0.9698\sigma_y - 0.342 \tau_{xy}$$

$$40.302 = 0.9698 \sigma_y - 0.342 \tau_{xy}$$
 (2)

Dividing equation (1) by 0.25 gives:

$$150 = \sigma_y + 3.464 \tau_{xy}$$
 (1)a

Dividing equation (2) by 0.9698 gives:

$$41.55 = \sigma_y - 0.353 \tau_{xy}$$
 (2)a

Taking equation (2)a from equation (1)a gives:

$$108.45 = 3.817 \tau_{xy}$$

i.e.  $\tau_{xy} = \mathbf{28.41 \text{ MN/m}^2}$

From equation (1)a:

$$\sigma_y = 150 - 3.464(28.41) = \mathbf{51.58 \text{ MN/m}^2}$$

$$\theta = \frac{1}{2} \tan^{-1} \left( \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right)$$

i.e.  $\theta = \mathbf{-21.35^\circ}$

$$\sigma_I = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2} \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$= 20.79 + 41.89$$

i.e.  $\sigma_1 = 62.68 \text{ MN/m}^2$

$$\sigma_I = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}$$

$$= 20.79 - 41.89$$

i.e.  $\sigma_2 = -21.10 \text{ MN/m}^2$

$$\tau = \frac{(\sigma_1 - \sigma_2)}{2} = \frac{62.68 - 21.1}{2}$$

$$= 41.89 \text{ MN/m}^2$$

**3.** A circular shaft of diameter 40 cm is subjected to a combined bending moment,  $M$ , of 3 MN m, together with a torsional moment,  $T$ , of 1 MN m. Determine the maximum stresses due to the combined effects of 'M' and 'T'.

$$M_e = \frac{1}{2}(M \pm \sqrt{M^2 + T^2})$$

$$= \frac{1}{2}(3 \pm \sqrt{3^2 + 1^2}) = \frac{1}{2}(3 \pm 3.162)$$

i.e.  $M_e = +3.081 \text{ MN m}$  or  $-0.081 \text{ MN m}$

$$T_e = \sqrt{M^2 + T^2} = \sqrt{3^2 + 1^2} = \pm 3.162 \text{ MN m}$$

$$\sigma_1 = 3.081 \times \frac{32}{\pi d^3} = 3.081 \times \frac{32}{\pi (0.4)^3}$$

$$= 3.081 \times 159.155$$

i.e.  $\sigma_1 = 490.4 \text{ MPa}$

and  $\sigma_1 = -0.081 \times 159.155$

i.e.  $\sigma_2 = -12.89 \text{ MPa}$

$$\hat{\tau} = \frac{16 \times T_e}{\pi d^3} = \frac{16 \times 3.162}{\pi \times 0.4^3}$$

i.e.  $\tau = 251.6 \text{ MPa}$

**4.** If for the shaft in Problem 3,  $M = 4 \text{ MN m}$  and  $T = 2 \text{ MN m}$ , what are the maximum stresses due to these effects?

$$M_e = \frac{1}{2} \left( M \pm \sqrt{M^2 + T^2} \right)$$

$$= \frac{1}{2} \left( 4 \pm \sqrt{4^2 + 2^2} \right) = \frac{1}{2} (4 \pm 4.472)$$

i.e.  $M_e = + 4.236 \text{ MN m}$  or  $- 0.236 \text{ MN m}$

$$T_e = \sqrt{M^2 + T^2} = \sqrt{4^2 + 2^2} = \pm 4.472 \text{ MN m}$$

$$\sigma_1 = 4.236 \times \frac{32}{\pi d^3} = 4.236 \times \frac{32}{\pi (0.4)^3}$$

$$= 4.236 \times 159.155$$

i.e.  $\sigma_1 = 674.2 \text{ MPa}$

and  $\sigma_1 = -0.236 \times 159.155$

i.e.  $\sigma_2 = - 37.56 \text{ MPa}$

$$\hat{\tau} = \frac{16 \times T_e}{\pi d^3} = \frac{16 \times 4.472}{\pi \times 0.4^3}$$

i.e.  $\tau = 355.9 \text{ MPa}$

**EXERCISE 42, Page 219**

1. In a two dimensional system of stress,  $\sigma_x = 100$  MPa,  $\sigma_y = 50$  MPa and  $\tau_{xy} = 40$  MPa.

Determine the magnitudes of the principal strains, and the magnitude of the maximum shear strain,

given that:  $E = 2 \times 10^{11} \text{ N/m}^2$  and  $\nu = 0.3$ .

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \frac{1}{2}(100 + 50) + \frac{1}{2}\sqrt{(100 - 50)^2 + 4(40)^2} \\ &= 75 + 47.17\end{aligned}$$

i.e.  $\sigma_1 = 122.17 \text{ MPa}$

and  $\sigma_2 = 75 - 47.17 = 27.83 \text{ MPa}$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{1}{2 \times 10^{11}}(122.17 - 0.3 \times 27.83) \times 10^6$$

i.e.  $\varepsilon_1 = 5.691 \times 10^{-4}$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) = \frac{1}{2 \times 10^{11}}(27.83 - 0.3 \times 122.17) \times 10^6$$

i.e.  $\varepsilon_2 = 5.260 \times 10^{-5}$

$$\gamma = \varepsilon_1 - \varepsilon_2 = 5.691 \times 10^{-4} - 5.260 \times 10^{-5}$$

$$= 5.165 \times 10^{-4}$$

2. In a two dimensional system of stress,  $\sigma_x = 60$  MPa,  $\sigma_y = -120$  MPa and  $\tau_{xy} = 50$  MPa. If

$E = 1 \times 10^{11} \text{ N/m}^2$  and  $\nu = 0.33$ , what is the magnitude of the principal strains, and the magnitude of the maximum shear strain?



$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \\ &= \frac{1}{2}(60 - 120) + \frac{1}{2}\sqrt{(60 - -120)^2 + 4(50)^2} \\ &= -30 + 102.96\end{aligned}$$

i.e.  $\sigma_1 = \mathbf{72.96 \text{ MPa}}$

and  $\sigma_2 = -30 - 102.96 = -\mathbf{132.96 \text{ MPa}}$

$$\varepsilon_1 = \frac{1}{E}(\sigma_1 - \nu\sigma_2) = \frac{1}{2 \times 10^{11}}(72.96 - 0.3 \times -132.96) \times 10^6$$

i.e.  $\varepsilon_1 = \mathbf{5.642 \times 10^{-4}}$

$$\varepsilon_2 = \frac{1}{E}(\sigma_2 - \nu\sigma_1) = \frac{1}{2 \times 10^{11}}(-132.96 - 0.3 \times 72.96) \times 10^6$$

i.e.  $\varepsilon_2 = \mathbf{-7.742 \times 10^{-4}}$

$$\begin{aligned}\gamma &= \varepsilon_1 - \varepsilon_2 = 5.642 \times 10^{-4} - -7.742 \times 10^{-4} \\ &= \mathbf{1.338 \times 10^{-3}}\end{aligned}$$

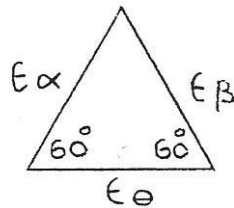

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### EXERCISE 43, Page 226

1. Prove that the following relationships apply to the  $60^\circ$  delta rosette of Figure 9.37 (page 224):

$$\tan 2\theta = \frac{\sqrt{3}(\varepsilon_\alpha - \varepsilon_\beta)}{(2\varepsilon_\theta - \varepsilon_\beta - \varepsilon_\alpha)}$$

$$\varepsilon_1, \varepsilon_2 = \frac{1}{3}(\varepsilon_\theta + \varepsilon_\alpha + \varepsilon_\beta) \pm \frac{\sqrt{2}}{3} \sqrt{(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2}$$



$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \cos 2\theta \quad (1)$$

$$\varepsilon_\alpha = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \cos(2\theta + 120^\circ) \quad (2)$$

$$\varepsilon_\beta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \cos(2\theta + 240^\circ) \quad (3)$$

Equations (2) and (3) can be rewritten in the following form:

$$\varepsilon_\alpha = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)(-\cos 2\theta \cos 60 - \sin 2\theta \sin 60) \quad (4)$$

$$\varepsilon_\beta = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)(-\cos 2\theta \cos 60 + \sin 2\theta \sin 60) \quad (5)$$

Adding together equations (1), (4) and (5) gives:

$$\varepsilon_\theta + \varepsilon_\alpha + \varepsilon_\beta = \frac{3}{2}(\varepsilon_1 + \varepsilon_2)$$

or 
$$(\varepsilon_1 + \varepsilon_2) = \frac{2}{3}(\varepsilon_\theta + \varepsilon_\alpha + \varepsilon_\beta) \quad (6)$$

Taking equation (4) from equation (5) gives:

$$\varepsilon_\beta - \varepsilon_\alpha = (\varepsilon_1 - \varepsilon_2) \sin 2\theta \sin 60 \quad (7)$$

Taking equation (1) from equation (5) gives:

$$\varepsilon_\beta - \varepsilon_\theta = \frac{1}{2} (\varepsilon_1 - \varepsilon_2) \left( -\frac{3}{2} \cos 2\theta + \sin 2\theta \frac{\sqrt{3}}{2} \right) 5 \quad (8)$$

Dividing equation (8) by equation (7) gives:

$$\frac{\varepsilon_\beta - \varepsilon_\theta}{\varepsilon_\beta - \varepsilon_\alpha} = \frac{1}{2} \frac{\left( -\frac{3}{2} \cos 2\theta + \sin 2\theta \frac{\sqrt{3}}{2} \right)}{\left( \sin 2\theta \frac{\sqrt{3}}{2} \right)}$$

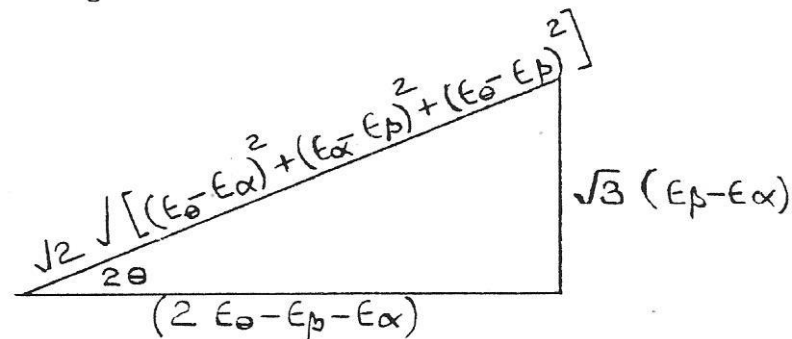
i.e. 
$$\frac{\varepsilon_\beta - \varepsilon_\theta}{\varepsilon_\beta - \varepsilon_\alpha} = \frac{1}{2} (-\sqrt{3} \cot 2\theta + 1)$$

or 
$$\begin{aligned} -\sqrt{3} \cot 2\theta &= \frac{2(\varepsilon_\beta - \varepsilon_\theta)}{\varepsilon_\beta - \varepsilon_\alpha} - 1 \\ &= \frac{2\varepsilon_\beta - 2\varepsilon_\theta - \varepsilon_\beta + \varepsilon_\alpha}{\varepsilon_\beta - \varepsilon_\alpha} \\ &= \frac{-2\varepsilon_\theta + \varepsilon_\beta + \varepsilon_\alpha}{\varepsilon_\beta - \varepsilon_\alpha} \end{aligned}$$

or 
$$\tan 2\theta = \frac{\sqrt{3}(\varepsilon_\beta - \varepsilon_\alpha)}{2\varepsilon_\theta - \varepsilon_\beta - \varepsilon_\alpha} \quad (9)$$

**To determine  $\varepsilon_1$  and  $\varepsilon_2$**

From the mathematical triangle



$$\cos 2\theta = \frac{(2\varepsilon_\theta - \varepsilon_\beta - \varepsilon_\alpha)}{\sqrt{2}\sqrt{[(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2]}} \quad (10)$$

$$\sin 2\theta = \frac{\sqrt{3}(\varepsilon_\beta - \varepsilon_\alpha)}{\sqrt{2}\sqrt{[(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2]}} \quad (11)$$

Substituting equation (11) into equation (7) gives:

$$\text{or} \quad \varepsilon_1 - \varepsilon_2 = 6 \frac{2}{3} \sqrt{[(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2]} \quad (12)$$

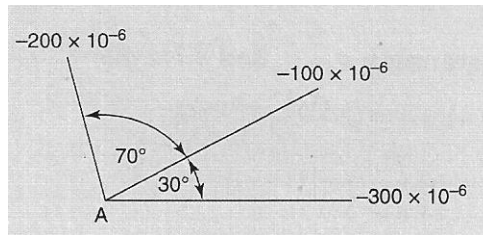
Adding equations (6) and (12) gives:

$$\varepsilon_1 = \frac{1}{3} \sqrt{[\varepsilon_\theta + \varepsilon_\alpha + \varepsilon_\beta]} + \frac{\sqrt{2}}{3} \sqrt{[(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2]}$$

$$\text{and} \quad \varepsilon_2 = \frac{1}{3} \sqrt{[\varepsilon_\theta + \varepsilon_\alpha + \varepsilon_\beta]} - \frac{\sqrt{2}}{3} \sqrt{[(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2]}$$

2. The web of a rolled steel joist has three linear strain gauges attached to a point A and indicating strains as shown below. Determine the magnitude and direction of the principal stresses, and the value of the maximum shear stress at this point, assuming the following to apply:

Elastic modulus =  $2 \times 10^{11} \text{ N/m}^2$  and Poisson's ratio = 0.3



$$\varepsilon_\theta = \frac{1}{2} (\varepsilon_x + \varepsilon_y) + \frac{1}{2} (\varepsilon_x - \varepsilon_y) \cos 2\theta + \frac{1}{2} \gamma_{xy} \sin 2\theta$$

Let  $\varepsilon_x = 300 \times 10^{-6}$

At  $\theta = 30^\circ$ ,

$$\varepsilon_{30} = -100 \times 10^{-6} = 150 \times 10^{-6} + 0.5 \varepsilon_y + 0.5 (300 \times 10^{-6} - \varepsilon_y) + 0.5 \gamma_{xy} \quad (1)$$

$$\text{i.e.} \quad -100 \times 10^{-6} = 225 \times 10^{-6} + 0.25 \varepsilon_y + 0.433 \gamma_{xy}$$

$$-325 \times 10^{-6} = 0.25 \varepsilon_y + 0.433 \gamma_{xy}$$

At  $\theta = 100^\circ$ ,

$$\begin{aligned} -200 \times 10^{-6} + 150 \times 10^{-6} + 0.5 \varepsilon_y &= 0.5(300 \times 10^{-6} - \varepsilon_y)(-0.9397) + 0.5 \gamma_{xy}(-0.342) \\ &= 9.05 \times 10^{-6} + 0.9699 \varepsilon_y - 0.171 \gamma_{xy} \\ -209.1 \times 10^{-6} &= 0.9699 \varepsilon_y - 0.171 \gamma_{xy} \quad (2) \\ -1300 \times 10^{-6} &= \varepsilon_y - 0.1732 \gamma_{xy} \quad (1)a \\ -215.6 \times 10^{-6} &= \varepsilon_y - 0.176 \gamma_{xy} \quad (2)a \end{aligned}$$

Take equation (2)a from equation (1)a gives:

$$-1084.4 - 1.908 \gamma_{xy}$$

from which,

$$\gamma_{xy} = -568.3 \times 10^{-6}$$

From equation (1)a,

$$\varepsilon_y = -315.6 \times 10^{-6}$$

$$\varepsilon_1 = \frac{(\varepsilon_x - \varepsilon_y)}{2} + \frac{1}{2} \sqrt{[(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2]} \quad 7$$

$$= -7.8 \times 10^{-6} + \frac{1}{2} \times 837.8 \times 10^{-6}$$

$$\varepsilon_1 = -7.8 \times 10^{-6} + 418.9 \times 10^{-6}$$

i.e.

$$\varepsilon_1 = 411.1 \times 10^{-6}$$

and

$$\varepsilon_2 = -426.7 \times 10^{-6}$$

$$\sigma_1 = \frac{2 \times 10^{11}}{0.91} 8(411.1 - 0.3 \times 426.7) \times 10^{-6}$$

$$\sigma_1 = 62.2 \text{ MN/m}^2$$

$$\sigma_2 = -66.67 \text{ MN/m}^2$$

$$\tau = 64.44 \text{ MN/m}^2$$

$$\theta = \frac{1}{2} \tan^{-1} \left( \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \right) 9$$

$$\theta = -21.36^\circ$$

3. A solid stainless steel propeller shaft of a power boat is of diameter 2 cm. A 45° rectangular rosette is attached to the shaft, where the central gauge (Gauge No.2) is parallel to the axis of the shaft. Assuming bending and thermal stresses are negligible, determine the thrust and torque that the shaft is subjected to, given that the recorded strains are as follows:

$$\varepsilon_1 = -300 \times 10^{-6} \quad \varepsilon_2 = -142.9 \times 10^{-6} \text{ (Gauge No.2)} \quad \varepsilon_3 = 200 \times 10^{-6}$$

$$E = 200 \times 10^6 \text{ N/m}^2 \quad \nu = 0.3$$

$$G = \frac{2 \times 10^{11}}{2.6} = 7.69 \times 10^{10}$$

$$\gamma = \varepsilon_1 - \varepsilon_3$$

$$= (-300 - 200) \times 10^{-6}$$

$$= -500 \times 10^{-6}$$

and

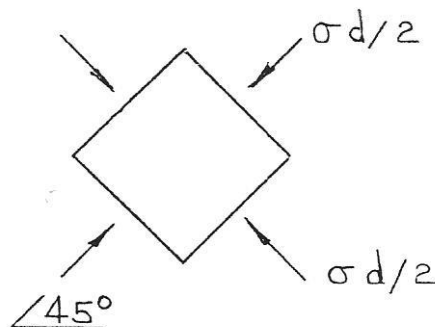
$$\tau = 38.45 \text{ MN/m}^2$$

$$J = \frac{\pi (2 \times 10^{-2})^4}{32} = 1.571 \times 10^{-8} \text{ m}^4$$

$$T = \frac{\tau \times J}{r} = \frac{38.45 \times 10^6 \times 1.571 \times 10^{-8}}{1 \times 10^{-2}} = 60.40 \text{ N m}$$

i.e.

$$T = 60.40 \text{ N m}$$



$$A = 3.142 \times 10^{-4} \text{ m}^2$$

$$\varepsilon_1 = \varepsilon_{IT} + \varepsilon_{45D}$$

$$\varepsilon_3 = -\varepsilon_{IT} + \varepsilon_{45D}$$

Adding together gives:

$$(-300 + 200) \times 10^{-6} = 2\varepsilon_{45D}$$

from which,  $\varepsilon_{45D} = -50 \times 10^{-6}$  at  $45^\circ$  to axis

$$= \frac{1}{2E} \sigma_d - \nu \sigma_d$$

Therefore, 
$$\frac{(100 \times 10^{-6})(2 \times 10^{11})}{0.7} = \sigma_d \quad 12$$

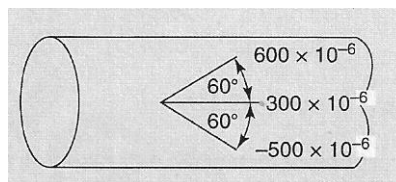
and 
$$\sigma_d = -28.57 \text{ MN/m}^2$$

$$\text{Thrust} = -8.98 \text{ kN}$$

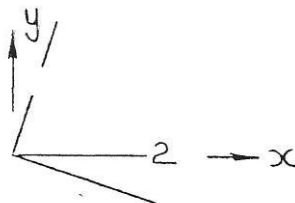
4. A solid circular-section steel shaft, of 0.03 m diameter, is simply supported at its ends and is subjected to a torque and a load that is radial to its axis. A small  $60^\circ$  strain gauge rosette is attached to the underside of the mid-span, as shown below.

Determine (a) the applied torque, (b) the length of the shaft, and (c) the direction and value of the maximum principal stress.

Assume that  $E = 2.1 \times 10^{11} \text{ N/m}^2$ ,  $\nu = 0.28$ ,  $\rho = 7860 \text{ kg/m}^3$  and  $g = 9.81 \text{ m/s}^2$



(a)



$$\varepsilon_\theta = \frac{1}{2} (\varepsilon_x + \varepsilon_y) + \frac{1}{2} (\varepsilon_x - \varepsilon_y) \cos 2\theta + \frac{1}{2} \gamma_{xy} \sin 2\theta$$

Let  $\varepsilon_2 = \varepsilon_x = 300 \times 10^{-6}$

$$\varepsilon_{-60} = \frac{1}{2} (300 \times 10^{-6} + \varepsilon_y) + \frac{1}{2} (300 \times 10^{-6} - \varepsilon_y) \cos (-120) + \frac{1}{2} \gamma_{xy} \sin (-120)$$

i.e.  $-500 \times 10^{-6} = 150 \times 10^{-6} + 0.5 \varepsilon_y - 0.25 (300 \times 10^{-6} - \varepsilon_y) - 0.433 \gamma_{xy}$

i.e.  $-500 \times 10^{-6} - 150 \times 10^{-6} + 75 \times 10^{-6} = 0.75 \varepsilon_y - 0.433 \gamma_{xy}$

and  $-575 \times 10^{-6} = 0.75 \varepsilon_y - 0.433 \gamma_{xy}$  (1)

$$\varepsilon_{+60} = \frac{1}{2} (300 \times 10^{-6} + \varepsilon_y) + \frac{1}{2} (300 \times 10^{-6} - \varepsilon_y) \cos (120) + \frac{1}{2} \gamma_{xy} \sin (120)$$

$$600 \times 10^{-6} = 150 \times 10^{-6} + 0.5 \varepsilon_y - 75 \times 10^{-6} + 0.25 \varepsilon_y + 0.433 \gamma_{xy}$$

$$525 \times 10^{-6} = 0.75 \varepsilon_y + 0.433 \gamma_{xy}$$
 (2)

Taking equation (1) from equation (2) gives:

$$1100 \times 10^{-6} = 0.866 \gamma_{xy}$$

from which,  $\gamma_{xy} = 1270 \times 10^{-6}$

$$G = \frac{E}{2(1+\nu)} = 8.20 \times 10^{10}$$

$$\tau_{xy} = 8.20 \times 10^{10} \times 12.70 \times 10^{-6} = 104.18 \text{ MN/m}^2$$

$$J = \frac{\pi (2 \times 10^{-2})^4}{32} = 7.952 \times 10^{-8} \text{ m}^4$$

$$T = \frac{\tau \times J}{r} = \frac{104.18 \times 10^6 \times 7.952 \times 10^{-8}}{0.015} = 552.3 \text{ N m}$$

i.e.  $T = 552.3 \text{ N m}$

(b)  $M_{max} = \frac{wl^2}{8} = 16 \text{ A} = 7.069 \times 10^{-4} \text{ m}^2$

$$w = \rho Ag = 54.5 \text{ N/m}$$

From equation (1),

$$\varepsilon_y = -\frac{575 \times 10^{-6} + 550 \times 10^{-6}}{0.75} = -17$$

i.e.  $\varepsilon_y = -33.3 \times 10^{-6}$

$$\theta = \frac{1}{2} \tan^{-1} \left( \frac{1270}{333} \right) = 18$$



i.e.  $\theta = 37.65^\circ$  anti-clockwise from the middle gauge

$$\sigma_x = \frac{E}{(1-\nu^2)}(\epsilon_x + \nu\epsilon_y) = \frac{2.1 \times 10^{11}}{0.922}(300 \times 10^{-6} - 9.32 \times 10^{-6}) \quad 19$$

i.e.  $\sigma_x = 66.21 \text{ MN/m}^2$

$$M_{\max} = \frac{\sigma_x I}{y} = 175.5 \text{ N m}$$

$$l = \sqrt{\frac{8M_{\max}}{w}} = \sqrt{\frac{8 \times 175.5}{54.5}}$$

i.e.  $l = 5.08 \text{ m}$

(c) 
$$\theta = \frac{1}{2} \tan^{-1} \left( \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \right) \quad 20$$

$$= \frac{1}{2} \tan^{-1} \left( \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \right) \quad 21$$

i.e.  $\theta = 37.65^\circ$  anti-clockwise from the middle gauge

$$\sigma_y = \frac{E}{(1-\nu^2)}(\epsilon_y + \nu\epsilon_x) = 2.278 \times 10^{11}(-33.3 \times 10^{-6} + 84 \times 10^{-6}) \quad 22$$

i.e.  $\sigma_y = 11.55 \text{ MN/m}^2$

$$\sigma_1 = \frac{1}{2}(66.2 + 11.5) + \frac{1}{2} \sqrt{(66.2 - 11.55)^2 + 4 \times 104.2^2} \quad 23$$

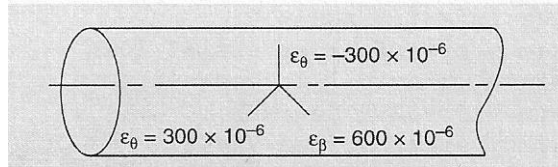
$$= 38.8 + 107.7$$

i.e.  $\sigma_1 = 146.5 \text{ MN/m}^2$

and  $\sigma_2 = -68.9 \text{ MN/m}^2$

**5.** A solid cylindrical aluminium-alloy shaft of 0.03 m diameter is subjected to a combined radial and axial load and a torque. A  $120^\circ$  strain gauge rosette is attached to the shaft and records the strains shown below. Determine the values of the axial load and torque, given the following:

$$E = 1 \times 10^{11} \text{ N/m}^2 \text{ and } \nu = 0.32$$

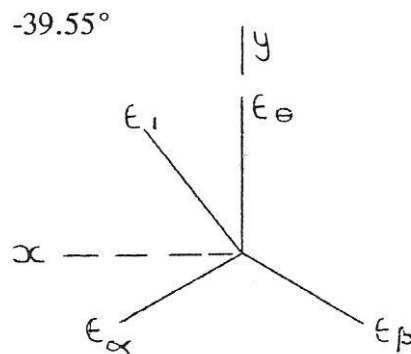


$$\tan 2\theta = \frac{\sqrt{3}(\varepsilon_\alpha - \varepsilon_\beta)}{(2\varepsilon_\theta - \varepsilon_\beta - \varepsilon_\alpha)}$$

$$= \frac{\sqrt{3}(-300 - 600)}{(600 - 600 + 300)} = -5.1962 \ 24$$

i.e.  $2\theta = -79.1^\circ$

from which,  $\theta = -39.55^\circ$



$$\varepsilon_1 = \frac{1}{3}(\varepsilon_\theta + \varepsilon_\alpha + \varepsilon_\beta) + \frac{\sqrt{2}}{3} \sqrt{[(\varepsilon_\theta - \varepsilon_\alpha)^2 + (\varepsilon_\alpha - \varepsilon_\beta)^2 + (\varepsilon_\theta - \varepsilon_\beta)^2]}$$

$$= 10^{-6} [200 + 0.4714 \sqrt{126000025}]$$

i.e.  $\varepsilon_1 = 729.1 \times 10^{-6}$

$$\varepsilon_2 = 10^{-6} (200 - 529.1) = -329.1 \times 10^{-6}$$

$$\varepsilon_\phi = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) + \frac{1}{2}(\varepsilon_1 - \varepsilon_2) \cos 2\phi$$

and  $\phi = 90 - 39.55 = 50.45^\circ$

Therefore,  $\varepsilon_x = \varepsilon_\phi = 10^{-6} (200 - 529.1 \times 0.1891)$

i.e.  $\varepsilon_x = 100 \times 10^{-6}$

and  $\varepsilon_y = \varepsilon_\theta = 300 \times 10^{-6}$

Check  $\varepsilon_y$

$$\varepsilon_y = 10^{-6} (200 + 529.1 \times 0.1891)$$

i.e.  $\varepsilon_y = 300 \times 10^{-6}$

$$\begin{aligned}\sigma_x &= \frac{E(\varepsilon_x + \nu\varepsilon_y)}{(1-\nu^2)} \\ &= \frac{1 \times 10^{11}}{0.8976} \times 10^{-6} (100 + 0.32 \times 300) \quad 26\end{aligned}$$

i.e.  $\sigma_x = 1.111 \times 10^5 \times 196 = 21.78 \text{ MPa}$

$$A = \frac{\pi \times 0.03^2}{4} = 7.0686 \times 10^{-4} \text{ m}^2 \quad 27$$

Therefore, axial load =  $21.78 \times 10^3 \times 7.0686 \times 10^{-4} \text{ kN}$

i.e. **Axial load = 15.40 kN**

$$\frac{\gamma_\phi}{2} = \frac{(\varepsilon_1 - \varepsilon_2)}{2} \sin 2\phi \quad 28$$

Therefore,  $\gamma_{xy} = 1058.2 \times 10^{-6} \sin 100.9^\circ$

i.e.  $\gamma_{xy} = 1039.1 \times 10^{-6}$

$$J = \frac{\pi \times 0.03^4}{32} = 7.952 \times 10^{-8} \text{ m}^4$$

$$\tau_{xy} = \frac{1 \times 10^{11}}{2.64} \times 1039.1 \times 10^{-6} = 39.24 \text{ MPa}$$

Therefore,  **$T = \tau_{xy} \times J/r = 208 \text{ N m}$**

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