

## CHAPTER 17 COMPOSITES

### EXERCISE 64, Page 373

1. Calculate the stiffness matrix ( $Q$ ) and derive the ( $A$ ), ( $B$ ) and ( $D$ ) matrices for a sheet of aluminium 1 mm thick. The following material properties can be assumed:

$$E = 70 \text{ GPa} \text{ and } \nu = 0.35 \text{ and for an isotropic material: } G = \frac{E}{2(1+\nu)}$$

$$(A) = (Q)h \quad \text{and} \quad (D) = (Q)\frac{h^3}{12}$$

$$G = \frac{E}{2(1+\nu)} = \frac{70}{2(1+0.35)} = 25.93 \text{ GPa}$$

$$1 - \nu^2 = 1 - 0.35^2 = 0.8775$$

$$(Q) = \left[ \frac{1}{1 - \nu^2} \right] \begin{pmatrix} E & \nu E & 0 \\ \nu E & E & 0 \\ 0 & 0 & (1 - \nu^2)G \end{pmatrix}$$

$$\text{i.e. } (Q) = \frac{1}{0.8775} \begin{pmatrix} 70 & 0.35(70) & 0 \\ 0.35(70) & 70 & 0 \\ 0 & 0 & 0.885(25.93) \end{pmatrix} \text{ in GPa}$$

$$\text{i.e. } (Q) = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \text{ in GPa}$$

$$(A) = (Q)h \text{ where } h = 1.0 \text{ mm}$$

$$\text{i.e. } (A) = (Q)(1.0)$$

**The units of ( $A$ ) are:** (GPa) mm =  $(10^3 \text{ MPa})\text{mm} = (10^3 \text{ N/mm}^2)\text{mm}$

$$(A) = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \text{ in kN/mm}$$

$(B) = 0$  due to symmetry

$$(D) = (Q) \frac{h^3}{12} = (Q) \left( \frac{1}{12} \times 1.0^3 \right) = \frac{1}{12} (Q)$$

**The units of  $(D)$  are:**  $(\text{GPa})(\text{mm}^3) = (10^3 \text{ MPa})(\text{mm}^3) = (10^3 \text{ N/mm}^2)(\text{mm}^3) = 10^3 \text{ N mm}$

$$(D) = \begin{pmatrix} 6.648 & 2.327 & 0 \\ 2.327 & 6.648 & 0 \\ 0 & 0 & 2.161 \end{pmatrix} \text{ in kNmm}$$

**2. Show that the inverse of  $(A)$  in Problem 1 is:**

$$(a) = \begin{pmatrix} 1.429 \times 10^{-5} & -5.0 \times 10^{-6} & 0 \\ -5.0 \times 10^{-6} & 1.429 \times 10^{-5} & 0 \\ 0 & 0 & 3.857 \times 10^{-5} \end{pmatrix} \text{ in mm/N}$$

From Problem 1,

$$(A) = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \text{ in kN/mm}$$

$$\text{hence, } [A^c] = \begin{pmatrix} + \begin{vmatrix} 79.77 & 0 \\ 0 & 25.93 \end{vmatrix} & - \begin{vmatrix} 27.92 & 0 \\ 0 & 25.93 \end{vmatrix} & + \begin{vmatrix} 27.92 & 79.77 \\ 0 & 0 \end{vmatrix} \\ - \begin{vmatrix} 27.92 & 0 \\ 0 & 25.93 \end{vmatrix} & + \begin{vmatrix} 79.77 & 0 \\ 0 & 25.93 \end{vmatrix} & - \begin{vmatrix} 79.77 & 27.92 \\ 0 & 0 \end{vmatrix} \\ + \begin{vmatrix} 27.92 & 0 \\ 79.77 & 0 \end{vmatrix} & - \begin{vmatrix} 79.77 & 0 \\ 27.92 & 0 \end{vmatrix} & + \begin{vmatrix} 79.77 & 27.92 \\ 27.92 & 79.77 \end{vmatrix} \end{pmatrix}$$

$$\text{i.e. } (A^c) = \begin{pmatrix} 2068 & -724.0 & 0 \\ -724.0 & 2068 & 0 \\ 0 & 0 & 5584 \end{pmatrix}$$

and 
$$(A^{CT}) = \begin{pmatrix} 2068 & -724.0 & 0 \\ -724.0 & 2068 & 0 \\ 0 & 0 & 5584 \end{pmatrix}$$

The determinate of A,  $|A| = 79.77(2068) + 27.92(-724)$   
 $= 144800$

$$(a) = \frac{1}{144800}(A)^a$$

i.e. 
$$(a) = \begin{pmatrix} 0.00001428 & -0.000005 & 0 \\ -0.000005 & 0.00001428 & 0 \\ 0 & 0 & 0.00003856 \end{pmatrix} \text{ in mm/N}$$

or 
$$(a) = \begin{pmatrix} 1.428 \times 10^{-5} & -5.0 \times 10^{-6} & 0 \\ -5.0 \times 10^{-6} & 1.428 \times 10^{-5} & 0 \\ 0 & 0 & 3.856 \times 10^{-5} \end{pmatrix} \text{ in mm/N}$$

**3.** An aluminium sheet is 1 mm thick 300 mm long and 200 mm wide. The sheet is clamped at one end, and a tensile load of 6 kN is applied across the 200 mm width (i.e. the x direction). Using the stiffness matrix (Q) (for question 1 above), determine the strains and hence the stress in the x and y directions, given the matrix (a). For the aluminium assume that  $E = 70 \text{ GPa}$ ,  $\nu = 0.35$ , and for an

isotropic material:  $G = \frac{E}{2(1+\nu)}$

Also given: 
$$(a) = \begin{pmatrix} 1.429 \times 10^{-5} & -5.0 \times 10^{-6} & 0 \\ -5.0 \times 10^{-6} & 1.429 \times 10^{-5} & 0 \\ 0 & 0 & 3.857 \times 10^{-5} \end{pmatrix} \text{ in mm/N}$$

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} \varepsilon^o \\ \kappa \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^o \\ \kappa \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} N \\ M \end{pmatrix}$$

and  $(B) = 0$  due to symmetry

Hence, 
$$(N) = (A)(\varepsilon^o) + (B)(\kappa)$$

Therefore, 
$$(\varepsilon^o) = (a)(N)$$

Now,  $N_x = 6000 / 200 = 30 \text{ N/mm}$

Hence,  $(N) = \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix} \text{ in N / mm}$

$$(\varepsilon^o) = (a) \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix} \text{ units: (mm/N)(N/mm)}$$

$$(\varepsilon^o) = \begin{pmatrix} 0.00001428 & -0.000005 & 0 \\ -0.000005 & 0.00001428 & 0 \\ 0 & 0 & 0.00003856 \end{pmatrix} \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix}$$

$$(\varepsilon^o) = \begin{pmatrix} 4.287 \times 10^{-4} \\ -1.5 \times 10^{-4} \\ 0 \end{pmatrix} \text{ in units of strain}$$

$$(\sigma) = (Q)(\varepsilon^o)$$

where (Q) is given by:

$$(Q) = \left[ \frac{1}{1-\nu^2} \right] \begin{pmatrix} E & \nu E & 0 \\ \nu E & E & 0 \\ 0 & 0 & (1-\nu^2)G \end{pmatrix}$$

i.e.  $(Q) = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \text{ in GPa}$

and

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \begin{pmatrix} 4.287 \times 10^{-4} \\ -1.5 \times 10^{-4} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} 30 \\ 0 \\ 0 \end{pmatrix} \text{ MPa}$$

4. The aluminium sheet described in Problem 1 above is 1 mm thick, 300 mm long and 200 mm wide. The sheet is clamped at one end forming a 300 mm long cantilever. A load is applied which produces a pure bending moment of 2000 N mm across the 200 mm width of the sheet (i.e.  $M_x$ ). Using the stiffness matrix (Q) (for Problem 1 above), determine the strain on the top and bottom of the sheet. Use this to determine the stress on the top and bottom of the sheet and sketch the graph of stresses in the  $x$  and  $y$  directions, given the matrix (d).

$$(d) = \begin{pmatrix} 0.0001714 & -0.00006 & 0 \\ -0.00006 & 0.0001714 & 0 \\ 0 & 0 & 0.0004628 \end{pmatrix} \text{ in units } \text{N}^{-1}\text{mm}^{-1}$$

$$\left[ (\kappa) = \begin{pmatrix} 0.001714 \\ -0.00006 \\ 0 \end{pmatrix} \text{ in units of } \text{mm}^{-1}, (\sigma)_{\text{TOP}} = \begin{pmatrix} -60.0 \\ 0 \\ 0 \end{pmatrix} \text{ in MPa}, (\sigma)_{\text{BOT}} = \begin{pmatrix} 60.0 \\ 0 \\ 0 \end{pmatrix} \text{ in MPa} \right]$$

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} \varepsilon^o \\ \kappa \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^o \\ \kappa \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} a & b \\ h & d \end{pmatrix} \begin{pmatrix} N \\ M \end{pmatrix}$$

Hence,  $(M) = (B)(\varepsilon^o) + (D)(\kappa)$

and  $(B) = 0$  due to symmetry

and  $(h) = 0$

$$\Rightarrow [\kappa] = [h] [N] + [d] [M] \quad \begin{matrix} \nearrow \\ = 0 \end{matrix}$$

Loading of 2000 N mm over 200 mm width

$M_x = 2000/200 = 10 \text{ N mm/mm}$  (i.e. Newtons)

$$[d] = \begin{pmatrix} 0.0001714 & -0.00006 & 0 \\ -0.00006 & 0.0001714 & 0 \\ 0 & 0 & 0.0004628 \end{pmatrix} \text{ N}^{-1}\text{mm}^{-1}$$

$$(\kappa) = (h)(N)$$

$$(\kappa) = \begin{pmatrix} 0.0001714 & -0.00006 & 0 \\ -0.00006 & 0.0001714 & 0 \\ 0 & 0 & 0.0004628 \end{pmatrix} \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

i.e.

$$(\kappa) = \begin{pmatrix} \mathbf{0.001714} \\ \mathbf{-0.00006} \\ 0 \end{pmatrix} \text{mm}^{-1}$$

$$(\varepsilon(z)) = (\varepsilon^0) + (z)(\kappa) \quad \nearrow = 0$$

Top surface  $\Rightarrow z = -0.5 \text{ mm}$  ( $z$  is +ve downwards)

$$(\varepsilon) = -0.5(\kappa)$$

$$(\varepsilon) = \begin{pmatrix} -0.000857 \\ 0.0003 \\ 0 \end{pmatrix} \text{strain}$$

$$(\sigma) = (Q)(\varepsilon^0)$$

$$(\sigma) = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \begin{pmatrix} -0.000857 \\ 0.0003 \\ 0 \end{pmatrix} \text{N/mm}^2 \times 10^3$$

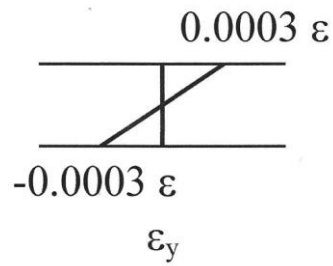
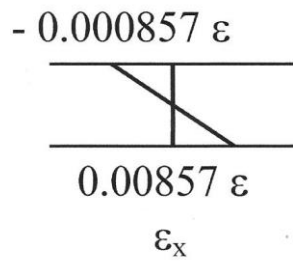
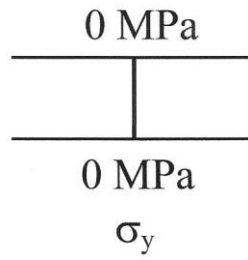
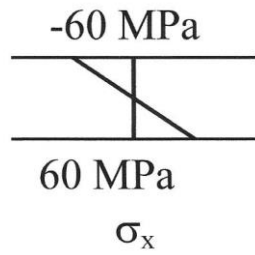
$$(\sigma)_{\text{TOP}} = \begin{pmatrix} \mathbf{-68.36 + 8.376} \\ \mathbf{-23.93 - 23.93} \\ \mathbf{0} \end{pmatrix} \times 10^3 = \begin{pmatrix} \mathbf{-60} \\ 0 \\ 0 \end{pmatrix} \text{MPa}$$

Mid-plane:  $z = 0 \Rightarrow (\kappa) = 0$  so  $(\sigma)_{\text{MIDPLANE}} = 0$

Bottom surface  $\Rightarrow z = +0.5 \text{ mm}$

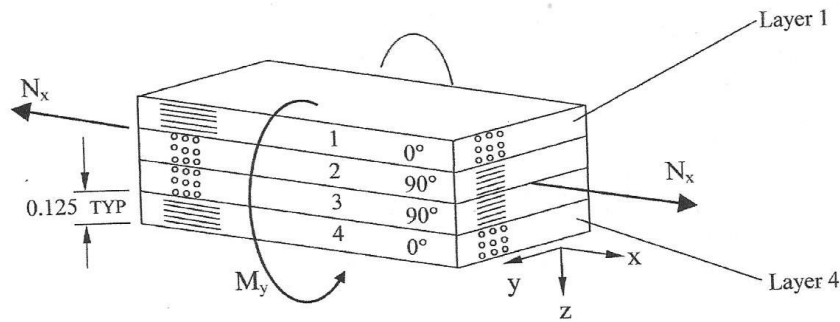
$$(\sigma) = \begin{pmatrix} 79.77 & 27.92 & 0 \\ 27.92 & 79.77 & 0 \\ 0 & 0 & 25.93 \end{pmatrix} \begin{pmatrix} 0.000857 \\ -0.0003 \\ 0 \end{pmatrix}$$

$$(\sigma)_{\text{BOTTOM}} = \begin{pmatrix} 68.36 - 8.376 \\ 23.93 + 23.93 \\ 0 \end{pmatrix} \times 10^3 = \begin{pmatrix} 60 \\ 0 \\ 0 \end{pmatrix} \text{ MPa}$$



## EXERCISE 65, Page 381

1. A laminate, shown below, consists of 4 plies of unidirectional high modulus carbon fibres in an epoxy resin, each 0.125 mm thick and arranged as  $(0/90)_s$ . Given that  $E_1 = 180$  GPa,  $E_2 = 8$  GPa,  $G_{12} = 5$  GPa and  $\nu_{12} = 0.3$ , calculate the reduced stiffness matrices ( $Q$ ) for the single plies along the fibres (i.e. at  $0^\circ$ ).



$$E_1 = 180 \text{ GPa}, \quad E_2 = 8 \text{ GPa}, \quad G_{12} = 5 \text{ GPa}, \quad \nu_{12} = 0.3, \quad \nu_{21} = \frac{E_2}{E_1} \nu_{12}$$

and 
$$\nu_{21} = \frac{8}{180}(0.3) = 0.01333$$

$$(Q)_0 = (Q)_{90}$$

$$= \left[ \frac{1}{1 - \nu_{12}\nu_{21}} \right] \begin{pmatrix} E_1 & \nu_{21}E_1 & 0 \\ \nu_{12}E_2 & E_2 & 0 \\ 0 & 0 & (1 - \nu_{12}\nu_{21})G_{12} \end{pmatrix}$$

$$= \left[ \frac{1}{0.996} \right] \begin{pmatrix} 180 & 0.01333(180) & 0 \\ 0.3(8) & 8 & 0 \\ 0 & 0 & 0.995(5) \end{pmatrix}$$

$$\text{i.e. } (Q)_0 = (Q)_{90} = \begin{pmatrix} 180.7 & 2.41 & 0 \\ 2.41 & 8.032 & 0 \\ 0 & 0 & 5 \end{pmatrix} \text{ in GPa}$$

2. For the laminate described in Problem 5, write out the transformation matrices ( $T$ ) and  $(T)^{-1}$



$$(\mathbf{T})_0 = \begin{pmatrix} c^2 & s^2 & +2sc \\ s^2 & c^2 & -2sc \\ -sc & +sc & c^2 - s^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[\mathbf{T}]_0^{-1} = \begin{pmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & +2sc \\ +sc & -sc & c^2 - s^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\mathbf{T})_{90} = \begin{pmatrix} c^2 & s^2 & +2sc \\ s^2 & c^2 & -2sc \\ -sc & +sc & c^2 - s^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$[\mathbf{T}]_{90}^{-1} = \begin{pmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & +2sc \\ +sc & -sc & c^2 - s^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

**3.** For the laminate described in Problem 5, calculate the reduced stiffness matrices  $(Q^*)_0$  and  $(Q^*)_{90}$  for the single plies with respect to the global x-axis (i.e.  $\theta = 0^\circ$ ).

In this case:

$(\mathbf{T})_0 = (\mathbf{I})$  the identity matrix (see Question 6)

$(\mathbf{T})_{90} = (\mathbf{T})_{90}^{-1}$  from inspection of  $(\mathbf{T})$  and  $(\mathbf{T})^{-1}$

$(Q^*)_{90} = (\mathbf{T})^{-1}(\mathbf{Q})(\mathbf{R})(\mathbf{T})(\mathbf{R})^{-1}$

$(Q^*)_0 = (Q)_0$

Since  $\theta = 0$  and  $(\mathbf{T})_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(\mathbf{Q})_{90} = \begin{pmatrix} 180.7 & 2.41 & 0 \\ 2.41 & 8.032 & 0 \\ 0 & 0 & 5.0 \end{pmatrix}$  in GPa

$$(T)(R)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -0.5 \end{pmatrix}$$

$$(R)(T)(R)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -0.5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(Q)(R)(T)(R)^{-1} = \begin{pmatrix} 180.7 & 2.41 & 0 \\ 2.41 & 8.032 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2.41 & 180.7 & 0 \\ 8.032 & 2.41 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$(Q^*)_{90} = (T)^{-1} (Q)(R)(T)(R)^{-1}$$

$$(Q^*)_{90} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2.41 & 180.7 & 0 \\ 8.032 & 2.41 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\text{i.e. } (Q^*)_{90} = \begin{pmatrix} 8.032 & 2.41 & 0 \\ 2.41 & 180.7 & 0 \\ 0 & 0 & 5 \end{pmatrix} \text{ in GPa}$$

**4.** A load of 1 N/mm is applied in the  $x$ -direction ( $N_x$ ) and a moment of 1 N mm/mm in the  $y$ -direction ( $M_y$ ), to the laminate described in Problem 1, Exercise 65. Using the ABD and the abhd matrices for the laminate described in Problem 1 given below:

- (i) Calculate the direct and the bending strains in the  $0^\circ$  and the  $90^\circ$  layers of the laminate.
- (ii) Sketch the graph of the direct and the bending strains in the  $x$  and  $y$  directions through each layer of the laminate.
- (iii) Calculate the direct and the bending stresses in the  $0^\circ$  and the  $90^\circ$  layers of the laminate.
- (iv) Sketch the graph of the direct and the bending stresses in the  $x$  and  $y$  directions through each layer of the laminate.

$$\begin{aligned}
 (ABD) &= \begin{pmatrix} 47.19 & 1.205 & 0 & 0 & 0 & 0 \\ 1.205 & 47.19 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.658 & 0.0251 & 0 \\ 0 & 0 & 0 & 0.0251 & 0.3085 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.05208 \end{pmatrix} \times 10^3 & \begin{array}{l} A \text{ in N/mm} \\ B \text{ in N} \\ D \text{ in Nmm} \end{array} \\
 (abd) &= \begin{pmatrix} 21.21 & -0.5414 & 0 & 0 & 0 & 0 \\ -0.5414 & 21.21 & 0 & 0 & 0 & 0 \\ 0 & 0 & 400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 604 & -49.14 & 0 \\ 0 & 0 & 0 & -49.14 & 32.45 & 0 \\ 0 & 0 & 0 & 0 & 0 & 19200 \end{pmatrix} \times 10^{-6} & \begin{array}{l} a \text{ in mm/N} \\ b \text{ in 1/N} \\ d \text{ in 1/(Nmm)} \end{array}
 \end{aligned}$$

(i)

$$\begin{pmatrix} N \\ M \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \begin{pmatrix} \varepsilon^o \\ \kappa \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon^o \\ \kappa \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix}^{-1} \begin{pmatrix} N \\ M \end{pmatrix}$$

and  $(B) = 0$  due to symmetry

Hence,  $(N) = (A)(\varepsilon^o) + (B)(\kappa)$

Now,  $N_x = 1 \text{ N/mm}$  and  $M_y = 1 \text{ N mm/mm}$

Therefore,  $(N) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  in N/mm and  $(M) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$  in N

The direct and bending strains in the  $0^\circ$  and the  $90^\circ$  layers must be compatible.

$$(\varepsilon(z)) = (\varepsilon^o) + (z)(\kappa)$$

**Mid-plane (direct) strain:**

$$(\varepsilon^o)_0 = (\varepsilon^o)_{90}$$

$$(\varepsilon^o) = (a)(N) + (b)(M) \quad \text{where } (b) = 0$$

Therefore,

$$(\epsilon^0) = (a) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ in (mm / N)(N / mm) i.e. strain}$$

$$(\epsilon^0) = \begin{pmatrix} 21.21 & -0.5414 & 0 \\ -0.5414 & 21.21 & 0 \\ 0 & 0 & 400 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times 10^{-6} \text{ strain}$$

i.e.  $(\epsilon^0) = \begin{pmatrix} \mathbf{21.21} \\ \mathbf{-0.5414} \\ \mathbf{0} \end{pmatrix} \text{ in units of } \mu\epsilon$

**Bending strain:**

$$(\epsilon^0)_{\text{BENDING}} = z(\kappa)$$

and  $(\kappa) = (h)(N) + (d)(M) \quad \text{where } (h) = 0$

Therefore,

$$(\kappa) = 0 + \begin{pmatrix} 604 & -49.14 & 0 \\ -49.14 & 3245 & 0 \\ 0 & 0 & 19200 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times 10^{-6}$$

i.e.  $(\kappa) = \begin{pmatrix} -49.14 \\ 3245 \\ 0 \end{pmatrix} \times 10^{-6} \frac{1}{\text{N mm}} (\text{N}) \text{ i.e. mm}^{-1}$

$$(\epsilon)_{\text{BENDING}} = z \begin{bmatrix} -49.14 \\ 3245 \\ 0 \end{bmatrix} \times 10^{-6} \text{ mm/mm i.e. strain}$$

### **Layer 1**

Top:  $z = -0.25 \text{ mm}$

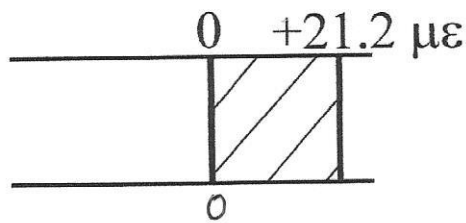
$$(\epsilon)_{\text{BENDING}} = (-0.25) \begin{pmatrix} \mathbf{-49.14} \\ \mathbf{3245} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{+12.48} \\ \mathbf{-811.3} \\ \mathbf{0} \end{pmatrix} \mu\epsilon$$

### **Layer 4**

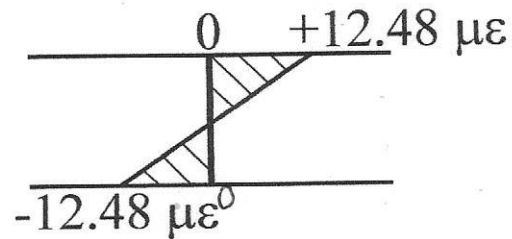
Bottom:  $z = 0.25 \text{ mm}$

$$(\epsilon)_{\text{BENDING}} = (+0.25) \begin{pmatrix} -49.14 \\ 3245 \\ 0 \end{pmatrix} = \begin{pmatrix} -12.48 \\ +811.3 \\ 0 \end{pmatrix} \mu\epsilon$$

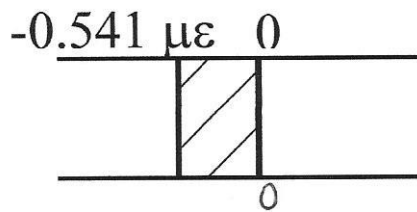
(ii)



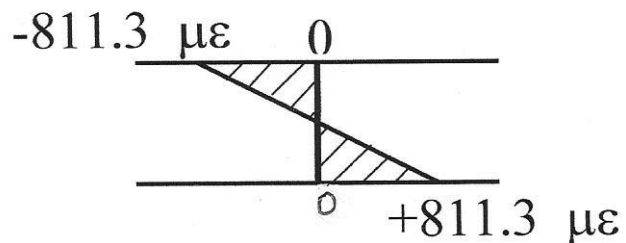
$\epsilon_x$  DIRECT



$\epsilon_x$  BENDING



$\epsilon_y$  DIRECT



$\epsilon_y$  BENDING

(iii)

**Mid-plane (direct) stress:**

$$(\sigma)_{\text{DIRECT}} = (Q^*)(\epsilon^0) \quad \text{layer dependent}$$

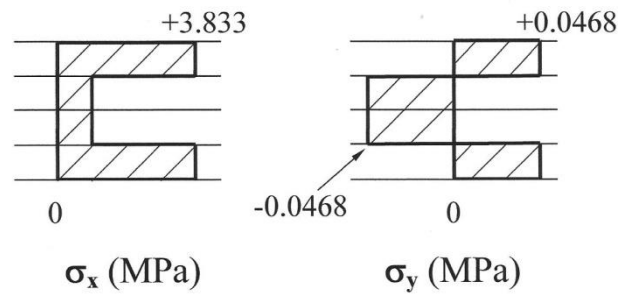
**Layer 1**

$$(\sigma)_{0 \text{ DIR}} = \begin{pmatrix} 180.7 & 2.41 & 0 \\ 2.41 & 8.032 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 21.21 \\ -0.5414 \\ 0 \end{pmatrix} = \begin{pmatrix} 3833 \\ 46.79 \\ 0 \end{pmatrix} \text{ kPa}$$

**Layer 2**

$$(\sigma)_{90 \text{ DIR}} = \begin{pmatrix} 8.032 & 2.41 & 0 \\ 2.41 & 180.7 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 21.21 \\ -0.5414 \\ 0 \end{pmatrix} = \begin{pmatrix} 169.1 \\ -46.79 \\ 0 \end{pmatrix} \text{ kPa}$$

Due to symmetry, layers 4 and 3 will be as for 1 and 2 respectively.



**Bending stress:**

$$(\sigma)_{0 \text{ BENDING}} = z(Q^*)_0(\kappa)$$

$$(\sigma)_{90 \text{ BENDING}} = z(Q^*)_{90}(\kappa)$$

This describes a linear variation of stress through the cross-section, with a different constant of proportionality for each layer (e.g.  $0^\circ$  and  $90^\circ$ ).

$$(\sigma)_{0 \text{ BENDING}} = z \begin{pmatrix} 180.7 & 2.41 & 0 \\ 2.41 & 8.032 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -49.14 \\ 3245 \\ 0 \end{pmatrix}$$

$$(\sigma)_{0 \text{ BENDING}} = z \begin{pmatrix} -1059 \\ 25950 \\ 0 \end{pmatrix} \text{ kPa}$$

$$(\sigma)_{90 \text{ BENDING}} = z \begin{pmatrix} 8.032 & 2.41 & 0 \\ 2.41 & 180.7 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} -49.14 \\ 3245 \\ 0 \end{pmatrix}$$

$$(\sigma)_{90 \text{ BENDING}} = z \begin{pmatrix} 7426 \\ 586300 \\ 0 \end{pmatrix} \text{ kPa}$$

**Top of layer 1:**

$z = -0.25 \text{ mm } (\theta = 0^\circ)$

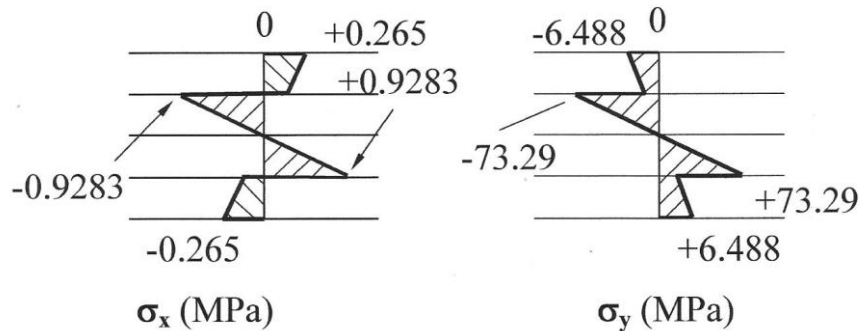
$$(\sigma)_{0 \text{ BEND}} = (-0.25) \begin{pmatrix} -1059 \\ 25950 \\ 0 \end{pmatrix} = \begin{pmatrix} +264.8 \\ -6488 \\ 0 \end{pmatrix} \text{ kPa}$$

**Top of layer 2:**

$$z = -0.125 \text{ mm } (\theta = 90^\circ)$$

$$(\sigma)_{90 \text{ BEND}} = (-0.125) \begin{pmatrix} 7426 \\ 586300 \\ 0 \end{pmatrix} = \begin{pmatrix} -928.3 \\ -73290 \\ 0 \end{pmatrix} \text{ kPa}$$

(iv)



**5.** Show, using the ABD matrix given in Problem 4, that the abdh matrix for the laminate is as given in Problem 4. Note that  $(B) = 0$ , hence:

$$(b) = (h) = 0 \quad (A) = (a)^{-1} \quad (d) = (D)^{-1}$$

$$(A) = \begin{pmatrix} 47.19 & 1.205 & 0 \\ 1.205 & 47.19 & 0 \\ 0 & 0 & 2.5 \end{pmatrix} \text{ in kN/mm}$$

$$(A)^{-1} = \frac{(A)^a}{|A|} = \frac{(A)^{c^T}}{|A|}$$

$$[A]^c = \begin{pmatrix} + \begin{vmatrix} 47.19 & 0 \\ 0 & 2.5 \end{vmatrix} & - \begin{vmatrix} 1.205 & 0 \\ 0 & 2.5 \end{vmatrix} & + \begin{vmatrix} 1.205 & 47.19 \\ 0 & 0 \end{vmatrix} \\ - \begin{vmatrix} 1.205 & 0 \\ 47.19 & 2.5 \end{vmatrix} & + \begin{vmatrix} 47.19 & 0 \\ 0 & 2.5 \end{vmatrix} & - \begin{vmatrix} 47.19 & 1.205 \\ 0 & 0 \end{vmatrix} \\ + \begin{vmatrix} 1.205 & 0 \\ 47.19 & 0 \end{vmatrix} & - \begin{vmatrix} 47.19 & 0 \\ 1.205 & 0 \end{vmatrix} & + \begin{vmatrix} 47.19 & 1.205 \\ 1.205 & 47.19 \end{vmatrix} \end{pmatrix}$$

$$(A)^c = \begin{pmatrix} 118.0 & -3.013 & 0 \\ -3.013 & 118.0 & 0 \\ 0 & 0 & 2225 \end{pmatrix}$$

$$(A)^{CT} = \begin{pmatrix} 118.0 & -3.013 & 0 \\ -3.013 & 118.0 & 0 \\ 0 & 0 & 2225 \end{pmatrix}$$

$$|A| = 47.19(118.0) - 1.205(3.013) + 0 = 5565$$

$$(a) = (A)^{-1} = \frac{1}{5565} \begin{pmatrix} 118.0 & -3.013 & 0 \\ -3.013 & 118.0 & 0 \\ 0 & 0 & 2225 \end{pmatrix}$$

$$(a) = \begin{pmatrix} \mathbf{0.02120} & \mathbf{-0.0005414} & \mathbf{0} \\ \mathbf{-0.0005414} & \mathbf{0.02120} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0.400} \end{pmatrix} \quad \text{in mm/kN}$$

$$(D) = \begin{pmatrix} 1658 & 25.1 & 0 \\ 25.1 & 308.5 & 0 \\ 0 & 0 & 52.08 \end{pmatrix} \text{ N mm}$$

$$(D)^{-1} = \frac{(D)^a}{|D|} = \frac{(D)^{CT}}{|D|}$$

$$(D)^C = \begin{pmatrix} + \begin{vmatrix} 308.5 & 0 \\ 0 & 52.08 \end{vmatrix} & - \begin{vmatrix} 25.1 & 0 \\ 1658 & 52.08 \end{vmatrix} & + \begin{vmatrix} 25.1 & 308.5 \\ 0 & 0 \end{vmatrix} \\ - \begin{vmatrix} 25.1 & 0 \\ 0 & 52.08 \end{vmatrix} & + \begin{vmatrix} 1658 & 0 \\ 0 & 52.08 \end{vmatrix} & - \begin{vmatrix} 1658 & 25.1 \\ 0 & 0 \end{vmatrix} \\ + \begin{vmatrix} 25.1 & 0 \\ 308.5 & 0 \end{vmatrix} & - \begin{vmatrix} 1658 & 0 \\ 25.1 & 0 \end{vmatrix} & + \begin{vmatrix} 1658 & 25.1 \\ 25.1 & 308.5 \end{vmatrix} \end{pmatrix}$$

Due to symmetry

$$(D)^a = (D)^{CT} = \begin{pmatrix} 16070 & -1307 & 0 \\ -1307 & 86350 & 0 \\ 0 & 0 & 510900 \end{pmatrix}$$

$$|D| = +1658(16070) + (25.1)(-1307) + 0$$

$$|D| = 26.61 \times 10^6$$

$$(d) = (D)^{-1} = \frac{1}{26.61 \times 10^6} \begin{pmatrix} 16070 & -1307 & 0 \\ -1307 & 86350 & 0 \\ 0 & 0 & 510900 \end{pmatrix}$$



$$(\mathbf{d}) = \begin{pmatrix} 603.9 & -49.11 & 0 \\ -49.11 & 3245 & 0 \\ 0 & 0 & 19200 \end{pmatrix} \times 10^{-6} \text{ in mm}^{-1}\text{N}^{-1}$$


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