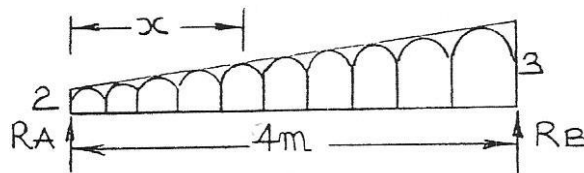
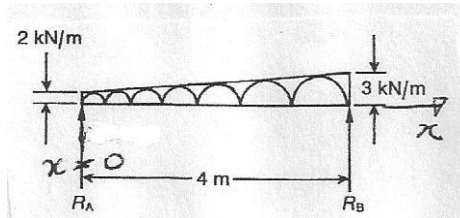


## CHAPTER 7 BEAM DEFLECTIONS DUE TO BENDING

### EXERCISE 33, Page 156

1. Obtain an expression for the deflection  $y$  at any distance  $x$  from the left end of the uniform section beam of shown.



$$w = 2 + x/4$$

Taking moments about B gives:

$$\begin{aligned} R_A \times 4 &= 2 \times 4 \times 2 + \frac{1}{2} \times 4 \times 1 \times \frac{1}{3} \times 4 \\ &= 16 + 2.667 \end{aligned}$$

i.e.  $R_A = 4.667 \text{ kN}$

Resolving vertically gives:

$$R_A + R_B = \frac{(2+3)}{2} \times 4 = 10 \quad 1$$

i.e.  $R_B = 10 - 4.667 = 5.333 \text{ kN}$

$$\begin{aligned} EI \frac{d^2 y}{dx^2} &= 4.667x - \frac{2 \times x}{2} \times \frac{2}{3}x - \frac{1}{2} \left( 2 + \frac{x}{4} \right) \times x \times \frac{1}{3} \times x \\ &= 4.667x - 0.667x^2 - \frac{1}{6} \left( 2x^2 + \frac{x^3}{4} \right) \\ &= 4.667x - 0.667x^2 - 0.333x^2 - 0.0417x^3 \end{aligned}$$

i.e.  $EI \frac{d^2 y}{dx^2} = 4.667x - x^2 - 0.0417x^3$

$$EI \frac{dy}{dx} = \left( \frac{4.667}{2} \right) x^2 - \frac{1}{3} x^3 - \left( \frac{0.0417}{4} \right) x^4 + A$$

i.e.  $EI \frac{dy}{dx} = 2.333x^2 - 0.333x^3 - 0.01043x^4 + A$

$$EIy = \left( \frac{2.333}{3} \right) x^3 - \left( \frac{0.333}{4} \right) x^4 - \left( \frac{0.01043}{5} \right) x^5 + Ax + B$$

i.e.  $EIy = 0.778x^3 - 0.0833x^4 - 0.002086x^5 + Ax + B$  (1)

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 4, y = 0$ ,

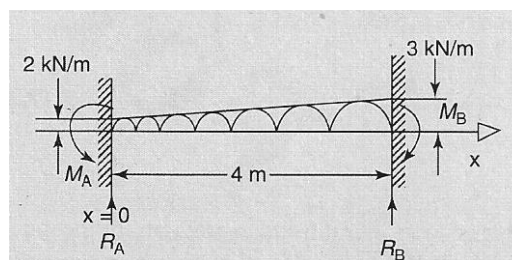
$$0 = 49.79 - 21.32 - 2.14 + 4A$$

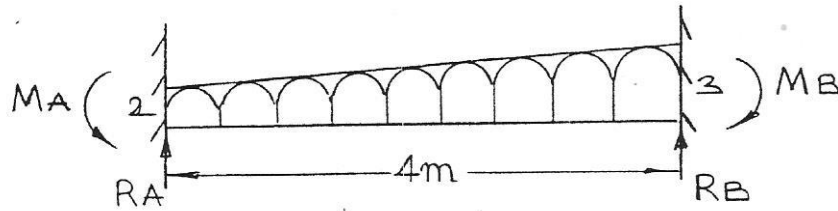
from which,  $A = -26.33/4 = \mathbf{6.583}$

i.e. from equation (1),

$$y = \frac{1}{EI} (0.778x^3 - 0.0833x^4 - 2.086 \times 10^{-3} x^5 - 6.583x)$$

**2.** Determine the value of the reactions and end fixing moments for the uniform-section beam shown. Hence, or otherwise, obtain an expression for the deflection  $y$  at any distance  $x$  from the left end of the beam by the double integration method.





$$EI \frac{d^2y}{dx^2} = R_A x - M_A - 0.667x^2 - 0.333x^2 - 0.0417x^3$$

i.e.  $EI \frac{d^2y}{dx^2} = R_A x - M_A - x^2 - 0.0417x^3$

and  $EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{x^3}{3} - \frac{0.0417x^4}{4} + A$

$$EI \frac{dy}{dx} = 0.5 R_A x^2 - M_A x - 0.333x^3 - 0.01043x^4 + A$$

and  $EI y = \frac{0.5 R_A x^3}{3} - \frac{M_A x^2}{2} - \frac{0.333x^4}{4} - \frac{0.01043x^5}{5} + Ax + B$

i.e.  $EI y = 0.167 R_A x^3 - 0.5 M_A x^2 - 0.0833x^4 - 2.086 \times 10^{-3} x^5 + Ax + B$  (i)

At  $x = 0, y = 0$  hence,  $B = 0$

At  $x = 0, \frac{dy}{dx} = 0$ , hence,  $A = 0$

Therefore, from equation (i),

$$y = \frac{1}{EI} \{0.77 x^3 - 1.62 x^2 - 0.0833x^4 - 2.086 \times 10^{-3} x^5\}$$

At  $x = 4, \frac{dy}{dx} = 0$ ,

hence,  $0 = 8 R_A - 4 M_A - 21.31 - 2.67$

i.e.  $8 R_A - 4 M_A - 24 = 0$  (1)

At  $x = 4, y = 0$ ,

hence,  $0 = 10.69 R_A - 8 M_A - 21.43 - 2.13$

i.e.  $5.345 R_A - 4 M_A - 11.73 = 0$  (2)

Taking equation (2) from equation (1) gives:

$$2.655 R_A - 12.27 = 0$$

from which,  $R_A = 4.62 \text{ kN}$

Substituting equation (3) into equation (1) gives:

$$36.97 - 4 M_A - 24 = 0$$

from which,  $M_A = 3.24 \text{ kN m}$

$$R_B = 5.38 \text{ kN}$$

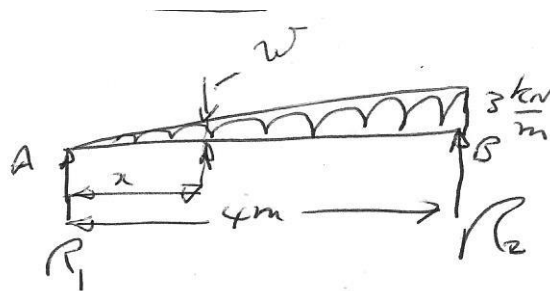
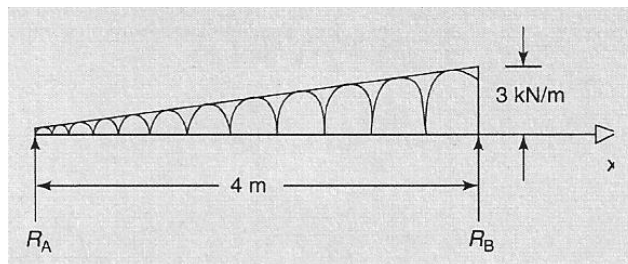
Taking moments about B gives:

$$M_B + 4.62 \times 4 = 3.24 + 2 \times 4 \times 2 + \frac{1}{2} \times 4 \times 1 \times \frac{1}{3} \times 4$$

i.e.  $M_B = -18.48 + 3.24 + 16 + 2.67$

from which,  $M_B = 3.43 \text{ kN m}$

3. Obtain an expression for the deflection  $y$  at any distance  $x$  from the left end of the uniform section beam shown.



$$w = \frac{3x}{l} = \frac{3x}{4} = 0.75x \text{ kN/m}$$

Taking moments about B gives:

$$R_1 \times 4 \text{ m} = 3 \frac{\text{kN}}{\text{m}} \times \frac{4 \text{ m}}{2} \times \frac{1}{3} \times 4 \text{ m}$$

i.e.  $4R_1 = 8$  and  $R_1 = 2 \text{ kN}$

Resolving vertically gives:

$$R_1 + R_2 = \frac{3 \times 4}{2} = 6 \text{ kN}$$

Hence,  $R_2 = 6 - 2 = 4 \text{ kN}$

$$EI \frac{d^2 y}{dx^2} = M = R_1 \times x - w \times \frac{x}{2} \times \frac{x}{3}$$

$$= 2x - \frac{0.75x^3}{6}$$

$$EI \frac{dy}{dx} = 2 \left( \frac{x^2}{2} \right) - \frac{0.75x^4}{4 \times 6} + A$$

i.e.  $EI \frac{dy}{dx} = x^2 - 0.03125x^4 + A$  (1)

and  $EIy = \frac{x^3}{3} - \frac{0.03125x^5}{5} + Ax + B$  (2)

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 4, y = 0$ , hence,  $0 = \frac{4^3}{3} - \frac{0.03125 \times 4^5}{5} + 4A$

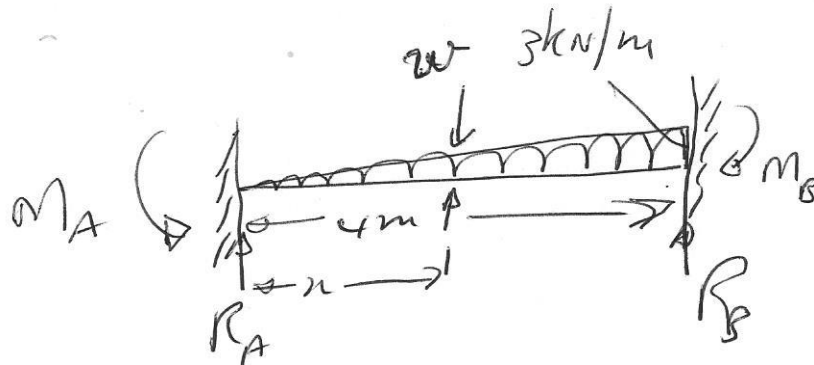
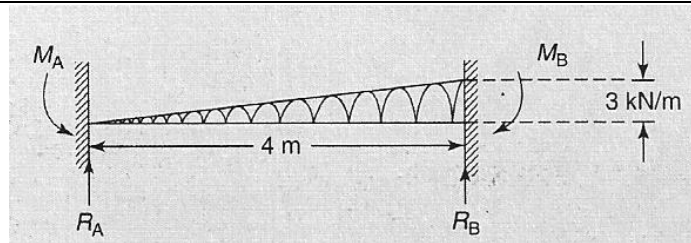
from which,  $4A = 6.4 - 21.33$

and  $A = \frac{-14.93}{4} = -3.7325$

Hence,  $EIy = \frac{x^3}{3} - \frac{0.03125x^5}{5} - 3.7325x$  (3)

or  $y = \frac{1}{EI} (0.333x^3 - 0.00643x^5 - 3.7325x)$

**4.** Determine the value of the reactions and end fixing moments for the uniform-section beam shown. Hence, or otherwise, obtain an expression for the deflection  $y$  at any distance  $x$  from the left end of the beam.



$$w = \frac{3x}{l} = \frac{3x}{4} = 0.75x \text{ kN/m}$$

Taking moments about B gives:

$$R_A \times 4 \text{ m} = M_A + 0.75 \times 4 \times \frac{4}{2} \times \frac{4}{3} - M_B$$

i.e.  $4R_A = M_A + 8 - M_B$

Resolving vertically gives:

$$R_A + R_B = \frac{3}{4} \times 4 \times \frac{4}{2} = 6$$

Hence,  $R_B = 6 - R_A$  (1)

$$\begin{aligned} EI \frac{d^2 y}{dx^2} &= R_A \times x - M_A - \frac{3}{4} \times x \times \frac{x}{2} \times \frac{x}{3} \\ &= R_A x - M_A - \frac{0.75x^3}{6} \end{aligned} \quad (2)$$

i.e.  $EI \frac{d^2 y}{dx^2} = R_A x - M_A - 0.125x^3$  (2a)

$$EI \frac{dy}{dx} = R_A \left( \frac{x^2}{2} \right) - M_A x - \frac{0.125x^4}{4} + A \quad (3)$$

At  $x = 0$ ,  $\frac{dy}{dx} = 0$ , hence,  $A = 0$

At  $x = 4$ ,  $\frac{dy}{dx} = 0$ , hence,  $0 = R_A \left( \frac{4^2}{2} \right) - 4M_A - \frac{0.125(4)^4}{4}$

from which,  $4M_A = 8R_A - 8$

and  $M_A = 2R_A - 2$  (4)

From equation (3):

$$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{0.125x^5}{20} + B \quad (5)$$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

At  $x = 4$ ,  $y = 0$ , hence,  $0 = \frac{R_A (4)^3}{6} - \frac{M_A (4)^2}{2} - \frac{0.125(4)^5}{20}$

i.e.  $0 = 10.667R_A - 8M_A - 6.4$

or  $M_A = 1.333R_A - 0.8$  (6)

Equating equations (4) and (6) gives:

$$2R_A - 2 = 1.333R_A - 0.8$$

i.e.  $0.667R_A = 1.2$

from which,  $R_A = \frac{1.2}{0.667} = 1.8 \text{ kN}$  (7)

From equation (1),  $R_B = 6 - R_A = 6 - 1.8$

i.e.  $R_B = 4.2 \text{ kN}$  (8)

Substituting equation (7) into equation (4) gives:

$$M_A = 2R_A - 2 = 2 \times 1.8 - 2$$

i.e.  $M_A = 1.6 \text{ kN m}$  (9)

At  $x = 4$ ,  $EI \frac{d^2 y}{dx^2} = M_B = 1.8 \times 4 - 1.6 - \frac{0.75(4)^3}{6}$  from equation (2a)

i.e.  $M_B = 7.2 - 1.6 - 8 = -2.4 \text{ kN m}$

From equation (5),  $EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{0.125x^5}{20} + B$

i.e.  $y = \frac{1}{EI} \left( \frac{1.8x^3}{6} - \frac{1.6x^2}{2} - \frac{0.125x^5}{20} \right)$

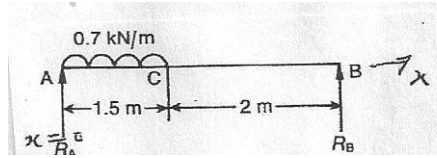
i.e.  $y = \frac{1}{EI} (0.3x^3 - 0.8x^2 - 0.00625x^5)$

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### EXERCISE 34, Page 160

1. Determine the position (from the left end) and the value of the maximum deflection for the uniform-section beam shown.



Taking moments about B gives:

$$R_A \times 3.5 = 0.7 \times 1.5 \times 2.75$$

from which,  $R_A = 0.825 \text{ kN}$

$EI \frac{d^2y}{dx^2} = 0.825x - \frac{0.7x^2}{2}$	$+ \frac{0.7}{2} (x - 1.5)^2$
$EI \frac{dy}{dx} = 0.413x^2 - 0.117x^3 + A$	$+ 0.117 (x - 1.5)^3$
$EIy = 0.138x^3 - 0.0292x^4 + Ax + B$	$+ 0.0292 (x - 1.5)^4$

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 3.5, y = 0$ ,

hence,  $0 = 5.917 - 4.38 + 3.5A + 0.467$

i.e.  $A = -0.573$

For  $\delta$ ,  $\frac{dy}{dx} = 0$  between C and B

$$0 = 0.413x^2 - 0.117x^3 - 0.573 + 0.117(x^3 + 2.25x - 1.5x^2 - 3.375)$$

i.e.  $0 = 0.238x^2 + 0.263x - 0.967$

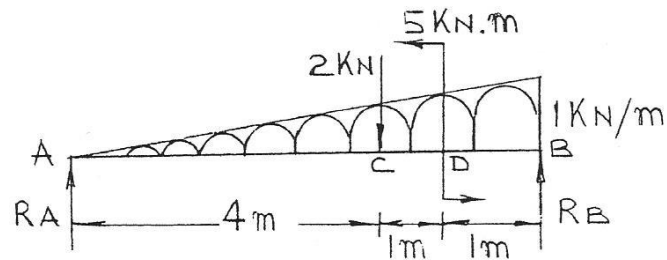
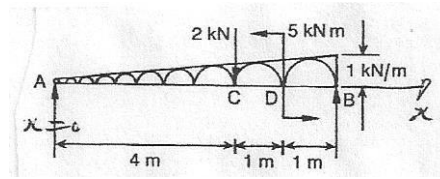
$$\text{and } x = \frac{-0.263 \pm \sqrt{(0.263)^2 - 4(0.238)(-0.967)}}{2(0.238)} = 1.54 \text{ m}$$

$$EIy = 0.138x^3 - 0.0292x^4 + Ax + B$$

and  $\delta = 0.543/EI$

2. Determine the deflections at the points C and D for the uniform-section beam shown, given that

$$EI = 4300 \text{ kNm}^2.$$



$$w = x/6$$

Taking moments about B gives:

$$R_A \times 6 = 2 \times 2 + 5 + \frac{1}{2} \times 6 \times 1 \times \frac{2}{3} \times 6$$

from which,  $R_A = 21/6 = 2.5 \text{ kN}$

Taking moments about A gives:

$$R_B \times 6 + 5 = 2 \times 4 + \frac{1}{2} \times 6 \times 1 \times \frac{2}{3} \times 6$$

from which,  $R_B = (20 - 5)/6 = 2.5 \text{ kN}$

$$2.5 + 2.5 = \frac{1}{2} \times 6 \times 1 + 2 = 5$$

$EI \frac{d^2y}{dx^2} = 2.5x - \frac{1}{2} \times x \times \frac{x}{6} \times \frac{x}{3}$	$-2(x-4)$	$-5(x-5)^0$
$EI \frac{d^2y}{dx^2} = 2.5x - \frac{x^3}{36}$	$-2(x-4)$	$-5(x-5)^0$
$EI \frac{dy}{dx} = 1.25x^2 - \frac{x^4}{144} + A$	$-(x-4)^2$	$-5(x-5)$
$EIy = 0.417x^3 - \frac{x^5}{720} + Ax + B$		

	$-\frac{(x-4)^3}{3}$	$-\frac{5}{2}(x-5)^2$
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At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 6, y = 0$

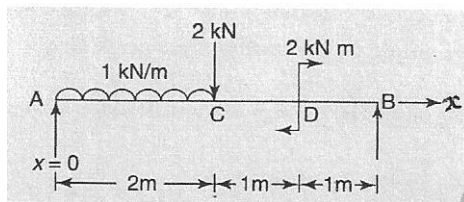
Hence,  $0 = 90 - 10.8 + 6A - 2.667 - 2.5$

from which,  $A = -12.34$

At  $x = 4$ ,  $\delta_c = \frac{1}{4300} (26.69 - 1.422 - 49.36) = -5.60 \times 10^{-3} \text{ m}$

At  $x = 5$ ,  $\delta_D = \frac{1}{4300} (52.1 - 4.34 - 61.7 - 0.3) = -3.32 \times 10^{-3} \text{ m}$

3. Determine the position and value of the maximum deflection of the simply supported beam shown, given that  $EI = 100 \text{ kN m}^2$ .



Taking moments about B gives:

$$R_A \times 4 + 2 = 2 \times 2 + 2 \times 1 \times 3$$

from which,  $R_A = (10 - 2)/4 = 2 \text{ kN}$

$EI \frac{d^2y}{dx^2} = 2x - \frac{x^2}{2}$	$-2(x-2) + \frac{(x-2)^2}{2}$	$+2(x-3)^0$
$EI \frac{dy}{dx} = x^2 - \frac{x^3}{6} + A$	$-(x-2)^2 + \frac{(x-2)^3}{6}$	$+2(x-3) \quad (1)$
$EIy = \frac{x^3}{3} - \frac{x^4}{24} + Ax + B$	$-\frac{(x-2)^3}{3} + \frac{(x-2)^4}{24}$	$+(x-3)^2 \quad (2)$

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 4, y = 0$ ,

hence,  $0 = 21.33 - 10.667 + 4A - 2.667 + 0.667 + 1$

from which,  $A = -9.663/4 = -2.416$

Therefore,  $EI \frac{dy}{dx} = x^2 - \frac{x^3}{6} - 2.416 - (x-2)^2 + \frac{(x-2)^3}{6} \cdot 4$

For maximum  $y$ ,  $\frac{dy}{dx} = 0$

Try second 'span':

$$x^2 - \frac{x^3}{6} - 2.416 - x^2 + 4x - 4 + \frac{x^3}{6} - x^2 + \frac{12x}{6} - \frac{8}{6} = 0$$

or  $2.416 + 4x - 4 - x^2 + 2x - 1.33 = 0$

i.e.  $-x^2 + 6x - 7.746 = 0$

from which,  $x = \frac{-6 \pm \sqrt{6^2 - 4(-1)(-7.746)}}{2(-1)} = \frac{-6 \pm 2.24}{-2}$

i.e.  $x = 1.88 \text{ m}$

As  $1.88 < 2$ , the maximum deflection does not occur in the second 'span'.

Consider first 'span':

$$0 = x^2 - \frac{x^3}{6} - 2.416$$

Try  $x = 1.87$ ,

then  $0 = 3.497 - 1.0899 - 2.416 = 0$

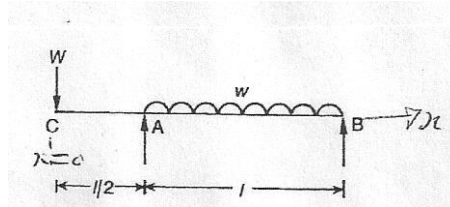
i.e.  $x = 1.87 \text{ m}$  (approximately) (3)

For maximum deflection, substituting equation (3) into equation (2) gives:

$$\delta = \frac{1}{EI} (2.18 - 0.51 - 4.518)7$$

i.e.  $\delta = -0.0285 \text{ m}$

4. The beam CAB is simply supported at the points A and B and is subjected to a concentrated load  $W$  at the point C, together with a uniformly distributed load  $w$  between the points A and B, as shown. Determine the relationship between  $W$  and  $wl$ , so that no deflection will occur at the point C.



$$R_A l = \frac{3Wl}{2} + \frac{wl^2}{2}$$

i.e. 
$$R_A = \frac{3W}{2} + \frac{wl}{2} + R_A \left( x - \frac{l}{2} \right) - \frac{w}{2} \left( x - \frac{l}{2} \right)^2$$

$EI \frac{d^2y}{dx^2} = -Wx$	$+R_A \left( x - \frac{l}{2} \right) - \frac{w}{2} \left( x - \frac{l}{2} \right)^2$
$EI \frac{dy}{dx} = -\frac{Wx^2}{2} + A$	$+\frac{R_A}{2} \left( x - \frac{l}{2} \right)^2 - \frac{w}{6} \left( x - \frac{l}{2} \right)^3$
$EIy = -\frac{Wx^3}{6} + Ax + B$	$+\frac{R_A}{6} \left( x - \frac{l}{2} \right)^3 - \frac{w}{24} \left( x - \frac{l}{2} \right)^4$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

At  $x = l/2$ ,  $y = 0$ ,

hence, 
$$0 = -\frac{Wl^3}{48} + \frac{Al}{2}$$

i.e. 
$$A = \frac{Wl^2}{24} \quad 8$$

At  $x = 3l/2$ ,  $y = 0$ ,

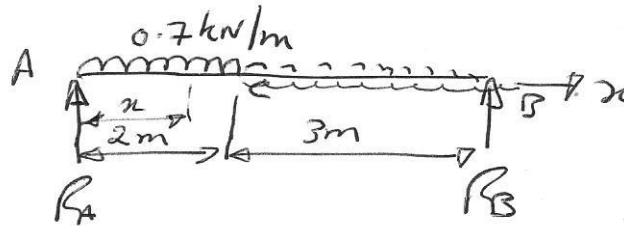
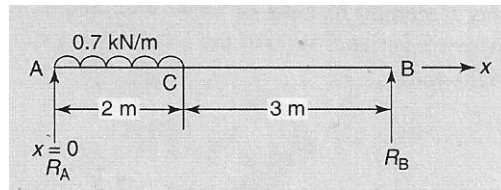
therefore, 
$$0 = -\frac{27Wl^3}{48} + \frac{3Wl^3}{48} + \frac{R_A}{6} l^3 - \frac{wl^4}{24}$$

i.e. 
$$0 = -\frac{27W}{48} + \frac{3W}{48} + \frac{3W}{12} + \frac{wl}{12} - \frac{wl}{24}$$

i.e. 
$$0 = -\frac{W}{4} + \frac{wl}{24}$$

i.e. 
$$w = \frac{6W}{l}$$

5. Determine the position (from the left end) and the value of the maximum deflection for the uniform-section beam shown.



Taking moments about  $B$  gives:

$$R_A \times 5 = 0.7 \times 2 \times 4$$

from which,  $R_A = 5.6 / 5 = \mathbf{1.12 \text{ kN}}$

$$\begin{array}{l|l} EI \frac{d^2 y}{dx^2} = 1.12x - \frac{0.7x^2}{2} & + \frac{0.7(x-2)^2}{2} \\ EI \frac{dy}{dx} = \frac{1.12x^2}{2} - \frac{0.7x^3}{6} + A & + \frac{0.7(x-2)^3}{6} \\ EIy = \frac{1.12x^3}{6} - \frac{0.7x^4}{24} + Ax + B & + \frac{0.7(x-2)^4}{24} \end{array}$$

At  $x = 0, y = 0$ , hence,  $B = 0$

$$\text{At } x = 5 \text{ m, } y = 0, \text{ hence, } 0 = \frac{1.12(5)^3}{6} - \frac{0.7(5)^4}{24} + 5A + \frac{0.7(3)^4}{24}$$

$$\text{i.e. } 0 = 23.333 - 18.229 + 5A + 2.363$$

$$\text{i.e. } 5A = 18.229 - 23.333 - 2.363 = -7.467$$

$$\text{and } A = -7.467/5 = -\mathbf{1.4934}$$

The maximum deflection,  $\delta$ , occurs where  $\frac{dy}{dx} = 0$

Since 
$$EI \frac{dy}{dx} = \frac{1.12x^2}{2} - \frac{0.7x^3}{6} + A + \frac{0.7(x-2)^3}{6}$$

i.e. 
$$EI \frac{dy}{dx} = \frac{1.12x^2}{2} - \frac{0.7x^3}{6} + A + \frac{0.7(x-2)(x^2-4x+4)}{6}$$

i.e. 
$$EI \frac{dy}{dx} = \frac{1.12x^2}{2} - \frac{0.7x^3}{6} + A + \frac{0.7(x^3-6x^2+12x-8)}{6}$$

Hence, at maximum  $\delta$ ,  $0 = 0.56x^2 - 0.1167x^3 - 1.4934 + 0.1167x^3 - 0.7x^2 + 1.4x - 0.9333$

i.e. 
$$-0.14x^2 + 1.4x - 2.4267 = 0$$

Using a calculator or the quadratic formula,

$$x = 7.769 \text{ (which is neglected) or } \mathbf{2.231 \text{ m}}$$

$$EIy = \frac{1.12x^3}{6} - \frac{0.7x^4}{24} + Ax + B + \frac{0.7(x-2)^4}{24}$$

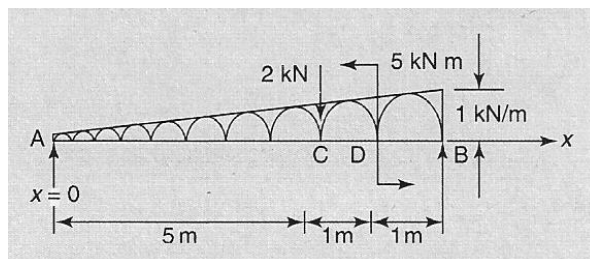
i.e. 
$$EIy = 0.1867x^3 - 0.029167x^4 - 1.4934x + 0.02917(x-2)^4$$

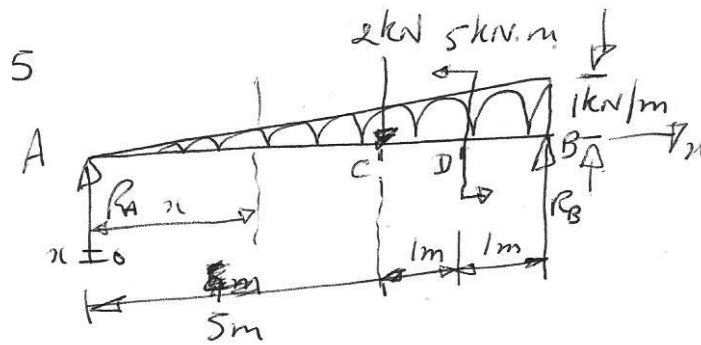
When  $x = 2.231$ ,  $\delta = \frac{1}{EI} \{0.1867(2.231)^3 - 0.029167(2.231)^4 - 1.4934(2.231) + 0.02917(0.231)^4\}$

$$= \frac{1}{EI} \{2.073 - 0.723 - 3.332 + 8.306 \times 10^{-5}\}$$

i.e. 
$$\delta = -\frac{\mathbf{1.98}}{EI}$$

**6.** Determine the deflections at the points C and D for the uniform-section shown, given that  $EI = 4300 \text{ kN/m}^2$ .





Taking moments about B gives:

$$R_A \times 7 = \frac{1 \times 7}{2} \times \frac{7}{3} + 2 \times 2 + 5$$

$$= 8.167 + 4 + 5 = 17.167$$

from which,  $R_A = 17.167 / 7 = 2.452 \text{ kN}$

Resolving vertically gives:

$$R_A + R_B = \frac{1 \times 7}{2} + 2 = 5.5$$

Hence,  $R_B = 5.5 - 2.452 = -3.048 \text{ kN}$

$$EI \frac{d^2 y}{dx^2} = 2.452x - \frac{1}{7} \times x \times \frac{x}{2} \times \frac{x}{3} \quad \left| \begin{array}{l} -2[x-5] \\ -\frac{2}{2}[x-5]^2 \\ -[x-5]^2 \end{array} \right. \quad \left| \begin{array}{l} -5[x-6]^0 \\ -5[x-6] \\ -5[x-6] \end{array} \right. \quad (1)$$

$$\text{or} \quad EI \frac{dy}{dx} = \frac{2.45x^2}{2} - \frac{x^4}{4 \times 42} + A \quad \left| \begin{array}{l} -\frac{2}{2}[x-5]^2 \\ -[x-5]^2 \end{array} \right. \quad \left| \begin{array}{l} -5[x-6] \\ -5[x-6] \end{array} \right.$$

$$EIy = \frac{1.225x^3}{3} - \frac{x^5}{5 \times 168} + Ax + B \quad \left| \begin{array}{l} -\frac{[x-5]^3}{3} \\ -\frac{[x-5]^3}{3} \end{array} \right. \quad \left| \begin{array}{l} -\frac{5}{2}[x-6]^2 \\ -\frac{5}{2}[x-6]^2 \end{array} \right. \quad (2)$$

$$\text{i.e.} \quad EIy = 0.408x^3 - \frac{x^5}{840} + Ax + B \quad \left| \begin{array}{l} -\frac{[x-5]^3}{3} \\ -\frac{[x-5]^3}{3} \end{array} \right. \quad \left| \begin{array}{l} -\frac{5}{2}[x-6]^2 \\ -\frac{5}{2}[x-6]^2 \end{array} \right.$$

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 7 \text{ m}, y = 0$ , hence,  $0 = 0.408(7)^3 - \frac{(7)^5}{840} + 7A - 2.667$



i.e.  $0 = 139.94 - 20.01 + 7A - 2.667$

i.e.  $7A = 20.01 + 2.667 - 139.94 = -117.26$

and  $A = -117.26/7 = -16.75$

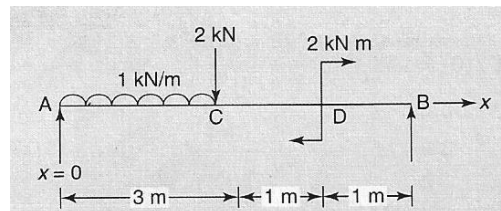
At  $x = 5$  m, 
$$\delta_C = \frac{1}{EI} \left\{ 0.408(5)^3 - \frac{(5)^5}{840} + 5A \right\} \quad \text{from equation (2)}$$
$$= \frac{1}{4300} \{ 51 - 3.72 - 83.75 \}$$

i.e.  $\delta_C = -8.48 \times 10^{-3} \text{ m}$

At  $x = 6$  m, 
$$\delta_D = \frac{1}{EI} \left\{ 0.408(6)^3 - \frac{(6)^5}{840} + 6A \right\} \quad \text{from equation (2)}$$
$$= \frac{1}{4300} \{ 88.13 - 9.26 - 100.5 \}$$

i.e.  $\delta_D = -5.03 \times 10^{-3} \text{ m}$

**7.** Determine the maximum deflection for the uniform-section beam shown, given that  $EI = 4300 \text{ kN m}^2$ .



Taking moments about  $B$  gives:

$$R_A \times 5 + 2 = 1 \times 3 \times 3.5 + 2 \times 2$$

i.e.  $5R_A = 10.5 + 4 - 2 = 12.5$

from which,  $R_A = 12.5/5 = 2.5 \text{ kN}$

$$\begin{array}{lcl}
EI \frac{d^2 y}{dx^2} = 2.5x - \frac{1 \times x^2}{2} & \left| \begin{array}{l} -2[x-3] + \frac{1 \times [x-3]^2}{2} \\ \end{array} \right| & +2[x-4]^0 \\
EI \frac{dy}{dx} = \frac{2.5x^2}{2} - \frac{x^3}{6} + A & \left| \begin{array}{l} -\frac{2}{2}[x-3]^2 + \frac{[x-3]^3}{6} \\ \end{array} \right| & +2[x-4] \quad (1) \\
EIy = \frac{2.5x^3}{6} - \frac{x^4}{24} + Ax + B & \left| \begin{array}{l} -\frac{[x-3]^3}{3} + \frac{[x-3]^4}{24} \\ \end{array} \right| & +\frac{2}{2}[x-4]^2 \quad (2)
\end{array}$$

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 5 \text{ m}, y = 0$ , hence,  $0 = \frac{2.5(5)^3}{6} - \frac{(5)^4}{24} + 5A - \frac{(2)^3}{3} + \frac{(2)^4}{24} + \frac{2}{2}(1)^2$

i.e.  $0 = 52.083 - 26.042 + 5A - 2.667 + 0.667 + 1$

i.e.  $5A = -52.083 + 26.042 + 2.667 - 0.667 - 1 = -25.041$

and  $A = -25.041/5 = -5.008$

For maximum deflection,  $\frac{dy}{dx} = 0$

Try the first span:  $0 = \frac{2.5x^2}{2} - \frac{x^3}{6} - 5.008$

Solving using a calculator or smart phone gives:

$x = 6.860 \text{ m}$  (which is impossible) or  $x = 2.436 \text{ m}$  or  $-1.80 \text{ m}$  (which is impossible)

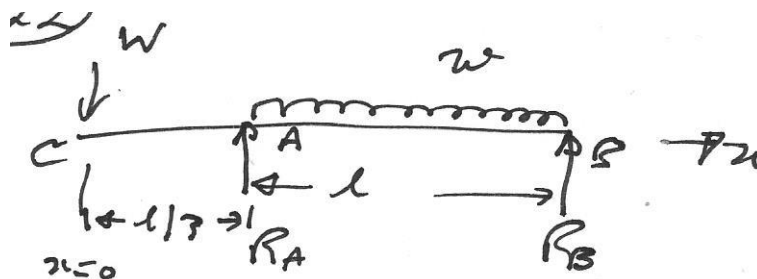
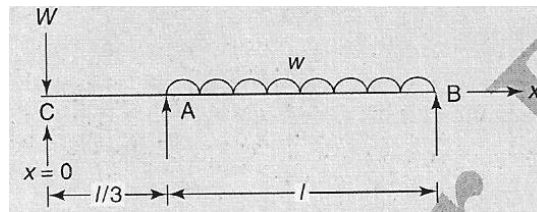
The value of  $2.436 \text{ m}$  is correct because it applies to the first quadrant – as the Macaulay bracket cannot be negative.

Hence, maximum deflection,  $\delta = \frac{1}{EI} \left( \frac{2.5x^3}{6} - \frac{x^4}{24} + 5.008x \right)$

$$\begin{aligned}
&= \frac{1}{4300} \left( \frac{2.5(2.436)^3}{6} - \frac{(2.436)^4}{24} + 5.008(2.436) \right) \\
&= \frac{1}{4300} (16.9308)
\end{aligned}$$

i.e. maximum, deflection,  $\delta = 0.00394 \text{ m}$

8. The beam CAB is simply supported at the points A and B and is subjected to a concentrated load  $W$  at the point C, together with a uniformly distributed load  $w$  between the points A and B, as shown. Determine the relationship between  $W$  and  $wl$ , so that no deflection will occur at the point B.



Taking moments about B gives:

$$R_A \times l = W(1.333l) + \frac{wl^2}{2}$$

i.e. 
$$R_A = 1.333W + \frac{wl}{2} \quad (1)$$

$$\begin{aligned} EI \frac{d^2 y}{dx^2} &= -Wx & \left| \begin{aligned} &+ R_A \left[ x - \frac{l}{3} \right] - \frac{w}{2} \left[ x - \frac{l}{3} \right]^2 \end{aligned} \right. \\ EI \frac{dy}{dx} &= -\frac{Wx^2}{2} + A & \left| \begin{aligned} &+ \frac{R_A}{2} \left[ x - \frac{l}{3} \right]^2 - \frac{w}{6} \left[ x - \frac{l}{3} \right]^3 \end{aligned} \right. \\ EIy &= -\frac{Wx^3}{6} + Ax + B & \left| \begin{aligned} &+ \frac{R_A}{6} \left[ x - \frac{l}{3} \right]^3 - \frac{w}{24} \left[ x - \frac{l}{3} \right]^4 \end{aligned} \right. \end{aligned}$$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

At  $x = \frac{l}{3}$ ,  $y = 0$ , hence,  $0 = -\frac{W\left(\frac{l}{3}\right)^3}{6} + A\left(\frac{l}{3}\right)$

or 
$$0 = -\frac{Wl^3}{162} + \frac{Al}{3}$$

from which, 
$$A = \frac{3Wl^2}{162} = \frac{Wl^2}{54}$$

At  $x = 1.333l$ ,  $y = 0$ , hence,

$$0 = -\frac{W(1.333l)^3}{6} + 1.333\left(\frac{Wl^2}{54}\right) + \frac{l^3}{6}\left(1.333W + \frac{wl}{2}\right) - \frac{wl^4}{24}$$

i.e. 
$$0 = -0.395Wl^3 + 0.0247l^3W + \frac{l^3}{6}\left(1.333W + \frac{wl}{2}\right) - \frac{wl^4}{24}$$

Dividing throughout by  $l^3$  gives:

$$0 = -0.395W + 0.0247W + \frac{1}{6}\left(1.333W + \frac{wl}{2}\right) - \frac{wl^3}{24}$$

i.e. 
$$0 = -0.395W + 0.0247W + 0.222W + 0.0833wl - 0.04166wl$$

i.e. 
$$0 = -0.1483W + 0.04164wl$$

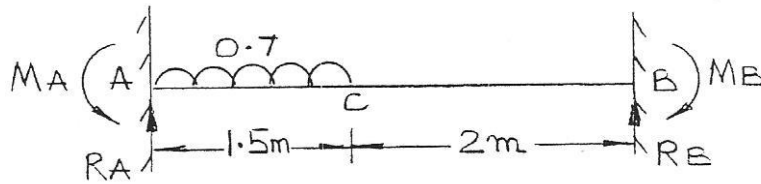
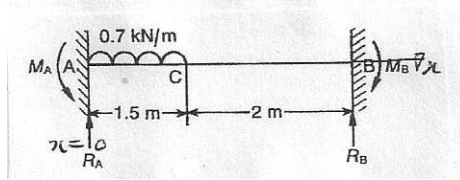
from which, 
$$w = \frac{-0.1483W}{0.0414l}$$

i.e. 
$$w = \frac{3.582W}{l}$$

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## EXERCISE 35, Page 164

1. Determine the position (from the left end) and the maximum deflection for the uniform-section beam shown, together with the reactions and end fixing forces by the double integration method.



$EI \frac{d^2y}{dx^2} = R_A x - M_A - \frac{0.7x^2}{2}$	$+ \frac{0.7}{2} (x - 1.5)^2$
$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - 0.1167x^3 + A$	$+ 0.1167 (x - 1.5)^3$
$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - 0.292x^4 + Ax + B$	$+ 0.0292 (x - 1.5)^4$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

At  $x = 0$ ,  $\frac{dy}{dx} = 0$ , hence,  $A = 0$

At  $x = 3.5$ ,  $\frac{dy}{dx} = 0$

hence,  $0 = 6.125 R_A - 3.5 M_A - 5 + 0.9336$

or  $M_A = 1.75 R_A - 1.162$  (1)

At  $x = 3.5$ ,  $y = 0$

hence,  $0 = 7.146 R_A - 6.125 M_A - 4.382 + 0.4672$

or  $M_A = 1.167 R_A - 0.6392$  (2)

Equating equations (1) and (2) gives:

$$(1.75 - 1.167) R_A = 1.162 - 0.6392$$

and 
$$R_A = \frac{0.5228}{0.583}$$

i.e. 
$$R_A = \mathbf{0.897 \text{ kN}}$$

From equation (1),

$$M_A = 1.75(0.897) - 1.162 = \mathbf{0.407 \text{ kN m}}$$

Now, 
$$R_B = (0.7 \times 1.5) - 0.897 = \mathbf{0.153 \text{ kN}}$$

and 
$$M_B = \mathbf{0.155 \text{ kN m}}$$

To find 'δ', try span CB:

$$0 = 0.4485x^2 - 0.407x - 0.1167x^3 + 0.1167(x^3 - 3.375 + 2.25x - 1.5x^2)$$

$$0 = 0.2735x^2 - 0.1444x - 0.394$$

i.e. 
$$x = \frac{- -0.1444 \pm \sqrt{(-0.1444)^2 - 4(0.2735)(-0.394)}}{2(0.2735)}$$

$$= 1.49 \text{ m}$$

This is not possible as  $1.49 < 1.5$

Try span AC:

$$0 = 0.4485x^2 - 0.407x - 0.1167x^3$$

i.e. 
$$0 = -0.1167x^2 + 0.4485x^2 - 0.407$$

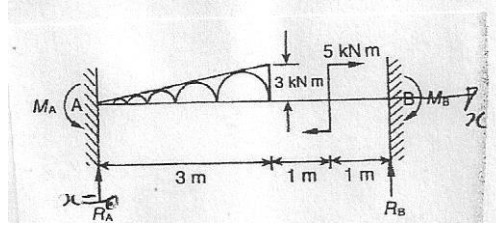
and 
$$x = \frac{- -0.407 \pm \sqrt{(-0.407)^2 - 4(0.4885)(-0.1167)}}{2(0.4885)}$$

i.e.  $x = 2.375$  or  $x = \mathbf{1.47 \text{ m}}$  which is OK

$$\delta = (0.4749 - 0.4397 - 0.1363)/EI$$

i.e. 
$$\delta = \mathbf{-0.101/EI}$$

**2.** Determine the end fixing moments and reactions for the encastré beam shown. The beam may be assumed to be of uniform section.



$EI \frac{d^2y}{dx^2} = R_A x - M_A = x \cdot \frac{x}{2} \cdot \frac{x}{3}$	$+ \frac{3(x-3)^2}{2} + \frac{(x-3)^3}{6}$	$+ 5(x-4)^0$
$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{x^4}{24} + A$	$+ \frac{3(x-3)^3}{6} + \frac{(x-3)^4}{24}$	$+ 5(x-4)$
@ $x = 0, \frac{dy}{dx} = 0$ hence, $A = 0$		
$EI y = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{x^5}{120} + B$	$+ \frac{3(x-3)^4}{24} + \frac{(x-3)^5}{120}$	$+ \frac{5(x-4)^2}{2}$

At  $x = 0, y = 0$ , hence,  $B = 0$

At  $x = 5, \frac{dy}{dx} = 0$ ,

$$0 = 12.5 R_A - 5 M_A - 26.04 + 4 + 0.667 + 5$$

$$= 12.5 R_A - 5 M_A - 16.373$$

or  $R_A = 1.31 + 0.4 M_A$  (1)

At  $x = 5, y = 0$ ,

$$0 = 20.833 R_A - 12.5 M_A - 26.04 + 2 + 0.27 + 2.5$$

$$= 20.833 R_A - 12.5 M_A - 21.27$$
 (2)

Substituting equation (1) into equation (2) gives:

$$0 = 27.29 - 4.167 M_A - 21.27$$

from which,  $M_A = \frac{6.02}{4.167} = 1.445 \text{ kN m}$  (3)

Substituting equation (3) into equation (1) gives:

$$R_A = 1.31 + 0.4(1.445) = 1.888 \text{ kN}$$

Resolving vertically gives:

$$R_A + R_B = \frac{3 \times 3}{2} = 4.5$$

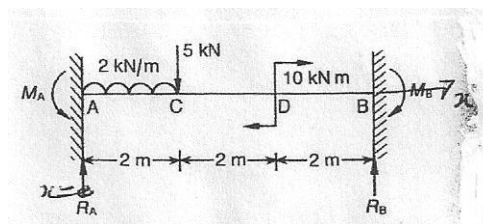
from which,  $R_B = 4.5 - 1.888 = 2.612 \text{ kN}$

Taking moments about A gives:

$$M_B + 5 + \frac{3 \times 3}{2} \times 2 = R_B \times 5 + M_A$$

from which,  $M_B = (2.612)(5) + 1.445 - 5 - 9 = 0.505 \text{ kN m}$

3. Determine the end fixing moments and reactions for the uniform-section encastré beam shown.



$EI \frac{d^2y}{dx^2} = R_A x - M_A - \frac{2x^2}{2}$	11	$-5(x-2) + \frac{(x-2)^2}{2} \cdot 2$	$+ 10(x-4)^0$
$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{x^3}{3} + A$		$-5 \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3}$	$+ 10(x-4)$
@ $x = 0 \frac{dy}{dx} = 0$ hence $A = 0$			
$EI y = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{x^4}{12} + B$		$-5 \frac{(x-2)^3}{6} + \frac{(x-2)^4}{12}$	$+ \frac{10(x-4)^2}{2}$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

At  $x = 6$ ,  $\frac{dy}{dx} = 0$ ,

$$\begin{aligned} \text{hence, } 0 &= 18 R_A - 6 M_A - 72 - 40 + 21.33 + 20 \\ &= 18 R_A - 6 M_A - 70.67 \end{aligned}$$

$$\text{or } R_A = 0.333 M_A + 3.926 \quad (1)$$

At  $x = 6$ ,  $y = 0$ ,



$$0 = 36 R_A - 18 M_A - 108 - 53.33 + 21.33 + 20$$

i.e.  $0 = 36 R_A - 18 M_A - 120$  (2)

Substituting equation (1) into equation (2) gives:

$$0 = 12 M_A + 141.33 - 18 M_A - 120$$

and  $M_A = 21.33/6 = \mathbf{3.556 \text{ kN m}}$  (3)

Substitute equation (3) into equation (1) gives:

$$R_A = 0.333(3.556) + 3.926 = \mathbf{5.11 \text{ kN}}$$

Resolving vertically gives:

$$R_A + R_B = 2 \times 2 + 5$$

from which,  $R_B = 9 - 5.11 = \mathbf{3.89 \text{ kN}}$

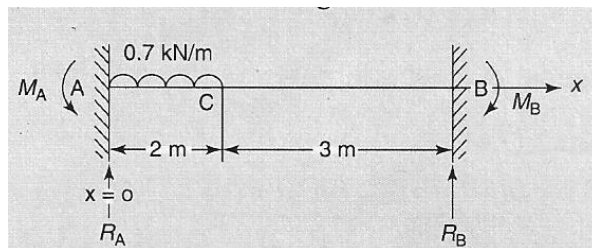
Taking moments about A gives:

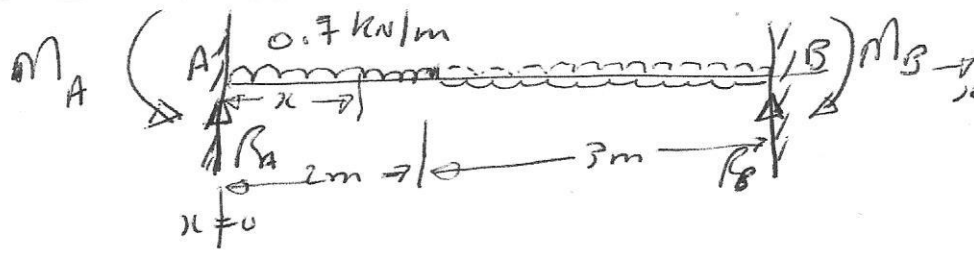
$$M_B + 10 + 5 \times 2 + 2 \times 2 \times 1 = R_B \times 6 + M_A$$

i.e.  $M_B = 3.89 \times 6 + 3.556 - 10 - 10 - 4$

i.e.  $M_B = \mathbf{2.896 \text{ kN m}}$

**4.** Determine the position (from the left end) and the maximum deflection for the uniform-section beam shown, together with the reactions and end fixing forces.





$$EI \frac{d^2 y}{dx^2} = R_A x - M_A - \frac{0.7x^2}{2} \quad \left| \quad + \frac{0.7}{2} [x-2]^2 \right.$$

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{0.7x^3}{6} + A \quad \left| \quad + \frac{0.7}{6} [x-2]^3 \right.$$

At  $x = 0$ ,  $\frac{dy}{dx} = 0$ , hence,  $A = 0$

$$EI y = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{0.7x^4}{24} + B \quad \left| \quad + \frac{0.7}{24} [x-2]^4 \right.$$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

At  $x = 5$  m,  $\frac{dy}{dx} = 0$

Hence, 
$$0 = \frac{R_A (5)^2}{2} - M_A (5) - \frac{0.7(5)^3}{6} + \frac{0.7}{6} [5-2]^3$$

i.e. 
$$0 = 12.5R_A - 5M_A - 14.583 + 3.15$$

or 
$$5M_A = 12.5R_A - 11.433$$

and 
$$M_A = 2.5R_A - 2.2866 \quad (1)$$

At  $x = 5$  m,  $y = 0$

Hence, 
$$0 = \frac{R_A (5)^3}{6} - \frac{M_A (5)^2}{2} - \frac{0.7(5)^4}{24} + \frac{0.7}{24} [5-2]^4$$

i.e. 
$$0 = 20.833R_A - 12.5M_A - 18.229 + 2.363$$

or 
$$12.5M_A = 20.833R_A - 15.866$$

i.e. 
$$M_A = 1.6666R_A - 1.2693 \quad (2)$$

Equating equations (1) and (2) gives:

$$2.5R_A - 2.2866 = 1.6666R_A - 1.2693$$

$$\text{i.e.} \quad R_A (2.5 - 1.6666) = 2.2866 - 1.2693$$

$$\text{i.e.} \quad 0.8334R_A = 1.0173$$

$$\text{and} \quad R_A = \frac{1.0173}{0.8334} = \mathbf{1.221 \text{ kN}} \quad (3)$$

Substituting equation (3) into equation (1) gives:

$$\begin{aligned} M_A &= 2.5R_A - 2.2866 \\ &= 2.5(1.221) - 2.2866 \end{aligned}$$

$$\text{i.e.} \quad M_A = \mathbf{0.766 \text{ kN m}} \quad (4)$$

Taking moments about  $B$  gives:

$$R_A \times 5 = 0.7 \times 2 \times 4 + M_A - M_B$$

$$\text{i.e.} \quad 5R_A = 5.6 + M_A - M_B \quad (5)$$

Substituting equations (3) and (4) into equation (5) gives:

$$5 \times 1.221 = 5.6 + 0.766 - M_B$$

$$\text{and} \quad M_B = 5.6 + 0.766 - 5 \times 1.221$$

$$\text{i.e.} \quad M_B = \mathbf{0.261 \text{ kN m}} \quad (6)$$

Resolving vertically gives:

$$R_A + R_B = 0.7 \times 2$$

$$\text{and from equation (3),} \quad 1.221 + R_B = 1.4$$

$$\text{from which,} \quad R_B = 1.4 - 1.221 = \mathbf{0.179 \text{ kN}}$$

$$\text{For maximum deflection,} \quad \frac{dy}{dx} = 0,$$

$$\text{i.e.} \quad 0 = \frac{R_A x^2}{2} - M_A x - \frac{0.7 x^3}{6} + \frac{0.7}{6} [x - 2]^3$$

$$\text{i.e.} \quad 0 = \frac{(1.221)x^2}{2} - 0.766x - \frac{0.7x^3}{6} + \frac{0.7}{6}(x^3 - 6x^2 + 12x - 8)$$

$$\text{i.e.} \quad 0 = 0.6105x^2 - 0.766x - 0.1167x^3 + 0.1167x^3 - 0.7x^2 + 1.4x - 0.9333$$

$$\text{i.e.} \quad -0.0895x^2 + 0.634x - 0.9333 = 0$$

Using a calculator or the quadratic formula,

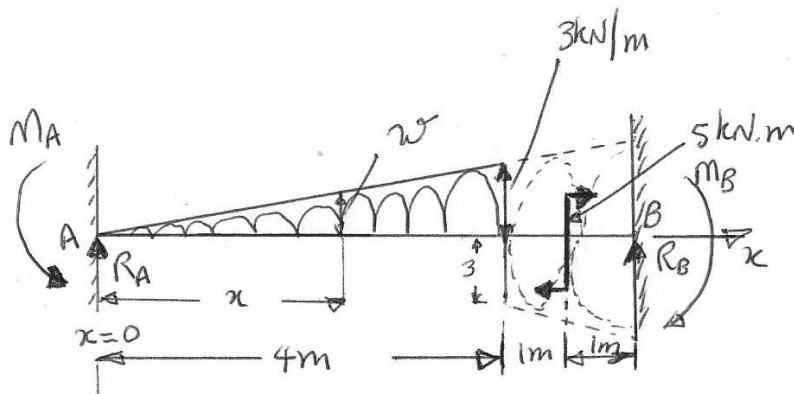
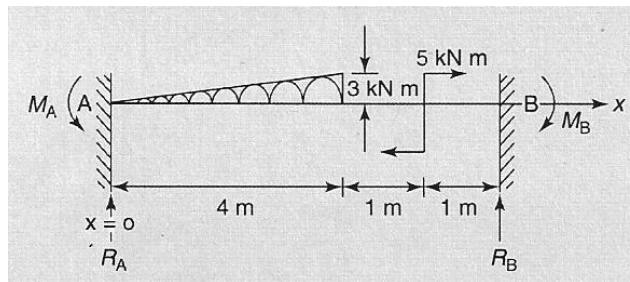
$$x = \mathbf{2.087 \text{ m}} \quad \text{or} \quad 4.997 \text{ m (which is ignored)}$$

Hence, maximum deflection,

$$\begin{aligned} \delta &= \frac{1}{EI} \left\{ \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{0.7x^4}{24} + B + \frac{0.7}{24}[x-2]^4 \right\} \\ &= \frac{1}{EI} \left\{ \frac{(1.221)(2.087)^3}{6} - \frac{(0.766)(2.087)^2}{2} - \frac{0.7(2.087)^4}{24} + \frac{0.7}{24}[2.087-2]^4 \right\} \end{aligned}$$

$$\text{i.e.} \quad \delta = -\frac{\mathbf{0.372}}{EI}$$

**5.** Determine the end fixing moments and reactions for the encastré beam shown. The beam may be assumed to be of uniform section.



$$\frac{w}{x} = \frac{3}{4} \quad \text{i.e.} \quad w = 0.75x$$

$$EI \frac{d^2 y}{dx^2} = R_A x - M_A - 0.75x \times \frac{x}{2} \times \frac{x}{3} \quad \left| \begin{array}{l} + [x-4]^2 \times \frac{3}{2} \\ \frac{3}{4} [x-4] \\ \frac{[x-4]}{2} \times \frac{[x-4]}{3} \end{array} \right| \quad \left| \begin{array}{l} 5[x-5] \end{array} \right|$$

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{0.75x^3}{6 \times 4} + A \quad \left| \begin{array}{l} + \frac{\frac{3}{4} [x-4]^4}{4 \times 6} + [x-4]^3 \times 0.5 \end{array} \right| \quad \left| \begin{array}{l} 5[x-5] \end{array} \right|$$

At  $x = 0$ ,  $\frac{dy}{dx} = 0$ , hence,  $A = 0$

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{0.75x^4}{24} \quad \left| \begin{array}{l} + \frac{[x-4]^4}{32} + 0.5[x-4]^3 \end{array} \right| \quad \left| \begin{array}{l} 5[x-5] \end{array} \right| \quad (1)$$

$$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{0.75x^5}{24 \times 5} + B \quad \left| \begin{array}{l} + \frac{[x-4]^5}{32 \times 5} + \frac{[x-4]^4}{8} \end{array} \right| \quad \left| \begin{array}{l} \frac{5[x-5]^2}{2} \end{array} \right|$$

At  $x = 0$ ,  $y = 0$ , hence,  $B = 0$

$$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{x^5}{160} \quad \left| \begin{array}{l} + \frac{[x-4]^5}{160} + \frac{[x-4]^4}{8} \end{array} \right| \quad \left| \begin{array}{l} 2.5[x-5]^2 \end{array} \right| \quad (2)$$

At  $x = 6$  m,  $y = 0$ , hence,

$$0 = \frac{R_A (6)^3}{6} - \frac{M_A (6)^2}{2} - \frac{(6)^5}{160} + \frac{(2)^5}{160} + \frac{(2)^4}{8} + 2.5(1)^2$$

i.e.  $0 = 36R_A - 18M_A - 48.6 + 0.2 + 2 + 2.5$

i.e.  $18M_A = 36R_A - 43.9$

or  $M_A = 2R_A - 2.439$  (3)

At  $x = 6$  m,  $\frac{dy}{dx} = 0$ , hence,

$$0 = \frac{R_A (6)^2}{2} - 6M_A - \frac{0.75(6)^4}{24} + \frac{[6-4]^4}{32} + 0.5[6-4]^3 + 5[6-5]$$

i.e.  $0 = 18R_A - 6M_A - 40.5 + 0.5 + 4 + 5$

i.e.  $6M_A = 18R_A - 31$

or  $M_A = 3R_A - 5.167$  (4)

Equating equations (3) and (4) gives:

$$2R_A - 2.439 = 3R_A - 5.167$$

from which,  $R_A = 5.167 - 2.439 = \mathbf{2.728 \text{ kN}}$

From equation (4),  $M_A = 3R_A - 5.167$

$$= 3(2.728) - 5.167$$

i.e.  $M_A = \mathbf{3.017 \text{ kN}}$

Resolving vertically gives:

$$R_A + R_B = 0.75 \times 4 \times \frac{4}{2} = 6$$

Hence,  $R_B = 6 - 2.728 = \mathbf{3.272 \text{ kN}}$

Taking moments about  $B$  gives:

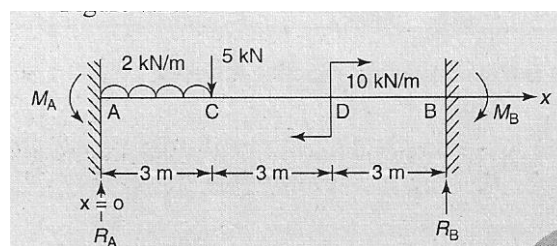
$$R_A \times 6 + 5 = M_A - M_B + \frac{3 \times 4}{2} \times \left( \frac{4}{3} + 2 \right)$$

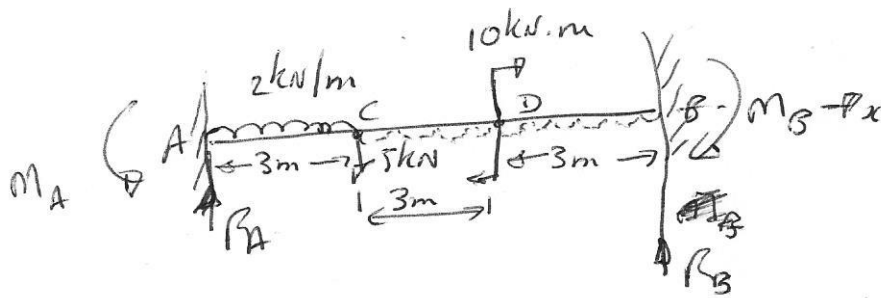
i.e.  $2.728 \times 6 + 5 = 3.017 - M_B + 20$

i.e.  $M_B = 3.017 + 20 - 2.728 \times 6 - 5$

i.e.  $M_B = \mathbf{1.649 \text{ kN}}$

**6. Determine the end fixing moments and reactions for the uniform-section encastré beam shown.**





$$w = 2$$

$$EI \frac{d^2 y}{dx^2} = R_A x - M_A - 0.2x \times \frac{x}{2}$$

$$-5[x-3]$$

$$+ \frac{2}{2}[x-3]^2$$

$$10[x-6]$$

$$EI \frac{dy}{dx} = \frac{R_A x^2}{2} - M_A x - \frac{x^3}{3} + A$$

$$-\frac{5}{2}[x-3]^2 + \frac{[x-3]^3}{3}$$

$$\frac{10}{2}[x-6]^2 \quad (1)$$

$$\text{At } x = 0, \frac{dy}{dx} = 0, \text{ hence, } A = 0$$

$$EIy = \frac{R_A x^3}{6} - \frac{M_A x^2}{2} - \frac{x^4}{12} + B$$

$$-\frac{5}{2}[x-3]^3 + \frac{[x-3]^4}{12}$$

$$\frac{5}{3}[x-6]^3 \quad (2)$$

$$\text{At } x = 0, y = 0, \text{ hence, } B = 0$$

$$\text{At } x = 6 \text{ m, } \frac{dy}{dx} = 0, \text{ hence,}$$

$$0 = \frac{R_A (6)^3}{6} - \frac{M_A (6)^2}{2} - \frac{(6)^4}{12} - \frac{5}{2} \times 3^2 + \frac{(3)^3}{3} + 0$$

$$\text{i.e. } 0 = 18R_A - 6M_A - 72 - 22.5 + 9$$

$$\text{i.e. } 6M_A = 18R_A - 85.5$$

$$\text{or } M_A = 3R_A - 14.25 \quad (3)$$

$$\text{At } x = 6 \text{ m, } y = 0, \text{ hence,}$$

$$0 = \frac{R_A (6)^3}{6} - M_A \times \frac{36}{2} - \frac{(6)^4}{12} - \frac{5(3)^3}{6} + \frac{(3)^3}{12}$$

$$\text{i.e. } 0 = 36R_A - 18M_A - 108 - 22.5 + 6.75$$

$$\text{i.e. } 18M_A = 36R_A - 123.75$$

or  $M_A = 2R_A - 6.875$  (4)

Equating equations (3) and (4) gives:

$$3R_A - 14.25 = 2R_A - 6.875$$

from which,  $R_A = 14.25 - 6.875 = \mathbf{7.375 \text{ kN}}$  (5)

From equations (4) and (5),

$$\begin{aligned} M_A &= 2R_A - 6.875 \\ &= 2(7.375) - 6.875 \end{aligned}$$

i.e.  $M_A = \mathbf{7.875 \text{ kN}}$  (6)

Resolving vertically gives:

$$R_A + R_B = 2 \times 3 + 5 = 11$$

Hence,  $R_B = 11 - 7.375 = \mathbf{3.625 \text{ kN}}$

Taking moments about  $B$  gives:

$$R_A \times 9 + 10 + M_B = M_A + 2 \times 3 \times 7.5 + 5 \times 6$$

i.e.  $7.375 \times 9 + 10 + M_B = 7.875 + 45 + 30$

i.e.  $M_B = 7.875 + 45 + 30 - 7.375 \times 9 - 10$

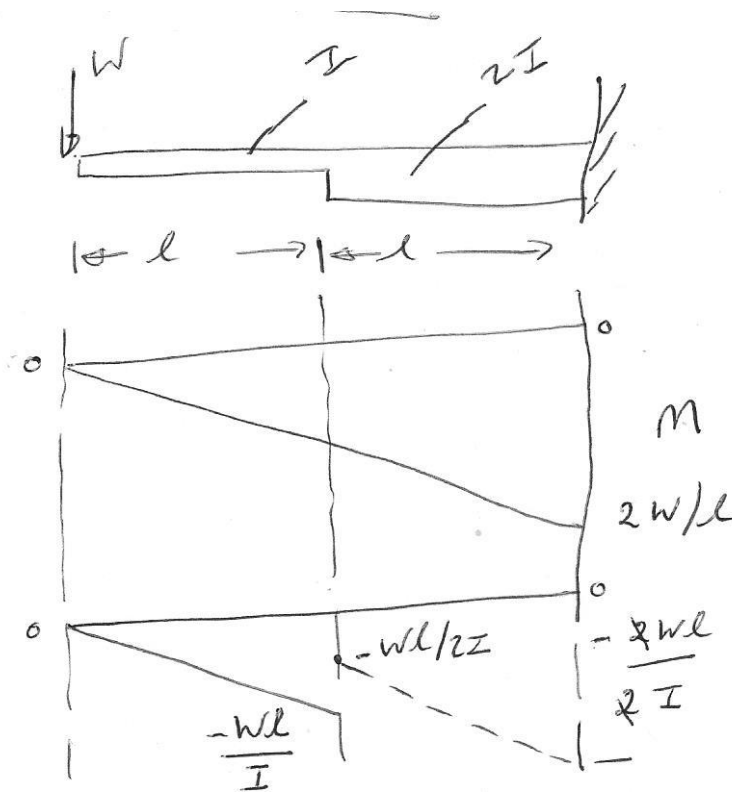
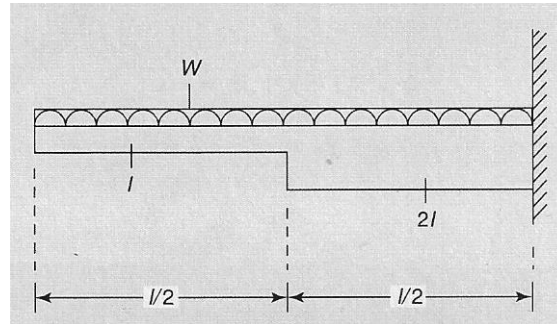
i.e.  $M_B = \mathbf{6.50 \text{ kN}}$

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# EXERCISE 36, Page 166

1. Determine the maximum deflection for the cantilever beam shown.



$$x \frac{dy}{dx} - y = \frac{1}{E} \times \text{moment of area of the } M/I \text{ diagram about A}$$

or

$$[(2l\theta_B - y_B) - (0 \times \theta_A - y_A)] = -\frac{1}{E} \left[ \frac{Wl}{I} \times \frac{2}{3} \times l \times \frac{l}{2} + \frac{Wl}{2I} \times \frac{3l}{2} \times l + \frac{Wl}{2I} \times \frac{l}{2} \times \left( l + \frac{2l}{3} \right) \right]$$

$$\theta_B = y_B = 0, \text{ hence, } y_A = -\frac{Wl^3}{EI} \left[ \frac{1}{3} + \frac{3}{4} + \frac{1}{4} \left( \frac{5}{3} \right) \right]$$

$$\text{i.e. maximum deflection, } y_A = -\frac{3Wl^3}{2EI}$$