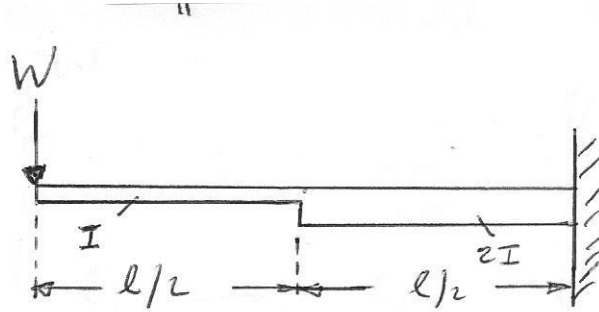


## CHAPTER 11 ENERGY METHODS

### EXERCISE 47, Page 249

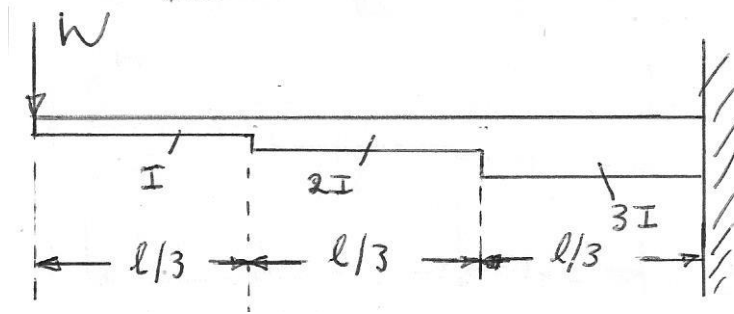
1. Determine the maximum deflection for the cantilever beam shown below.



$$\begin{aligned}
 U_e &= \int \frac{M^2}{2EI} dx \\
 &= \int_0^{l/2} \frac{(-Wx)^2}{2EI} dx + \int_{l/2}^l \frac{(-Wx)^2}{4EI} dx \\
 &= \frac{1}{2EI} \int_0^{l/2} W^2 x^2 dx + \frac{1}{4EI} \int_{l/2}^l W^2 x^2 dx \\
 &= \frac{W^2}{2EI} \left[ \frac{x^3}{3} \right]_0^{l/2} + \frac{W^2}{4EI} \left[ \frac{x^3}{3} \right]_{l/2}^l \\
 &= \frac{W^2}{6EI} \left[ \left( \frac{l}{2} \right)^3 - 0 \right] + \frac{W^2}{12EI} \left[ l^3 - \left( \frac{l}{2} \right)^3 \right] \\
 &= \frac{W^2 l^3}{48EI} + \frac{W^2}{12EI} \left[ l^3 - \frac{l^3}{8} \right] \\
 &= \frac{W^2 l^3}{48EI} + \frac{7W^2 l^3}{96EI} = \frac{W^2 l^3}{EI} \left( \frac{1}{48} + \frac{7}{96} \right) \\
 &= \frac{3W^2 l^3}{32EI}
 \end{aligned}$$

Hence, **maximum deflection**,  $\delta = \frac{\partial U_e}{\partial W} = \frac{3Wl^3}{16EI}$

2. Determine the maximum deflection for the cantilever beam shown below.

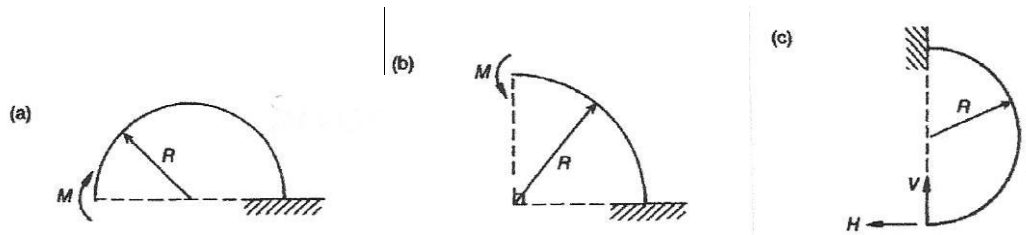


$$\begin{aligned}
 U_e &= \int \frac{M^2}{2EI} dx \\
 &= \int_0^{l/3} \frac{(-Wx)^2}{2EI} dx + \int_{l/3}^{2l/3} \frac{(-Wx)^2}{4EI} dx + \int_{2l/3}^l \frac{(-Wx)^2}{6EI} dx \\
 &= \frac{1}{2EI} \int_0^{l/3} W^2 x^2 dx + \frac{1}{4EI} \int_{l/3}^{2l/3} W^2 x^2 dx + \frac{1}{6EI} \int_{2l/3}^l W^2 x^2 dx \\
 &= \frac{W^2}{2EI} \left[ \frac{x^3}{3} \right]_0^{l/3} + \frac{W^2}{4EI} \left[ \frac{x^3}{3} \right]_{l/3}^{2l/3} + \frac{W^2}{6EI} \left[ \frac{x^3}{3} \right]_{2l/3}^l \\
 &= \frac{W^2}{6EI} \left[ \left( \frac{l}{3} \right)^3 - 0 \right] + \frac{W^2}{12EI} \left[ \left( \frac{2l}{3} \right)^3 - \left( \frac{l}{3} \right)^3 \right] + \frac{W^2}{18EI} \left[ l^3 - \left( \frac{2l}{3} \right)^3 \right] \\
 &= \frac{W^2 l^3}{162EI} + \frac{W^2 l^3}{12EI} \left[ \frac{8}{27} - \frac{1}{27} \right] + \frac{W^2 l^3}{18EI} \left[ 1 - \frac{8}{27} \right] \\
 &= \frac{W^2 l^3}{162EI} + \frac{7W^2 l^3}{324EI} + \frac{19W^2 l^3}{486EI} \\
 &= \frac{W^2 l^3}{EI} \left( \frac{1}{162} + \frac{7}{324} + \frac{19}{486} \right) = \frac{W^2 l^3}{EI} \left( \frac{1}{162} + \frac{7}{324} + \frac{19}{486} \right) \\
 &= \frac{65W^2 l^3}{972EI}
 \end{aligned}$$

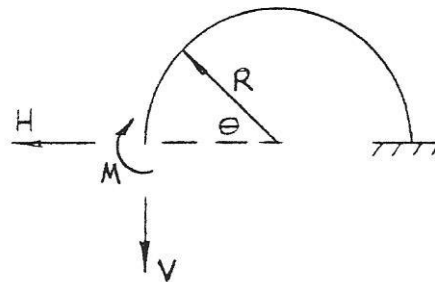
Hence, maximum deflection,  $\delta = \frac{\partial U_e}{\partial W} = \frac{65Wl^3}{486EI}$

# EXERCISE 48, Page 257

1. Determine expressions for the horizontal and vertical deflections of the free end of the thin curved beams shown in Figures (a) to (c).



(a)



$$M = M + HR \sin \theta - VR (1 - \cos \theta)$$

$$\begin{aligned} \delta V &= \frac{1}{EI} \int M \frac{\partial M}{\partial V} dx \\ &= \frac{1}{EI} \int [(M + HR \sin \theta - VR (1 - \cos \theta))] [-R (1 - \cos \theta)] R d\theta \end{aligned}$$

However,  $H = V = 0$

Therefore, 
$$\delta V = -\frac{MR^2}{EI} \int (1 - \cos \theta) d\theta \quad 1$$

$$= -\frac{MR^2}{EI} [\theta - \sin \theta]_0^\pi$$

i.e. 
$$\delta V = -\frac{\pi MR^2}{EI} \quad 2 \text{ i.e. upwards}$$

$$\delta H = \frac{1}{EI} \int M \frac{\partial M}{\partial H} R d\theta \quad 3$$

$$= \frac{1}{EI} \int (M) R \sin \theta R d\theta$$

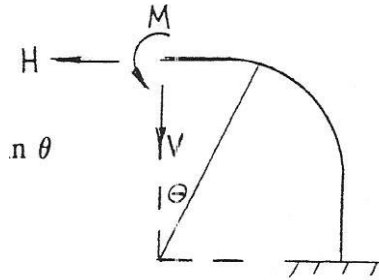
$$= -\frac{MR^2}{EI} [\cos \theta]_0^\pi$$

$$= -\frac{MR^2}{EI} [-1-1]$$

i.e.

$$\delta H = \frac{2MR^2}{EI} \quad \text{4 to the left}$$

(b)



$$M = M + HR(1 - \cos \theta) + VR \sin \theta$$

$$\delta V = \frac{1}{EI} \int (M) R \sin \theta R d\theta$$

$$= \frac{1}{EI} \int [M + HR(1 - \cos \theta) + VR \sin \theta] R \sin \theta R d\theta$$

$$= \frac{MR^2}{EI} [-\cos \theta]_0^{\pi/2}$$

$$= \frac{MR^2}{EI} [0+1]$$

$$\delta V = \frac{MR^2}{EI} \quad \text{5 downwards}$$

$$\delta H = \frac{1}{EI} \int (M) R(1 - \cos \theta) R d\theta$$

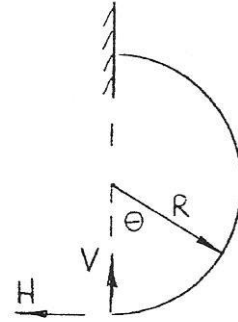
$$= \frac{1}{EI} \int [M + HR(1 - \cos \theta) + VR \sin \theta] R \sin \theta R d\theta$$

$$= \frac{MR^2}{EI} [\theta - \sin \theta]_0^{\pi/2}$$

$$= \frac{MR^2}{EI} \left[ \left( \frac{\pi}{2} - 1 \right) - (0) \right]$$

$$\delta V = \frac{MR^2}{EI} \left( \frac{\pi}{2} - 1 \right) \quad \text{6 to the left}$$

(c)



$$M = HR(1 - \cos \theta) + VR \sin \theta$$

$$\delta V = \frac{1}{EI} \int_0^\pi [HR(1 - \cos \theta) + VR \sin \theta] R \sin \theta R d\theta$$

$$= \frac{R^2}{EI} \int_0^\pi [HR(\sin \theta - \sin \theta \cos \theta) + VR \sin^2 \theta] d\theta$$

$$= \frac{R^2}{EI} \left[ HR(-\cos \theta - \sin^2 \theta / 2) + VR \int \frac{1 - \cos 2\theta}{2} d\theta \right]$$

$$= \frac{R^2}{EI} \left[ HR(-\cos \theta - \sin^2 \theta / 2) + \frac{VR}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right] \right]_0^\pi$$

$$= \frac{R^3}{EI} \left[ H(+1-0) - (-1-0) + \frac{V}{2} [(\pi-0) - (0-0)] \right]$$

i.e.

$$\delta V = \frac{R^3}{EI} \left[ 2H + \frac{\pi V}{2} \right]$$

$$\delta H = \frac{R^3}{EI} \int_0^\pi [H(1 - \cos \theta) + V \sin \theta] R(1 - \cos \theta) d\theta$$

$$= \frac{R^3}{EI} \int_0^\pi [H(1 - 2\cos \theta + \cos^2 \theta) + V(\sin \theta - \sin \theta \cos \theta)] d\theta$$

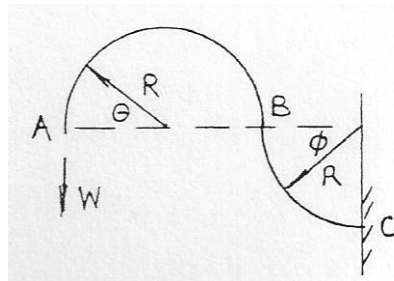
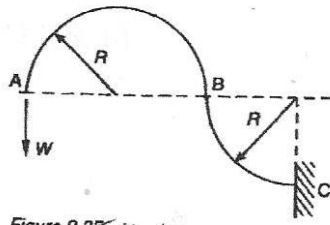
$$= \frac{R^3}{EI} \left[ H(\theta - 2\sin \theta) + V \left[ -\cos \theta - \frac{\sin^2 \theta}{2} \right] + H \int \frac{(1 + \cos 2\theta)}{2} d\theta \right]_0^\pi$$

$$= \frac{R^3}{EI} \left[ H \left( \theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) + V \left( -\cos\theta - \frac{\sin^2\theta}{2} \right) \right]$$

$$= \frac{R^3}{EI} \left[ H \left( \frac{3\pi}{2} - 0 \right) - (0) + V [(+1 - 0) - (-1 - 0)] \right]$$

i.e. 
$$\delta H = \frac{R^3}{EI} \left[ \frac{3\pi H}{2} + 2V \right]$$

2. A thin curved beam consists of a length AB, which is of semi-circular form, and a length BC, which is a quadrant, as shown below. Determine the vertical deflection at A due to a downward load  $W$ , applied at this point. The end C is firmly fixed, and the beam's cross-section may be assumed to be uniform.



**AB**

$$M = WR(1 - \cos\theta)$$

**BC**

$$M = W[2R + R(1 - \cos\phi)]$$

$$\delta W = \frac{1}{EI} \int_0^\pi WR(1 - \cos\theta) R(1 - \cos\theta) R d\theta + \frac{W}{EI} \int_0^{\pi/2} [2R + R(1 - \cos\phi)]^2 R d\theta$$

$$= \frac{WR^3}{EI} \int_0^\pi (1 - 2\cos\theta + \cos^2\theta) d\theta + \frac{WR^3}{EI} \int_0^{\pi/2} [4 + (1 - 2\cos\phi + \cos^2\phi) + 4(1 - \cos\phi)] d\phi$$

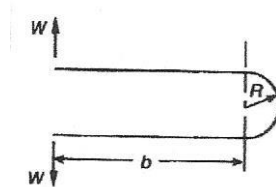
$$= \frac{WR^3}{EI} \left\{ \left[ 1 - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^\pi + \int_0^{\pi/2} [4 + 1 - 2\cos\phi + \cos^2\phi + 4 - 4\cos\phi] d\phi \right\}$$

$$= \frac{WR^3}{EI} \left\{ \left[ \pi - 0 + \frac{\pi}{2} + 0 \right] - [0] + \left[ 9\phi - 6\sin\phi + \frac{\phi}{2} + \frac{\sin 2\phi}{4} \right]_0^{\pi/2} \right\}$$

$$= \frac{WR^3}{EI} \left\{ \frac{3\pi}{2} + \left[ \frac{9\pi}{2} - 6 + \frac{\pi}{4} + 0 \right] - (0) \right\}$$

i.e.  $\delta W = \frac{WR^3}{EI} \left( \frac{25\pi}{4} - 6 \right)$

**3.** A clip, which is made from a length of wire of uniform section, is subjected to the load shown below. Determine an expression for the distance by which the free ends of the clip separate, when it is subjected to this load.

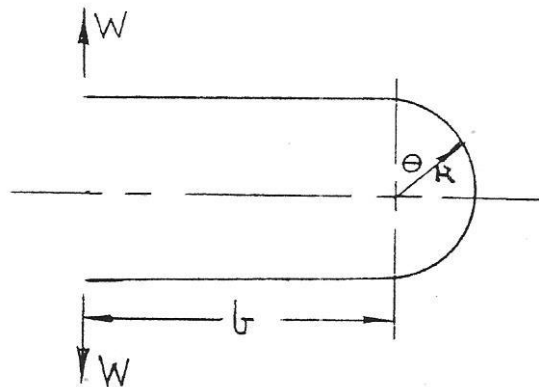


**Straight**

$$M = Wx$$

**Curved**

$$M = W(b + R \sin \theta)$$



$$\delta W = \frac{1}{EI} \int_0^b (Wx) x dx + \frac{W}{EI} \int (b + R \sin \theta)^2 R d\theta$$

$$= \frac{Wb^3}{3EI} + \frac{W}{EI} \int [b^2 + 2bR \sin \theta + R^2 \sin^2 \theta] R d\theta$$

$$= \frac{Wb^3}{3EI} + \frac{W}{EI} \left[ b^2 \theta - 2bR \cos \theta + \frac{R^2 \theta}{2} + \frac{R^2 \sin 2\theta}{4} \right]_0^{\pi/2}$$

$$= \frac{Wb^3}{3EI} + \frac{W}{EI} \left\{ \left[ \frac{b^2\pi}{2} - 0 + \frac{R^2\pi}{2} + 0 \right] + R \right\}$$

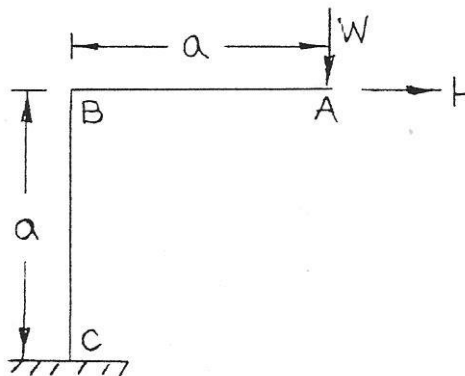
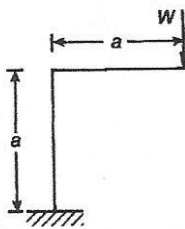
$$= \frac{Wb^3}{3EI} + \frac{\pi WR}{EI} \left( \frac{b^2 + R^2}{2} \right) + \frac{2bRW}{EI}$$

i.e. 
$$\delta W = \frac{W}{EI} \left[ \frac{b^3}{3} + R \left( \frac{b^2\pi}{2} + \frac{R^2\pi}{4} \right) + 2bR \right]$$

**Distance by which the two end separate**

$$= \frac{2W}{EI} \left[ \frac{b^3}{3} + R \left( \frac{b^2\pi}{2} + \frac{R^2\pi}{4} \right) + 2bR \right]$$

**4.** Determine the vertical and horizontal displacements at the free end of the rigid-jointed frame shown.



**AB**

$$M = Wx$$

**BC**

$$M = Wa + Hy$$

$$\delta V = \frac{1}{EI} \int_0^a Wx^2 dx + \frac{1}{EI} \int_0^a (Wa + Hy) a dx$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{EI} [y]_0^a$$



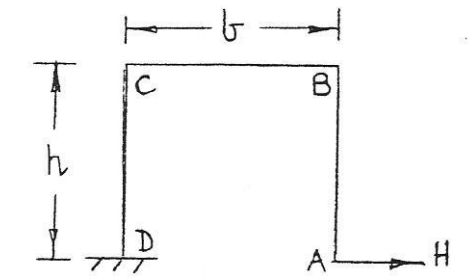
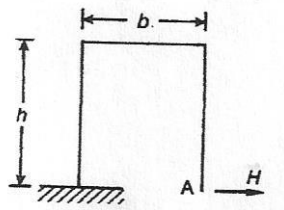
$$\delta V = \frac{4Wa^3}{3EI} \text{ downwards}$$

$$\delta H = \frac{1}{EI} \int (Wx) 0 dx + \frac{1}{EI} \int (Wa + Hy) y dy$$

$$= 0 + \frac{1}{EI} \left[ \frac{Way^2}{2} + \frac{Hy^3}{3} \right]_0^a$$

i.e.  $\delta H = \frac{Wa^3}{2EI} \text{ to the right}$

5. Determine the horizontal deflection at the free end A of the rigid-jointed frame shown.



**AB**

**BC**

**CD**

$$M = Hx$$

$$M = Hh$$

$$M = H(h - x)$$

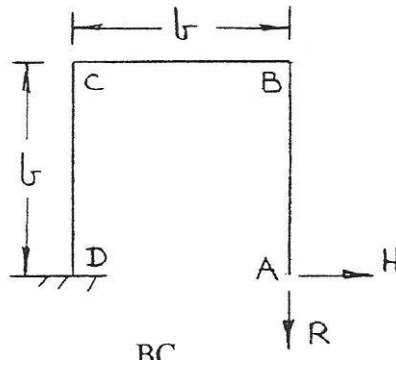
$$\delta H = \frac{1}{EI} \int_0^h Hx^2 dx + \frac{1}{EI} \int_0^b Hh^2 dx + \frac{1}{EI} \int_0^h H(h - x)^2 dx$$

$$= \frac{H}{EI} \left\{ \frac{h^3}{3} + h^2b + \int_0^h (h^2 - 2hx + x^2) dx \right\}$$

$$= \frac{H}{EI} \left\{ \frac{h^3}{3} + h^2b + h^3 - h^3 + \frac{h^3}{3} \right\}$$

i.e.  $\delta H = \frac{Hh^2}{EI} \left( \frac{2h}{3} + b \right)$

6. If the framework shown in question 5 were prevented from moving vertically at A, so that it was statically indeterminate to the first degree and  $h = b$ , what force would be required to prevent this movement?



<b>AB</b>	<b>BC</b>	<b>CD</b>
$M = (Hx)$	$M = Hb - Rx$	$M = H(b - x) - Rb$

$$\frac{\partial u}{\partial R} = 0 = \frac{1}{EI} \int M \frac{\partial M}{\partial R} dx$$

i.e. 
$$0 = \frac{1}{EI} \left\{ \int [Hx] 0 dx + \int [Hb - Rx] (-x dx) + \int [H(b - x) - Rb] (-b dx) \right\}$$

i.e. 
$$0 = \left[ -\frac{Hbx^2}{2} + \frac{Rx^3}{3} - Hb^2x + \frac{Hbx^2}{2} + Rb^2x \right]_0^b$$

i.e. 
$$0 = -\frac{Hb^3}{2} + \frac{Rb^3}{3} - Hb^3 + \frac{Hb^3}{2} + Rb^3$$

from which, 
$$\frac{Hb^3}{2} + Hb^3 - \frac{Hb^3}{2} = \frac{Rb^3}{3} + Rb^3$$

i.e. 
$$H = \frac{4}{3} R$$

or 
$$R = \frac{3}{4} H \quad \text{or} \quad R = 0.75 H \text{ acting downwards}$$

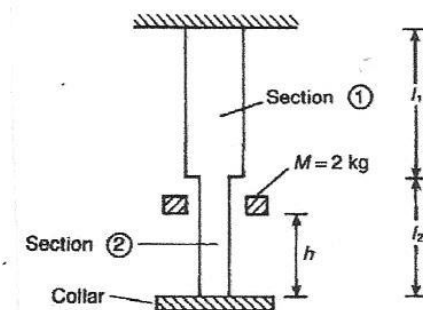
### EXERCISE 49, Page 263

1. A rod, composed of two elements of different cross-sectional areas, is firmly fixed at its top end and has an inextensible collar, firmly secured to its bottom end, as shown below. If a mass of magnitude 2 kg is dropped from a height of 0.4 m above the top of the collar, determine the maximum deflection of the rod, and also the maximum stress. Assume that:

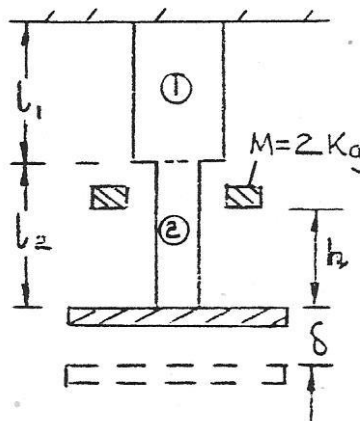
$$E = 2 \times 10^{11} \text{ N/m}^2 \text{ and } g = 9.81 \text{ m/s}^2$$

$$\text{For section 1: } A_1 = 500 \text{ mm}^2 \text{ and } l_1 = 1.2 \text{ m}$$

$$\text{For section 2: } A_2 = 300 \text{ mm}^2 \text{ and } l_2 = 0.8 \text{ m}$$



$$\sigma_1 A_1 = \sigma_2 A_2 \text{ from which, } \sigma_1 = \sigma_2 A_2 / A_1$$



$$Mg(h + \delta) = \frac{\sigma_1^2}{2E} \times A_1 l_1 + \frac{\sigma_2^2}{2E} \times A_2 l_2$$

or

$$Mg(h + \delta_1 + \delta_2) = \frac{\sigma_2^2 \times A_2^2 l_1}{2A_1 E} + \frac{\sigma_2^2}{2E} \times A_2 l_2$$

$$\text{i.e.} \quad Mg(h + \varepsilon_1 l_1 + \varepsilon_2 l_2) = \frac{\sigma_2^2}{2E} \left( \frac{A_2^2 l_1}{A_1} + A_2 l_2 \right)$$

$$\text{i.e.} \quad Mg \left[ h + \frac{\sigma_2 A_2 l_2}{A_1 E} + \frac{\sigma_2 l_2}{E} \right] = \frac{\sigma_2^2}{2E} \left( \frac{A_2^2 l_1}{A_1} + A_2 l_2 \right)$$

$$\text{i.e.} \quad 2 \times 9.81 \left[ 0.4 + \frac{\sigma_2 \times 0.72}{E} + \frac{\sigma_2 \times 0.8}{E} \right] = \frac{\sigma_2^2}{2E} \left( \frac{216}{1 \times 10^6} + 2.4 \times 10^{-4} \right)$$

$$\text{i.e.} \quad 19.62(0.4 + 3.6 \times 10^{-12} \sigma_2 + 4 \times 10^{-12} \sigma_2) = 2.5 \times 10^{-12} \sigma_2^2 (4.56 \times 10^{-4})$$

$$\text{and} \quad 1.14 \times 10^{-15} \sigma_2^2 - 1.491 \times 10^{-10} \sigma_2 - 7.848 = 0$$

Using a calculator or a smartphone or the quadratic formula,  $\sigma_2 = \mathbf{83.04 \text{ MN/m}^2}$

$$\sigma_1 = \sigma_2 A_2 / A_1 = \mathbf{49.82 \text{ MN/m}^2}$$

$$\begin{aligned} \delta &= \frac{\sigma_1}{E} l_1 + \frac{\sigma_2}{E} l_2 = \frac{49.82 \times 10^6 \times 1.2}{2 \times 10^{11}} + \frac{83.04 \times 10^6 \times 0.8}{2 \times 10^{11}} \\ &= 2.989 \times 10^{-4} + 3.32 \times 10^{-4} \\ &= 6.31 \times 10^{-4} \text{ m} \end{aligned}$$

**i.e. maximum deflection,  $\delta = 0.631 \text{ mm}$**

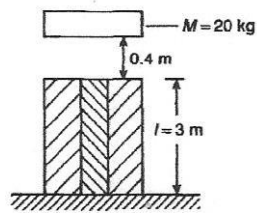
**2.** A reinforced concrete pillar, of length 3 m, is fixed firmly at its base and is free at the top, on to which a 20 kg mass is dropped, as shown below. Determine the maximum stresses in the steel and the concrete, assuming the following apply:

$$A_s = \text{cross-sectional area of steel reinforcement} = 0.01 \text{ m}^2$$

$$E_s = \text{elastic modulus in steel} = 2 \times 10^{11} \text{ N/m}^2$$

$$A_c = \text{cross-sectional area of concrete reinforcement} = 0.2 \text{ m}^2$$

$$E_c = \text{elastic modulus in concrete} = 1.4 \times 10^{10} \text{ N/m}^2$$



Let  $W_e$  = equivalent static load

$$W_e = \sigma_c A_c + \sigma_s A_s \quad (1)$$

$$\delta = \epsilon_c = \epsilon_s$$

$$\frac{\sigma_c}{E_c} = \frac{\sigma_s}{E_s}$$

$$\sigma_c = \frac{E_c \sigma_s}{E_s} = 0.07 \sigma_s \quad (2)$$

Therefore,  $W_e = 0.07 \sigma_s \times 0.2 + \sigma_s \times 0.01$

i.e.  $W_e = 0.014 \sigma_s = 0.01 \sigma_s$

and  $W_e = 0.024 \sigma_s \quad (3)$

$$Mg(h + \delta) = \frac{\sigma_c^2}{2E_c} \times A_c l + \frac{\sigma_s^2}{2E_s} \times A_s l$$

i.e.  $20 \times 9.81 \left( 0.4 + \frac{\sigma_s}{E_s} l \right) = \frac{0.07^2 \sigma_s^2}{2E_c} \times A_c l + \frac{\sigma_s^2}{2E_s} \times A_s l$

i.e.  $196.2 (0.4 + 1.5 \times 10^{-11} \sigma_s) = 1.05 \times 10^{-13} \sigma_s^2 + 7.5 \times 10^{-14} \sigma_s^2$

i.e.  $78.48 + 2.943 \times 10^{-9} \sigma_s = 1.8 \times 10^{-13} \sigma_s^2$

i.e.  $1.8 \times 10^{-13} \sigma_s^2 - 2.943 \times 10^{-9} \sigma_s - 78.48 = 0$

Using a calculator or a smartphone or the quadratic formula,

$$\sigma_s = -20.89 \text{ MN/m}^2$$

From equation (2),  $\sigma_c = 0.07 \sigma_s = 0.07 \times -20.89$

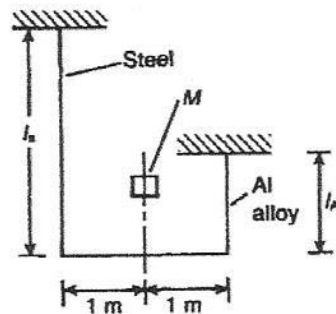
i.e.  $\sigma_c = -1.46 \text{ MN/m}^2$

3. An initially horizontal beam, of length 2 m, is supported at its ends by two wires, one made from aluminium alloy and the other from steel, as shown below. If a mass of 203.87 kg is dropped a distance of 0.1 m above the mid-span of the beam, determine the deflection of the beam under this load. The following may be assumed to apply:

For steel:  $E_s = 2 \times 10^{11} \text{ N/m}^2$ ,  $l_s = 2 \text{ m}$ ,  $A_s = 2 \times 10^{-4} \text{ m}^2$

For aluminium alloy:  $E_{Al} = 7 \times 10^{10} \text{ N/m}^2$ ,  $l_{Al} = 1 \text{ m}$ ,  $A_{Al} = 4 \times 10^{-4} \text{ m}^2$

$EI$  (for beam)  $= 2 \times 10^6 \text{ N/m}^2$



$EI$  for beam  $= 2 \times 10^6 \text{ N/m}^2$

$$\sigma_s = \frac{W_e}{2A_s} = 2500 W_e$$

$$\sigma_a = \frac{W_e}{2A_a} = 1250 W_e$$

or  $W_e = 8 \times 10^{-4} \sigma_a$

$$\sigma_s = 2\sigma_a \quad (1)$$

Let  $\delta_b$  = deflection of the beam due to the flexure alone

$$= \frac{W_e l^3}{48EI} = 8.333 \times 10^{-8} W_e$$

i.e.  $\delta_b = 6.6667 \times 10^{-11} \sigma_a$

$$\delta_a = \frac{\sigma_a}{E_a} l_a = 1.4286 \times 10^{-11} \sigma_a$$

and

$$\delta_s = \frac{\delta_s l_s}{E_s} = 2 \times 10^{-11} \sigma_a$$

$$Mg \left( h + \delta_b + \frac{\delta_a}{2} + \frac{\delta_s}{2} \right) = \frac{\sigma_s^2}{2E_s} \times A_s l_s + \frac{\sigma_a^2}{2E_a} \times A_a l_a + \frac{W_e^2 l^3}{96EI}$$

i.e.

$$2000(0.1 + 6.6667 \times 10^{-11} \sigma_a + 7.143 \times 10^{-12} \sigma_a + 1 \times 10^{-11} \sigma_a) \\ = 4 \times 10^{-15} \sigma_a^2 + 2.857 \times 10^{-15} \sigma_a^2 + 2.667 \times 10^{-14} \sigma_a^2$$

i.e.

$$200 + 1.67 \times 10^{-8} \sigma_a = 3.353 \times 10^{-14} \sigma_a^2$$

or

$$3.353 \times 10^{-14} \sigma_a^2 - 1.676 \times 10^{-8} \sigma_a - 200 = 0$$

Using a calculator or a smartphone or the quadratic formula,

$$\sigma_a = 77.48 \text{ MN/m}^2$$

From equation (1),  $\sigma_s = 2 \times \sigma_a = 154.96 \text{ MN/m}^2$

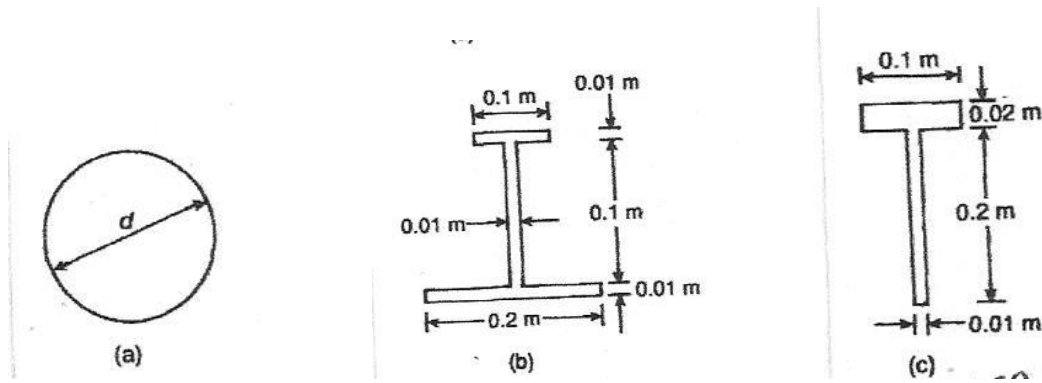
$$\delta = \delta_b + \frac{\delta_a}{2} + \frac{\delta_s}{2} \quad 8 \\ = 5.16 \text{ mm} + \frac{1.107 \text{ mm}}{2} + \frac{1.55 \text{ mm}}{2} \quad 9 \\ = 5.16 + 0.55 + 0.775$$

i.e. **deflection of beam,  $\delta = 6.489 \text{ mm}$**

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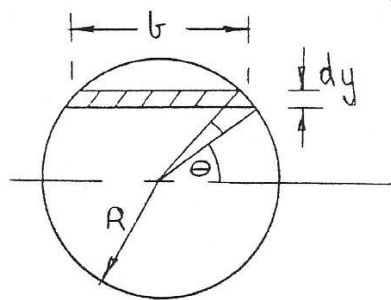
## EXERCISE 50, Page 268

1. Determine the shape factors for the cross-sections shown in figures (a) to (c).

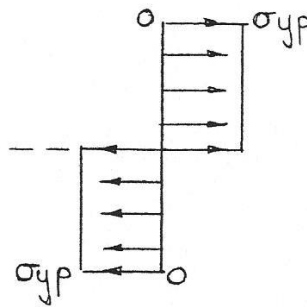


(a) The stress distribution of the solid circular section is shown below, where it can be seen that

$$M_p = \int_{-R}^R \sigma_{yp} \times b \, dy \times y$$



(a) Section



(b) Stress distribution

However,  $b = 2R \cos \theta$

and  $y = R \sin \theta$

and by differentiating  $y$  w.r.t.  $\theta$ ,

$$dy = R \cos \theta \, d\theta$$

Therefore,

$$M_p = \sigma_{yp} \int_{-\pi/2}^{\pi/2} 2R^3 \cos^2 \theta \sin \theta \, d\theta$$

$$= -4R^3 \sigma_{yp} \int_0^{\pi/2} \cos^2 \theta \, d(\cos \theta)$$



$$= -4R^3\sigma_{yp}\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/2}$$

i.e. 
$$M_p = \frac{4R^3\sigma_{yp}}{3}$$

From elementary elastic theory,

$$M_{yp} = \frac{\sigma_{yp}}{R} \times I = \frac{\sigma_{yp}}{R} \times \frac{\pi R^4}{4}$$

i.e. 
$$M_{yp} = \frac{\pi\sigma_{yp}R^3}{4}$$

Now, 
$$S = \frac{M_p}{M_{yp}}$$

i.e. 
$$S = \frac{\frac{4R^3\sigma_{yp}}{3}}{\frac{\pi\sigma_{yp}R^3}{4}} = \frac{16}{3\pi}$$

i.e. 
$$S = 1.70$$

(b)

Section	a	y	ay	ay <sup>2</sup>	i <sub>o</sub>
1	1×10 <sup>-3</sup>	0.115	1.15×10 <sup>-4</sup>	1.323×10 <sup>-5</sup>	8.3×10 <sup>-9</sup>
2	1×10 <sup>-3</sup>	0.600	6×10 <sup>-5</sup>	3.6×10 <sup>-8</sup>	8.33×10 <sup>-7</sup>
3	2×10 <sup>-3</sup>	0.005	1×10 <sup>-5</sup>	5×10 <sup>-8</sup>	1.67×10 <sup>-8</sup>
Σ	4×10 <sup>-3</sup>	—	1.85×10 <sup>-4</sup>	1.688×10 <sup>-5</sup>	8.58×10 <sup>-7</sup>

$$\bar{y} = \frac{1.85 \times 10^{-4}}{4 \times 10^{-3}} = 0.0463 \text{ m}$$

$$I_{xx} = 1.688 \times 10^{-5} + 8.58 \times 10^{-7} = 1.774 \times 10^{-5}$$

$$I_{NA} = 1.7738 \times 10^{-5} - 0.0463^3 \times 4 \times 10^{-3}$$

$$= 9.163 \times 10^{-6} \text{ m}^4$$

$$M_{yp} = \frac{\sigma_{yp}}{\bar{y}} \times I = \frac{\sigma_{yp} \times 9.163 \times 10^{-6}}{0.0737} \quad \text{where } \bar{y} = 0.12 - \bar{y} = 0.12 - 0.0463 = 0.0737 \text{ m}$$

i.e.  $M_{yp} = 1.2433 \times 10^{-4} \sigma_{yp}$

$$M_p = (1 \times 10^{-3} \times 0.105 + 1 \times 10^{-3} \times 0.05 + 2 \times 10^{-3} \times 0.005) \sigma_{yp}$$

$$= 1.65 \times 10^{-4} \sigma_{yp}$$

$$S = \frac{M_p}{M_{yp}} = \frac{1.65 \times 10^{-4}}{1.2433 \times 10^{-4}} = \mathbf{1.327}$$

(c)

Section	a	y	ay	ay <sup>2</sup>	i <sub>o</sub>
1	2 × 10 <sup>-3</sup>	0.21	4.2 × 10 <sup>-4</sup>	8.82 × 10 <sup>-5</sup>	6.67 × 10 <sup>-8</sup>
2	2 × 10 <sup>-3</sup>	0.10	2 × 10 <sup>-4</sup>	2 × 10 <sup>-5</sup>	6.667 × 10 <sup>-6</sup>
Σ	4 × 10 <sup>-3</sup>	—	6.2 × 10 <sup>-4</sup>	10.82 × 10 <sup>-5</sup>	6.734 × 10 <sup>-6</sup>

$$\bar{y} = \frac{6.2 \times 10^{-4}}{4 \times 10^{-3}} = 0.155$$

$$I_{xx} = 10.82 \times 10^{-5} + 6.734 \times 10^{-6} = 1.1493 \times 10^{-4}$$

$$I_{NA} = 1.1493 \times 10^{-4} - 0.155^3 \times 4 \times 10^{-3}$$

$$= 1.883 \times 10^{-5} \text{ m}^4$$

$$M_{yp} = \frac{\sigma_{yp}}{y} \times I = \frac{\sigma_{yp} \times 1.883 \times 10^{-5}}{0.155}$$

i.e.  $M_{yp} = 1.2151 \times 10^{-4} \sigma_{yp}$

$$M_p = \sigma_{yp} (2 \times 10^{-3} \times 0.01 + 2 \times 10^{-3} \times 0.1)$$

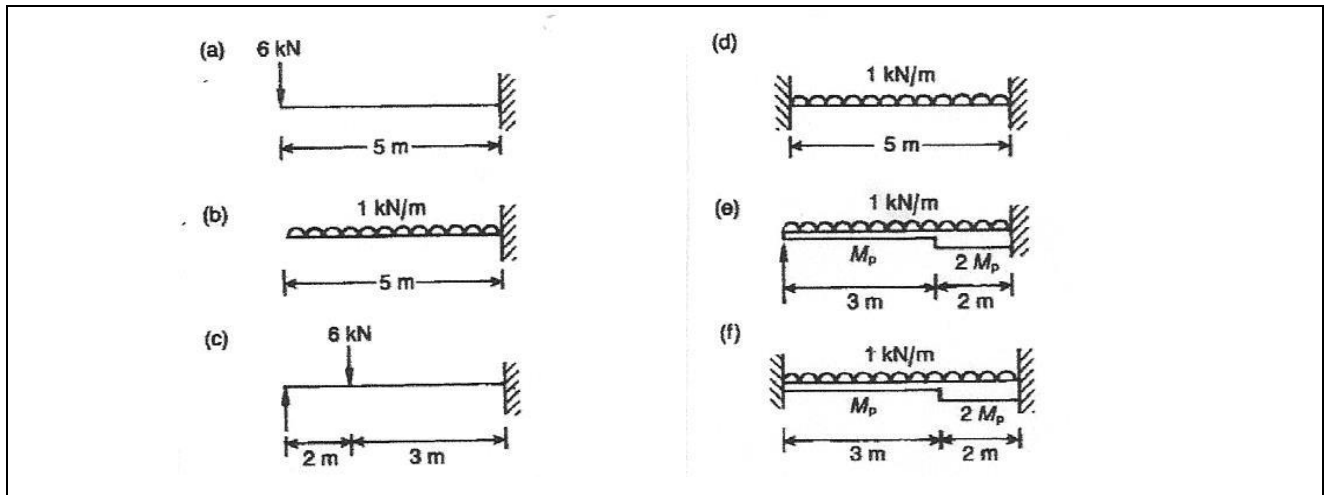
$$= 2.2 \times 10^{-4} \sigma_{yp}$$

$$S = \frac{M_p}{M_{yp}} = \frac{2.2 \times 10^{-4}}{1.2151 \times 10^{-4}} = \mathbf{1.811}$$

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**2.** Using the plastic hinge theory, determine suitable sectional moduli for the beams of below, given that:  $\lambda = 3$ ,  $S = 1.14$  and  $\sigma_{yp} = 300 \text{ MPa}$



(a)  $6 \times 5\theta = M_p\theta$

i.e.  $M_p = 30 \text{ kN m}$

Design  $M_p = 30 \times 3 = 90 \text{ kN m}$

$$M_{yp} = \frac{90}{1.14} = 78.95 \text{ kN m} \quad 11$$

$$Z = \frac{M_{yp}}{\sigma_{yp}} = \frac{78.95 \times 10^3}{300 \times 10^6} \quad 12$$

i.e.  $Z = 2.632 \times 10^{-4} \text{ m}^3$

(b)  $5 \times 1 \times 2.5 \theta = M_p\theta$

i.e.  $M_p = 12.5 \text{ kN m}$

Design  $M_p = 12.5 \times 3 = 37.5 \text{ kN m}$

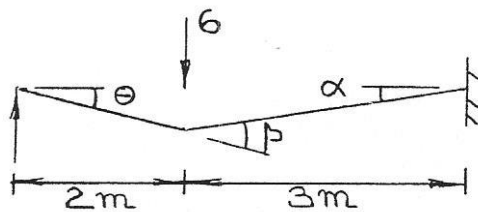
$$M_{yp} = \frac{37.5}{1.14} = 32.89 \text{ kN m} \quad 13$$

$$Z = \frac{M_{yp}}{\sigma_{yp}} = \frac{32.89 \times 10^3}{300 \times 10^6}$$

i.e.  $Z = 1.096 \times 10^{-4} \text{ m}^3$

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(c)



$$2\theta = 3\alpha$$

$$\alpha = 0.667 \theta$$

$$\beta = \alpha + \theta = 1.667\theta$$

$$M_p\beta + M_p\alpha = 6 \times 2\theta$$

or 
$$M_p = \frac{12}{2.333} = 5.143 \text{ kN m}$$

Design  $M_p = 5.143 \times 3 = 15.43 \text{ kN m}$

$$M_{yp} = \frac{15.43}{1.14} = 13.535 \text{ kN m} \quad 15$$

$$Z = \frac{M_{yp}}{\sigma_{yp}} = \frac{13.535 \times 10^3}{300 \times 10^6}$$

i.e.  $Z = 4.511 \times 10^{-5} \text{ m}^3$

(d)  $M_p(\theta + 2\theta + \theta) = 5 \times 1 \times 1.25 \theta$

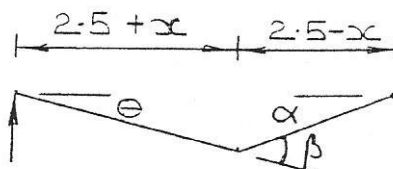
i.e.  $M_p = 1.563 \text{ kN m}$

Design  $M_p = 4.688 \text{ kN m}$

$$Z = \frac{M_{yp}}{\sigma_{yp}} = \frac{4.688 \times 10^3}{300 \times 10^6}$$

i.e.  $Z = 1.563 \times 10^{-5} \text{ m}^3$

(e)



$$(2.5 + x)\theta - (2.5 - x)\alpha$$

$$\alpha = \left( \frac{2.5 + x}{2.5 - x} \right) \theta$$

$$\beta = \alpha + \theta = \left( \frac{2.5+x}{2.5-x} \right) \theta + \left( \frac{2.5-x}{2.5-x} \right) \theta$$

$$\beta = \frac{5\theta}{2.5-x}$$

$$M_p \beta + 2M_p \alpha = 5 \times 1 \times \frac{(2+5+x)}{2} \theta$$

or 
$$M_p \left( \frac{5\theta}{2.5-x} \right) + 2M_p \left( \frac{2.5+x}{2.5-x} \right) \theta = 2.5(2.5+x) \theta$$

i.e. 
$$M_p \left( \frac{5+5+2x}{2.5-x} \right) = 2.5(2.5+x)$$

i.e. 
$$M_p = \frac{2.5(2.5+x)(2.5-x)}{10+2x}$$

i.e. 
$$M_p = \frac{2.5(6.25-x^2)}{10+2x}$$

For maximum  $M_p$ ,  $\frac{dM_p}{dx} = 0$  16

Therefore, 
$$(10+2x) 2.5 (-2x) - 2.5 (6.25-x^2) \times 2 = 0$$

i.e. 
$$-20x - 4x^2 - 12.5 + 2x^2 = 0$$

or 
$$2x^2 + 20x + 12.5 = 0$$

Using a calculator or a smartphone or the quadratic formula,

$$x = -0.67 \text{ m}$$

Hence, 
$$M_p = \frac{2.5(6.25-x^2)}{10+2x}$$

When  $x = -0.67$ ,  $M_p = 1.675 \text{ kN m}$

$$\text{Design } M_p = 5.024 \text{ kN.m}$$

$$M_{yp} = 5.024/1.14 = 4.407 \text{ kN m}$$

$$Z = \frac{M_{yp}}{\sigma_{yp}} = \frac{4.407 \times 10^3}{300 \times 10^6}$$

i.e.  $\mathbf{Z = 1.469 \times 10^{-5} m^3}$

(f)  $M_p \theta + M_p \beta + 2M_p \alpha = 5 \times 1 \times \frac{(2.5 + x)}{2} \theta$

or  $M_p \frac{(2.5 - x)}{(2.5 - x)} + \frac{5}{(2.5 - x)} + \frac{2(2.5 + x)}{(2.5 - x)} = 2.5(2.5 + x)$

i.e.  $M_p \left( \frac{2.5 - x + 5 + 5 + 2x}{2.5 - x} \right) = 2.5(2.5 + x)$

i.e.  $\left( \frac{12.5 + x}{2.5 - x} \right) M_p = 2.5(2.5 + x)$

Now,  $\frac{dM_p}{dx} = 0$

Hence,  $0 = (12.5 + x)2.5(-2x) - 2.5(6.25 - x^2)$

i.e.  $-25 - 2x^2 - 6.25 + x^2 = 0$

or  $x^2 + 25x + 6.25 = 0$

By calculator or a smartphone or the quadratic formula,

$$x = -0.253 \text{ m}$$

Therefore,  $M_p = 1.263 \text{ kN m}$

Design  $M_p = 3.788 \text{ kN m}$

$M_{yp} = 3.323 \text{ kN m}$

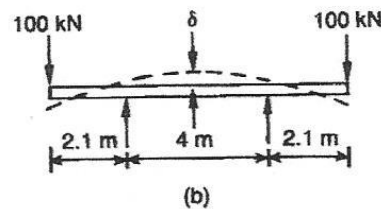
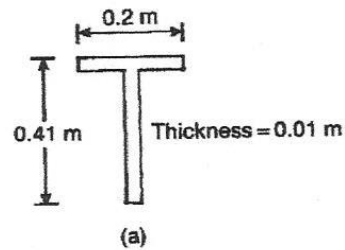
$$Z = \frac{M_{yp}}{\sigma_{yp}} = \frac{3.232 \times 10^3}{300 \times 10^6}$$

i.e.  $\mathbf{Z = 1.108 \times 10^{-5} m^3}$

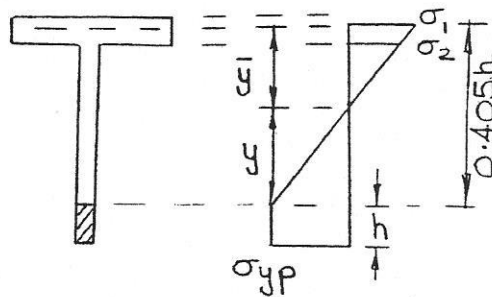
## EXERCISE 51, Page 271

1. A uniform-section tee beam, with the sectional properties shown in figure (a), is subjected to four-point bending, as shown in figure (b). Determine the central deflection on application of the load, and the residual central deflection on its removal. Assume that:

$$\sigma_{yp} = 300 \text{ MN/m}^2 \text{ and } E = 2 \times 10^{11} \text{ N/m}^2$$



$$M = 100 \times 2.1 = 210 \text{ kN m} = M_e + M_p$$



Resolving forces gives:

$$\begin{aligned} \sigma_1 \times 2 \times 10^{-3} + \frac{\sigma_2}{2} \times (\bar{y} - 0.005) \times 0.01 \\ = \sigma_{yp} \times 0.01h + \frac{\sigma_{yp}}{2} \times (0.405 - h - \bar{y}) \times 0.0117 \end{aligned} \quad (1)$$

By similar triangles,

$$\frac{\sigma_1}{\bar{y}} = \frac{\sigma_{yp}}{(0.405 - h - \bar{y})} \quad (2)$$

Therefore, 
$$\sigma_1 = \frac{\sigma_{yp} \bar{y}}{(0.405 - h - \bar{y})}$$

and 
$$\frac{\sigma_2}{(\bar{y} - 0.005)} = \frac{\sigma_{yp}}{(0.405 - h - \bar{y})}$$

or 
$$\sigma_2 = \frac{\sigma_{yp} (\bar{y} - 0.005)}{(0.405 - h - \bar{y})} \quad (3)$$

Substituting equation (2) and (3) into equation (1) gives:

$$\frac{2 \times 10^{-3} \times \bar{y}}{(0.405 - h - \bar{y})} + \frac{(\bar{y} - 0.005)^2 \times 0.01}{2(0.405 - h - \bar{y})} = 0.01h + \frac{(0.405 - h - \bar{y})}{2} \times 0.01$$

or 
$$2 \times 10^{-3} \bar{y} + 0.005 (\bar{y}^2 - 0.01\bar{y} + 2.5 \times 10^{-5}) = 0.01h (0.405 - h - \bar{y}) + 0.005 (0.405 - h - \bar{y})^2$$

or 
$$1.95 \times 10^{-3} \bar{y} + 0.005 \bar{y} + 1.25 \times 10^{-7} = 4.05 \times 10^{-3} h - 0.01h - 0.01h\bar{y} + 0.005 (0.164 + h^2 + \bar{y}^2 - 0.81h - 0.81\bar{y} + 2h\bar{y})$$

or 
$$1.95 \times 10^{-3} \bar{y} + 0.005 \bar{y} + 1.25 \times 10^{-7} = 4.05 \times 10^{-3} h - 0.01h^2 - 0.01h\bar{y} + 8.2 \times 10^{-4} + 0.005h^2 + 0.005\bar{y}^2 - 4.05 \times 10^{-3} h - 4.05 \times 10^{-3} \bar{y} + 0.01h\bar{y}$$

or 
$$1.95 \times 10^{-3} \bar{y} + 1.25 \times 10^{-7} + 4.05 \times 10^{-3} \bar{y} - 8.2 \times 10^{-4} = -0.005h^2$$

i.e. 
$$6 \times 10^{-3} \bar{y} - 8.2 \times 10^{-4} = -0.005h^2$$

i.e. 
$$6 \times 10^{-3} \bar{y} = 8.2 \times 10^{-4} - 0.005h^2$$

Now, 
$$\frac{\sigma_{yp}}{Y} = \frac{\sigma_1}{\bar{y}} \quad \text{or} \quad Y = \frac{\sigma_{yp} \bar{y}}{\sigma_1}$$

$$\bar{y} = 0.1367 - 0.833h^2$$

$h$	$y$	$\sigma_1$	$Y$	$Y + y + H$
0.05	0.1346	$0.61\sigma_{yp}$	0.220	0.404
0.1	0.1284	$0.727\sigma_{yp}$	0.1766	0.405

Therefore,  $\bar{y} = 0.1284 \text{ m}$  and  $h = 0.1 \text{ m}$



Now,  $M_p = \sigma_{yp} \times 0.01 \times h \times (Y + h/2)$   
 $= 300 \times 0.01 \times 0.1 \times (0.2266)$

i.e.  $M_p = 68 \text{ kN m}$

Since  $210 \text{ kN m} = M_e + M_p$

then,  $M_e = 142 \text{ kN m}$

$$I_{NA} = \frac{0.2 \times 0.01^3}{12} 18 + 0.2 \times 0.01 \times 0.1284^2 + \frac{0.01}{3} 19 (0.1234^3 + 0.1766^3)$$

$$= 1.67 \times 10^{-8} + 3.297 \times 10^{-5} + 2.462 \times 10^{-5}$$

i.e.  $I_{NA} = 5.76 \times 10^{-5} \text{ m}^4$

Now,  $\frac{M_e}{I} = \frac{E}{R}$

and  $R = \frac{EI}{M_e} = \frac{2 \times 10^{11} \times 5.76 \times 10^{-5}}{142 \times 10^3}$

i.e.  $R = 81.1 \text{ m}$

From the properties of a chord

$$(2R - \delta) \times \delta = 2^2$$

i.e.  $162.2\delta - \delta^2 = 4$  neglecting  $\delta^2$

from which, **central deflection,  $\delta = 0.0247 \text{ m}$**

On removal of load, the total  $I$  has to be used.

Section	a	y	ay	ay <sup>2</sup>	i <sub>o</sub>
1	$2 \times 10^{-3}$	0.405	$8.1 \times 10^{-4}$	$3.28 \times 10^{-4}$	$1.67 \times 10^{-8}$
2	$4 \times 10^{-3}$	0.2	$8 \times 10^{-4}$	$1.6 \times 10^{-4}$	$5.333 \times 10^{-5}$
$\Sigma$	$6 \times 10^{-3}$	—	$16.1 \times 10^{-4}$	$4.88 \times 10^{-4}$	$5.335 \times 10^{-5}$

$$\bar{y} = \frac{16.1 \times 10^{-4}}{6 \times 10^{-3}} = 0.2683$$

$$I_{xx} = 4.88 \times 10^{-4} + 5.34 \times 10^{-5}$$

$$= 5.413 \times 10^{-4}$$

$$I_{MA} = 5.413 \times 10^{-4} - 0.2683^3 \times 6 \times 10^{-3}$$

i.e.  $I_{MA} = 1.094 \times 10^{-4} \text{ m}^4$

$$R = \frac{EI}{M} = \frac{2 \times 10^{11} \times 1.094 \times 10^{-4}}{210 \times 10^3}$$

i.e.  **$R = 104.2 \text{ m}$**

$$\delta = \frac{4}{104.2 \times 2} = 0.019 \text{ m}$$

Hence, **residual deflection** =  $0.247 - 0.019$

$$= \mathbf{0.0055 \text{ m}}$$


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