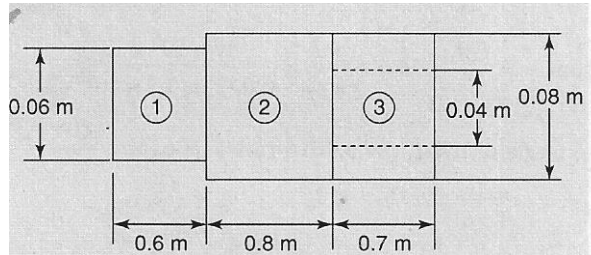


CHAPTER 8 TORSION

EXERCISE 37, Page 174

1. A circular-section steel shaft consists of three elements, two solid and one hollow, as shown, and it is subjected to a torque of 3 kN m.



Determine (a) the angle of twist of one end relative to the other, and (b) the maximum shearing stresses in each element. It may be assumed that $G = 7.7 \times 10^{10} \text{ N/m}^2$

(a) Polar second moment of area of section 1, $J_1 = \frac{\pi D_1^4}{32} = \frac{\pi (0.06)^4}{32} = 1.272 \times 10^{-6} \text{ m}^4$

Polar second moment of area of section 2, $J_2 = \frac{\pi D_2^4}{32} = \frac{\pi (0.08)^4}{32} = 4.021 \times 10^{-6} \text{ m}^4$

Polar second moment of area of section 3, $J_3 = \frac{\pi (0.08^4 - 0.04^4)}{32} = 3.770 \times 10^{-6} \text{ m}^4$

Now, $\frac{T}{J} = \frac{G\theta}{L}$ from which, $\theta = \frac{T}{G} \times \frac{L}{J}$

Hence,
$$\theta = \frac{3 \times 10^3}{7.7 \times 10^{10}} \times \left(\frac{0.6}{1.272 \times 10^{-6}} + \frac{0.8}{4.021 \times 10^{-6}} + \frac{0.7}{3.770 \times 10^{-6}} \right)$$

$$= 3.896 \times 10^{-8} \times (471698 + 198944 + 185676)$$

i.e. angle of twist, $\theta = 3.336 \times 10^{-2} \text{ rads} = 1.91^\circ$

(b) Since $\frac{\tau}{R} = \frac{T}{J}$ then $\tau = \frac{TR}{J}$

Maximum shearing stress in element 1, $\tau_1 = \frac{TR_1}{J_1} = \frac{3 \times 10^3 \times \frac{0.06}{2}}{1.272 \times 10^{-6}} = 70.75 \text{ MN/m}^2$

$$\text{Maximum shearing stress in element 2, } \tau_2 = \frac{TR_2}{J_2} = \frac{3 \times 10^3 \times \frac{0.08}{2}}{4.021 \times 10^{-6}} = \mathbf{29.84 \text{ MN/m}^2}$$

$$\text{Maximum shearing stress in element 3, } \tau_3 = \frac{TR_3}{J_3} = \frac{3 \times 10^3 \times \frac{0.08}{2}}{3.770 \times 10^{-6}} = \mathbf{31.83 \text{ MN/m}^2}$$

2. A circular-section steel shaft consists of three elements, two solid and one hollow, as shown in the diagram in Problem 1, and it is subjected to a torque of 4 kN m. Determine (a) the angle of twist of one end relative to the other, and (b) the maximum shearing stresses in each element. It may be assumed that $G = 7.7 \times 10^{10} \text{ N/m}^2$.

$$\text{(a) Polar second moment of area of section 1, } J_1 = \frac{\pi D_1^4}{32} = \frac{\pi (0.06)^4}{32} = 1.272 \times 10^{-6} \text{ m}^4$$

$$\text{Polar second moment of area of section 2, } J_2 = \frac{\pi D_2^4}{32} = \frac{\pi (0.08)^4}{32} = 4.021 \times 10^{-6} \text{ m}^4$$

$$\text{Polar second moment of area of section 1, } J_3 = \frac{\pi (0.08^4 - 0.04^4)}{32} = 3.770 \times 10^{-6} \text{ m}^4$$

$$\text{Now, } \frac{T}{J} = \frac{G\theta}{L} \quad \text{from which, } \theta = \frac{T}{G} \times \frac{L}{J}$$

$$\begin{aligned} \text{Hence, } \theta &= \frac{4 \times 10^3}{7.7 \times 10^{10}} \times \left(\frac{0.6}{1.272 \times 10^{-6}} + \frac{0.8}{4.021 \times 10^{-6}} + \frac{0.7}{3.770 \times 10^{-6}} \right) \\ &= 5.195 \times 10^{-8} \times (471698 + 198944 + 185676) \end{aligned}$$

$$\text{i.e. angle of twist, } \theta = \mathbf{4.448 \times 10^{-2} \text{ rads} = 2.55^\circ}$$

$$\text{(b) Since } \frac{\tau}{R} = \frac{T}{J} \quad \text{then } \tau = \frac{TR}{J}$$

$$\text{Maximum shearing stress in element 1, } \tau_1 = \frac{TR_1}{J_1} = \frac{4 \times 10^3 \times \frac{0.06}{2}}{1.272 \times 10^{-6}} = \mathbf{94.34 \text{ MN/m}^2}$$

$$\text{Maximum shearing stress in element 2, } \tau_2 = \frac{TR_2}{J_2} = \frac{4 \times 10^3 \times \frac{0.08}{2}}{4.021 \times 10^{-6}} = \mathbf{39.79 \text{ MN/m}^2}$$

$$\text{Maximum shearing stress in element 3, } \tau_3 = \frac{TR_3}{J_3} = \frac{4 \times 10^3 \times \frac{0.08}{2}}{3.770 \times 10^{-6}} = \mathbf{42.44 \text{ MN/m}^2}$$

3. If the shaft of Problem 1 were constructed from three separate materials, namely steel in element (1), aluminium alloy in element (2) and manganese bronze in element (3), determine (a) the total angle of twist, and (b) the maximum shearing stresses in each element. Assume that $G_1 = 7.7 \times 10^{10} \text{ N/m}^2$, $G_2 = 2.5 \times 10^{10} \text{ N/m}^2$ and $G_3 = 3.9 \times 10^{10} \text{ N/m}^2$.

Since $\theta = \frac{T}{G} \times \frac{L}{J}$

then
$$\theta = \frac{3 \times 10^3}{7.7 \times 10^{10}} \times \frac{0.6}{1.272 \times 10^{-6}} + \frac{3 \times 10^3}{2.5 \times 10^{10}} \times \frac{0.8}{4.021 \times 10^{-6}} + \frac{3 \times 10^3}{3.9 \times 10^{10}} \times \frac{0.7}{3.770 \times 10^{-6}}$$

$$= 0.0184 + 0.0239 + 0.0143$$

i.e. angle of twist, $\theta = \mathbf{0.0566 \text{ rads} = 3.24^\circ}$

$$\text{Maximum shearing stress in element 1, } \tau_1 = \frac{TR_1}{J_1} = \frac{3 \times 10^3 \times \frac{0.06}{2}}{1.272 \times 10^{-6}} = \mathbf{70.75 \text{ MN/m}^2}$$

$$\text{Maximum shearing stress in element 2, } \tau_2 = \frac{TR_2}{J_2} = \frac{3 \times 10^3 \times \frac{0.08}{2}}{4.021 \times 10^{-6}} = \mathbf{29.84 \text{ MN/m}^2}$$

$$\text{Maximum shearing stress in element 3, } \tau_3 = \frac{TR_3}{J_3} = \frac{3 \times 10^3 \times \frac{0.08}{2}}{3.770 \times 10^{-6}} = \mathbf{31.83 \text{ MN/m}^2}$$

4. If the shaft of Problem 2 were constructed from three separate materials, namely steel in element (1), aluminium alloy in element (2) and manganese bronze in element (3), determine (a) the total angle of twist, and (b) the maximum shearing stresses in each element. Assume that

$$G_1 = 7.7 \times 10^{10} \text{ N/m}^2, G_2 = 2.5 \times 10^{10} \text{ N/m}^2 \text{ and } G_3 = 3.9 \times 10^{10} \text{ N/m}^2.$$

(a) Polar second moment of area of section 1, $J_1 = \frac{\pi D_1^4}{32} = \frac{\pi (0.06)^4}{32} = 1.272 \times 10^{-6} \text{ m}^4$

Polar second moment of area of section 2, $J_2 = \frac{\pi D_2^4}{32} = \frac{\pi (0.08)^4}{32} = 4.021 \times 10^{-6} \text{ m}^4$

Polar second moment of area of section 3, $J_3 = \frac{\pi (0.08^4 - 0.04^4)}{32} = 3.770 \times 10^{-6} \text{ m}^4$

Now, $\frac{T}{J} = \frac{G\theta}{L}$ from which, $\theta = \frac{T}{G} \times \frac{L}{J}$

Hence,
$$\theta = \frac{4 \times 10^3}{7.7 \times 10^{10}} \times \frac{0.6}{1.272 \times 10^{-6}} + \frac{4 \times 10^3}{2.5 \times 10^{10}} \times \frac{0.8}{4.021 \times 10^{-6}} + \frac{4 \times 10^3}{3.9 \times 10^{10}} \times \frac{0.7}{3.770 \times 10^{-6}}$$

$$= 0.0245 + 0.0318 + 0.0190$$

i.e. **angle of twist, $\theta = 0.0753 \text{ rads} = 4.314^\circ$**

(b) Since $\frac{\tau}{R} = \frac{T}{J}$ then $\tau = \frac{TR}{J}$

Maximum shearing stress in element 1, $\tau_1 = \frac{TR_1}{J_1} = \frac{4 \times 10^3 \times \frac{0.06}{2}}{1.272 \times 10^{-6}} = 94.34 \text{ MN/m}^2$

Maximum shearing stress in element 2, $\tau_2 = \frac{TR_2}{J_2} = \frac{4 \times 10^3 \times \frac{0.08}{2}}{4.021 \times 10^{-6}} = 39.79 \text{ MN/m}^2$

Maximum shearing stress in element 3, $\tau_3 = \frac{TR_3}{J_3} = \frac{4 \times 10^3 \times \frac{0.08}{2}}{3.770 \times 10^{-6}} = 42.44 \text{ MN/m}^2$

5. Determine the output torque of an electric motor which supplies 5 kW at 25 rev/s.

(a) If this torque is transmitted through a tube of external diameter 20 mm, determine the internal diameter if the maximum permissible shearing stress in the shaft is 35 MN/m^2 .

(b) If this shaft is to be connected to another one, via a flanged coupling, determine a suitable bolt diameter if four bolts are used on a pitch circle diameter of 30 mm, and the maximum permissible shearing stresses in the bolts equal 45 MN/m^2 .

$$(a) \quad \omega = 2\pi \times 25 = 157.1 \text{ Hz}$$

$$T\omega = \text{Power}$$

$$\text{from which, torque, } T = \frac{P}{\omega} = \frac{5 \times 10^3}{157.1} = \mathbf{31.83 \text{ N m}}$$

$$\text{Now,} \quad \frac{\tau}{r} = \frac{T}{J}$$

$$\text{i.e.} \quad \frac{35 \times 10^6}{\frac{0.02}{2}} = \frac{31.83}{\frac{\pi}{2} \left((0.01)^4 - r^4 \right)}$$

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$$\text{i.e.} \quad 1 \times 10^{-8} - r^4 = \frac{31.83 \times 0.01}{\frac{\pi}{2} \times 35 \times 10^6} = 5.7896 \times 10^{-9}$$

$$\text{and} \quad r^4 = 1 \times 10^{-8} - 5.7896 \times 10^{-9} = 4.210 \times 10^{-9}$$

$$\text{from which,} \quad r = \sqrt[4]{4.210 \times 10^{-9}} = 8.055 \times 10^{-3} \text{ m} = 8.055 \text{ mm}$$

Hence, **the internal diameter, $d = 16.11 \text{ mm}$, say, 16 mm**

(b) For the bolts:

$$\delta F = \frac{\pi d^2}{4} \times \tau_F = \frac{\pi \times d^2}{4} \times 45 \times 10^6 = 35.34 \times 10^6 d^2 \quad 3$$

$$\text{Torque, } T = n \times \delta F \times R = 4 \times 35.34 \times 10^6 d^2 \times 0.015$$

$$= 2.1204 \times 10^6 d^2 = 31.83 \quad \text{from part (a)}$$

and **the bolt diameter, $d = \sqrt{\frac{31.83}{2.1204 \times 10^6}} = 3.87 \times 10^{-3} \text{ m} = 3.87 \text{ mm}$, say, 4 mm**

6. Determine the output torque of an electric motor which supplies 8 kW at 25 rev/s.

(a) If this torque is transmitted through a tube of external diameter 22 mm, determine the internal diameter if the maximum permissible shearing stress in the shaft is 35 MN/m^2 .

(b) If this shaft is to be connected to another one, via a flanged coupling, determine a suitable bolt diameter if four bolts are used on a pitch circle diameter of 32 mm, and the maximum permissible shearing stresses in the bolts equal 45 MN/m^2 .

(a) $\omega = 2\pi \times 25 = 157.08 \text{ Hz}$

$T\omega = \text{Power}$

from which, **torque, $T = \frac{P}{\omega} = \frac{8 \times 10^3}{157.08} = 50.93 \text{ N m}$**

Now, $\frac{\tau}{r} = \frac{T}{J}$

i.e. $\frac{35 \times 10^6}{\frac{0.022}{2}} = \frac{50.92}{\frac{\pi}{2}((0.011)^4 - r^4)}$

45

i.e. $1.464 \times 10^{-8} - r^4 = \frac{50.92 \times 0.011}{\frac{\pi}{2} \times 35 \times 10^6} = 1.0188 \times 10^{-8}$

and $r^4 = 1.464 \times 10^{-8} - 1.0188 \times 10^{-8} = 4.452 \times 10^{-9}$

from which, $r = \sqrt[4]{4.452 \times 10^{-9}} = 8.168 \times 10^{-3} \text{ m} = 8.168 \text{ mm}$

Hence, **the internal diameter, $d = 16.34 \text{ mm}$, say, 16.5 mm**

(b) For the bolts:

$$\delta F = \frac{\pi d^2}{4} \times \tau_F = \frac{\pi \times d^2}{4} \times 45 \times 10^6 = 35.34 \times 10^6 d^2$$

Torque, $T = n \times \delta F \times R = 4 \times 35.34 \times 10^6 d^2 \times 0.016$

$$= 2.2618 \times 10^6 d^2 = 50.92 \quad \text{from part (a)}$$

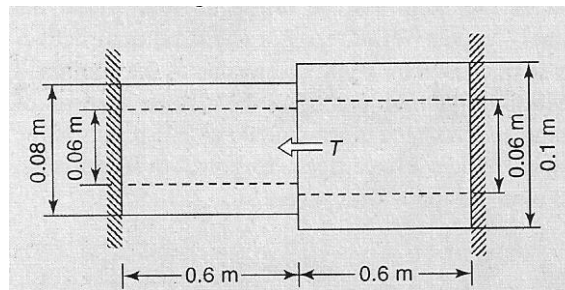
$$\text{and the bolt diameter, } d = \sqrt{\frac{50.92}{2.2618 \times 10^6}} = 4.745 \times 10^{-3} \text{ m} = \mathbf{4.745 \text{ mm, say, 5.0 mm}}$$

EXERCISE 38, Page 179

1. A compound shaft consists of two equal length hollow shafts joined together in series and subjected to a torque T , as shown below. If the shaft on the left is made from steel, and the shaft on the right of the figure is made from aluminium alloy, determine the maximum permissible value of T , given the following:

For steel: $G = 7.7 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 140 MN/m^2

For aluminium alloy: $G = 2.6 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 90 MN/m^2



$$\text{Polar second moment of area of steel section, } J_1 = \frac{\pi(0.08^4 - 0.06^4)}{32} = 2.749 \times 10^{-6} \text{ m}^4$$

$$\text{Polar second moment of area of aluminium section, } J_2 = \frac{\pi(0.1^4 - 0.06^4)}{32} = 8.545 \times 10^{-6} \text{ m}^4$$

$$\text{Now, } \frac{\tau}{r} = \frac{T}{J} \text{ from which, } T = \frac{\tau J}{r}$$

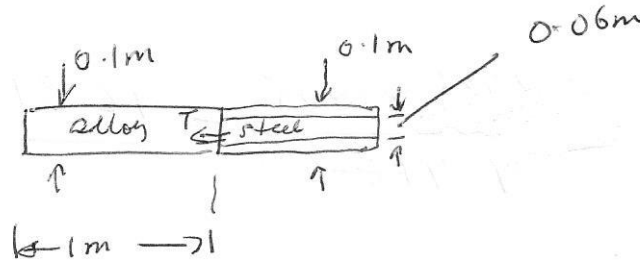
$$T_s = \frac{140 \times 10^6 \times 2.749 \times 10^{-6}}{0.04} = 9621 \text{ N m} = 9.62 \text{ kN m}$$

$$T_a = \frac{90 \times 10^6 \times 8.545 \times 10^{-6}}{0.05} = 15381 \text{ N m} = 15.38 \text{ kN m}$$

The maximum permissible value of $T = 9.62 \text{ kN m}$

2. A compound shaft consists of a solid aluminium-alloy cylinder of length 1 m and diameter 0.1 m, connected in series to a steel tube of the same length and external diameter, and of thickness 0.02 m. The shaft is fixed at its ends and is subjected to an intermediate torque of 9 kN m at the joint.

Determine the angle of twist and the maximum shear stress in the two halves. Assume that $G_{(\text{steel})} = 7.7 \times 10^{10} \text{ N/m}^2$ and $G_{(\text{Al alloy})} = 2.6 \times 10^{10} \text{ N/m}^2$.



$$\theta_c = \frac{T_1 l_1}{G_1 J_1} \quad \text{and} \quad \theta_c = \frac{T_2 l_2}{G_2 J_2}$$

Hence,
$$\frac{T_1 l_1}{G_1 J_1} = \frac{T_2 l_2}{G_2 J_2}$$

However,
$$l_1 = l_2$$

from which,
$$T_1 = \frac{T_2 G_1 J_1}{G_2 J_2} \quad (1)$$

Polar second moment of area of aluminium section, $J_a = \frac{\pi D_1^4}{32} = \frac{\pi (0.1)^4}{32} = 9.8175 \times 10^{-6} \text{ m}^4$

Polar second moment of area of steel section, $J_s = \frac{\pi (0.1^4 - 0.06^4)}{32} = 8.5451 \times 10^{-6} \text{ m}^4$

$$T_1 = \frac{T_2 \times 2.6 \times 10^{10} \times 9.817 \times 10^{-6}}{7.7 \times 10^{10} \times 8.545 \times 10^{-6}} = 0.388 T_2 \quad (1a)$$

$$T_1 + T_2 = T = 9 \times 10^3 \quad (2)$$

From equation (1a),
$$0.388 T_2 + T_2 = 9 \times 10^3$$

from which,
$$T_2 = \frac{9000}{1.388} = 6484 \text{ N m} \quad (3)$$

From equation (1a),
$$T_1 = 0.388 \times 6484 = 2516 \text{ N m}$$

Now, $\frac{\tau}{r} = \frac{T}{J}$ from which,
$$\tau = \frac{T r}{J}$$

Alloy: $\tau_1 = \frac{T_1 R_1}{J_1} = \frac{2516 \times 0.05}{9.817 \times 10^{-6}} = \mathbf{12.81 \text{ MPa}}$

Steel: $\tau_2 = \frac{T_2 R_2}{J_2} = \frac{6484 \times 0.05}{8.545 \times 10^{-6}} = \mathbf{37.94 \text{ MPa}}$

$$\frac{T}{J} = \frac{G\theta}{l} \quad \text{i.e.} \quad \theta = \frac{Tl}{GJ}$$

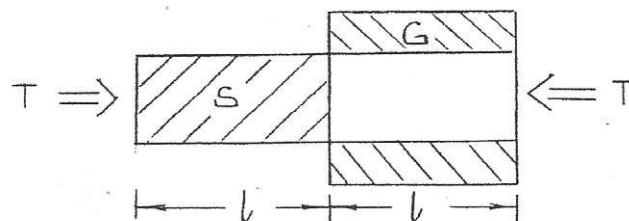
Alloy: $\theta_1 = \frac{T_1 l_1}{G_1 J_1} = \frac{2516 \times 1}{2.6 \times 10^{10} \times 9.8175 \times 10^{-6}} = \mathbf{9.857 \times 10^{-3} \text{ rad} = 0.565^\circ}$

Steel: $\theta_2 = \frac{T_2 l_2}{G_2 J_2} = \frac{6484 \times 1}{7.7 \times 10^{10} \times 8.5451 \times 10^{-6}} = \mathbf{9.855 \times 10^{-3} \text{ rad} = 0.565^\circ}$

3. A compound shaft consists of two elements of equal length, joined together in series. If one element of the shaft is constructed from gunmetal tube and the other from solid steel, where the external diameter of the steel shaft and the internal diameter of the gunmetal shaft equal 50 mm, determine the external diameter of the gunmetal shaft if the two shafts are to have the same torsional stiffness. Determine also the maximum permissible torque that can be applied to the shaft, given the following:

For gunmetal: $G = 3 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 45 MN/m^2

For steel: $G = 7.5 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 90 MN/m^2



Polar second moment of area of aluminium section, $J_s = \frac{\pi D_1^4}{32} = \frac{\pi (0.05)^4}{32} = 6.136 \times 10^{-7} \text{ m}^4$

Polar second moment of area of steel section, $J_G = \frac{\pi (D_2^4 - 0.05^4)}{32} \text{ m}^4 \quad (1)$

Now, $\frac{GJ}{l} = \text{constant}$

i.e. $\frac{G_s J_s}{l_s} = \frac{G_G J_G}{l_G}$

i.e. $\frac{7.5 \times 10^{10} \times 6.136 \times 10^{-7}}{l} = \frac{3 \times 10^{10} J_G}{l}$

and $J_G = \frac{7.5 \times 10^{10} \times 6.136 \times 10^{-7}}{3 \times 10^{10}} = 1.534 \times 10^{-6}$

From equation (1), $1.534 \times 10^{-6} = \frac{\pi(D_2^4 - 0.05^4)}{32}$

Hence, $\frac{1.534 \times 10^{-6} \times 32}{\pi} = (D_2^4 - 0.05^4)$

i.e. $15.625 \times 10^{-6} = (D_2^4 - 0.05^4)$

and $D_2^4 = 15.625 \times 10^{-6} + 0.05^4 = 2.1875 \times 10^{-5}$

from which, **external diameter of gunmetal,**

$$D_2 = \sqrt[4]{2.1875 \times 10^{-5}} = 0.06839 \text{ m} = \mathbf{68.4 \text{ mm}}$$

The **maximum permissible torque of gunmetal,**

$$T_G = \frac{\tau_G \times J_G}{r_G} = \frac{45 \times 10^6 \times 1.534 \times 10^{-6}}{0.025} = \mathbf{2761 \text{ N m}}$$

The **maximum permissible torque of steel,**

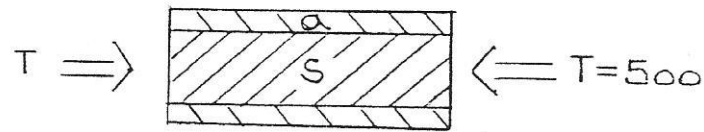
$$T_s = \frac{\tau_s \times J_s}{r_s} = \frac{90 \times 10^6 \times 6.136 \times 10^{-7}}{0.025} = \mathbf{2209 \text{ N m}}$$

Therefore, **maximum permissible torque, $T = T_s = 2209 \text{ N m}$** which is the smallest of the two torques.

4. A compound shaft consists of a solid steel core, which is surrounded co-axially by aluminium bronze sheath. Determine suitable values for the diameters of the shafts if the steel core is to carry two thirds of a total torque of 500 N m and if the limiting stresses are not exceeded.

For aluminium alloy: $G = 3.8 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 18 MN/m^2

For steel: $G = 7.7 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 36 MN/m^2



$$T_s = \frac{2}{3} \times 500 = 333.3 \text{ N m}$$

$$T_{al} = \frac{1}{3} \times 500 = 166.7 \text{ N m}$$

Now,
$$\frac{T}{J} = \frac{G\theta}{l}$$

Hence,
$$\theta = \frac{T_s l}{G_s J_s} = \frac{T_a l}{G_a J_a}$$

Therefore,
$$\frac{333.3}{7.7 \times 10^{10} J_s} = \frac{166.7}{3.8 \times 10^{10} J_a}$$

from which,
$$J_a = 1.0135 J_s$$

Now,
$$J_a = \frac{\pi(D^4 - d^4)}{32}$$

and
$$J_s = \frac{\pi d^4}{32}$$

Hence,
$$\frac{\pi(D^4 - d^4)}{32} = 1.0135 \times \frac{\pi d^4}{32}$$

i.e.
$$(D^4 - d^4) = 1.0135 d^4$$

and
$$D^4 = 2.0135 d^4$$

or
$$D = 1.191 d$$

Now,
$$\frac{\tau}{r} = \frac{G\theta}{l}$$

For steel,
$$\theta = \frac{\tau_s l}{G_s r_s} = \frac{36 \times 10^6 \times l}{7.7 \times 10^{10} \times \frac{d}{2}}$$

and
$$\frac{\theta}{l} = \frac{9.351 \times 10^{-4}}{d} \quad (1)$$

For aluminium bronze alloy,

$$\theta = \frac{\tau_a l}{G_a r_a} = \frac{18 \times 10^6 \times l}{3.8 \times 10^{10} \times \frac{D}{2}}$$

and
$$\frac{\theta}{l} = \frac{9.474 \times 10^{-4}}{D} \quad (2)$$

From equations (1) and (2),
$$\frac{9.351 \times 10^{-4}}{d} = \frac{9.474 \times 10^{-4}}{D}$$

and
$$D = 1.0132 d \quad (3)$$

The aluminium bronze alloy is the design criterion

$$J_a = \frac{\pi(1.191^4 - 1^4)d^4}{32} = 0.0994 d^4$$

Now,
$$\frac{\tau_a}{r_a} = \frac{T_a}{J_a}$$

i.e.
$$\frac{18 \times 10^6}{1.191d/2} = \frac{166.7}{0.0994d^4}$$

from which,
$$d^3 = \frac{166.7 \times 1.191}{2 \times 0.0994 \times 18 \times 10^6} = 5.5483 \times 10^{-5}$$

and
$$d = 0.0381 \text{ m} = \mathbf{38.1 \text{ mm}}$$

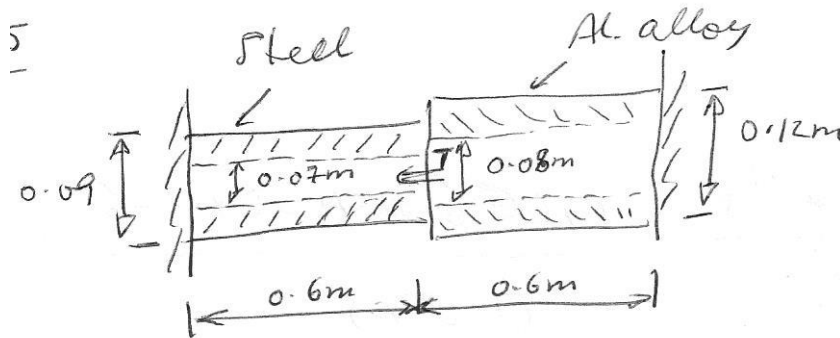
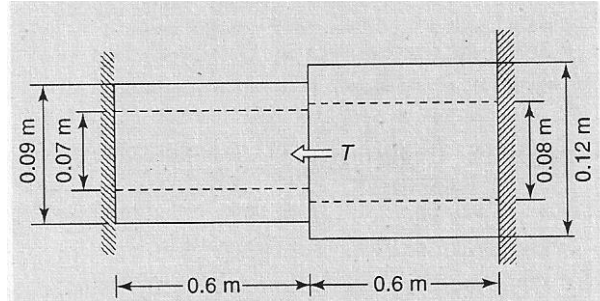
From equation (3),
$$D = 1.0132 d = 1.0132 \times 38.1 = \mathbf{38.6 \text{ mm}}$$

5. A compound shaft consists of two equal length hollow shafts joined together in series and subjected to a torque T , as shown below. If the shaft on the left is made from steel, and the shaft on

the right of the figure is made from aluminium alloy, determine the maximum permissible value of T , given the following:

For steel: $G = 7.7 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 140 MN/m^2

For aluminium alloy: $G = 2.6 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 90 MN/m^2



Polar second moment of area of steel section, $J_1 = \frac{\pi(0.09^4 - 0.07^4)}{32} = 4.084 \times 10^{-6} \text{ m}^4$

Polar second moment of area of aluminium section, $J_2 = \frac{\pi(0.12^4 - 0.08^4)}{32} = 1.634 \times 10^{-5} \text{ m}^4$

From equation (1) of Problem (2), $T_s = \frac{T_a G_s J_s}{G_a J_a} = \frac{T_2 \times 7.7 \times 10^{10} \times 4.084 \times 10^{-6}}{2.6 \times 10^{10} \times 1.634 \times 10^{-5}}$

i.e. $T_s = 0.74 T_a$ (1)

From equation (2) of Problem 2, $T_s + T_a = T$ (2)

From equation (1), $0.74 T_a + T_a = T$

i.e. $T_a = \frac{T}{1.74} = 0.575 T$

Now, $\frac{\tau}{r} = \frac{T}{J}$ from which, $T = \frac{\tau J}{r}$

Alloy: $T_a = \frac{\tau_a J_a}{R_a} = \frac{90 \times 10^6 \times 1.632 \times 10^{-5}}{0.06} = \mathbf{1470.6 \text{ N m}}$

Steel: $T_s = \frac{\tau_s J_s}{R_s} = \frac{140 \times 10^6 \times 4.084 \times 10^{-6}}{0.045} = \mathbf{12706 \text{ N m}}$

From equation (1), $T_a = \frac{T_s}{0.74} = \frac{12706}{0.74} = 17170 \text{ N m},$

which is impossible as the aluminium alloy tube will fail; thus, $T_a = 1471 \text{ N m}$ is the design criterion, and from equation (1), $T_s = 0.74 \times 1471 = 1088.5 \text{ N m}$

The maximum permissible value of $T = T_a + T_s = 1471 + 1088.5 = 2.56 \text{ kN m}$

6. A compound shaft consists of a solid aluminium-alloy cylinder of length 1 m and diameter 0.1 m, connected in series to a steel tube of the same length and external diameter, and of thickness 0.02 m. The shaft is fixed at its ends and is subjected to an intermediate torque of 10 kN m at the joint. Determine the angle of twist and the maximum shear stress in the two halves. Assume that $G_{(\text{steel})} = 7.7 \times 10^{10} \text{ N/m}^2$ and $G_{(\text{Al alloy})} = 2.6 \times 10^{10} \text{ N/m}^2$.

Polar second moment of area of aluminium section, $J_a = \frac{\pi D_1^4}{32} = \frac{\pi (0.1)^4}{32} = 9.8175 \times 10^{-6} \text{ m}^4$

Polar second moment of area of steel section, $J_s = \frac{\pi (0.1^4 - 0.06^4)}{32} = 8.5451 \times 10^{-6} \text{ m}^4$

Now, $\frac{\tau}{r} = \frac{G\theta}{L}$ from which, $\theta = \frac{\tau L}{G r}$

Hence, $\theta_{C_a} = \frac{T_a l_a}{G_a J_a} = \frac{T_a \times 1}{2.6 \times 10^{10} \times 9.8175 \times 10^{-6}} = \mathbf{3.918 \times 10^{-6} T_a}$

and $\theta_{C_s} = \frac{T_s l_s}{G_s J_s} = \frac{T_s \times 1}{7.7 \times 10^{10} \times 8.5451 \times 10^{-6}} = \mathbf{1.5198 \times 10^{-6} T_s}$

Since $\theta_{C_a} = \theta_{C_s}$ then $3.918 \times 10^{-6} T_a = 1.5198 \times 10^{-6} T_s$

from which, $T_s = \frac{3.918}{1.5198} T_a$ i.e. $T_s = 2.578 T_a$

Now, $T_s + T_a = 10 \times 10^3$ (1)

i.e. $2.578 T_a + T_a = 10000$

i.e. $T_a = \frac{10000}{3.578} = 2.795 \text{ kN m}$

From equation (1), $T_s = 10 \times 10^3 - 2.795 \times 10^3 = 7.205 \text{ kN m}$

Angle of twist, $\theta_c = 3.918 \times 10^{-6} \times 2.795 \times 10^3 = 0.01095 \text{ rad} = 0.627^\circ$

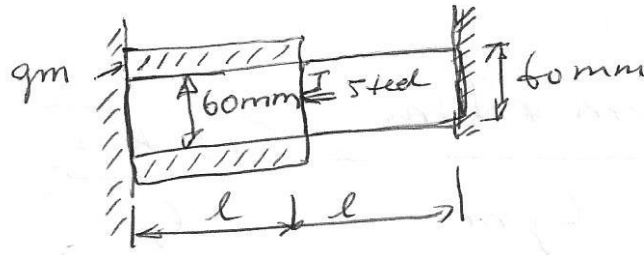
Shear stress in the aluminium alloy, $\tau_{al} = \frac{T_a r}{J_a} = \frac{2.795 \times 10^3 \times 0.05}{9.8175 \times 10^{-6}} = 14.23 \text{ MN/m}^2$

Shear stress in the steel, $\tau_{steel} = \frac{T_s r}{J_s} = \frac{7.205 \times 10^3 \times 0.05}{8.5451 \times 10^{-6}} = 42.16 \text{ MN/m}^2$

7. A compound shaft consists of two elements of equal length, joined together in series. If one element of the shaft is constructed from gunmetal tube and the other from solid steel, where the external diameter of the steel shaft and the internal diameter of the gunmetal shaft equal 60 mm, determine the external diameter of the gunmetal shaft if the two shafts are to have the same torsional stiffness. Determine also the maximum permissible torque that can be applied to the shaft, given the following:

For gunmetal: $G = 3 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 45 MN/m^2

For steel: $G = 7.5 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 90 MN/m^2



From equation (1) of Problem 2, $T_{gm} = \frac{T_s G_{gm} J_{gm}}{G_s J_s}$ (1)

$$J_{gm} = \frac{\pi [D^4 - (60 \times 10^{-3})^4]}{32} = \frac{\pi}{32} (D^4 - 1.296 \times 10^{-5})$$

i.e. $J_{gm} = 0.098D^4 - 1.272 \times 10^{-6} \text{ m}^4$ (2)

$$J_s = \frac{\pi (60 \times 10^{-3})^4}{32} = 1.272 \times 10^{-6} \text{ m}^4$$

Substituting equations (2) into equation (1) gives:

$$T_{gm} = \frac{T_s \times 3 \times 10^{10} \times (0.098D^4 - 1.272 \times 10^{-6})}{7.5 \times 10^{10} \times 1.272 \times 10^{-6}} \quad (3)$$

i.e. $T_{gm} = \frac{T_s \times (2940 \times 10^6 D^4 - 38160)}{95400}$ (4)

Torsional stiffness = $k = \frac{GJ}{l}$ (5)

$$\frac{G_{gm} \times J_{gm}}{l_{gm}} = \frac{G_s \times J_s}{l_s}$$

However, $l_{gm} = l_s$

thus, $G_{gm} \times J_{gm} = G_s \times J_s$

i.e. $3 \times 10^{10} \times (0.098D^4 - 1.272 \times 10^{-6}) = 7.5 \times 10^{10} \times 1.272 \times 10^{-6}$

i.e. $0.294D^4 - 3.816 \times 10^{-6} = 9.54 \times 10^{-6}$

and $D^4 = \frac{9.54 \times 10^{-6} + 3.816 \times 10^{-6}}{0.294}$

and
$$D = \sqrt[4]{\frac{9.54 \times 10^{-6} + 3.816 \times 10^{-6}}{0.294}} = 0.082 \text{ m}$$

i.e. the external diameter, **D = 82 mm**

From equation (4),
$$T_{gm} = \frac{T_s \times (2940 \times 10^6 (0.082)^4 - 38160)}{95400}$$

$$= \frac{T_s \times (132924 - 38160)}{95400} = 0.9933 T_s$$

$$T_s = \frac{\tau_s J_s}{r_s} = \frac{90 \times 10^6 \times 1.272 \times 10^{-5}}{0.03} = \mathbf{3816 \text{ N m}}$$

and
$$T_{gm} = \frac{\tau_{gm} J_{gm}}{r_{gm}} = \frac{45 \times 10^6 \times 3.122 \times 10^{-6}}{0.041} = \mathbf{3427 \text{ N m}}$$

The design criterion = $T_{gm} = 3427 \text{ N m}$

$$T_s = \frac{T_{gm}}{0.9933} = \frac{3427}{0.9933} = 3450$$

Therefore, **maximum permissible torque, $T = 3427 + 3450 = 6877 \text{ N m}$**

8. A compound shaft consists of a solid steel core, which is surrounded co-axially by aluminium bronze sheath. Determine suitable values for the diameters of the shafts if the steel core is to carry two thirds of a total torque of 700 N m and if the limiting stresses are not exceeded.

For aluminium alloy: $G = 3.8 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 18 MN/m^2

For steel: $G = 7.7 \times 10^{10} \text{ N/m}^2$ Maximum permissible shear stress = 36 MN/m^2

Total torque, $T = 700 \text{ N m}$

$$T_s = \frac{2}{3} \times T = 466.7 \text{ N m} \quad \text{and} \quad T_a = 700 - T_s = 233.3 \text{ N m}$$

$$\theta_s = \frac{T_s l}{G_s J_s} = \frac{T_a l}{G_a J_a} \quad \text{i.e.} \quad \frac{T_s}{G_s J_s} = \frac{T_a}{G_a J_a}$$

Hence,
$$\frac{466.7}{7.7 \times 10^{10} \times J_s} = \frac{233.3}{3.8 \times 10^{10} \times J_a}$$

from which, $J_a = 1.0129J_s$

$$J_s = \frac{\pi d^4}{32} \quad \text{and} \quad J_a = \frac{\pi(D^4 - d^4)}{32}$$

from which, $D^4 - d^4 = 1.0129d^4$

i.e. $D^4 = 2.0129d^4$

and $D = 1.191d$

$$\frac{\tau}{r} = \frac{G\theta}{l}$$

For steel: $\theta = \frac{\tau_s l}{G_s r_s} = \frac{36 \times 10^6 l}{7.7 \times 10^{10} \times d / 2}$

i.e. $\frac{\theta}{l} = \frac{36 \times 10^6 \times 2}{7.7 \times 10^{10} \times d} = \frac{9.35 \times 10^{-4}}{d}$ (1)

For aluminium bronze alloy:

$$\theta = \frac{\tau_a l}{G_a r_a} = \frac{18 \times 10^6 l}{3.8 \times 10^{10} \times D / 2}$$

However, $D = 1.191d$

therefore, $\frac{\theta}{l} = \frac{18 \times 10^6 \times 2}{3.8 \times 10^{10} \times 1.191d} = \frac{7.954 \times 10^{-4}}{d}$ (2)

From equations (1) and (2), **the aluminium bronze alloy is the design criterion.**

$$J_a = \frac{\pi(D^4 - d^4)}{32} = \frac{\pi(1.191^4 - 1^4)d^4}{32} = 0.0994d^4$$

$$\frac{\tau_a}{r_a} = \frac{T_a}{J_a} \quad \text{i.e.} \quad \frac{18 \times 10^6}{1.191d / 2} = \frac{233.3}{0.0994d^4}$$

from which, $d^3 = \frac{18 \times 10^6}{1.191 / 2} = \frac{233.3 \times 1.191}{18 \times 10^6 \times 0.0994 \times 2} = 7.7649 \times 10^{-5}$

and $d = \sqrt[3]{7.7649 \times 10^{-5}} = 0.0427 \text{ m} = \mathbf{42.66 \text{ mm}}$

and $D = 1.191 d = \mathbf{50.86 \text{ mm}}$

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1. A circular-section shaft of diameter 0.2 m and length 1 m is subjected to a torque that causes an angle of twist of 3.5° . Determine this torque and the residual angle of twist on removal of this torque. Assume that $G = 7.7 \times 10^{10} \text{ N/m}^2$ and $\tau_{yp} = 180 \text{ MN/m}^2$.

$$\frac{\tau}{r} = \frac{G\theta}{l}$$

from which,
$$r = \frac{\tau l}{G\theta} = \frac{180 \times 10^6 \times 1}{7.7 \times 10^{10} \times 3.5 \times \frac{\pi}{180}} = 0.03827 \text{ m}$$

$$\frac{T_e}{J} = \frac{\tau}{r}$$

from which,
$$T_e = \frac{\tau J}{r} = \frac{180 \times 10^6 \times 3.3687 \times 10^{-6}}{0.03827} = 15844 \text{ N m} = 0.0158 \text{ MN m}$$

$$\begin{aligned} T_p &= \frac{2\pi}{3} \tau_{yp} (R^3 - r^3) \\ &= \frac{2\pi}{3} \times 180 \times 10^6 \times (0.1^3 - 0.03827^3) \\ &= 355861 \text{ N m} = 0.356 \text{ MN m} \end{aligned}$$

Hence, **torque, $T = T_e + T_p = 0.0158 + 0.356 = 0.372 \text{ MN m}$**

To determine the residual angle of twist:

Now $\theta = 3.5^\circ$ (given)

On release of torque,
$$\theta_1 = \frac{\hat{\tau} \times l}{G \times r} = \frac{180 \times 10^6 \times 1}{7.7 \times 10^{10} \times 0.1} = 0.02338 \text{ rads}$$

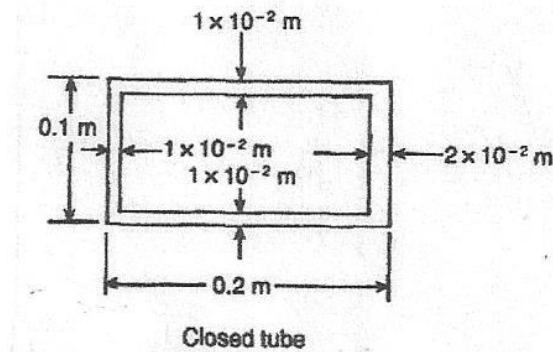
$$= 0.0338 \times \frac{180^\circ}{\pi} = 1.339^\circ$$

$$\theta_R = \theta - \theta_1 = 3.5^\circ - 1.339^\circ$$

i.e. the residual angle of twist on removal of the torque, $\theta_R = 2.161^\circ$

2. Determine the maximum possible torque that the cross-section shown below can withstand, given that yield shear stress = 170 MPa.

What would be the angle of twist per unit length if $G = 7.7 \times 10^{10}$ Pa?



$$\tau = \frac{T}{2At}$$

Hence,

$$170 \times 10^6 = \frac{T}{2 \times 0.1 \times 0.2 \times 1 \times 10^{-2}}$$

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from which,

$$T = 170 \times 10^6 \times 2 \times 0.1 \times 0.2 \times 1 \times 10^{-2}$$

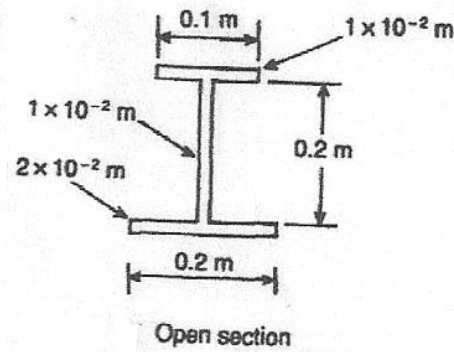
i.e. **maximum stress, $T = 68000$ N m**

$$\begin{aligned} \theta &= \frac{TL}{4A^2G} \oint \frac{ds}{t} \\ &= \frac{68000 \times 1}{4 \times 4 \times 10^{-4} \times 7.7 \times 10^{10}} \times \frac{180}{\pi} \left(\frac{0.5}{1 \times 10^{-2}} + \frac{0.1}{2 \times 10^{-2}} \right) \\ &= 0.0316 \times 55 \end{aligned}$$

i.e. **angle of twist, $\theta = 1.738^\circ$**

3. Determine the maximum possible torque that the cross-section shown below can withstand given that yield shear stress = 170 MPa.

What would be the angle of twist per unit length if $G = 7.7 \times 10^{10}$ Pa?



$$J = \sum \frac{bt^3}{3} = \frac{0.1 \times (1 \times 10^{-2})^3 + 0.2 \times (1 \times 10^{-2})^3 + 0.2 \times (2 \times 10^{-2})^3}{3}$$

$$= \frac{3 \times 10^{-7} + 1.6 \times 10^{-6}}{3}$$

i.e. $J = 6.333 \times 10^{-7} \text{ m}^4$

$$\tau_{\max} = \frac{T t_{\max}}{J}$$

from which, **maximum torque**,

$$T = \frac{\tau_{\max} J}{t_{\max}} = \frac{170 \times 10^6 \times 6.333 \times 10^{-7}}{2 \times 10^{-2}} = 5383 \text{ N m}$$

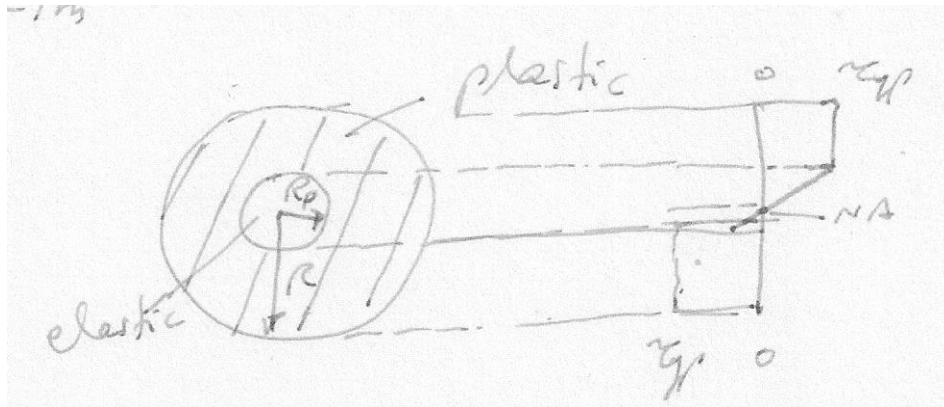
$$T = GJ \left(\frac{\theta}{l} \right)$$

from which, angle of twist per unit length,

$$\frac{\theta}{l} = \frac{T}{GJ} = \frac{5383}{7.7 \times 10^{10} \times 6.333 \times 10^{-7}} \times \frac{180}{\pi}$$

$$= 6.325^\circ/\text{metre}$$

4. A circular-section shaft of diameter 0.25 m and length 1 m is subjected to a torque that causes an angle of twist of 3.5° . Determine this torque and the residual angle of twist on removal of this torque. Assume that $G = 7.7 \times 10^{10} \text{ N/m}^2$ and $\tau_{yp} = 180 \text{ MN/m}^2$.



Radius, $R = 0.25/2 = 0.125$ m

R_p = outer radius of elastic core

T = applied torque = $T_e + T_{p1}$

T_e = torsional resistance of elastic core

T_{p1} = torsional resistance of shaded plastic core

J_e = polar 2nd moment of area of elastic core

τ_{yp} = shear stress at yield

To calculate θ :

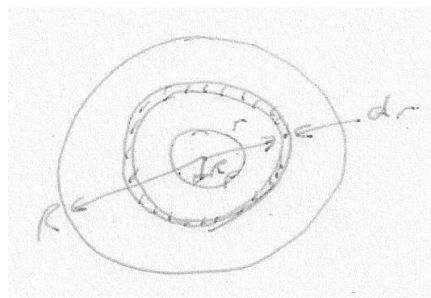
$$\frac{\tau_{yp}}{R_p} = \frac{T_e}{J_e} = \frac{G\theta}{l} \quad \text{where} \quad J_e = \frac{\pi R_p^4}{2}$$

$$T_e = \frac{\tau_{yp}}{R_p} \times J_e = \frac{\tau_{yp}}{R_p} \times \frac{\pi R_p^4}{2}$$

$$= \frac{180}{R_p} \times \frac{\pi R_p^4}{2} = 90\pi \times R_p^3$$

i.e.

$$T_e = 282.7 R_p^3 \quad (1)$$



From diagram,

$$\begin{aligned}T_{p'} &= \int_{R_p}^R \tau_{yp} \times (2\pi r dr) \times r \\&= 2\pi \tau_{yp} \int_{R_p}^R r^2 dr = 2\pi \tau_{yp} \left[\frac{r^3}{3} \right]_{R_p}^R \\&= \frac{2\pi}{3} \times 180 [R^3 - R_p^3]\end{aligned}$$

However, $R = 0.125$

hence,

$$T_{p'} = \frac{2\pi}{3} \times 180 [0.125^3 - R_p^3]$$

i.e.

$$T_{p'} = 377 [1.953 \times 10^{-3} - R_p^3] \quad (2)$$

$$\theta = 3.5^\circ = 0.0611 \text{ rads}$$

$$\frac{G\theta}{l} = \frac{\tau}{r} \quad \text{or} \quad r = \frac{\tau l}{G\theta}$$

In this case,

$$R_p = \frac{\tau_{yp} \times l}{G\theta} = \frac{180 \times 10^6 \times 1}{7.7 \times 10^{10} \times 0.0611}$$

i.e.

$$R_p = 0.0383 \text{ m} \quad (3)$$

Substituting equation (3) into equation (1) gives:

$$T_e = 282.7 R_p^3 = 282.7 \times 0.0383^3$$

i.e.

$$T_e = 0.0159 \text{ MN m}$$

Substituting equation (3) into equation (2) gives:

$$\begin{aligned}T_{p'} &= 377 [1.953 \times 10^{-3} - R_p^3] \\&= 377 [1.953 \times 10^{-3} - 0.0383^3] \\&= 377 \times 1.8968 \times 10^{-3}\end{aligned}$$

i.e.

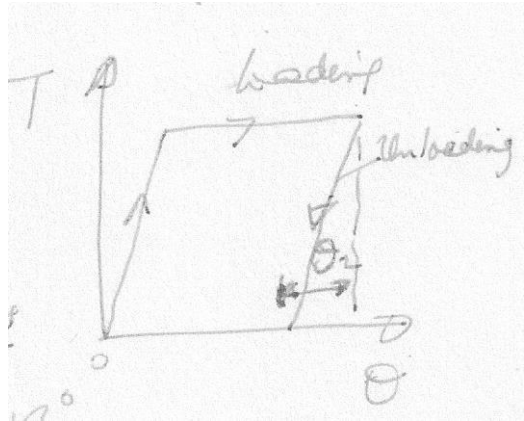
$$T_{p'} = 0.715 \text{ MN m}$$

$$T = T_e + T_{p'} = 0.715 + 0.0159$$

i.e.

torque, $T = 0.731 \text{ MN m}$

To determine the residual angle of twist:



$$\theta_2 = \frac{Tl}{GJ} = \frac{0.731 \times 10^6 \times 1}{7.7 \times 10^{10} \times J}$$

$$J = \frac{\pi R^4}{2} = \frac{\pi \times 0.125^4}{2} = 3.835 \times 10^{-4} \text{ m}^4$$

Hence,

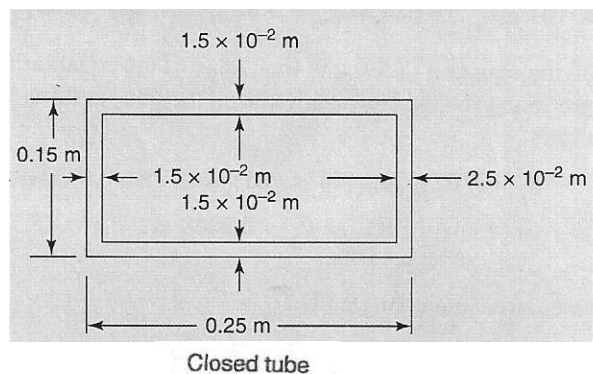
$$\theta_2 = \frac{Tl}{GJ} = \frac{0.731 \times 10^6 \times 1}{7.7 \times 10^{10} \times 3.835 \times 10^{-4}} = 1.418^\circ$$

The residual angle of twist, $\theta_R = \theta_1 - \theta_2$

$$= 3.5^\circ - 1.418^\circ = \mathbf{1.58^\circ}$$

5. Determine the maximum possible torque that the cross-section shown can withstand, given that yield shear stress = 170 MPa.

What would be the angle of twist per unit length if $G = 7.7 \times 10^{10} \text{ Pa}$?



$$\tau = \frac{T}{2At}$$

i.e. $170 \times 10^6 = \frac{T}{2 \times 0.15 \times 0.25 \times 1.5 \times 10^{-2}}$

from which, $T = 170 \times 10^6 \times 2 \times 0.15 \times 0.25 \times 1.5 \times 10^{-2}$

i.e. **maximum torque, $T = 191250 \text{ N m} = 191.25 \text{ kN m}$**

$$\begin{aligned}\theta &= \frac{Tl}{4A^2G} \int \frac{ds}{t} \\ &= \frac{191250 \times 1}{4 \times 1.406 \times 10^{-3} \times 7.7 \times 10^{10}} \times \frac{180}{\pi} \left(\frac{0.65}{1.5 \times 10^{-2}} + \frac{0.15}{2.5 \times 10^{-2}} \right) \\ &= \frac{34425000}{433^6} (43.33 + 6)\end{aligned}$$

i.e. **angle of twist, $\theta = 3.92^\circ/\text{m}$**
