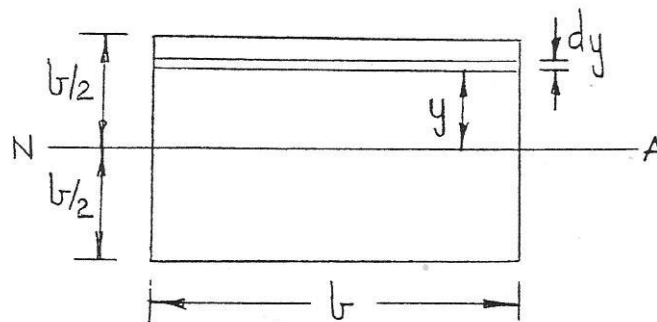
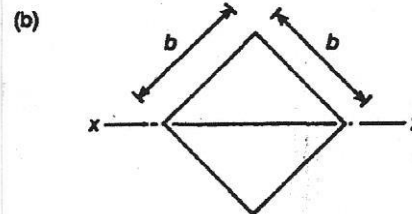
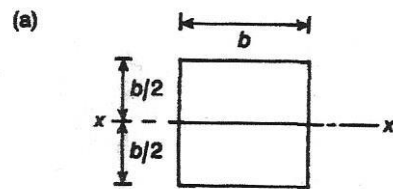


CHAPTER 5 GEOMETRICAL PROPERTIES OF SYMMETRICAL SECTIONS

EXERCISE 27, Page 120

1. Determine the second moments of area about the centroid of the squares shown below.



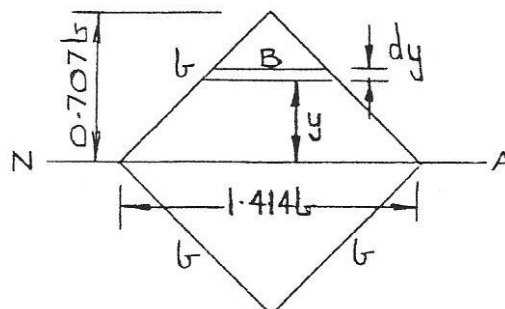
(a)

$$I_{NA} = \int_{-b/2}^{+b/2} b \cdot dy \cdot y^2$$

$$= b \left[\frac{y^3}{3} \right]_{-b/2}^{+b/2} = b \left[\left(\frac{(b/2)^3}{3} \right) - \left(\frac{(-b/2)^3}{3} \right) \right]$$

$$= b \left[\left(\frac{b^3}{24} \right) - \left(-\frac{b^3}{24} \right) \right] = b \left[\frac{b^3}{12} \right] = \frac{b^4}{12}$$

1
(b)



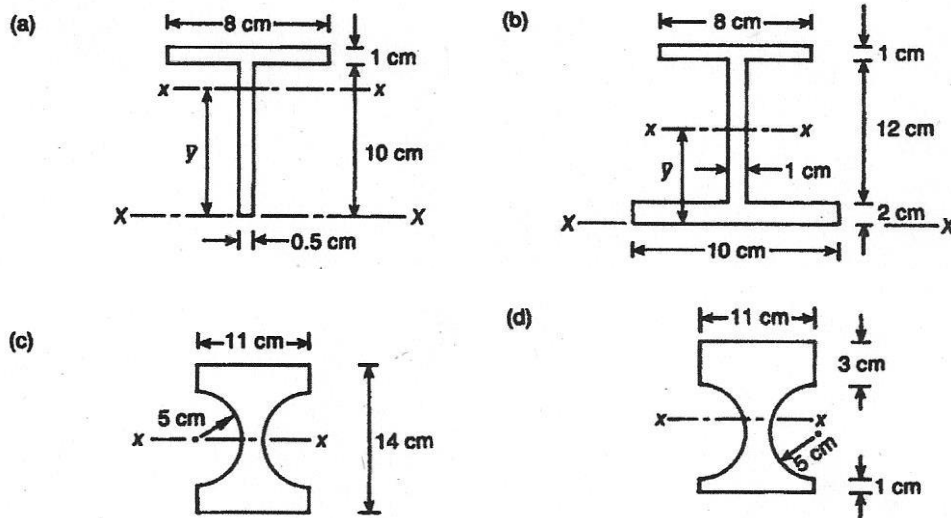
From similar triangles

$$\frac{1.414b}{0.707b} = \frac{B}{0.707b - y}$$

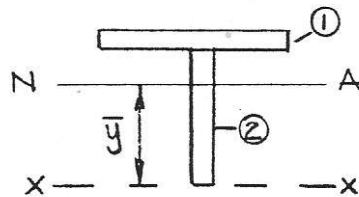
i.e. $2(0.707b - y) = B$

$$\begin{aligned}
 I_{NA} &= \int_{-0.707b}^{0.707b} B \cdot dy \cdot y^2 \\
 &= \int_{-0.707b}^{0.707b} 2(0.707b - y) dy \cdot y^2 = \int_{-0.707b}^{0.707b} (1.414by^2 - 2y^3) dy \\
 &= \left[\frac{1.414by^3}{3} - \frac{2y^4}{4} \right]_{-0.707b}^{0.707b} = \left[\left\{ \left(\frac{1.414b(0.707b)^3}{3} \right) - \left(\frac{2(0.707b)^4}{4} \right) \right\} \right. \\
 &\quad \left. - \left\{ \left(\frac{1.414b(-0.707b)^3}{3} \right) - \left(\frac{2(-0.707b)^4}{4} \right) \right\} \right] \\
 &= \left[\left\{ \left(\frac{\sqrt{2}b \left(\frac{\sqrt{2}}{2}b \right)^3}{3} \right) - \left(\frac{2 \left(\frac{\sqrt{2}}{2}b \right)^4}{4} \right) \right\} \right. \\
 &\quad \left. - \left\{ \left(\frac{\sqrt{2}b \left(-\frac{\sqrt{2}}{2}b \right)^3}{3} \right) - \left(\frac{2 \left(-\frac{\sqrt{2}}{2}b \right)^4}{4} \right) \right\} \right] \\
 &= \left[\left\{ \left(\frac{4}{24}b^4 \right) - \left(\frac{1}{8}b^4 \right) \right\} - \left\{ \left(-\frac{4}{24}b^4 \right) + \left(\frac{1}{8}b^4 \right) \right\} \right] \\
 &= \left[\frac{1}{3}b^4 - \frac{1}{4}b^4 \right] = \frac{b^4}{12}
 \end{aligned}$$

2. Determine the positions of the centroidal axes xx and the second moments of area about these axes, for the sections of figures (a) to (d).



(a)



Section	a	y	ay	ay^2	i_o
1	8	10.5	84	882	0.67
2	5	5	25	125	41.67
Σ	13	—	109	1007	42.34

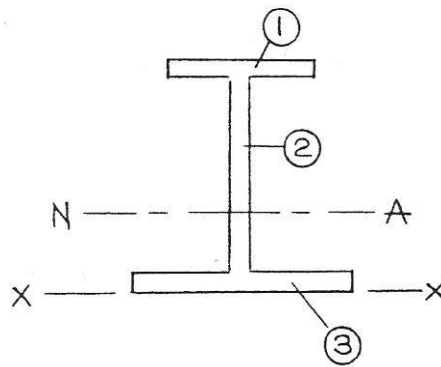
$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{109}{13} = 8.385 \text{ cm}$$

$$I_{xx} = \sum ay^2 + \sum i_o = 1007 + 42.34 = 1049.34$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 1049.34 - (8.385)^2 \times 13$$

i.e. $I_{NA} = 135.3 \text{ cm}^4 = 1.353 \times 10^{-6} \text{ m}^4$

(b)



Section	a	y	ay	ay^2	i_o
1	8	14.5	116	1682	0.67
2	12	8	96	768	144
3	20	1	20	20	6.67
Σ	40	—	232	2470	151.34

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{232}{40} = \mathbf{5.80 \text{ cm}}$$

$$I_{xx} = \sum ay^2 + \sum i_o = 2470 + 151.34 = 2621.34$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 2621.34 - (5.80)^2 \times 40$$

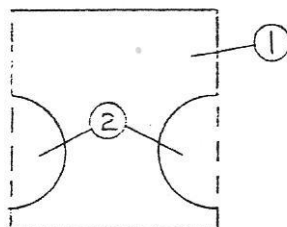
i.e. $I_{NA} = 1275.7 \text{ cm}^4 = \mathbf{1.276 \times 10^{-5} \text{ m}^4}$

(c) $I_{NA} = \frac{11 \times 14^3}{12} - \frac{\pi \times 10^4}{64} \quad 2$

i.e. $I_{NA} = 2515.33 - 490.87$

$$= 2024 \text{ cm}^4 = \mathbf{2.024 \times 10^{-5} \text{ m}^4}$$

(d)



Section	a	y	ay	ay^2	u_o
1	154	7	1078	7546	2515.3
2	-78.54	6	-471.2	-2827.4	-490.9
Σ	75.46	-	606.8	4718.6	2024.4

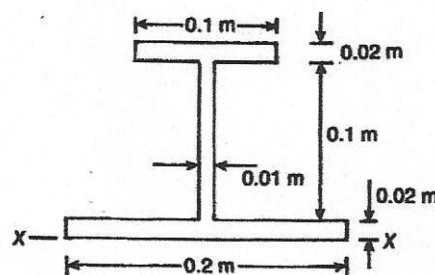
$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{606.8}{75.46} = \mathbf{8.04 \text{ cm}}$$

$$I_{xx} = \sum ay^2 + \sum i_o = 4718.6 + 2024.4 = 6743$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 6743 - (8.04)^2 \times 75.46$$

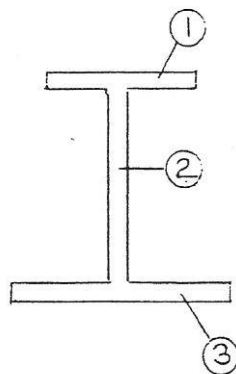
i.e. $I_{NA} = 1865 \text{ cm}^4 = \mathbf{1.865 \times 10^{-5} \text{ m}^4}$

3. Determine the second moment of area of the section shown about an axis passing through the centroid and parallel to the xx axis.



What would be the percentage reduction in second moment of area if the bottom flange were identical to the top flange?

First case



Section	a	y	ay	ay^2	i_o
1	2×10^{-3}	0.13	2.6×10^{-4}	3.38×10^{-5}	6.7×10^{-8}
2	1×10^{-3}	0.07	7×10^{-5}	4.9×10^{-6}	8.33×10^{-7}
3	4×10^{-3}	0.01	4×10^{-5}	4×10^{-7}	1.33×10^{-7}
Σ	7×10^{-3}	—	3.7×10^{-4}	3.91×10^{-5}	1.03×10^{-6}

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{3.7 \times 10^{-4}}{7 \times 10^{-3}} = \mathbf{0.0529 \text{ m}}$$

$$I_{xx} = \sum ay^2 + \Sigma i_o = 3.91 \times 10^{-5} + 1.03 \times 10^{-6} = 4.013 \times 10^{-5}$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 4.013 \times 10^{-5} - (0.0529)^2 \times 7 \times 10^{-3}$$

i.e. $I_{NA} = 1865 \text{ cm}^4 = \mathbf{2.054 \times 10^{-5} \text{ m}^4}$

Second case

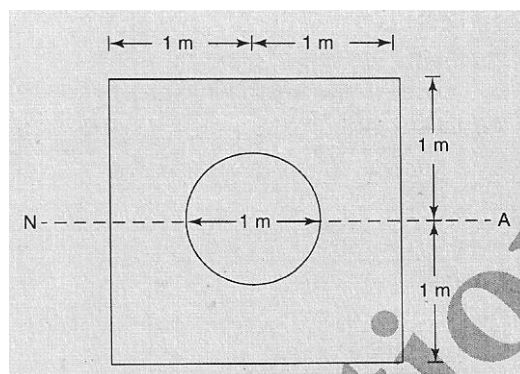
$$I_{NA} = \frac{0.1 \times 0.14^3}{12} - \frac{0.09 \times 0.1^3}{12} \quad 3$$

i.e. $I_{NA} = 2.2867 \times 10^{-5} - 7.5 \times 10^{-6}$

$$= \mathbf{1.537 \times 10^{-5} \text{ m}^4}$$

$$\text{Percentage reduction} = \left(\frac{2.054 - 1.537}{2.054} \right) \times 100\% = \mathbf{25.2\%}$$

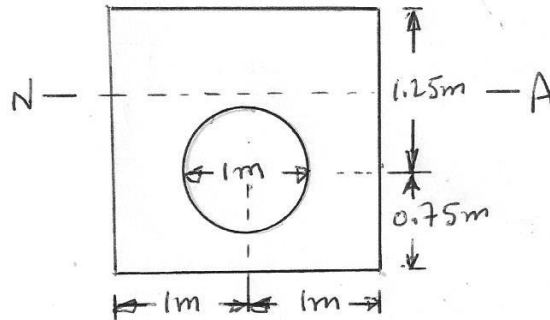
4. Determine the second moment of area for the cross-section shown below about the neutral axis, NA.



$$\text{Second moment of area about the neutral axis, } I_{NA} = \frac{bd^3}{12} - \frac{\pi d^4}{64}$$

$$= \frac{2 \times 2^3}{12} - \frac{\pi 1^4}{64} = \mathbf{1.284 \text{ m}^4}$$

5. Determine the second moment of area for the cross-section shown below about the neutral axis, NA.



Section	a	y	ay	ay^2	i_o
1	4	1	4	4	1.333
2	-0.785	0.75	-0.589	-0.442	-0.0491
Σ	3.215	-	3.411	3.558	1.281

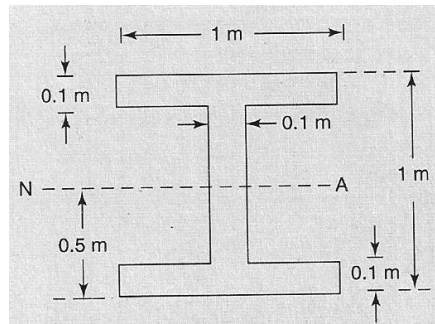
$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{3.411}{3.215} = \mathbf{1.061 \text{ m}}$$

$$I_{xx} = \sum ay^2 + \Sigma i = 3.558 + 1.281 = 4.839$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 4.839 - (1.061)^2 \times 3.215$$

i.e. $I_{NA} = \mathbf{1.220 \text{ m}^4}$

6. Determine the second moment of area for the cross-section shown below about the neutral axis, NA.



Section	a	y	ay	ay^2	i_o
1	0.10	0.95	0.095	0.09025	8.333×10^{-5}
2	0.08	0.50	0.040	0.0200	4.267×10^{-3}
3	0.10	0.05	0.005	2.5×10^{-4}	8.333×10^{-5}
Σ	0.28	—	0.140	0.1105	4.434×10^{-3}

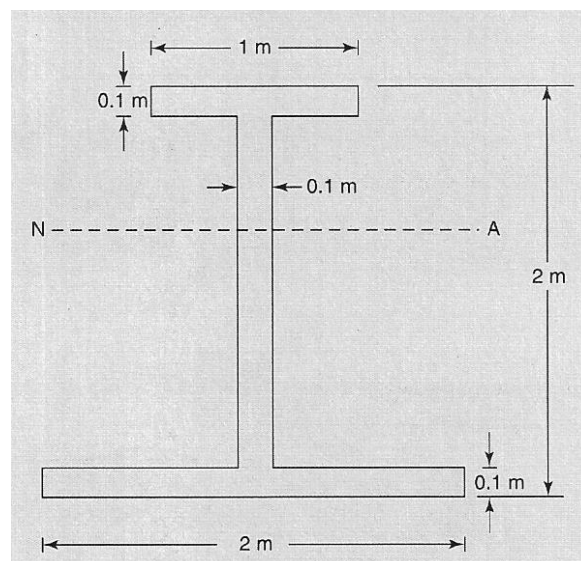
$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{0.140}{0.28} = 0.5 \text{ m}$$

$$I_{xx} = \sum ay^2 + \sum i_o = 0.1105 + 4.434 \times 10^{-3} = 0.114934$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 0.114934 - (0.5)^2 \times 0.28$$

i.e. $I_{NA} = \mathbf{0.0449 \text{ m}^4}$

7. Determine the second moment of area for the cross-section shown below about the neutral axis, NA.



Section	a	y	ay	ay^2	i_o
1	0.10	1.95	0.195	0.3803	8.333×10^{-5}
2	0.18	1.00	0.180	0.1800	0.0486
3	0.20	0.05	0.010	5×10^{-4}	1.667×10^{-4}
Σ	0.48	—	0.385	0.5608	0.04885

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{0.385}{0.48} = 0.8021 \text{ m}$$

$$I_{xx} = \sum ay^2 + \sum i_o = 0.5608 + 0.04885 = 0.6097$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \sum a = 0.6097 - (0.8021)^2 \times 0.48$$

i.e. $I_{NA} = 0.301 \text{ m}^4$

8. Determine the second moment of area for the cross-section shown in Figure 5.24 about the neutral axis, NA.

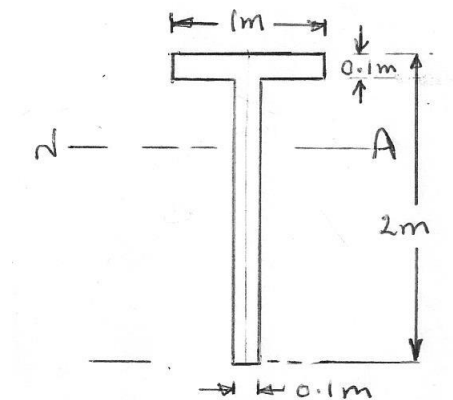


Figure 5.24

Section	a	y	ay	ay^2	i_o
1	0.10	1.95	0.195	0.3803	8.333×10^{-5}
2	0.19	0.95	0.1805	0.1715	0.05716
Σ	0.29	—	0.3755	0.5518	0.05724

$$\bar{y} = \frac{\sum ay}{\sum a} = \frac{0.3755}{0.29} = 1.295 \text{ m}$$

$$I_{xx} = \Sigma ay^2 + \Sigma i_o = 0.5518 + 0.05724 = 0.6090$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \Sigma a = 0.6090 - (1.295)^2 \times 0.29$$

i.e. $I_{NA} = \mathbf{0.123 \text{ m}^4}$

9. Determine the second moment of area for the cross-section shown in Figure 5.25 about the neutral axis, NA.

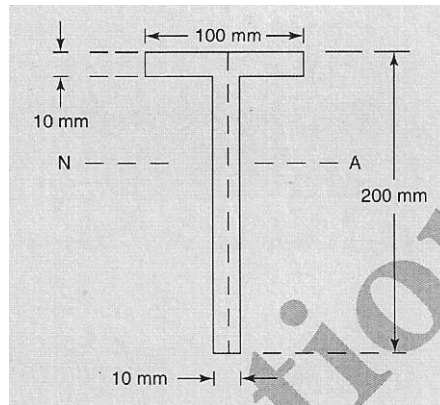


Figure 5.25

Section	a	y	ay	ay^2	i_o
1	1000	195	195000	38.03×10^6	8333.33
2	1900	95	180500	17.15×10^6	5.716×10^6
Σ	2900	—	375500	55.18×10^6	5.724×10^6

$$\bar{y} = \frac{\Sigma ay}{\Sigma a} = \frac{375500}{2900} = \mathbf{129.48 \text{ mm}}$$

$$I_{xx} = \Sigma ay^2 + \Sigma i_o = 55.18 \times 10^6 + 5.724 \times 10^6 = 60.90 \times 10^6$$

$$I_{NA} = I_{xx} - (\bar{y})^2 \Sigma a = 60.90 \times 10^6 - (129.48)^2 \times 2900$$

i.e. $I_{NA} = 12.28 \times 10^6 \text{ mm}^4 = \mathbf{12.28 \times 10^{-6} \text{ m}^4}$

10. Determine the second moment of area for the cross-section shown in Figure 5.26 about the neutral axis, NA, together with its polar moments of area.

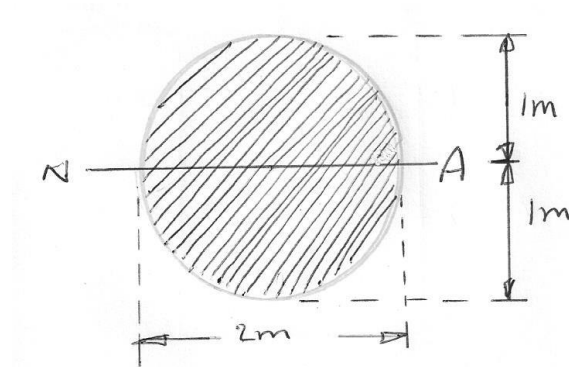
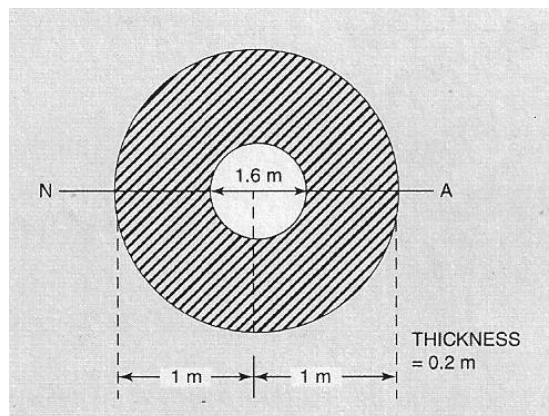


Figure 5.26

$$\begin{aligned} \text{Second moment of area about the neutral axis, } I_{NA} &= \frac{\pi d^4}{64} \\ &= \frac{\pi(2)^4}{64} = \mathbf{0.785 \, m^4} \end{aligned}$$

$$\begin{aligned} \text{Polar second moment of area, } I_{polar} &= \frac{\pi d^4}{32} \\ &= \frac{\pi(2)^4}{32} = \mathbf{1.571 \, m^4} \end{aligned}$$

11. Determine the second moment of area for the cross-section shown about the neutral axis, NA, together with its polar moments of area.

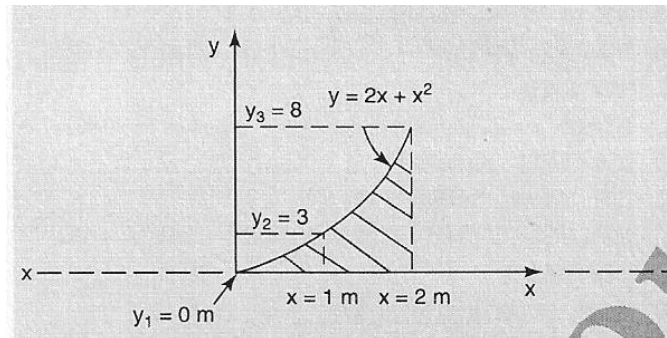


$$\begin{aligned}\text{Second moment of area about the neutral axis, } I_{NA} &= \frac{\pi(d_1^4 - d_2^4)}{64} \\ &= \frac{\pi(2^4 - 1.6^4)}{64} = \mathbf{0.464 \text{ m}^4}\end{aligned}$$

$$\begin{aligned}\text{Polar second moment of area, } I_{polar} &= \frac{\pi(d_1^4 - d_2^4)}{32} \\ &= \frac{\pi(2^4 - 1.6^4)}{32} = \mathbf{0.927 \text{ m}^4}\end{aligned}$$

EXERCISE 28, Page 129

1. Determine the second moment of area about XX of the section enclosed by $y = 2x + x^2$ and the x-axis between $x = 0$ m and $x = 2$ m as shown below by (a) the Naval architects' method, and (b) by Ross's method. Compare the results with the exact solution obtained by integration.



The area is shown in Figure 5.33 where x varies from 0 m to 2 m.

At $x = 0$, $y_1 = 0$ m

At $x = 1$ m, $y_2 = 3$ m

At $x = 2$ m, $y_3 = 8$ m

(a) **By the Naval architects' method,**

$$\text{From equation (5.37), } I_{xx} = \frac{h}{9} (y_1^3 + 4y_2^3 + y_3^3)$$

where Figure 5.33, $h = 1$ m, $y_1 = 0$, $y_2 = 3$ m and $y_3 = 8$ m

$$\text{Hence, } I_{xx} = \frac{1}{9} (0^3 + 4(3)^3 + 8^3) = \frac{1}{9} (0^3 + 108 + 512) = \mathbf{68.89 \text{ m}^4}$$

(b) **By Ross's method,** from equation (5.38),

$$\begin{aligned} I_{xx} &= \frac{2h}{3} \left\{ \frac{13}{140} (y_1^3 + y_3^3) + \frac{16}{35} y_2^3 + \frac{1}{7} (y_1^2 y_2 + y_2 y_3^2) - \frac{3}{140} (y_1^2 y_3 + y_1 y_3^2) \right. \\ &\quad \left. + \frac{4}{35} (y_1 y_2^2 + y_2^2 y_3) - \frac{4}{35} (y_1 y_2 y_3) \right\} \\ &= \frac{2(1)}{3} \left\{ \frac{13}{140} (0^3 + 8^3) + \frac{16}{35} 3^3 + \frac{1}{7} (0^2(3) + (3)(8)^2) - \frac{3}{140} (0^2(8) + (0)(8)^2) \right. \\ &\quad \left. + \frac{4}{35} ((0)(3)^2 + (3)^2(8)) - \frac{4}{35} ((0)(3)(8)) \right\} \end{aligned}$$

$$= \frac{2}{3} \left\{ \frac{13}{140}(512) + \frac{16}{35}(27) + \frac{1}{7}(0+192) - \frac{3}{140}(0+0) + \frac{4}{35}(0+72) - \frac{4}{35}(0) \right\}$$

$$= \frac{2}{3} \{ 47.543 + 12.343 + 27.429 - 0 + 8.229 - 0 \} = \mathbf{63.70 \, m^4}$$

By integration, $I_{xx} = \int y \, dx \frac{y^2}{3} = \int \left(\frac{y^3}{3} \right) dx$

and from Figure 5.33, $I_{xx} = \int_0^2 \left(\frac{(2x+x^2)^3}{3} \right) dx = \frac{1}{3} \int_0^2 (8x^3 + 12x^4 + 6x^5 + x^6) dx$

$$= \frac{1}{3} \left[\frac{8x^4}{4} + \frac{12x^5}{5} + \frac{6x^6}{6} + \frac{x^7}{7} \right]_0^2 = \frac{1}{3} [(32 + 76.8 + 64 + 18.2857) - (0)]$$

$$= \mathbf{63.70 \, m^4}$$

Integration gives an exact value.

Hence, **Ross' method is exact and the Naval architects' method is in error by**

$$\left(\frac{68.89 - 63.70}{63.70} \right) \times 100\% = \mathbf{8.15\%}$$

2. Calculate the second moment of area, about the base XX (where $y = 0$), of the area enclosed by $y = e^x$ and the x-axis between the limits of $x = 0$ and $x = 2$ m, by the 'exact' integration method. Hence, or otherwise, obtain the approximate numerical values of this equation using (a) the Naval architects' method, and (b) Ross's method.

By integration, $I_{xx} = \int y \, dx \frac{y^2}{3} = \int \left(\frac{y^3}{3} \right) dx$

and from Figure 5.33, $I_{xx} = \int_0^2 \left(\frac{(e^x)^3}{3} \right) dx = \frac{1}{3} \int_0^2 (e^{3x}) dx = \frac{1}{3} \left[\frac{e^{3x}}{3} \right]_0^2 = \frac{1}{9} [e^{3x}]_0^2$

$$= \frac{1}{9} [(e^6) - (e^0)]$$

$$= \mathbf{44.714 \, m^4}$$

Integration gives an exact value.

(a) **By the Naval architects' method,**

$$\text{From equation (5.37), } I_{xx} = \frac{h}{9} (y_1^3 + 4y_2^3 + y_3^3)$$

where Figure 5.33, $h = 1 \text{ m}$, $y_1 = 0$, $y_2 = e^1 \text{ m}$ and $y_3 = e^2 \text{ m}$

$$\begin{aligned} \text{Hence, } I_{xx} &= \frac{1}{9} (0^3 + 4(e^1)^3 + (e^2)^3) \\ &= \frac{1}{9} (0^3 + 80.342 + 403.429) = \mathbf{53.752 \text{ m}^4} \end{aligned}$$

(b) **By Ross's method,** from equation (5.38),

$$\begin{aligned} I_{xx} &= \frac{2h}{3} \left\{ \frac{13}{140} (y_1^3 + y_3^3) + \frac{16}{35} y_2^3 + \frac{1}{7} (y_1^2 y_2 + y_2 y_3^2) - \frac{3}{140} (y_1^2 y_3 + y_1 y_3^2) \right. \\ &\quad \left. + \frac{4}{35} (y_1 y_2^2 + y_2^2 y_3) - \frac{4}{35} (y_1 y_2 y_3) \right\} \\ &= \frac{2(1)}{3} \left\{ \frac{13}{140} (0^3 + (e^2)^3) + \frac{16}{35} (e^1)^3 + \frac{1}{7} (0^2 (e^1) + (e^1)(e^2)^2) - \frac{3}{140} (0^2 (e^2) + (0)(e^2)^2) \right. \\ &\quad \left. + \frac{4}{35} ((0)(e^1)^2 + (e^1)^2 (e^2)) - \frac{4}{35} ((0)(e^1)(e^2)) \right\} \\ &= \frac{2}{3} \left\{ \frac{13}{140} (403.429) + \frac{16}{35} (20.086) + \frac{1}{7} (0 + 148.413) - \frac{3}{140} (0 + 0) + \frac{4}{35} (0 + 54.598) - \frac{4}{35} (0) \right\} \\ &= \frac{2}{3} \{ 37.461 + 9.182 + 21.202 - 0 + 6.240 - 0 \} = \mathbf{49.39 \text{ m}^4} \end{aligned}$$
