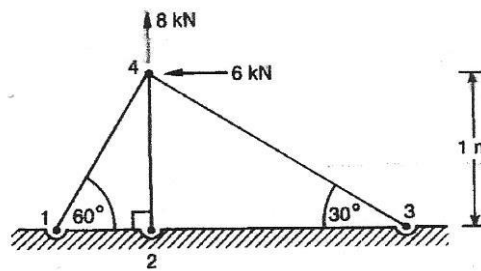


# CHAPTER 18 THE MATRIX DISPLACEMENT METHOD

## EXERCISE 66, Page 389

1. Determine the nodal displacements and member forces in the plane pin-jointed truss shown. For all members,  $AE$  is constant.



$$1-4 \quad \alpha = 60^\circ \quad c = 0.5 \quad s = 0.866$$

$$l_{1-4} = 1.155 \text{ m}$$

$$(k_{1-4}^o) = \frac{AE}{1.155}$$

$$\begin{pmatrix} u_4^0 & v_4^0 \\ 0.25 & 0.433 \\ 0.433 & 0.75 \end{pmatrix} \begin{pmatrix} u_4^0 \\ v_4^0 \end{pmatrix}$$

$$= AE \begin{pmatrix} 0.216 & 0.375 \\ 0.375 & 0.649 \end{pmatrix}$$

$$2-4 \quad \alpha = 90^\circ \quad c = 0 \quad s = 1$$

$$l_{2-4} = 1 \text{ m}$$

$$(k_{2-4}^o) = AE \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$3-4 \quad \alpha = 150^\circ \quad c = -0.866 \quad s = 0.5$$

$$l_{3-4} = 2 \text{ m}$$

$$(k_{3-4}) = AE \begin{pmatrix} 0.375 & -0.217 \\ -0.217 & 0.125 \end{pmatrix}$$

$$(K_{11}) = AE \begin{pmatrix} 0.591 & 0.158 \\ 0.158 & 1.774 \end{pmatrix}$$

$$(q_F) = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

or  $AE \begin{pmatrix} 0.591 & 0.158 \\ 0.158 & 1.774 \end{pmatrix} \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

i.e.  $\begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = \frac{1}{AE} \begin{pmatrix} 1.774 & -0.158 \\ -0.158 & 0.591 \end{pmatrix} \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

$$= \frac{1}{AE} \begin{pmatrix} 1.733 & -0.154 \\ -0.154 & 0.577 \end{pmatrix} \begin{pmatrix} -6 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = \frac{1}{AE} \begin{pmatrix} -11.63 \\ 5.54 \end{pmatrix}$$

**1–4**

Resolving in the direction 1–4 gives:

$$u_4 = (C \quad S) \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix}$$

i.e.  $u_4 = (0.5 \quad 0.866) \frac{1}{AE} \begin{pmatrix} -11.63 \\ 5.54 \end{pmatrix}$

$$= \frac{-1.0174}{AE}$$

$$F_{1-4} = \frac{-1.0174}{AE} \frac{AE}{1.155}$$

i.e.  $F_{1-4} = -0.88 \text{ kN}$

**2–4**

Resolving in the direction 2–4 gives:

$$u_4 = (C \quad S) \begin{pmatrix} u_4^0 \\ v_4^0 \end{pmatrix}$$

i.e. 
$$u_4 = (0 \quad 1) \frac{1}{AE} \begin{pmatrix} -11.63 \\ 5.54 \end{pmatrix}$$

$$= \frac{5.54}{AE}$$

and  $F_{2-4} = 5.54 \text{ kN}$

**3-4**

Resolving in the direction 3-4 gives:

$$u_4 = (C \quad S) \begin{pmatrix} u_4^0 \\ v_4^0 \end{pmatrix}$$

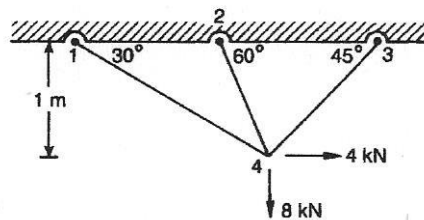
$$u_4 = (-0.866 \quad 0.5) \frac{1}{AE} \begin{pmatrix} -11.63 \\ 5.54 \end{pmatrix}$$

$$= \frac{12.84}{AE}$$

$$F_{3-4} = \frac{12.84}{AE} \times \frac{AE}{2}$$

i.e.  $F_{3-4} = 6.42 \text{ kN}$

**2.** Determine the nodal displacements and member forces in the plane pin-jointed truss shown. For all members, AE is constant.



**1-4**

$$\alpha = -30^\circ$$

$$c = 0.866$$

$$s = -0.5$$

$$l_{1-4} = 2 \text{ m}$$

$$\begin{pmatrix} k_{1-4}^o \end{pmatrix} = AE \begin{pmatrix} 0.375 & -0.217 \\ -0.217 & 0.125 \end{pmatrix}$$

$$\mathbf{2-4} \quad \alpha = -60^\circ \quad c = 0.5 \quad s = -0.866$$

$$l_{2-4} = 1.155 \text{ m}$$

$$\begin{pmatrix} k_{2-4}^o \end{pmatrix} = AE \begin{pmatrix} 0.216 & -0.375 \\ -0.375 & 0.649 \end{pmatrix}$$

$$\mathbf{3-4} \quad \alpha = -135^\circ \quad c = -0.707 \quad s = -0.707$$

$$l_{3-4} = 1.414 \text{ m}$$

$$\begin{pmatrix} k_{3-4}^o \end{pmatrix} = AE \begin{pmatrix} 0.354 & 0.354 \\ 0.354 & 0.354 \end{pmatrix}$$

$$\begin{pmatrix} K_{11} \end{pmatrix} = AE \begin{pmatrix} 0.945 & -0.238 \\ -0.238 & 1.128 \end{pmatrix}$$

$$\begin{pmatrix} K_{11} \end{pmatrix}^{-1} = \frac{1}{AE} \begin{pmatrix} 1.118 & 0.236 \\ 0.236 & 0.936 \end{pmatrix}$$

$$\begin{pmatrix} q_F \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = \begin{pmatrix} K_{11} \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ -8 \end{pmatrix} = \frac{1}{AE} \begin{pmatrix} 1.118 & 0.236 \\ 0.236 & 0.936 \end{pmatrix} \begin{pmatrix} 4 \\ -8 \end{pmatrix}$$

$$= \frac{1}{AE} \begin{pmatrix} 2.584 \\ -6.544 \end{pmatrix}$$

Hence,  $u_4^o = 2.584/AE$  and  $v_4^o = -6.544/AE$

**1-4**

$$u_4 = \begin{pmatrix} c & s \end{pmatrix} \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = \begin{pmatrix} \cos(-30^\circ) & \sin(-30^\circ) \end{pmatrix} \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix}$$

i.e.  $u_4 = (0.866 \quad -0.5) \frac{1}{AE} \begin{pmatrix} 2.584 \\ -6.544 \end{pmatrix}$

$$= \frac{5.51}{AE}$$

$$F_{1-4} = \frac{(u_4 - u_1)}{l_{1-4}} \times AE = \frac{\left(\frac{5.51}{AE} - 0\right)}{2} \times AE$$

$$F_{1-4} = \mathbf{2.755 \text{ kN}}$$

**2-4**

$$u_4 = (c \quad s) \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = (\cos(-60^\circ) \quad \sin(-60^\circ)) \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix}$$

$$u_4 = (0.5 \quad -0.866) \frac{1}{AE} \begin{pmatrix} 2.584 \\ -6.544 \end{pmatrix}$$

$$= \frac{6.959}{AE}$$

$$F_{2-4} = \frac{(u_4 - u_2)}{l_{2-4}} \times AE = \frac{\left(\frac{6.959}{AE} - 0\right)}{1.155} \times AE$$

and

$$F_{2-4} = \mathbf{6.025 \text{ kN}}$$

**3-4**

$$u_4 = (c \quad s) \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix} = (\cos(-135^\circ) \quad \sin(-135^\circ)) \begin{pmatrix} u_4^o \\ v_4^o \end{pmatrix}$$

$$u_4 = (-0.707 \quad -0.707) \frac{1}{AE} \begin{pmatrix} 2.584 \\ -6.544 \end{pmatrix}$$

$$= \frac{2.80}{AE}$$

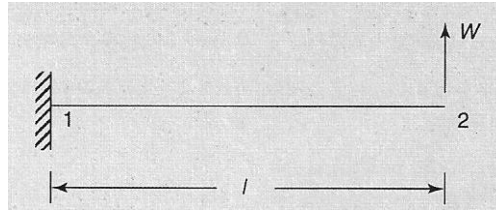
$$F_{3-4} = \frac{(u_4 - u_3)}{l_{3-4}} \times AE = \frac{\left(\frac{2.80}{AE} - 0\right)}{1.414} \times AE$$

i.e.

$$F_{3-4} = \mathbf{1.980 \text{ kN}}$$

# EXERCISE 67, Page 395

1. Using the matrix displacement method, determine the nodal displacements and bending moments for the beam shown.



There are two degrees of freedom:  $v_2$  and  $\theta_2$

$$(k_{1-2}) = EI \begin{pmatrix} & & & \\ & & & \\ & & 12/l^3 & 6/l^2 \\ & & 6/l^2 & 4/l \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

It may be seen that  $v_1 = \theta_1 = 0$

As there is only 1 element,  $(K_{11}) = (k_{1-2})$

$$(K_{11}) = EI \begin{pmatrix} 12/l^3 & 6/l^2 \\ 6/l^2 & 4/l \end{pmatrix}$$

and by inspection,  $(q) = \begin{pmatrix} W \\ 0 \end{pmatrix} \begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix}$

Therefore, 
$$(K_{11})^{-1} = \frac{\frac{1}{EI} \begin{pmatrix} 4/l & -6/l^2 \\ -6/l^2 & 12/l^3 \end{pmatrix}}{\frac{12}{l^3} \times \frac{4}{l} - \frac{6}{l^2} \times \frac{6}{l^2}} = \frac{\frac{1}{EI} \begin{pmatrix} 4/l & -6/l^2 \\ -6/l^2 & 12/l^3 \end{pmatrix}}{\frac{12}{l^4}}$$

i.e. 
$$(K_{11})^{-1} = \frac{1}{EI} \begin{pmatrix} l^3/3 & -l^2/2 \\ -l^2/2 & l \end{pmatrix}$$

and 
$$\begin{pmatrix} v_2 \\ \theta_2 \end{pmatrix} = \frac{1}{EI} \begin{pmatrix} l^3/3 & -l^2/2 \\ -l^2/2 & l \end{pmatrix} \begin{pmatrix} W \\ 0 \end{pmatrix}$$

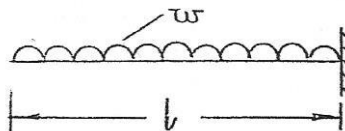
Hence, 
$$v_2 = \frac{W l^3}{3EI} \quad \text{and} \quad \theta_2 = -\frac{W l^2}{2EI}$$

$$\begin{aligned}
 M_2 &= \frac{6EI}{l^2} \left[ (v_2 - v_1) + (\theta_1 + 2\theta_2)l/3 \right] \\
 &= \frac{6EI}{l^2} \left[ \left( \frac{Wl^3}{3EI} - 0 \right) + \left( 0 + 2 \left\{ -\frac{Wl^2}{2EI} \right\} \right) l/3 \right] \\
 &= 2Wl - 2Wl = 0 \\
 M_1 &= \frac{6EI}{l^2} \left[ (v_2 - v_1) + (2\theta_1 + \theta_2)l/3 \right] \\
 &= \frac{6EI}{l^2} \left[ \left( \frac{Wl^3}{3EI} - 0 \right) + \left( 0 + \left\{ -\frac{Wl^2}{2EI} \right\} \right) l/3 \right] \\
 &= 2Wl - Wl
 \end{aligned}$$

i.e.

$$M_1 = Wl$$

**2.** Using the matrix displacement method, determine the nodal displacements and bending moments for the beam shown.



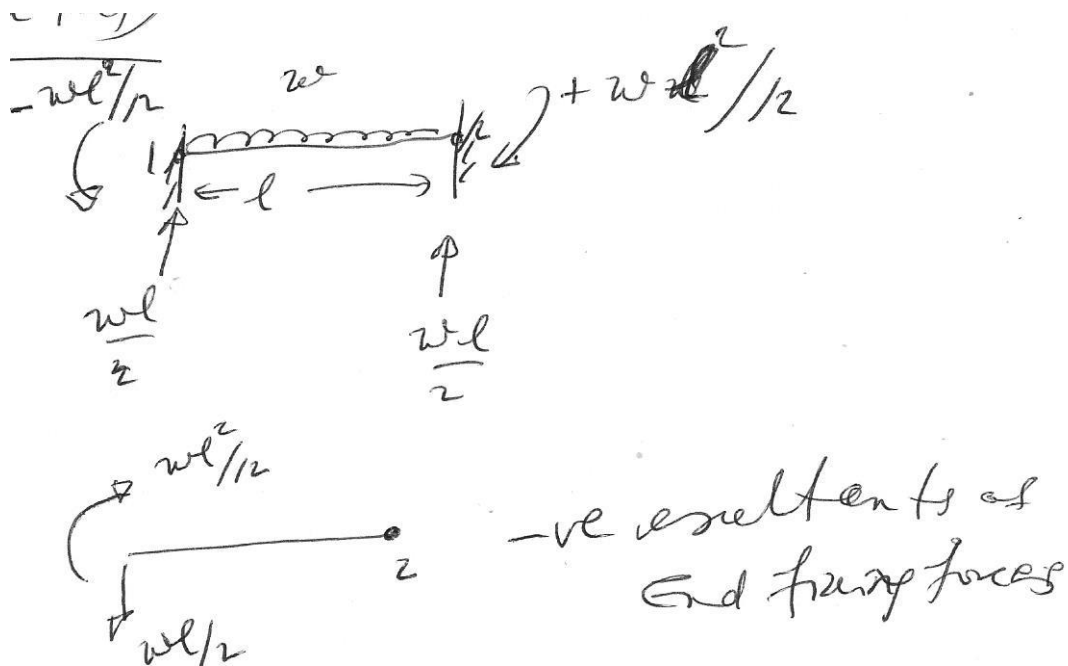
$$(K_{11}) = EI \begin{pmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ 12/l^3 & -6/l^2 & & \\ -6/l^2 & 4/l & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

$$(K_{11}) = EI \begin{pmatrix} 12/l^3 & -6/l^2 \\ -6/l^2 & 4/l \end{pmatrix}$$

$$(K_{11})^{-1} = \frac{\frac{1}{EI} \begin{pmatrix} 4/l & 6/l^2 \\ 6/l^2 & 12/l^3 \end{pmatrix}}{\frac{12}{l^3} \times \frac{4}{l} - \frac{6}{l^2} \times \frac{6}{l^2}} = \frac{\frac{1}{EI} \begin{pmatrix} 4/l & 6/l^2 \\ 6/l^2 & 12/l^3 \end{pmatrix}}{\frac{12}{l^4}}$$

i.e.

$$(K_{11})^{-1} = \frac{1}{EI} \begin{pmatrix} l^3/3 & l^2/2 \\ l^2/2 & l \end{pmatrix}$$



$$(q) = \begin{pmatrix} -wl/2 \\ wl^2/12 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ \theta_1 \end{pmatrix} = \frac{1}{EI} \begin{pmatrix} l^3/3 & l^2/2 \\ l^2/2 & l \end{pmatrix} \begin{pmatrix} -wl/2 \\ wl^2/12 \end{pmatrix}$$

$$= \frac{1}{EI} \begin{pmatrix} -wl^4/6 + wl^4/24 \\ -wl^3/4 + wl^3/12 \end{pmatrix}$$

$$= \frac{w}{EI} \begin{pmatrix} -l^4/8 \\ -l^3/6 \end{pmatrix}$$

$$M_2 = EI \begin{pmatrix} -6/l^2 & 2/l & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= EI \begin{pmatrix} -6/l^2 & 2/l & 0 & 0 \end{pmatrix} \begin{pmatrix} -wl^4/8EI \\ -wl^3/6EI \\ 0 \\ 0 \end{pmatrix}$$

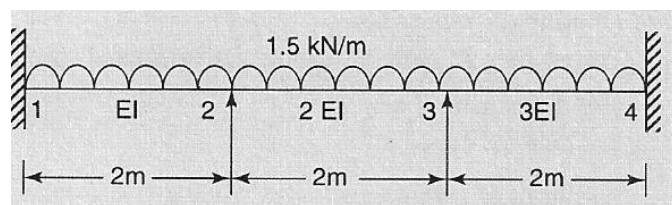


i.e. 
$$M_2 = -\frac{6}{l^2} \times -\frac{wl^4}{8} + \frac{2}{l} \times -\frac{wl^3}{6} + \frac{wl^2}{12}$$
 the final term being the end fixing moment

$$= \frac{3wl^2}{4} - \frac{wl^2}{3} + \frac{wl^2}{12}$$

i.e. 
$$M_2 = \frac{wl^2}{2}$$

**3.** Using the matrix displacement method, determine the nodal displacements and bending moments for the continuous beam shown.



End turning moments:

$$M_{1-2} = M_{2-3} = M_{3-4} = -wl^2 / 12 = -1.5 \times 2^2 / 12 = -0.5 \text{ kN m}$$

$$M_{2-1} = M_{3-2} = +wl^2 / 12 = 1.5 \times 2^2 / 12 = +0.5 \text{ kN m}$$

Hence, 
$$(q_F) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

**1-2**

$$(k_{1-2}) = EI \begin{pmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ \left( \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \right) \end{pmatrix}$$

**2-3**

$$(k_{2-3}) = 2EI \begin{pmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ \left( \begin{array}{cc} 4/2 & 2/2 \\ 2/2 & 4/2 \end{array} \right) \begin{array}{c} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{array} \end{pmatrix}$$

3-4

$$(k_{3-4}) = 3EI \begin{pmatrix} v_1 & \theta_1 & v_2 & \theta_2 \\ & 4/2 & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

Therefore,

$$(K_{11}) = EI \begin{pmatrix} \theta_2 & \theta_3 \\ 2+4 & 4/2 \\ 4/2 & 8/2+12/2 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}$$

i.e.

$$(K_{11}) = EI \begin{pmatrix} \theta_2 & \theta_3 \\ 6 & 2 \\ 2 & 10 \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix}$$

$$(K_{11})^{-1} = \frac{1}{EI} \begin{pmatrix} 10 & -2 \\ -2 & 6 \end{pmatrix} = \frac{1}{EI} \begin{pmatrix} 0.1786 & -0.03571 \\ -0.03571 & 0.1071 \end{pmatrix}$$

$$\begin{pmatrix} \theta_2 \\ \theta_3 \end{pmatrix} = \frac{1}{EI} \begin{pmatrix} 0.1786 & -0.03571 \\ -0.03571 & 0.1071 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} M_1 \\ M_{2-} \\ M_{2+} \\ M_{3-} \\ M_{3+} \\ M_{4-} \end{pmatrix} = \begin{pmatrix} -0.5 \\ +0.5 \\ -0.5 \\ +0.5 \\ -0.5 \\ +0.5 \end{pmatrix}$$

Hence,

$$M_1 = -0.5 \text{ kN m}$$

$$M_2 = \pm 0.5 \text{ kN m}$$

$$M_3 = \pm 0.5 \text{ kN m}$$

$$M_4 = +0.5 \text{ kN m}$$

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = v_1 = v_2 = v_3 = v_4 = 0$$