

COMPANION FOR CHAPTER 9

OF

**FUNDAMENTALS
OF PROBABILITY**
WITH STOCHASTIC PROCESSES

FOURTH EDITION

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Chapter 9

Multivariate Distributions

9A ADDITIONAL EXAMPLES

Example 9a Suppose that mine explosions in a certain country occur at $X_1, X_1 + X_2, X_1 + X_2 + X_3, \dots$, where X_i 's are identically distributed and independent random variables with probability density function.

$$f(x) = \lambda^2 x e^{-\lambda x} \quad x \geq 0.$$

Let $N(t)$ be the total number of mine explosions in that country by time t . Find the probability mass function of $N(t)$.

Solution: Note that for $i \geq 1$, X_i is a gamma random variable with parameters 2 and λ . Hence X_i is the sum of two independent exponential random variables. Consider a Poisson process $\{N_1(t) : t \geq 0\}$ with parameter λ . The times between consequent events of this process are exponential with parameter λ . Thus $N(t) = n$ if and only if $N_1(t) = 2n$ or $2n + 1$. This observation yields

$$P(N(t) = n) = \frac{e^{-\lambda t} (\lambda t)^{2n}}{(2n)!} + \frac{e^{-\lambda t} (\lambda t)^{2n+1}}{(2n)!}, \quad n = 0, 1, 2, \dots \quad \blacklozenge$$

Example 9b Let X_1, X_2, \dots, X_n be independent exponential random variables with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively. There are $n!$ possible rank orderings of X_1, X_2, \dots, X_n , one of them being $X_1 < X_2 < X_3 < \dots < X_n$. Show that the rank ordering of $\{X_1, X_2, \dots, X_n\}$ is independent of $\min(X_1, X_2, \dots, X_n)$. That is, if (i_1, i_2, \dots, i_n) is a permutation of $(1, 2, \dots, n)$, then

$$P(X_{i_1} < X_{i_2} < \dots < X_{i_n} \mid \min_{1 \leq i \leq n} X_i > t) = P(X_{i_1} < X_{i_2} < \dots < X_{i_n}).$$

Solution: Given that $\min_{1 \leq i \leq n} X_i > t$, each X_i is greater than t . Due to the memoryless property of exponential random variables, after t units of time, $X_{i_1} - t, X_{i_2} - t, \dots, X_{i_n} - t$ have the same distributions as $X_{i_1}, X_{i_2}, \dots, X_{i_n}$, respectively. So

$$\begin{aligned} P(X_{i_1} < X_{i_2} < \dots < X_{i_n} \mid \min_{1 \leq i \leq n} X_i > t) &= P(X_{i_1} - t < X_{i_2} - t < \dots < X_{i_n} - t) \\ &= P(X_{i_1} < X_{i_2} < \dots < X_{i_n}). \quad \blacklozenge \end{aligned}$$

Example 9c At an institution, the registration process consists of three steps. First, students discuss their options for courses with their faculty advisors. Second, students check in at the Administration Services to receive clearance on their financial and health status. Third, they meet with a registrar who inputs their courses into the system. Suppose that the amounts of time spent at each step are independent exponential random variables with parameters λ_1 , λ_2 , and λ_3 , respectively. Find the probability density function of the total time a student spends on the entire registration process if

- (a) $\lambda_i \neq \lambda_j$, if $i \neq j$;
 (b) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

Solution: Let X_1 , X_2 , and X_3 be the amounts of time the student spends at steps 1, 2, and 3, respectively. We are given that X_1 , X_2 , and X_3 are independent exponential random variables with parameters λ_1 , λ_2 , and λ_3 , respectively. Let f_1 , f_2 , and f_3 be the probability density functions of X_1 , X_2 , and X_3 , respectively. We will now find the probability density function of the desired quantity, $X_1 + X_2 + X_3$, in cases (a) and (b).

- (a) Using the convolution theorem, we will first find, g , the probability density function of $X_1 + X_2$. We have

$$\begin{aligned} g(t) &= \int_0^\infty f_1(x)f_2(t-x) dx \\ &= \int_0^t \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(t-x)} dx \\ &= \lambda_1 \lambda_2 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1 - \lambda_2)x} dx \\ &= \lambda_1 \lambda_2 e^{-\lambda_2 t} \left[\frac{1}{\lambda_1 - \lambda_2} (1 - e^{-(\lambda_1 - \lambda_2)t}) \right] \\ &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \lambda_2 e^{-\lambda_2 t} + \frac{\lambda_2}{\lambda_2 - \lambda_1} \lambda_1 e^{-\lambda_1 t}. \end{aligned}$$

Let h be the probability density function of $X_1 + X_2 + X_3$; applying the convolution theorem again, straightforward calculations give

$$\begin{aligned} h(t) &= \int_0^\infty g(x)f_3(t-x) dx \\ &= \frac{\lambda_2 \lambda_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} \lambda_1 e^{\lambda_1 t} + \frac{\lambda_1 \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)} \lambda_2 e^{-\lambda_2 t} \\ &\quad + \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \lambda_3 e^{-\lambda_3 t}. \end{aligned}$$

- (b) In this case also, by the convolution theorem, g , the probability density function of $X_1 + X_2$, is given by

$$g(t) = \int_{-\infty}^\infty f_1(x)f_2(t-x) dx = \int_0^t \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(t-x)} dx = \lambda^2 t e^{-\lambda t}.$$

This is the density function of a gamma random variable with parameters 2 and λ . Hence $X_1 + X_2$ is gamma with parameters 2 and λ . Let h be the probability density function of $X_1 + X_2 + X_3$; applying the convolution theorem again, straightforward calculations give

$$h(t) = \int_0^\infty g(x)f_3(t-x)dx = (1/2)\lambda^3 t^2 e^{-\lambda t}.$$

Hence if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, then $X_1 + X_2 + X_3$ is gamma with parameters 3 and λ .

Remark Let X_1, X_2, \dots, X_n be n independent exponential random variables with parameters $\lambda_1, \lambda_2, \dots, \lambda_n$, respectively. Using induction and the result of this example, we can show that,

- (a) if $\lambda_i \neq \lambda_j$, for $i \neq j$, then, f , the probability density function of $X_1 + X_2 + \dots + X_n$ is given by

$$f(t) = \sum_{i=1}^n \left(\prod_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \right) \lambda_i e^{-\lambda_i t};$$

- (b) if $\lambda_i = \lambda$, for $1 \leq i \leq n$, then $X_1 + X_2 + \dots + X_n$ is gamma with parameters n and λ . ♦

Example 9d Michelle and Fred are shopping at a mall. Michelle decides to buy a leather coat and Fred, knowing that it will take her a while, goes to buy an ice cream. At the ice cream parlor, there is only one employee behind the counter. When Fred arrives, there is one customer currently being waited on and 6 more are in line. Suppose that the time that it takes to buy an ice cream is exponentially distributed with mean 3 minutes. Furthermore, the time it takes to choose and buy a leather coat is also exponentially distributed, but with mean 28 minutes. Also suppose that Michelle is the only customer in the coat shop, and, lastly, it takes 2 minutes to travel from the coat shop to the ice cream parlor. What is the probability that Fred will return before Michelle has finished buying her coat?

Solution: When Fred enters the ice cream parlor, there are 7 customers ahead of him. Let us call the customer who is currently being waited on Customer 1, and the 6 customers in line Customers 2 to 7, respectively. Fred is Customer 8. Let X_1 be the period starting when Fred enters the ice cream parlor and ending when Customer 1 leaves. By the memoryless property of the exponential random variables, X_1 is exponential with mean 3 minutes. For $2 \leq i \leq 8$, let X_i be the time that it takes for Customer i to buy an ice cream; X_i is exponential with mean 3 minutes. Finally, let Y be the time until Michelle buys her favorite leather coat. Then, it is reasonable to assume that $\{Y, X_1, X_2, \dots, X_8\}$ is a set of independent random variables. To find the desired probability,

$$P(4 + X_1 + X_2 + \dots + X_8 < Y) = P(X_1 + X_2 + \dots + X_8 < Y - 4),$$

we will successively calculate

$$\begin{aligned}
&P(X_1 < Y - 4), \\
&P(X_1 + X_2 < Y - 4), \\
&P(X_1 + X_2 + X_3 < Y - 4), \\
&\vdots \\
&P(X_1 + X_2 + \cdots + X_8 < Y - 4).
\end{aligned}$$

To do so, note that since the random variables X_1 and Y are independent, $f(x, y)$, the joint probability density function of these two random variables is given by

$$f(x, y) = \frac{1}{3}e^{-x/3} \cdot \frac{1}{28}e^{-y/28}, \quad x \geq 0, \quad y \geq 0.$$

Therefore,

$$\begin{aligned}
P(X_1 < Y - 4) &= \int_0^\infty \left(\int_{x+4}^\infty \frac{1}{3}e^{-x/3} \cdot \frac{1}{28}e^{-y/28} dy \right) dx \\
&= \int_0^\infty \frac{1}{3}e^{-x/3} \left(\int_{x+4}^\infty \frac{1}{28}e^{-y/28} dy \right) dx \\
&= \int_0^\infty \frac{1}{3}e^{-x/3} \cdot e^{-(x+4)/28} dx = \frac{1}{3}e^{-4/28} \int_0^\infty e^{-31x/84} dx \\
&= \frac{84}{93}e^{-4/28} = 0.783.
\end{aligned}$$

Thus, $P(X_1 < Y - 4) = P(2 + X_1 < Y - 2)$, the probability that Customer 1 leaves at least 2 minutes before Michelle has bought her leather coat is 0.783. Next, we calculate, $P(X_1 + X_2 < Y - 4)$, the probability that Customer 2 also leaves at least 2 minutes before Michelle is done. To do so, note that by the memoryless property of the exponential, the period starting when Customer 1 leaves, and ending when Michelle is done, is still exponentially distributed with mean 28 minutes. Therefore repeating the same argument yields

$$P(X_1 + X_2 < Y - 4 \mid X_1 < Y - 4) = 0.783.$$

Thus

$$\begin{aligned}
P(X_1 + X_2 < Y - 4) &= \\
P(X_1 + X_2 < Y - 4 \mid X_1 < Y - 4)P(X_1 < Y - 4) &= (0.783)^2.
\end{aligned}$$

Similarly,

$$P(X_1 + X_2 + X_3 < Y - 4 \mid X_1 + X_2 < Y - 4) = 0.783.$$

Hence,

$$\begin{aligned}
P(X_1 + X_2 + X_3 < Y - 4) &= \\
P(X_1 + X_2 + X_3 < Y - 4 \mid X_1 + X_2 < Y - 4) \times \\
P(X_1 + X_2 < Y - 4) &= (0.783)(0.783)^2 = (0.783)^3.
\end{aligned}$$

Continuing this argument, we have that the probability that Fred (Customer 8) leaves the ice cream parlor at least 2 minutes before Michelle has bought her leather coat is

$$P(X_1 + X_2 + \cdots + X_8 < Y - 4) = (0.783)^8 = 0.141. \quad \blacklozenge$$

Example 9e Suppose that shocks occur to a system according to a Poisson process with parameter λ . Furthermore, suppose that k types of shocks are identified to occur and, independently from other shocks, the probability that a shock is of type i , $1 \leq i \leq k$, is p_i . For $1 \leq i \leq k$, let X_i be the number of shocks of type i occurring between 0 and t . Find the joint probability mass function of X_1, X_2, \dots, X_k .

Solution: Let j_1, j_2, \dots, j_k be non-negative integers. For $1 \leq i \leq k$, let $q_i = P(X_i = j_i)$. Then

$$P(X_1 = j_1, X_2 = j_2, \dots, X_k = j_k) = \frac{(j_1 + j_2 + \cdots + j_k)!}{j_1! j_2! \cdots j_k!} q_1^{j_1} q_2^{j_2} \cdots q_k^{j_k}.$$

To find $q_i = P(X_i = j_i)$, let $N(t)$ be the number of shocks occurring to the system between 0 and t . We have

$$\begin{aligned} q_i &= P(X_i = j_i) = \sum_{n=0}^{\infty} P(X_i = j_i \mid N(t) = n) P(N(t) = n) \\ &= \sum_{n=j_i}^{\infty} \binom{n}{j_i} p_i^{j_i} (1 - p_i)^{n-j_i} \cdot \frac{e^{-\lambda t} (\lambda t)^n}{n!} \\ &= p_i^{j_i} e^{-\lambda t} \sum_{n=j_i}^{\infty} \frac{n!}{j_i! (n - j_i)!} (1 - p_i)^{n-j_i} \cdot \frac{(\lambda t)^{n-j_i} (\lambda t)^{j_i}}{n!} \\ &= \frac{(p_i \lambda t)^{j_i} e^{-\lambda t}}{j_i!} \sum_{n=j_i}^{\infty} \frac{[\lambda t (1 - p_i)]^{n-j_i}}{(n - j_i)!} \\ &= \frac{(p_i \lambda t)^{j_i} e^{-\lambda t}}{j_i!} \cdot e^{\lambda t (1 - p_i)} = \frac{(p_i \lambda t)^{j_i} e^{-p_i \lambda t}}{j_i!}. \end{aligned}$$

Therefore, the joint probability mass function of X_1, X_2, \dots, X_k is

$$\begin{aligned} &P(X_1 = j_1, X_2 = j_2, \dots, X_k = j_k) \\ &= \frac{(j_1 + j_2 + \cdots + j_k)!}{j_1! j_2! \cdots j_k!} \left[\frac{(p_1 \lambda t)^{j_1} e^{-p_1 \lambda t}}{j_1!} \right] \left[\frac{(p_2 \lambda t)^{j_2} e^{-p_2 \lambda t}}{j_2!} \right] \cdots \left[\frac{(p_k \lambda t)^{j_k} e^{-p_k \lambda t}}{j_k!} \right], \end{aligned}$$

where j_1, j_2, \dots, j_k are non-negative integers. \blacklozenge

Example 9f (Genetics) As we know, in humans, for blood type, there are three alleles A, B, and O. The alleles A and B are codominant to each other and dominant to O. A man of genotype AB marries a woman of genotype BO. If they have six children, what is the probability that three will have type B blood, two will have type A blood, and one will have type AB blood?

Solution: The probability is $1/4$ that the blood type of a child of this man and this woman is AB. The probability is $1/4$ that it is A, and the probability is $1/2$ that it is B. The desired probability is equal to

$$\frac{6!}{3! 2! 1!} \left(\frac{1}{2}\right)^3 \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)^1 = \frac{15}{128} = 0.117. \quad \blacklozenge$$

Example 9g (Genetics) Let p and q be positive numbers with $p + q = 1$. For a gene with dominant allele A and recessive allele a , let p^2 , $2pq$, and q^2 be the probabilities that a randomly selected person from a population has genotype AA , Aa , and aa , respectively. A group of six members of the population is selected randomly. Determine the value of p that maximizes the probability of the event of obtaining two AA 's, two Aa 's, and two aa 's.

Solution: The probability of two AA 's, two Aa 's, and two aa 's is

$$g(p) = \frac{6!}{2! 2! 2!} (p^2)^2 [2p(1-p)]^2 [(1-p)^2]^2 = 360p^6(1-p)^6.$$

To find the maximum of this function, set $g'(p) = 0$ to obtain $p = 1/2$. \blacklozenge

9B TRANSFORMATIONS OF $n > 2$ RANDOM VARIABLES

Theorem 8.8 of the book can be generalized in the following obvious way for functions of more than two random variables.

Theorem 9a *Let X_1, X_2, \dots, X_n be n random variables with the joint probability density function $f(x_1, x_2, \dots, x_n)$. Let g_1, g_2, \dots, g_n be n real valued functions of n variables and $Y_1 = g_1(X_1, X_2, \dots, X_n), Y_2 = g_2(X_1, X_2, \dots, X_n), \dots, Y_n = g_n(X_1, X_2, \dots, X_n)$. If*

(i) *the system of n equations in n unknowns*

$$\begin{cases} g_1(x_1, x_2, \dots, x_n) = y_1 \\ g_2(x_1, x_2, \dots, x_n) = y_2 \\ \vdots \\ g_n(x_1, x_2, \dots, x_n) = y_n \end{cases} \quad (1)$$

has a unique solution for x_1, x_2, \dots, x_n in terms of y_1, y_2, \dots, y_n , and

(ii) *the functions g_1, g_2, \dots, g_n have continuous partial derivatives and the Jacobian of the transformation*

$$\begin{cases} y_1 = g_1(x_1, x_2, \dots, x_n) \\ y_2 = g_2(x_1, x_2, \dots, x_n) \\ \vdots \\ y_n = g_n(x_1, x_2, \dots, x_n), \end{cases}$$

is nonzero at all points (x_1, x_2, \dots, x_n) ; that is, the following $n \times n$ determinant is nonzero

$$\frac{\partial(g_1, g_2, \dots, g_n)}{\partial(x_1, x_2, \dots, x_n)} = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \frac{\partial g_n}{\partial x_2} & \dots & \frac{\partial g_n}{\partial x_n} \end{vmatrix} \neq 0,$$

then, the random variables Y_1, Y_2, \dots, Y_n are jointly continuous with the joint probability density function $h(y_1, y_2, \dots, y_n)$ given by

$$h(y_1, y_2, \dots, y_n) = f(x_1, x_2, \dots, x_n) \left| \frac{\partial(g_1, g_2, \dots, g_n)}{\partial(x_1, x_2, \dots, x_n)} \right|^{-1},$$

where in this formula, for given (y_1, y_2, \dots, y_n) , we have that (x_1, x_2, \dots, x_n) is the unique solution of the system (1).

Example 9h Let $f(x_1, x_2, x_3)$ be the joint probability density function of X_1, X_2 , and X_3 . Find the joint probability density function of $Y_1 = X_1 + X_2 + X_3$, $Y_2 = X_1 - X_2 + X_3$, and $Y_3 = X_1 - X_2 - X_3$.

Solution: Let $g_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $g_2(x_1, x_2, x_3) = x_1 - x_2 + x_3$, and $g_3(x_1, x_2, x_3) = x_1 - x_2 - x_3$. Then the system of equations

$$\begin{cases} x_1 + x_2 + x_3 = y_1 \\ x_1 - x_2 + x_3 = y_2 \\ x_1 - x_2 - x_3 = y_3 \end{cases}$$

has the unique solution $x_1 = (y_1 + y_3)/2$, $x_2 = (y_1 - y_2)/2$, $x_3 = (y_2 - y_3)/2$. Also

$$\frac{\partial(g_1, g_2, g_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 4 \neq 0.$$

Hence, by Theorem 9a, $h(y_1, y_2, y_3)$, the joint probability density function of Y_1, Y_2, Y_3 is given by

$$h(y_1, y_2, y_3) = \frac{1}{4}f(x_1, x_2, x_3) = \frac{1}{4}f\left(\frac{y_1 + y_3}{2}, \frac{y_1 - y_2}{2}, \frac{y_2 - y_3}{2}\right). \quad \blacklozenge$$