THE IMPEDANCES, ANGULAR VELOCITIES AND FREQUENCIES OF OSCILLATING-CURRENT CIRCUITS*

By

A. E. Kennelly

INTRODUCTION

It is the object of this paper to disclose a simple yet powerful proposition, recently discovered by the writer, which applies to transient currents, charges, discharges, or temporary disturbances, in electric circuits. This proposition, which is believed to be new, may be briefly stated in the following terms: The impedance of any closed circuit or group of circuits, to free oscillations, is zero.† The angular velocity, or velocities, of the oscillations are such as will bring about this condition.

Simple Resistanceless Oscillating-Current Circuits

The simplest type of oscillating-current circuit consists of a condenser in series with a reactor of negligible resistance, as indicated in Figure 1. It is known that the angular velocity of the free oscillations in this simple resistanceless circuit is such as

$$p_c = i^2 z_c = \mp i^2 j\sqrt{\frac{l}{c}} \qquad p_l = i^2 z_l = \pm i^2 j\sqrt{\frac{l}{c}} \qquad \text{watts}\angle$$

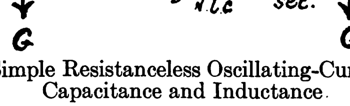

Figure 1—Simple Resistanceless Oscillating-Current Circuit of Capacitance and Inductance.

* Presented before the Institute of Radio Engineers, New York, November 3, 1915. Manuscript received 30th August, 1915.

† See, however, a paper by George A. Campbell in "Trans. A. I. E. E.," April, 1911; on "Cisoidal Oscillations," Vol. XXX, part II, page 902, to which paper the attention of the present writer has been called since this paper was set in type.

(The notation used in this paper is tabulated at the end of the paper.—Editor.)

makes the total reactance zero.* Thus, if l is the inductance of the reactor in henrys, c the capacitance of the condenser in farads, ω the angular velocity of free oscillations thru the circuit, in radians per second, and $j = \sqrt{-1}$; then the reactance of the reactor at this velocity will be $j l \omega$ ohms, that of the condenser $\frac{1}{j c \omega}$ ohms, and the total reaction of the circuit will be $j l \omega + \frac{1}{j c \omega}$ ohms. Equating this total reactance to zero, we obtain:

$$l . j\omega + \frac{1}{c . j\omega} = 0 \qquad \dagger \text{ ohms } \angle \quad (1)$$

$$(j\omega)^2 l c + 1 = 0 \qquad \textit{numeric} \angle \quad (2)$$

$$(j\omega)^2 = -\frac{1}{l c} \qquad \left(\frac{\text{radians}}{\text{second}}\right)^2 \angle \quad (3)$$

$$j\omega = \pm j \frac{1}{\sqrt{l c}} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (4)$$

Selecting the positive sign, the reactance of the reactor is:

$$X_{.l} = j\sqrt{\frac{l}{c}} = j l \omega \qquad \text{ohms } \angle \quad (5)$$

And that of the condenser is:

$$X_c = -j\sqrt{\frac{l}{c}} = \frac{1}{j c \omega} \qquad \text{ohms } \angle \quad (6)$$

Thus, a condenser of 0.01 microfarad ($c = 10^{-8}$), is connected in series with a reactor of 0.01 henry or 10 millihenrys ($l = 10^{-2}$). The angular velocity of the free oscillations of this circuit is $\omega = 10^5$ radians per second. The reactance of the reactor will then be $j1000$ ohms, and that of the condenser $-j\,1000$ ohms, making the total reactance zero.

It is furthermore known that if we count time t in seconds from a suitable epoch, either the instantaneous voltage of the condenser, or the instantaneous current in the reactor, may be represented by the instantaneous projection $O p$, of a vector $O P$, Figure 2, on a straight line of reference $X'O X$, the vector revolving with the angular velocity ω radians per second, and therefore describing in time t, a circular angle $X O P = \varepsilon^{j\omega t}$ radians; where ε is the Napierian base 2.71828. . . . Knowing the angular velocity ω, we can thus predict the electrical condition of the system at any assigned subsequent instant.

*Bibliography (6) page 375.

† The angle sign ∠ attached to the unit of an equation indicates a complex quantity or "plane vector." By this means the use of special vector symbols in the equations is dispensed with. They are to be interpreted vectorially, or treated as complex quantities.

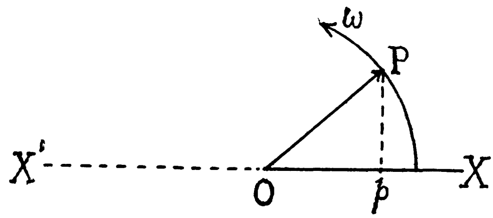

FIGURE 2—Vector Revolving with Angular Velocity ω. Its Projection Op indicates either the instantaneous voltage or the instantaneous current in System of Figure 1, according to the position of the Reference Line $X'OX$

The coefficient $j\omega$ of t, in the angular expression of the characteristic radius vector $\varepsilon^{\pm j\omega t}$, is thus the characteristic angular velocity for the oscillations of the system. The imaginary quantity $j\omega$ indicates oscillation, and is significant of angular velocity in a circle. The double sign indicates that either direction of rotation is possible, and that their sum* is equivalent to a sinusoidal quantity. But suppose that the coefficient of t in the general case is denoted by n; so that n is a generalized angular velocity, which may be either real, imaginary or complex; and so that e^{nt} is the characteristic angular exponential of the system at time t seconds, which determines the voltage or current then existing. Moreover, let us suppose that the impedance offered by an inductance l henrys to an electrical discharge of angular velocity n, is ln ohms, in general a complex quantity; while the impedance offered by a capacitance c farads to the same is $\frac{1}{cn}$ ohms. Then, for the case already considered of a simple resistanceless circuit containing a condenser and reactor in series, if the total impedance is to be zero, we have:

$$ln + \frac{1}{cn} = 0 \qquad \text{ohms } \angle \quad (7)$$

or

$$n^2 . lc = -1 \qquad \text{numeric } \angle \quad (8)$$

and

$$n = \pm\sqrt{\frac{-1}{lc}} = \pm j \frac{1}{\sqrt{lc}} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (9)$$

But this is precisely the coefficient of t which we have found to exist in the simple resistanceless oscillating-current circuit.

According to the above assumptions, therefore, which we shall proceed to justify, an inductance of l henrys offers an impedance of ln ohms, to any generalized alternating current of generalized angular velocity n radians per second, of the type

* The sum of two oppositely directed rotations having the same frequency is well known to be a sinusoidally varying quantity in a straight line.

$a + j\,\omega$, where a is a real quantity, which may be regarded as a hyperbolic angular velocity, or uniform angular velocity in a hyperbola, expressible in hyperbolic radians per second;* while $j\,\omega$ is an "imaginary" angular velocity, which may be regarded as a circular angular velocity, or uniform angular velocity in a circle, expressible in circular radians per second. Similarly, the above assumptions lead to the conclusion that a capacitance of c farads, offers an impedance of $\frac{1}{c\,n}$ ohms to any generalized angular velocity of n radians per second. A pure resistance of r ohms, with negligible inductance or capacity, continues to offer an impedance of r ohms to oscillations of any angular velocity.

It will be noticed that in the case of any simple and sustained alternating current of angular velocity $j\,\omega$ circular radians per second, the assumption above stated reduces to the well known proposition that the impedance of an inductance l henrys is $j\,l\,\omega$ ohms; while that of a capacitance c farads is $\frac{1}{j\,c\,\omega}$ ohms. These impedances, whose sum is, in general, finite, then obey all the laws of direct-current resistances, following the rules of complex quantities. This proposition concerning sustained oscillations was discovered by the writer in 1893, and was first published by him in that year,† forming the basis of our ordinary complex algebra of the alternating-current circuit, in general use at the present day.

The new proposition, here presented, may be looked upon as an extension of the writer's earlier proposition, from sustained alternating currents or oscillations, to unsustained oscillations. From an algebraic standpoint, the value of n is extended from the pure imaginary quantity $j\,\omega$, to the complex quantity $a + j\,\omega$. Altho in engineering practice, sustained oscillations, or simple alternating currents, form the rule, and unsustained oscillations, or transients, form the exception; yet, from a physical point

* (It can be readily shown that, just as

$$\varepsilon^{j\omega t} = \cos \omega t + j \sin \omega t$$

so also

$$\varepsilon^{a t} = \cosh a t + \sinh a t.$$

There is therefore an analogy between the hyperbolic cosine (*cosh*) and the hyperbolic sine (*sinh*) and their corresponding trigonometric functions. Tables and charts permitting the ready use of the hyperbolic functions in engineering calculations have been published by Professor Kennelly, "Tables of Complex Hyperbolic and Circular Functions" and "Chart Atlas of Complex Hyperbolic and Circular Functions," Harvard University Press, Cambridge, Mass., 1914. —EDITOR.)

† Bibliography (2).

of view, the unsustained oscillations of complex angular velocity may be regarded as the general case, and the proposition of 1893, for sustained oscillations, reduces to a mere particular instance of the new proposition.

UNSUSTAINED OSCILLATIONS OF REAL ANGULAR VELOCITY

CONDENSER IN SIMPLE CIRCUIT WITH NON-INDUCTIVE RESISTANCE

Let a condenser of capacitance c farads, be connected in a simple circuit with resistance r ohms and negligible inductance as indicated in Figure 3. Let n be the angular velocity of the

$p_c = i^2 z_c = -i^2 \times 10^6 \qquad p_r = i^2 z_r = i^2 \times 10^6$ watts.

$10^{-5} f$

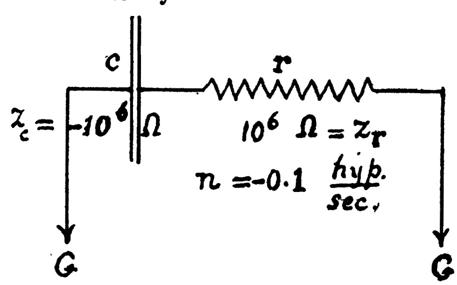

FIGURE 3—Simple Circuit of Capacitance and Resistance

discharge, as yet undetermined. Then, by assumption, $\frac{1}{cn}$ will be the impedance of the condenser, to this angular velocity, in ohms. The remaining impedance in the discharging circuit will be the resistance r ohms. The total impedance in the circuit will then be $\frac{1}{cn} + r$ ohms. But, by assumption, this total impedance of the circuit during free discharge must be zero: or

$$\frac{1}{cn} + r = 0 \qquad \text{ohms} \quad (10)$$

whence

$$n = -\frac{1}{cr} \qquad \text{hyps. per second} \quad (11)$$

That is, the angular velocity n of discharge is a negative real quantity, which may be regarded, therefore, as a negative angular velocity in a hyperbola, expressible in hyperbolic radians per second or, by abbreviation, in hyps. per sec. The angle described in t seconds of discharge will then be nt or $\frac{-t}{cr}$ hyps., which is the well known exponent of the discharge factor of the system, such that if U is the initial potential difference at condenser terminals, in volts, just before closing the circuit, the potential difference u remaining after t seconds, is

$$u = U \varepsilon^{nt} = U \varepsilon^{-\frac{t}{cr}} \qquad \text{volts} \quad (12)$$

As an example, we may consider a condenser of ten microfarads ($c = 10^{-5}$), charged to a potential difference of 500 volts ($U = 500$), and allowed to discharge thru a non-inductive resistance of one megohm ($r = 10^6$). Then $n = -0.1$ hyp. per second. The hyperbolic angle described during 5 seconds would be -0.5 hyp. and the voltage across the condenser, remaining at that time, would be $500\,\varepsilon^{-0.5} = 303.3$ volts, the instantaneous current 0.3033 milliampere, and the instantaneous dissipated power 0.092 watt. The impedance offered by the condenser during discharge is -10^6 ohms, or one megohm negative. We may regard a negative resistance $(-r)$ in a discharge element as involving a dissipative *absorption* of power *into* the circuit of $-i^2 r$ watts, under an instantaneous current strength of i amperes. This is just equal to the dissipative *liberation* of power *out of* the circuit of $+i^2 r$ watts, in heat or in electromagnetic radiation. In any discharging oscillation system, the instantaneous sum of the negative or absorbed powers, and the positive or liberated powers is zero; or $\Sigma\, i_n^2 z_n = 0$ watts; where i_n is the instantaneous current in the oscillation impedance z_n of branch n. In general, the instantaneous power $i_n^2 z_n$ is a complex quantity. The real component is dissipative power. The imaginary component is reactive or non-dissipative power; i. e., the power of storing energy. In the case considered, the instantaneous power absorbed into the circuit from the dielectric of the condenser is -0.092 watt, and the instantaneous thermally liberated power in the resistance r is $+0.092$ watt.

The same conditions apply to the sustained oscillations of alternating-current circuits; but with $a = 0$, negative real components of impedance and power do not present themselves.

Reactor in Simple Circuit with Non-Condensive Resistance

If a reactor of inductance l henrys, (Figure 4) is connected in a simple discharge circuit of total resistance r ohms, with negligible side-capacitance, then, according to assumption, the impedance of the reactor to any generalized angular velocity n, will be ln ohms, and the impedance of the remainder of the circuit will be r ohms; so that the total discharge impedance

$$p_l = i^2 z_l = -i^2 \times 10^2 \qquad p_r = i^2 z_r = i^2 \times 10^2 \text{ watts.}$$

0.1 h

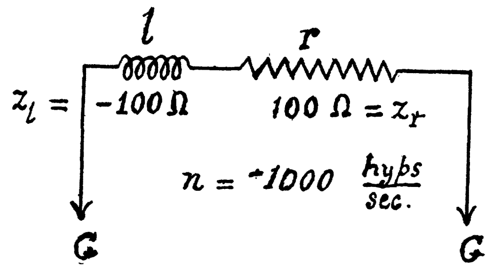

FIGURE 4—Simple Circuit of Inductance and Resistance

will be $l n + r$ ohms. If now the value of n so adjusts itself in free discharge that this sum total reduces to zero,

$$l n + r = 0 \qquad \text{ohms } \angle \qquad (12a)$$

whence

$$n = -\frac{r}{l} \qquad \frac{\text{hyps.}}{\text{sec.}} \qquad (13)$$

The angle described in t seconds of discharge will be $n t$ or $-\frac{r t}{l}$ hyps. which is the well known exponent of the discharge factor $\varepsilon^{-\frac{rt}{l}}$ of the system, such that if I is the initial current in the reactor just before discharge, the current remaining after t seconds is

$$i = I \varepsilon^{nt} = I \varepsilon^{-\frac{rt}{l}} \qquad \text{amperes} \qquad (14)$$

Thus, if a reactor of inductance 100 millihenrys ($l = 0.1$) discharges thru a total resistance of 100 ohms ($r = 100$), with an initial current of 2 amperes, the angular velocity of discharge will be $n = -1000$ hyps. per second. The hyperbolic angle described in 0.5 millisecond will be -0.5 hyp. and the current flowing at that instant will be $2\varepsilon^{-0.5} = 1.213$ amperes. The discharge impedances of the reactor, assumed resistanceless, will be $ln = -100$ ohms. The instantaneous rate of dissipative energy absorption from the reactor's magnetic field into the circuit is $-100 i^2 = -147.2$ watts, and the corresponding rate of dissipative energy liberation from the circuit in the resistor r is $+100 i^2 = +147.2$ watts. This power is affected with a damping factor $\varepsilon^{-\frac{2rt}{l}}$. There is no reactive power component and therefore no oscillatory storage of energy.

Hyperbolic Angles and Their Exponential Symbols

We have seen that our proposition leads to the deduction that the discharges of either condensers or reactors, thru simple resistances, involve real negative values of n; or what may be represented as uniform angular velocities in a hyperbola. We may consider briefly the geometry of this representation.

Let the rectangular coördinate axes OX, OY, Figure 5, be

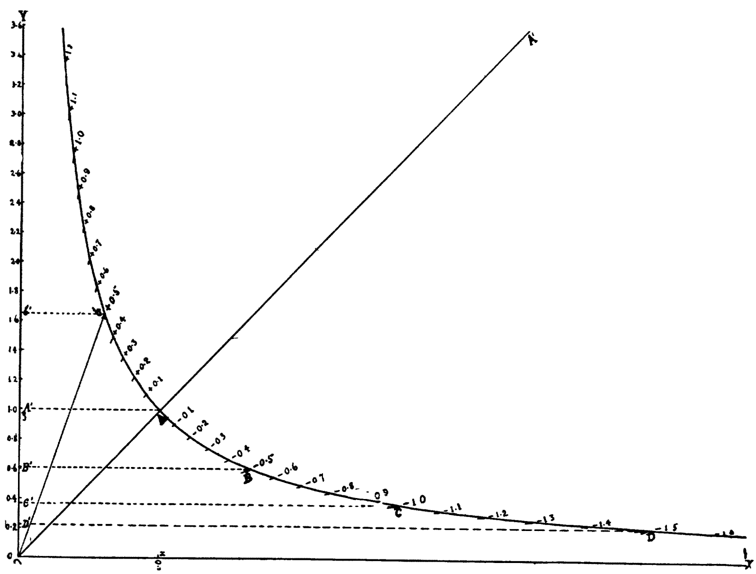

Figure 5—Rectangular Hyperbola with Successive Equal Increments of Hyperbolic Angle and Their Exponential Projections

the asymptotes of the rectangular hyperbola $c\ b\ A\ B\ C\ D$. Let this hyperbola have the axis OAA' and pass through the point A, whose coordinates are $x = 1, y = 1$. Then let a radius vector Ob, starting from the position OA, move with center O over the curve in the positive direction, so as to include an assigned hyperbolic angle θ, defined by the area of the sector OAb, these angles being marked off in Figure 5 along the curve. The ordinate Ob' of the extremity of the vector Ob, is known to be ε^{θ} units in length. In the case presented, $\theta = 0.5$ hyp., and $O\,b' = \varepsilon^{0.5} = 1.649$. In this sense, the value of ε^{θ} may be said to define the hyperbolic angle θ. Similarly, if starting from the initial position OA, the radius vector moves over the curve in a negative or clockwise direction, so as to occupy successively the positions $OB, OC, O\,D$, which include respectively -0.5, -1.0, and -1.5

hyps., the corresponding ordinates OB', OC', OD', measure $\varepsilon^{-0.5}$, $\varepsilon^{-1.0}$, and $\varepsilon^{-1.5}$ units. An exponential $\varepsilon^{-\theta}$, in this sense defines projectively a negative hyperbolic angle $-\theta$. As, therefore, a radius vector OB moves over the hyperbola with uniform hyperbolic velocity $+n$ hyps. per second, describing equal areas in equal times, the ordinate of the moving extremity of the radius vector follows the exponential ε^{+nt} units of length. Conversely, the expression ε^{-nt} may be interpreted geometrically as defining a radius vector which moves over a hyperbola with a uniform negative hyperbolic angular velocity $-n$ hyps. per second. The quantity $-n$ is often called a "damping constant" and the expression ε^{-nt} a damping factor, or damping coefficient; but the conception of n as an angular velocity seems better adapted for our present purposes.

Circular Angles and their Exponential Symbols

It is well known that the exponential quantity $\varepsilon^{j\beta}$ defines the position B (Figure 6) of a point situated in a plane, and on a

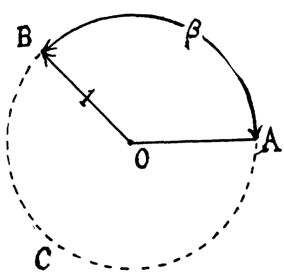

Figure 6—Representation of the Exponential $\varepsilon^{j\beta}$

circle $A\ B\ C$ of unit radius, which is at a distance of β units of length from A, the initial point on the circle. It therefore likewise defines the number of circular radians β, which the radius vector OB has described in its rotation or rotations, about the center O, from the initial position $O\ A$. The exponential $\varepsilon^{j\beta}$ thus defines the total circular radians described in passing from OA to OB. Consequently the exponential ε^{jnt} may be interpreted geometrically as defining a radius vector which moves over a circle of unit radius with a uniform circular angular velocity n circular radians per second.

Complex Angular Velocities and Their Exponential Symbols

Since $\varepsilon^{(\pm a \pm j\omega)t} = \varepsilon^{\pm at} \times \varepsilon^{\pm j\omega t}$, it follows that the exponential $\varepsilon^{(\pm a \pm j\omega)t}$ may be interpreted as representing the product of two angles, each increasing uniformly with time, one a plus or minus hyperbolic angle, and the other a plus or minus circular angle. Or, we may interpret it as a radius vector rotating in a plane with uniform circular angular velocity $\pm \omega$ circular radians per second, the radius vector at the same time changing in length, in projective accordance with uniform hyperbolic angular velocity $\pm \alpha$ hyperbolic radians per second. The path of such a moving point is known to be an equiangular spiral. Such angular velocity may be described as complex, being defined by the complex quantity $(\pm \alpha \pm j\,\omega)$ radians per second.

Proof of Oscillation Impedance Theorems

The following considerations will probably suffice to establish the three propositions: (1) that the oscillation impedance of an inductance l henrys is $l\,n$ ohms; (2) that the oscillation impedance of a capacitance c farads is $1/(c\,n)$ ohms; (3) that the total oscillation impedance of a circuit or path to free oscillation is zero.

(1) Let pure inductance of l henrys, assumed devoid of either resistance or capacitance, carry an instantaneous oscillating current of i amperes, which obeys the law

$$i = I\,\varepsilon^{nt} \qquad \text{amperes } \angle \quad (15)$$

where I is an initial current at time $t = 0$ and $n = -(\alpha \pm j\,\omega)$, a generalized angular velocity. Then the instantaneous back emf. of self induction, opposed to the current, is:

$$-e_l = -l\frac{d\,i}{d\,t} = -I\,n\,\varepsilon^{nt} = -l\,n\,i \qquad \text{volts } \angle \quad (16)$$

The instantaneous driving emf. which is therefore necessary to overcome this back emf. is:

$$e_l = l\,n\,i \qquad \text{volts } \angle \quad (17)$$

The instantaneous apparent resistance offered by the inductance at its terminals is then:

$$z_l = \frac{e_l}{i} = l\,n \qquad \text{ohms } \angle \quad (18)$$

(2) Let a pure capacitance of c farads carry the same instantaneous oscillating current i, above considered under (1). Then the instantaneous back emf. of condensance, opposing the current, is:

$$-e_c = -\frac{1}{c}\int i\,d\,t = -\frac{1}{c\,n}I\,\varepsilon^{nt} = -\frac{i}{c\,n} \qquad \text{volts } \angle \quad (19)$$

The instantaneous driving emf. to overcome this is:

$$e_c = \frac{i}{c\,n} \qquad \text{volts} \angle \quad (20)$$

The instantaneous apparent resistance of the capacitance is then:

$$z_c = \frac{e_c}{i} = \frac{1}{c\,n} \qquad \text{ohms} \angle \quad (21)$$

(3) At any instant the total driving emf. of disturbed energy must be equal to the total back emf. in the circuit, including $r\,i$ drop in the circuit. Otherwise the unsatisfied driving emf. would create a greater current than actually flows at the instant considered. That is, at each and every instant,

$$i\,.\,\Sigma z = 0 \qquad \text{volts} \angle \quad (22)$$

where $i\,.\,\Sigma z$ signifies the vector sum of all the oscillation impedances drops in the circuit, including simple ohmic drops of the type $i\,r$ volts.

Dividing (22) by i, we obtain:

$$\Sigma z = 0 \qquad \text{volts} \angle \quad (23)$$

Or the sum of all the oscillation impedances in the path of the current i is zero at every instant.

In any closed loop of an oscillating-current system, the total instantaneous emf., including $i\,r$ drops, must be zero; or, if z_n is the oscillation impedance of conductor n carrying instantaneous current i_n and forming part of a closed loop, $\Sigma\, i_n\, z_r = 0$ volts. In the case of sustained oscillations, or impressed alternating currents $(a = 0)$, this reduces to the extended complex or two-dimensional form of Kirchoff's loop-voltage law* first published by Steinmetz in 1893.

Case of a Simple Circuit of Capacitance, Inductance and Resistance

We may now proceed to consider more complicated cases of unsustained oscillations. In Figure 7, the reactor of inductance l henrys is in series with a total resistance of r ohms (including that within the reactor) and the condenser of capacitance c farads. If n is the generalized angular velocity of unsustained oscillation, the condenser will have an impedance of $1/(c\,n)$ ohms, and the reactor an impedance of $l\,n$ ohms. Equating the total impedance to zero, we have:

$$\frac{1}{c\,n} + r + l\,n = 0 \qquad \text{ohms} \angle \quad (24)$$

* Bibliography (3).

or $$n^2 . cl + ncr + 1 = 0 \qquad \text{numeric } \angle \qquad (25)$$

whence $$n = -\frac{r}{2l} \pm \sqrt{\left(\frac{r}{2l}\right)^2 - \frac{1}{cl}} \qquad \frac{\text{hyp. radians}}{\text{second}} \qquad (26)$$

or $$n = -\frac{r}{2l} \pm j\sqrt{\frac{1}{cl} - \left(\frac{r}{2l}\right)^2} \qquad \frac{\text{radians}}{\text{second}} \angle \qquad (27)$$

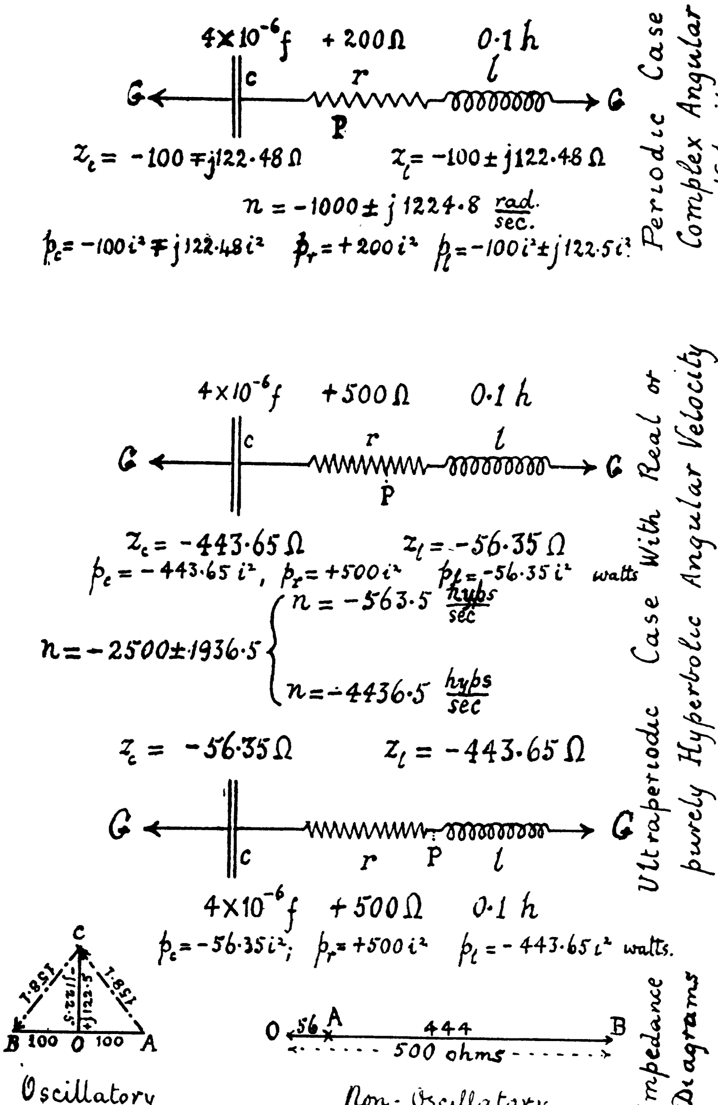

FIGURE 7—Condenser and Resistive Reactor. Periodic and Ultraperiodic Cases

according as $\left(\frac{r}{2l}\right)^2$ is greater or less than $\frac{1}{cl}$. In the former case, the discharge is ultraperiodic, and the angular velocity wholly hyperbolic.† In the latter case, the discharge is periodic, and the angular velocity is partly hyperbolic and partly circular. In the intermediate condition, with $\frac{r}{2} = \sqrt{\frac{l}{c}}$, the discharge is aperiodic. If we denote $r/(2l)$ by α, and $\sqrt{\frac{1}{cl} - \left(\frac{r}{2l}\right)^2}$ by ω, then, in the periodic case,

$$n = -\alpha \pm j\,\omega \qquad \frac{\text{radians}}{\text{second}} \angle \tag{28}$$

As before, the dual sign of the circular angular velocity $j\,\omega$ indicates that either direction of rotation may be adopted, and the sum of two opposite rotations gives rise to a sinusoidal quantity.

The quantity of electricity, q coulombs, in the condenser at any instant t seconds from the beginning of the discharge is known to be:

$$q = A\,\varepsilon^{(-\alpha + j\omega)t} + B\,\varepsilon^{(-\alpha - j\omega)t} \qquad \text{coulombs} \tag{29}$$

where A and B are arbitrary constants depending upon the initial conditions. This general and well known result was first published in 1853* by Lord Kelvin, from an analysis based on energy relations. It is evident that the impedance equation (24) leads directly to the angular velocity of discharge. The oscillation frequency is $f = \frac{\omega}{2\pi}$ cycles per second, and the damping constant $-\alpha$.

As an example, let $c = 4 \times 10^{-6}$ farad, $l = 0.1$ henry, and $r = 200$ ohms. Then $n = -1000 \pm j\,1224.75$ radians per second, and the frequency of oscillation is $f = \frac{1224.75}{2\pi} = 194.92$ cycles per second, accompanied by a damping constant of 1000; or a damping factor of ε^{-1000t}. The discharge impedance, or oscillation impedance of the reactor, assumed resistanceless, is $-100 \pm j\,122.48$ ohms, and that of the condenser $1/(cn) = -100 \mp j\,122.48$ ohms. If the resistance $r = 0$; or were entirely removed from the circuit, the angular velocity would, by (9), become sustained

† Bibliography (5).
* Bibliography (1).

at $n = \pm j\,1581.1$ radians per second, the impedance of the reactor $\pm j\,158.11$ ohms, and that of the condenser $\mp j\,158.11$ ohms. The frequency would then be sustained at $1581.1/(2\pi)$ or 251.65 cycles per second. If this frequency were sustained by an independent alternator or impressing source, only the upper signs would be applicable under international notation; i. e., the reactor's impedance would be $+j\,158.11$ and the condenser's impedance $-j\,158.11$ ohms. The dual signs presenting themselves in the solutions of free oscillations may be attributed to the absence of an independent source of impressed current. Either the condenser or the reactor may become the source of discharges, and either direction of current the direction of reference. With this understanding, the dual signs of imaginary (circular) angular velocities need give rise to no ambiguity or uncertainty.

If the resistance r of the circuit were increased to say 500 ohms, then (26) would apply, and $n = -2500 \pm 1936.5$ radians per second $= -563.5$ or -4436.5 hyps. per second. There are thus two hyperbolic angular velocities present, and two damping factors, $\varepsilon^{-563.5t}$ and $\varepsilon^{-4436.5t}$. The impedance of the reactor to the lower angular velocity is shown in Figure 7, to be -56.35 ohms, and that of the condenser -443.65. At the higher velocity, these values interchange, the reactor taking -443.65, and the condenser -56.35 ohms. In the complete analysis of this ultra-periodic case, it is optional either to assign a certain share of the discharge to each independent hyperbolic angular velocity: or to combine them into the single hyperbolic angular velocity 1936.5 hyps. per second, associated with the damping factor ε^{-2500t}. The results in either case are the same.*

Combination of Condensers and Reactors in Series Circuit

If a circuit contains a plurality of condensers in simple series with a plurality of reactors and resistances, the angular velocity of disturbance in the circuit is readily found.

Let C_1, C_2, C_3 . . . be the respective capacitances in the circuit (farads).

Let l_1, l_2, l_3 . . . be the respective inductances in the circuit (henrys).

Let r_1, r_2, r_3 . . . be the respective resistances in the circuit (ohms).

Let the capacitance-reciprocals, or elastances, of the condensers be found, $s_1 = 1/c_1$, $s_2 = 1/c_2$, $s_3 = 1/c_3$ · · · These may be ex-

* See Bibliography (6), Page 411, for a more detailed analysis of this case.

pressed in darafs. Then the total elastance of the circuit is $S = s_1 + s_2 + s_3 + \cdots$ darafs. The total inductance is $L = l_1 + l_2 + l_3 + \cdots$ henrys, and the total resistance $R = r_1 + r_2 + r_3 + \cdots$ ohms. Then the oscillation impedance of the total elastance is $\frac{S}{n}$ ohms, that of the total inductance $L n$ ohms, and of the total resistance R ohms.

Consequently $$\frac{S}{n} + L n + R = 0 \qquad \text{ohms} \angle \quad (30)$$

whence, as in (27), $$n = -\frac{R}{2L} \pm j\sqrt{\frac{S}{L} - \left(\frac{R}{2L}\right)^2} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (31)$$

assuming that the resistance R is less than $2\sqrt{LS}$ ohms; i. e., that the disturbance is oscillatory; otherwise the roots of (30) are real, as in (26).

As an example let $S = 25$ darafs; $L = 0.1$ henry; $R = 200$ ohms; then $n = -1000 \pm j\,1224.75$ radians per second.

Oscillation Angular Velocity of Resistanceless Discharging Elements in Parallel

The simplest case of discharging elements in parallel, producing oscillations, is perhaps that indicated in Figure 8. A discharging element may be defined as an element capable of containing electromagnetic energy, and therefore capable of having the amount of its energy content disturbed. A discharging element may therefore be a reactor of inductance l henrys, which may contain magnetic energy of $li^2/2$ joules, when traversed by a current of i amperes. It may also be a condenser, of capacitance c farads, which may contain electric energy of $ce^2/2$ joules, when charged to a potential difference of e volts. The oscillations here considered may be those accompanying either an increase, or a decrease of energy in any element; i. e. accompanying either charge or discharge; but discharge is the easier phenomenon to analyze; because in charge, a final steady state has ordinarily to be superposed upon that transient state of disturbance which is the immediate subject of discussion. We may, therefore, confine our discussion to cases of discharge, with the understanding that the results apply also, with reversal of currents and powers, to cases of charge, if the subsequent steady state is independently superposed.

In Figure 8, let a number of condensers of capacitances $c_1, c_2, c_3 \ldots$ farads, respectively, be connected in parallel to common bus-bars BB', $b\,b'$. Let any number of reactors be also

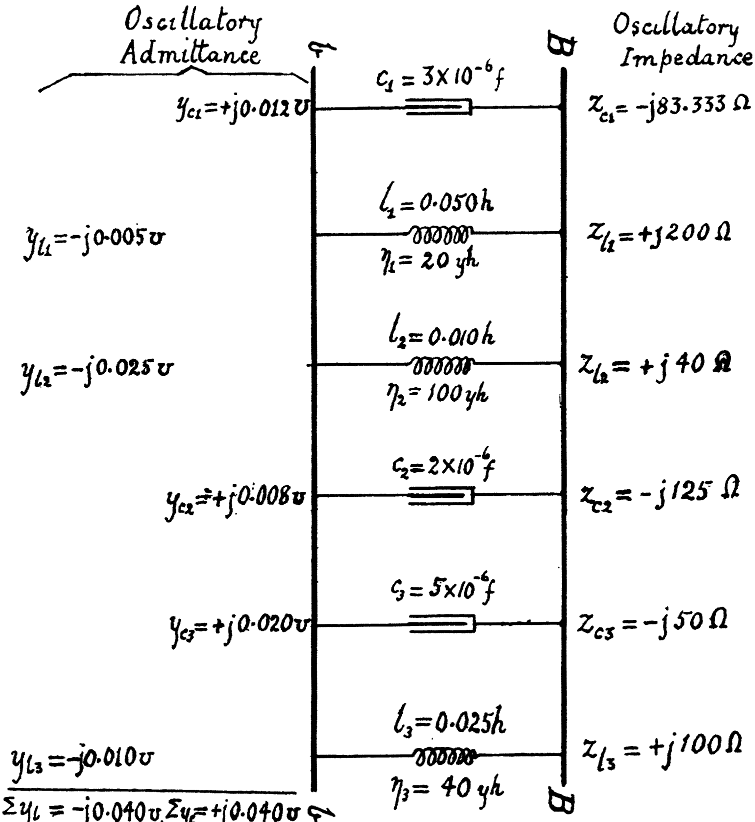

FIGURE 8—Parallel Connection of a Number of Discharge Elements, with Negligible Resistance

connected in parallel to the same bars, and let the resistance of each and all the elements of the circuit be negligibly small. When reactors of inductances l_1, l_2, l_3 . . . henrys are connected in parallel, it is convenient to use the reciprocals of these values $\eta_1 = 1/l_1$, $\eta_2 = 1/l_2$, $\eta_3 = 1/l_3$, . . . for arithmetical purposes. These reciprocal inductances may be called *ductances*, for want of a better term. A ductance may be expressed in (henrys)$^{-1}$; or, as Karapetoff suggested, say, in yrnehs. An inductance of 0.1 henry is therefore a ductance of 10 yrnehs. The total ductance of a number of ductances in parallel is then their numerical sum, just as the total capacitance of a number of capacitances in parallel is their numerical sum.

It is evident that the system of Figure 8, assumed resistanceless, is equivalent to a single condenser of capacitance $C = c_1 +$

$c_2 + c_3 +$ farads, in simple series with a single ductance $E = \gamma_1 + \gamma_2 + \gamma_3 +$ yrnehs. The case of Figure 8 thus reduces to that of Figure 1, with C in place of c farads, and $1/E$ in place of l henrys. The discharge impedance of the combined condenser is then $1/(C\,n)$ ohms, and that of the combined ductance n/E ohms. Consequently

$$\frac{1}{C\,n} + \frac{n}{E} = 0 \qquad \text{ohms} \angle \quad (32)$$

and

$$n^2 C + E = 0 \qquad \text{yrnehs} \quad (33)$$

or

$$n = \pm j\sqrt{\frac{E}{C}} \qquad \text{radians/sec.} \angle \quad (34)$$

which result is in agreement with (9), and is almost self-evident in view of (9). Our proposition states, however, that the discharge impedance of the system to any element must be zero. Consider the element c_1 as having its energy suddenly disturbed, and as discharging thru the rest of the system. The discharge impedance of c_1 is $1/(c_1 n)$ ohms. That of the remaining condensers is $1/(C - c_1)\,n$ ohms, and that of the ductances n/E ohms, as already considered. The remaining condensers are in parallel with the ductances, and their joint impedance must be taken in relation to c_1; so that

$$\frac{1}{c_1 n} + \frac{\dfrac{n}{E} \times \dfrac{1}{n(C-c_1)}}{\dfrac{n}{E} + \dfrac{1}{n(C-c_1)}} = 0 \qquad \text{ohms} \angle \quad (35)$$

from which

$$n^2 = -\frac{E}{C} \qquad \left(\frac{\text{radians}}{\text{sec.}}\right)^2 \angle \quad (36)$$

or

$$n = \pm j\sqrt{\frac{E}{C}} \qquad \frac{\text{radians}}{\text{sec.}} \angle \quad (37)$$

This is the same result as was reached in (34). It means that the angular velocity of discharge oscillations is the same in each individual condenser as in the system as a whole; so that there is one and only one oscillation frequency $f = n/(j\,2\pi)$ cycles per second. Moreover, this frequency is such that the impedance of the system is zero, taking each condenser in turn as the main path of discharge.

Similarly, taking any one ductance, say γ_1, as the main path of discharge, this element only having its energy suddenly disturbed; then its impedance is n/γ_1 ohms, and that of the remaining ductances, in parallel, $n/(E - \gamma_1)$ ohms.

Consequently

$$\frac{n}{\gamma_1}+\frac{\dfrac{n}{E-\gamma_1}\times\dfrac{1}{nC}}{\dfrac{n}{E-\gamma_1}+\dfrac{1}{nC}}=0 \qquad \text{ohms} \angle \quad (38)$$

from which

$$n^2 = -\frac{E}{C} \qquad \left(\frac{\text{radians}}{\text{second}}\right)^2 \angle \quad (39)$$

or

$$n = \pm j\sqrt{\frac{E}{C}} \qquad \frac{\text{radians}}{\text{second}} \angle \quad (40)$$

again the same result as in (34) and (37). There is thus one and the same oscillation frequency in all branches of the system. If resistances are injected into the various branches, this simple relation is destroyed, and altho the same principles and method of procedure apply, the result is usually an equation of the nth degree, for n discharging elements, giving on solution, n roots, every one root corresponding to the angular velocity of each discharge element, considered in turn as the main path. The number of distinct oscillation frequencies is, however, usually distinctly less than n; because each pair of conjugate complex roots gives rise to but a single oscillatory frequency.

If we take as an example the following values:— $c_1 = 3\times10^{-6}$, $c_2 = 2\times10^{-6}$, $c_3 = 5\times10^{-6}$, $C = 10^{-5}$, $\gamma_1 = 20$, $\gamma_2 = 100$, $\gamma_3 = 40$, $E = 160$; then $n = \pm j\,4000$ circular radians per second, and the oscillation frequency $f = 4000/(2\pi) = 636.6$ cycles per second. The impedances and admittances of the various elements at this frequency are indicated in Figure 8, just as if the frequency were independently sustained in an alternating-current circuit. It may be observed that the total admittance of the branches of the system is zero, and this we shall find to be a general law, whether resistances are present in the various branches, or not.

Condenser, Reactor, and Resistance, in Star Connection

We may next consider the case represented in Figure 9, of a condenser, reactor and non-inductive resistance in star connection; or, what is of course the same, connected in parallel between bus-bars. Here we have two discharge elements and an inert or energyless resistance leak, all in parallel. It is optional to consider either discharge element as the main path and the two others, in joint connection, closed on it. Let c be the capacitance in farads, l the inductance in henrys containing a resistance of r ohms, and g the conductance of the leak in

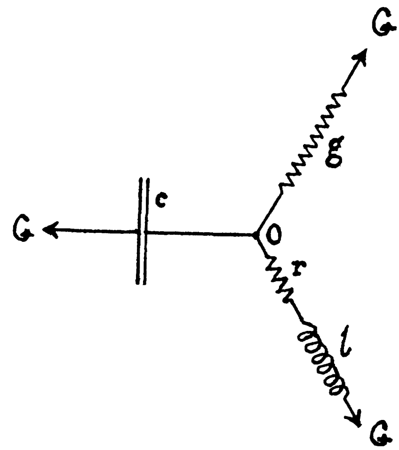

FIGURE 9—Fleming's Case of a Condenser c Shunted by a Leak g and Discharging thru a Reactor l

mhos. Then taking the condenser as the main path, we obtain:—

$$\frac{1}{cn} + \frac{1}{g + \frac{1}{r + ln}} = 0 \qquad \text{ohms} \angle \quad (41)$$

whence $\quad n^2 c\, l + n\,(c\,r + g\,l) + (1 + g\,r) = 0 \qquad$ numeric $\angle$ (42)

$$\text{and } n = -\left(\frac{r}{2l} + \frac{g}{2c}\right) \pm j\sqrt{\frac{1+g\,r}{c\,l} - \left(\frac{r}{2l} + \frac{g}{2c}\right)^2} \quad \frac{\text{radians}}{\text{second}} \angle \quad (43)$$

$$= -\left(\frac{r}{2l} + \frac{g}{2c}\right) \pm j\sqrt{\frac{1}{c\,l} - \left(\frac{r}{2l} - \frac{g}{2c}\right)^2} \qquad \text{“} \quad \angle \quad (44)$$

Formula (44) was derived by Fleming, from a different method, in 1913.* If we prefer to take the reactor as the main path of discharge: then

$$l\,n + r + \frac{1}{c\,n + g} = 0 \qquad \text{ohms} \angle \quad (45)$$

whence $n^2 c\, l + n\,(c\,r + g\,l) + (1 + g\,r) = 0 \qquad$ numeric $\angle$ (46)

which is identical with (42) and therefore leads to the same result.

In view of the practical instance cited by Fleming, no example of this case needs to be discussed arithmetically.

Two Resistive Reactors and a Condenser, in Star or Parallel Connection

In the case presented in Figure 10, we have three discharge elements in parallel, two of them reactors of l_1 and l_2 henrys,

* Bibliography (7).

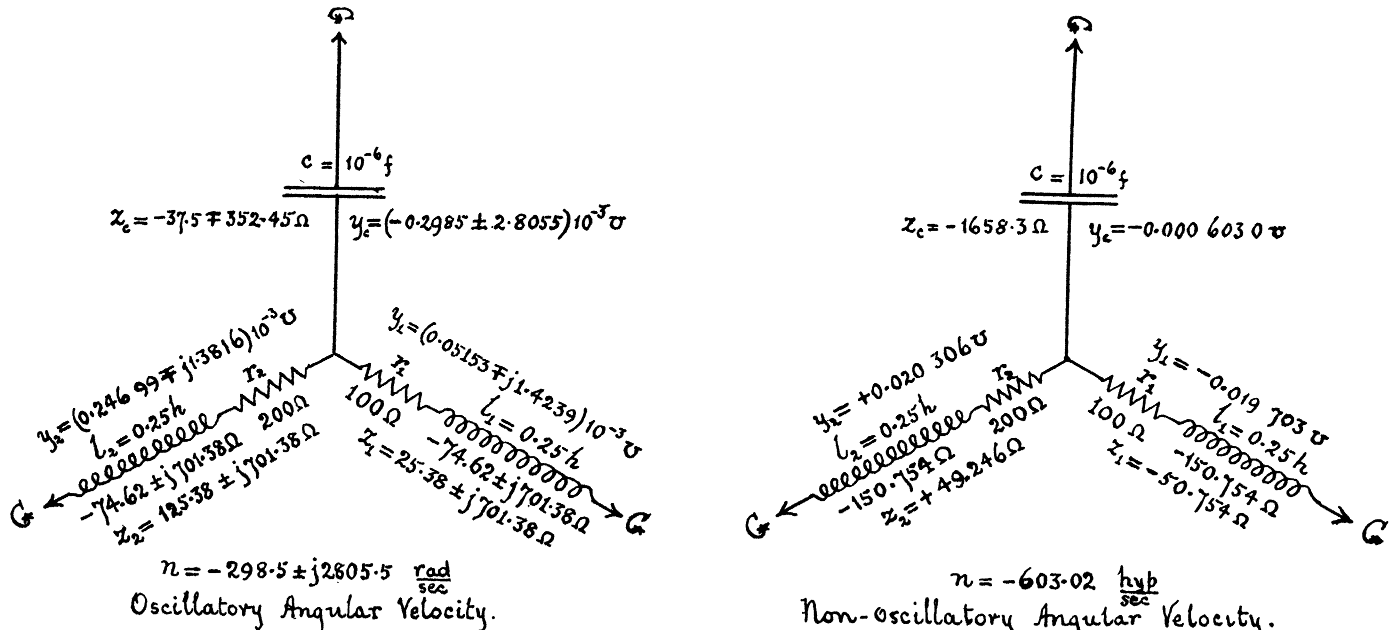

FIGURE 10—Case of Condenser Connected to Pair of Unlike Reactors in Parallel

r_1 and r_2 ohms respectively, and the third a condenser of capacitance c farads. It is a matter of indifference which element we take as the main path of discharge; but taking the condenser element, we have:

$$\frac{1}{cn} + \frac{(r_1 + l_1 n) \times (r_2 + l_2 n)}{(r_1 + l_1 n) + (r_2 + l_2 n)} = 0 \qquad \text{ohms} \angle \quad (47)$$

whence

$$n^3 + n^2 \left(\frac{r_1}{l_1} + \frac{r_2}{l_2}\right) + n \left(\frac{1}{c\,l_1} + \frac{1}{c\,l_2} + \frac{r_1 r_2}{l_1 l_2}\right) + \frac{r_1 + r_2}{c\,l_1 l_2} = 0 \qquad \left(\frac{\text{radians}}{\text{second}}\right)^3 \angle \quad (48)$$

a cubic equation having one real and two complex roots. There are thus three values of the angular velocity n, which, substituted in (47), will enable that equation to hold. The real value may be regarded as pertaining to the discharge from one reactor thru the other, and thru the resistance $r_1 + r_2$, in their circuit. The two conjugate complex values may be regarded as pertaining to the discharge from the condenser into a certain single resistance and inductance, equivalent to the pair of parallel reactors.

As an example, we may take $c = 10^{-6}$ farad, $r_1 = 100$ ohms, $l_1 = 0.25$ henry, $r_2 = 200$ ohms, $l_2 = 0.25$ henry. Then (48) becomes

$$n^3 + 1200\,n^2 + 8.32 \times 10^6 n + 4.8 \times 10^9 = 0 \qquad \left(\frac{\text{radians}}{\text{second}}\right)^3 \quad (49)$$

This equation may be solved by the use of an auxiliary hyperbolic angle in the well known manner; but it is easy to find the roots by first plotting the value of (49), as ordinates, against arbitrarily selected values of n, as abscissas, in the regular way, as indicated in Figure 11, which shows that the graph passes through the zero line of ordinates near $n = -600$. A few more arithmetical trials, close to this value of n, will give a more nearly correct value of -603.02. This is the numerical value of the real root. Dividing (49) by $(n + 603.02)$, we obtain as the quotient:

$$n^2 + 596.98\,n + 7.96 \times 10^6 = 0 \qquad \left(\frac{\text{radians}}{\text{second}}\right)^2 \angle \quad (50)$$

an ordinary quadratic equation, of which the solution is:—

$$n = -298.49 \pm j\,2805.5 \qquad \frac{\text{radians}}{\text{second}} \angle \quad (51)$$

This equation gives the two remaining complex roots of (49).

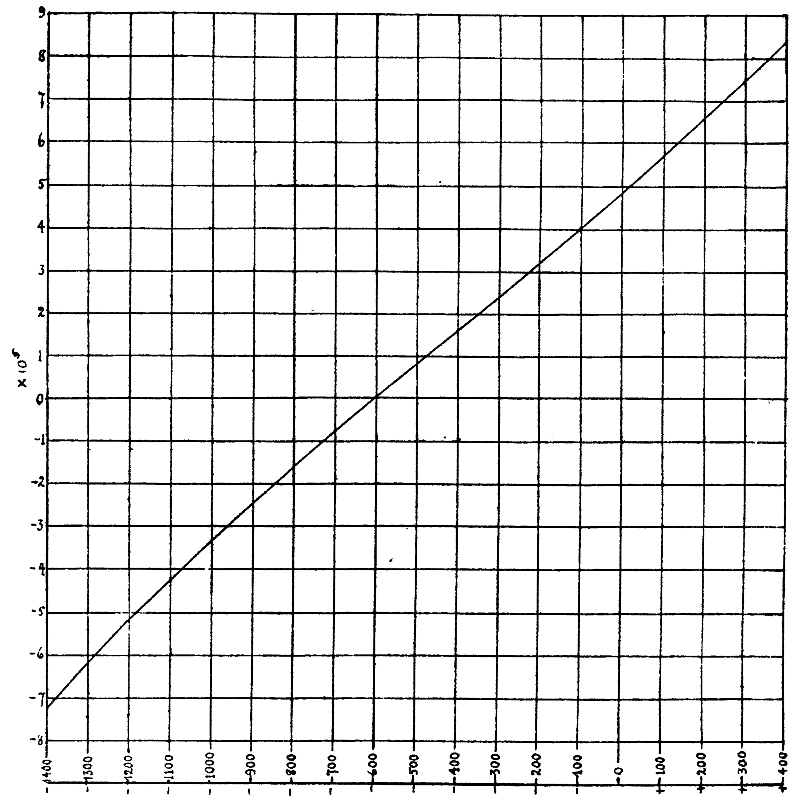

FIGURE 11—Graph of Expression $n^3 + 1200n^2 + 8.32 \times 10^6 n + 4.8 \times 10^9$ Between the Values $n = -1400$ and $n = +400$

If the condenser were disconnected from the system, leaving the two reactors connected thru $r_1 + r_2 = 300$ ohms, we find, by (13), that their free angular velocity would be $n = -600$ hyps. per second; so that the presence of the condenser merely modifies this to -603.02. Again, if the two resistances were removed ($r_1 = r_2 = 0$), leaving the condenser in series with the reactor, we find, by (9), that the free angular velocity would be $n = \pm j$ 2828.4 radians per second. The presence of the resistances reduces this to $-298.49 \pm j\, 2805.5$.

The system of Figure 10 in the example considered, thus dissipates disturbance energy in two different modes. One is a non-oscillatory discharge of -603.02 hyps. per second, or accompanied by a damping factor of $\varepsilon^{-603.02t}$. This is the discharge between the two reactors, slightly modified by the presence of the condenser. The other is an oscillatory discharge of angular velocity $-298.49 \pm j\,2805.5$ radians per second, having a frequency of $2805.5/(2\pi) = 446.5$ cycles per second, accompanied by a damping constant of 298.49, or a damping factor of $\varepsilon^{-298.49t}$. This damped oscillatory discharge is between the condenser and the joint ductance, as modified by the presence of the resistances.

It is shown in Figure 10 that at $n = -603.02$, the condenser has an impedance of -1658.3 ohms, one reactor -50.754 ohms, and the other $+49.246$. Taking the admittances, or reciprocals of these quantities, the condenser has -0.60302 millimhos, one reactor -19.703 millimhos, and the other $+20.306$. The sum of these admittances is zero.

Similarly, at $n = -298.49 \pm j\,2805.5$ radians per second, Figure 10 shows that the sum of the three branch admittances is zero. We may proceed to establish this proposition generally.

The Sum of the Oscillation Admittances About any Branch Point is Zero

In Figure 12, a number of branches, to ground or common connection, meet at the point O Each branch may contain a

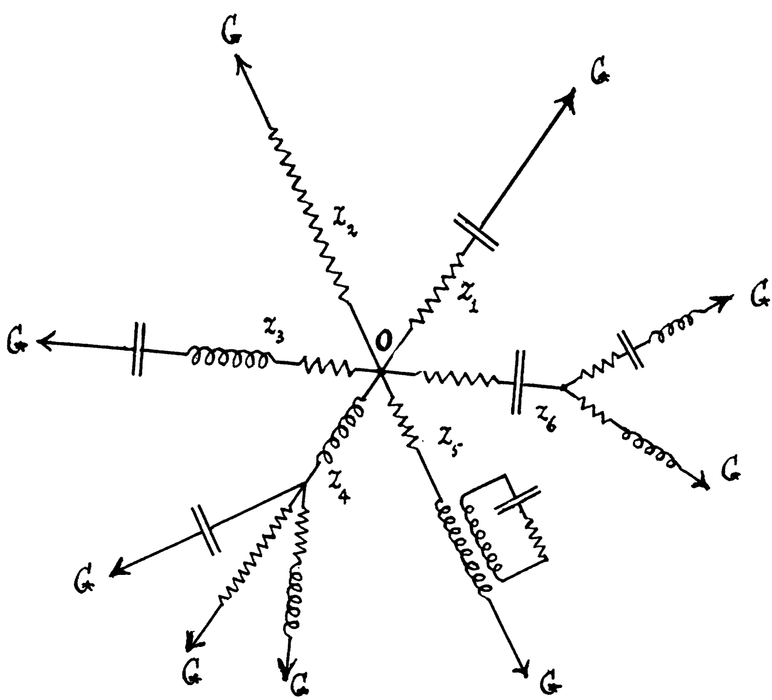

Figure 12—Group of Oscillatory Impedances meeting at a Knot Point O

plurality of discharge elements, or resistances, or sub-branches. Then let the discharge impedance of these branches be z_1, z_2, z_3 . . . etc., each being formed on the understanding that an inductance has ln ohms, and a capacitance $1/(c\,n)$ ohms, n being the subsequently determined generalized angular velocity. Then, in order that the total discharge impedance in the path of any one branch, say z_1, shall be zero, we must have:

$$z_1 + \frac{1}{\frac{1}{z_2} + \frac{1}{z_3} + \frac{1}{z_4} + \cdot\ \cdot} = 0 \qquad \text{ohms} \angle \quad (52)$$

or if $y_1 = 1/z_1$, $y_2 = 1/z_2$, $y_3 = 1/z_3$, $y_4 = 1/z_4$. . are the respective discharge admittances,

$$z_1 + \frac{1}{y_2 + y_3 + y_4 + \cdot\ \cdot} = 0 \qquad \text{ohms} \angle \quad (53)$$

whence $$y_2 + y_3 + y_4 + \quad = -y_1 \qquad \text{mhos} \angle \quad (54)$$

or $$y_1 + y_2 + y_3 + y_4 + \quad = 0 \qquad \text{"} \quad (55)$$

or $$\Sigma y = 0 \qquad \text{"} \quad (56)$$

This relation must hold for each and all values of n which may satisfy (52). It applies not merely to a subdivided circuit: but also to a single undivided circuit; such as that in Figure 7, if any available point P be selected as a branch point of two branches. Knot-point cases* may often be solved advantageously by using this rule.

A number of less simple oscillating-current networks have been worked out by the methods here presented, and checked by independent means. No discrepancies have yet been found.

Inductively Coupled Circuits

If two circuits are inductively coupled by a mutual inductance μ henrys, as in Figure 13, the primary having constants c_1, l_1, r_1, and the secondary c_2, l_2, r_2, it was shown by the writer in 1893† that the impedance z'_{12} ohms of the primary circuit to sustained oscillations in the presence of the closed secondary circuit, is:

$$z'_{12} = z'_1 - \frac{(\mu j \omega)^2}{z'_2} \qquad \text{ohms} \angle \quad (57)$$

---

*Thus, the case presented in Figure 10, with equation (47), may be stated as follows:—

$$c\,n + \frac{1}{r_1 + l_1 n} + \frac{1}{r_2 + l_2 n} = 0 \qquad \text{mhos} \angle \quad (56a)$$

which reduces to (48).

†Bibliography (4).

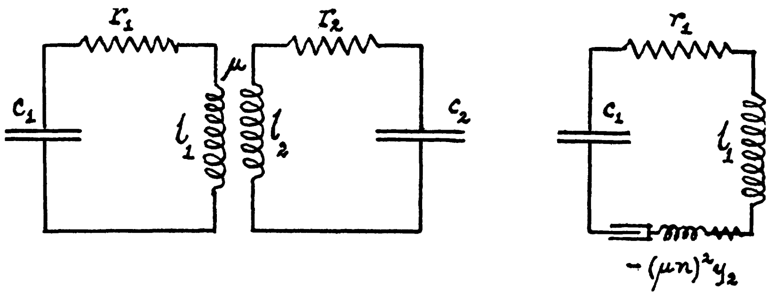

FIGURE 13—Pair of Inductively Connected Oscillatory Circuits and the Equivalent Single Primary Circuit

where z'_1 is the impedance of the primary circuit with the secondary circuit open, z'_2 the impedance of the secondary circuit with the primary circuit open (ohms $\angle$), and ω is the angular velocity of the impressed alternating current, in radians per second. Knowing the impedance of the primary circuit from (57), the current in that circuit to any impressed alternating emf. is immediately obtained. The emf. induced in the secondary circuit is then found by multiplying the primary current with $-\mu j \omega$ ohms.

The corresponding rule for free oscillations of generalized angular velocity n radians per second is:

$$z_{12} = z_1 - \frac{(\mu n)^2}{z_2} = z_1 - \mu^2 n^2 y_2 \qquad \text{ohms } \angle \quad (58)$$

so that, considering the primary circuit as the discharging circuit, with zero oscillation impedance, z_{12} must be equated to zero; or:

$$z_1 - \frac{(\mu n)^2}{z_2} = 0 \qquad \text{ohms } \angle \quad (59)$$

is the condition for determining n, it being understood that z_1 is the impedance of the primary circuit to angular velocity n, when the secondary circuit is open, and z_2 the impedance of the secondary circuit to angular velocity n, when the primary circuit is open.

The proposition may be proved as follows:

Let the instantaneous oscillating current i_1 in the primary circuit follow the law

$$i_1 = I_1 \varepsilon^{nt} \qquad \text{amperes } \angle \quad (60)$$

where n is a generalized or complex angular velocity, and I_1 the initial value of the primary current when $t = 0$. Then the instantaneous induced emf. in the secondary circuit will be

$$e_2 = -\mu \frac{d\,i_1}{d\,t} = -\mu\, n\, I_1\, \varepsilon^{nt} = -\mu\, n\, i_1 \qquad \text{volts} \angle \quad (61)$$

and if the oscillation impedance of the secondary circuit is z_2 ohms, the instantaneous secondary current strength will be

$$i_2 = \frac{e_2}{z_2} = -\frac{\mu\, n\, i_1}{z_2} = -\mu\, n\, i_1\, y_2 \qquad \text{amperes} \angle \quad (62)$$

The instantaneous emf. induced in the primary circuit by the rate of change in secondary current will be

$$-e_1 = -\mu \frac{d\,i_2}{d\,t} = \mu^2 n^2 i_1 y_2 \qquad \text{volts} \angle \quad (63)$$

The instantaneous driving emf. in the primary circuit, needed to overcome $-e_1$ will be

$$e_1 = -\mu^2 n^2 i_1 y_2 \qquad \text{volts} \angle \quad (64)$$

The instantaneous impedance in the primary circuit due to the reaction of the secondary will be

$$z_{12} = \frac{e_1}{i_1} = -\mu^2 n^2 y_2 = -\frac{\mu^2 n^2}{z_2} \qquad \text{ohms} \angle \quad (65)$$

a result which includes (57), when $\alpha = 0$ and $n = j\,\omega$.

General Considerations

An equation in n of the second degree can be satisfied either by two real roots ($-\alpha_1$, $-\alpha_2$, non-oscillating angular velocities) or by a pair of conjugate complex roots, of the type $(-\alpha \pm j\,\omega)$, entailing one oscillation frequency. An equation in n of the third degree, or of any odd degree, indicates at least one real root $-\alpha_1$, which can ordinarily be evaluated in the manner exemplified by Figure 11. The remaining two roots, if conjugate, represent one oscillation frequency. Similarly, an equation in n, with lowest terms, and of the fourth degree, may indicate the presence of two independent oscillation frequencies, and an equation of the sixth degree, three oscillation frequencies. An inspection of the oscillation system connection-diagram may help in forming a judgment as to the number of independent oscillation frequencies present.

In view of the close analogy which exists between the arithmetics of electric oscillations in oscillatory-current circuits, and of small mechanical oscillations in mechanically vibrating systems,* it is evident that the rules above discussed apply, in general, also to mechanical oscillation-systems, provided the

* Bibliography (8).

elastic forces are proportional to the corresponding displacements and the frictional forces to the first powers of the velocities.*

In the case represented in Figure 13, the full expression of (59) yields an equation of the fourth degree in n, with two pairs of conjugate complex roots, corresponding to two oscillation frequencies and damping constants. The complete solution of this fourth-degree equation is, in general, very tedious; but full results for engineering purposes may be obtained by abbreviated methods. The detailed discussion of oscillation frequencies in mutually coupled circuits calls however, for a separate paper, and need not be continued here.

Conclusions

(1) The oscillation impedance of a circuit traversed by free electric oscillations is zero.

(2) The oscillation impedance of a pure resistance is equal to its ohmic resistance.

(3) The oscillation impedance of a capacitance c farads, to angular velocity n, is $1/(c\,n)$ ohms $\angle$. In other words, its oscillation admittance is $c\,n$ mhos $\angle$.

(4) The oscillation impedance of an inductance of l henrys, to angular velocity n, is $l\,n$ ohms $\angle$.

(5) The oscillation impedances of the elements of a circuit or system of circuits follow the laws of resistances in such circuits when traversed by continuous currents, subject to the rules of complex quantities, or of plane-vector arithmetic.

(6) The impedance of a circuit, or system of circuits, to sustained oscillations, or impressed alternating currents, is a particular case under the general laws above stated, ($a = 0$, $\Sigma z \neq 0$).

(7) Any free oscillation in a circuit, or system of circuits, selects such an angular velocity, n radians per second, as will reduce its total impedance to zero.

(8) A generalized angular velocity n is a complex quantity containing a real and an imaginary component. The real component is the damping constant, and may be regarded as the projection of a hyperbolic angular velocity. The imaginary component is a circular angular velocity, of 2π times the oscillation frequency. Its projection, on an axis of reference, gives a sinusoidal quantity.

* Since the printing of this paper, the author's attention has been directed to a statement by Mr. H. W. Nichols, which indicates that certain mathematical propositions concerning mechanical oscillations, bearing closely on this matter, are already known to physicists.

(9) The sum of the oscillation admittances of the branches of a multiple oscillation circuit, at a knot point, is zero.

(10) The oscillation angular velocities of mutually coupled circuits can be expressed in terms of their mutual impedances.

(11) The sum of the instantaneous oscillation-impedance drops ($\Sigma i_n z_n$) around any closed loop in an oscillation system is zero. In the case of sustained oscillations, i. e., alternating currents, with $a = 0$, this reduces to Steinmetz's extension of Kirchoff's law into two dimensions.

(12) The instantaneous power of discharge in an oscillation impedance z is $i^2 z$ watts $\angle$, the phase angle of the instantaneous current i being taken as zero. Negative power values signify powers absorbed into the circuit. Positive values signify powers liberated out of the circuit. Real components signify dissipative powers. Imaginary components signify non-dissipative and transformed or reactive powers. The same conditions apply to the sustained oscillations in alternating-current circuits, except that with $a = 0$, negative real components do not present themselves.

(13) The total instantaneous discharge power in an oscillation system ($\Sigma i_n^2 z_n$) is zero.

(14) The share of oscillating current which a discharging element delivers to any one of a group of oscillation admittances in parallel is proportional to the oscillation admittance of that path, computed according to the rules of complex quantities or plane vectors.

BIBLIOGRAPHY

(1) Professor William Thomson, "On Transient Electric Currents," "*Phil. Mag.*", Series 4, Vol. 5, p. 393, 1853; Also, "Mathematical and Physical Papers." Vol. 1, p. 540.

(2) "Impedance" by A. E. Kennelly. "*Trans. Am. Inst. Elect. Engineers.*" April, 1893. Vol. X., p. 175.

(3) "Complex quantities and their use in Electrical Engineering" by C. P. Steinmetz, "*Proc. International Electrical Congress,*" *Chicago*, August, 1893. pp. 33–76.

(4) "Impedances of Mutually Inductive Circuits," by A. E. Kennelly. "*The Electrician,*" London, Vol. XXXI, October 27th, 1893, pp. 699–700.

(5) "Application of Hyperbolic Analysis to the Discharge of a Condenser." Alex Macfarlane, "*Trans. Am. Inst. Elect. Engineers.*" Vol. 14, 1897, p. 163.

(6) "Vector Diagrams of Oscillating-Current Circuits," by A. E. Kennelly. "*Proc. Am. Acad. of Arts and Sciences.*" Vol. 46, No. 17, p. 375. Jan., 1911.
(7) "Some Oscillograms of Condenser Discharges and a Simple Theory of Coupled Oscillatory Circuits," by J. A. Fleming. "*Proc. Phys. Soc. London,*" p. 217, Apr., 1913.
(8) "The Impedance of Telephone Receivers as Affected by the Motion of their Diaphragms." A. E. Kennelly and G. W. Pierce. "*Proc. Am. Acad. Arts and Sciences.*" December, 1913.

LIST OF SYMBOLS EMPLOYED

A, B	Integration constants of initial electric quantity (coulombs).
a	Projected hyperbolic angular velocity; (hyps. per sec.) or damping constant.
β	Circular angle described by a rotating unit radius vector (radians).
$C = c_1 + c_2 + c_3 +$	Sum of capacitances in parallel (farads).
c	Capacitance of a condenser (farads).
c_1, c_2, c_3	Capacitances of individual condensers (farads).
d	Sign of differentiation.
$E = \gamma_1 + \gamma_2 + \gamma_3 +$	Sum of pure ductances in parallel (yrnehs).
$\gamma = 1/l$	Ductance, or reciprocal of a pure inductance (yrnehs).
$\gamma_1, \gamma_2, \gamma_3$	Ductances of individual pure reactors (yrnehs).
e	Instantaneous emf. of a discharging element (volts $\angle$).
$-e_1, e_2$	Instantaneous primary and secondary induced emfs. (volts $\angle$).
$-e_c$	Instantaneous back emf. of a capacitance (volts $\angle$).
$-e_l$	Instantaneous back emf. of an inductance (volts $\angle$).
$\varepsilon = 2.71828\ldots$	The Napierian base (numeric).
$f = \omega/2\pi$	Oscillation frequency (cycles per second). In drawings, the symbol for a farad.
G	In drawings, the symbol for a ground connection, assumed perfect.

g Conductance of a leak (mhos).

h In drawings, the symbol for a henry.

θ Hyperbolic angle (hyperbolic radian or hyp.).

I Initial current strength (amperes $\angle$).

I_1 Initial primary current strength (amperes $\angle$).

i Instantaneous current strength (amperes $\angle$).

i_1, i_2 Instantaneous primary and secondary currents (amperes $\angle$).

$j = \sqrt{-1}$ Sign of "imaginary" quantity.

$L = l_1 + l_2 + l_3 +$ Sum of individual inductances in series (henrys).

l An inductance (henrys).

l_1, l_2, l_3 Individual inductances (henrys).

μ Mutual inductance between two circuits (henrys).

$n = -(a \pm j\,\omega)$ A generalized complex angular velocity (radians per second $\angle$).

p_c, p_l, p_r Instantaneous powers in a capacitance, inductance, or resistance (watts $\angle$).

$\pi = 3.14159\ldots$ (Numeric).

q Quantity of electricity in a condenser (coulombs).

$R = r_1 + r_2 + r_3 +$ Sum of a number of pure resistances in series (ohms).

r A pure resistance (ohms).

$S = s_1 + s_2 + s_3 +$ The sum of individual elastances in series (darafs).

$s = 1/c$ Elastance of a condenser of capacitance c (darafs).

s_1, s_2, s_3 Elastances of individual condenser (darafs).

Σ Sign of summation.

t Time elapsed from an epoch, or original condition (seconds).

U Initial difference of potential across a discharging condenser (volts).

u Instantaneous difference of potential across a condenser (volts).

$X_l = l\,\omega$ Reactance of an inductance to sustained angular velocity ω (ohms).

$X_c = 1/(c\,\omega)$ Reactance of a capacitance to sustained angular velocity ω (ohms).

x, y Rectangular Cartesian coordinate of a point in a plane (cm).

$y = 1/z$ Admittance of an impedance z (mhos $\angle$).

$y\,h$ In drawings, a symbol for yrnehs.

y_2 Admittance of a secondary oscillation circuit (mhos $\angle$).

y_1, y_2, y_3 Individual admittances in parallel (mhos $\angle$).

z An oscillation impedance (ohms $\angle$).

z_1, z_2, z_3 Individual oscillation impedances (ohms $\angle$).

z_1, z'_1 Oscillation impedance of a primary circuit with the secondary open or removed (ohms $\angle$).

z_2, z'_2 Oscillation impedance of a secondary circuit with the primary open or removed (ohms $\angle$).

z_{12}, z'_{12} Oscillation impedance of a primary circuit with the secondary closed or present (ohms $\angle$).

z_c Oscillation impedance of a capacitance c (ohms $\angle$).

z_l Oscillation impedance of an inductance l (ohms $\angle$).

z_r Oscillation impedance of a resistance r (ohms).

Ω In drawings a symbol for ohms.

$\mho$ In drawings a symbol for mhos.

ω Circular angular velocity (radians per second).

$\angle$ Angle sign appended to a unit, indicating the existence of a complex quantity or plane vector.

SUMMARY: Corresponding to the usual angular velocity (2π times the frequency) of an alternating current is the generalized angular velocity of an oscillating current. The generalized velocity is a complex quantity; the real portion determining the damping constant, the imaginary portion the frequency of the current. The author shows that the oscillation impedances of resistances, inductances and capacities are formed in the same way from generalized angular velocities as from the usual angular velocity. The oscillation impedance of any circuit or system of circuits is found by the usual law of resistances for continuous currents, due regard being paid to the rules of complex quantities. It is then shown that free oscillations of any system of circuits select such angular velocities as to reduce the total oscillation impedance to zero. A number of cases of parallel and series oscillating circuits are treated by this method with much simplicity. The total oscillation admittance at a knot point is shown to be zero, as also is the sum of the instan-

taneous oscillation-impedance drops around a closed loop. The instantaneous discharge power in any oscillation impedance is readily derived and shown to be zero in a pure oscillation system. The problem of coupled circuits is given a preliminary treatment by these methods.