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SYMBOLIC REPRESENTATION OF GENERAL ALTERNATING WAVES AND OF DOUBLE FREQUENCY VECTOR PRODUCTS.

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PART I.

a. Graphically alternating currents and E. M. F.'s are usually represented by vectors. A vector is a quantity having length and direction. The length represents the intensity, the direction the phase of the alternating wave. The vectors generally issue from the center of co-ordinates.

In the topographical method, however, which is more convenient for complex networks, as interlinked polyphase circuits, the alternating wave is represented by the straight line between two points, these points representing the absolute values of potential (with regard to any reference point chosen as co-ordinate center) and their connection the difference of potential, in phase and intensity. Algebraically these vectors are represented by complex quantities. The impedance, admittance, etc., of the circuit is a complex quantity also, in symbolic denotation.

Thus current, E. M. F., impedance and admittance are related by multiplication and division of complex quantities similar as current, E. M. F., resistance and conductance are related by Ohm's law in direct current circuits.

In direct current circuits, power is the product of current into E. M. F. In alternating current circuits, if:

$$E = e^{1} + j\, e^{11}$$

$$I = i^{1} + j\, i^{11}$$

The product:

$$P_0 = E\,I = (e^1\,i^1 - e^{11}\,i^{11}) + j\,(e^{11}\,i^1 + e^1\,i^{11})$$

is not the power, that is multiplication and division which are correct in the inter-relation of current, E. M. F., impedance, do not give a correct result in the inter-relation of E. M. F., current, power. The reason is that E I are vectors of the same frequency, and Z a constant numerical factor which thus does not change the frequency.

The power P, however, is of double frequency compared with E and I, and thus cannot be represented by a vector in the same diagram with E and I.

$P_0 = E\,I$ is a quantity of the same frequency with E and I and thus cannot represent the power.

b. Since the power is a quantity of double frequency of E and I, and thus the phase angle ω in E and I corresponds to a phase angle $2\,\omega$ in the power, it is of interest to investigate the product $E\,I$ formed by doubling the phase angle.

Algebraically it is:

$$P = E\,I = (e^1 + j\,e^{11})\,(i^1 + j\,i^{11}) =$$

$$(e^1\,i^1 + j^2\,e^{11}\,i^{11}) + (j\,e^{11}\,i^{11} + e^1\,j\,i^{11})$$

Since $j^2 = -1$, that is $180°$ rotation for E and I, for the double frequency vector, P, $j^2 = +1$, or $360°$ rotation, and

$$j \times 1 = j$$

$$1 \times j = -j$$

Hence, substituting these values, it is:

$$P = [E\,I] = (e^1\,i^1 + e^{11}\,i^{11}) + j\,(e^{11}\,i^1 - e^1\,i^{11})$$

The symbol $[E\,I]$ here denotes the transfer from the frequency of E and I to the double frequency of P.

The product: $P = [E\,I]$ consists of two components; the real component:

$$P^1 = [E\,I]^1 = (e^1\,i^1 + e^{11}\,i^{11})$$

and the imaginary component:

$$j\,P^j = j\,[E\,I]^j = j\,(e^{11}\,i^1 - e^1\,i^{11})$$

The component:

$$P^1 = [E\,I]^1 = (e^1\,i^1 + e^{11}\,i^{11})$$

is the power of the circuit, $=$ E I cos (E I)

The component:

$$P^j = [E\,I]^j = (e^{11}\,i^1 - e^1\,i^{11})$$

is what may be called the "wattless power," or the powerless or quadrature volt-amperes of the circuit, $=$ E I sin (E I).

The real component will be distinguished by the index 1, the imaginary or wattless component by the index j.

By introducing this symbolism, the power of an alternating circuit can be represented in the same way as in the direct current circuit, as the symbolic product of current and E.M.F.

Just as the symbolic expression of current and E.M.F. as complex quantity does not only give the mere intensity, but also the phase:

$$E = e^1 + j\,e^{11}$$

$$\text{E} = \sqrt{{e^1}^2 + {e^{11}}^2}$$

$$\tan\varphi = \frac{e^{11}}{e^1}$$

so the double frequency vector product $P = [E\,I]$ denotes more than the mere power, by giving with its two components $P^1 = [E\,I]^1$ and $P^j = [E\,I]^j$ the true energy volt-amperes, and the wattless volt-amperes.

If:

$$E = e^1 + j\,e^{11}$$

$$I = i^1 + j\,i^{11}$$

thus:

$$\text{E} = \sqrt{{e^1}^2 + {e^{11}}^2}$$

$$\text{I} = \sqrt{{i^1}^2 + {i^{11}}^2}$$

it is:

$$P^1 = [E\,I]^1 = (e^1\,i^1 + e^{11}\,i^{11})$$

$$P^j = [E\,I]^j = (e^{11}\,i^1 - e^1\,i^{11})$$

and:

$$\begin{aligned} {P^1}^2 + {P^j}^2 &= {e^1}^2\,{i^1}^2 + {e^{11}}^2\,{i^{11}}^2 + {e^{11}}^2\,{i^1}^2 + {e^1}^2\,{i^{11}}^2 \\ &= ({e^1}^2 + {e^{11}}^2)\,({i^1}^2 + {i^{11}}^2) \\ &= (\mathrm{E\,I})^2 \\ &= Q^2 \end{aligned}$$

where $Q =$ total volt-amperes of circuit.

That is:

The true power P^1 and the wattless power P^j are the two rectangular components of the total apparent power Q of the circuit.

Herefrom it follows:

In symbolic representation as double frequency vector products, powers can be combined and resolved by the parallelogram of vectors just as currents and E.M.F.'s in graphical or symbolic representation.

Hereby the graphical methods of treatment of alternating current phenomena are extended to include double frequency quantities, as power, torque, etc.

It is:

$$\frac{P^1}{Q} = p = \cos\omega = \text{power factor.}$$

$$\frac{P^j}{Q} = q = \sin\omega = \text{inductance factor}$$

of the circuit, and the general expression of power is:

$$\begin{aligned} P &= Q\,(p + j\,q) \\ &= Q\,(\cos\omega + j\sin\omega) \end{aligned}$$

(*c.*) The introduction of the double frequency vector product $P = [E\,I]$ brings us outside of the limits of algebra, however, and the commutative principle of algebra: $a \times b = b \times a$, does not apply any more, but it is:

$$[E\,I] \text{ unlike } [I\,E]$$

since:

$$[E\,I] = [E\,I]^1 + j\,[E\,I]^j$$

$$[I\,E] = [I\,E]^1 + j\,[I\,E]^j$$

$$= [E\,I]^1 - j\,[E\,I]^j$$

it is:

$$[E\,I]^1 = [I\,E]^1$$

$$[E\,I]^j = -\,[I\,E]^j$$

that is, the imaginary component reverses its sign by the interchange of factors.

The physical meaning hereof is: if the wattless power $[E\,I]^j$ is lagging with regard to E, it is leading with regard to I.

The wattless component of power is absent, or the total apparent power is true power, if:

$$[E\,I]^j = (e^{11}\,i^1 - e^1\,i^{11}) = 0.$$

that is:

$$\frac{e^{11}}{e^1} = \frac{i^{11}}{i^1}$$

or:

$$\tan\,(E) = \tan\,(I),$$

that is, E and I are in phase or in opposition.

The true power is absent, or the total apparent power wattless, if:

$$[E\,I]^1 = (e^1\,i^1 + e^{11}\,i^{11}) = 0$$

that is:

$$\frac{e^{11}}{e^1} = -\,\frac{i^1}{i^{11}}$$

or:

$$\tan E = -\cot I$$

that is, E and I are in quadrature.

The wattless power is lagging (with regard to E, or leading with regard to I) if:

$$[E\,I]^j > 0$$

and leading if:

$$[E\,I]^j < 0$$

The true power is negative, that is, power returns, if:

$$[E\,I]^1 < 0$$

It is:

$$[E, -I] = [-E\,I] = -[E\,I]$$

$$[-E, -I] = +[E\,I]$$

that is, when representing the power of a circuit or a part of a circuit, current and E.M.F. must be considered in their proper *relative* phases, but their phase relation with the remaining part of the circuit is immaterial.

(*d.*) If:

$$P_1 = [E_1\,I_1], \qquad P_2 = [E_2\,I_2] \ldots P_n = [E_n\,I_n]$$

are the symbolic expressions of the power of the different parts of a circuit or network of circuits, the total power of the whole circuit or network of circuits is:

$$P = P_1 + P_2 + \ldots + P_n$$

and it is:

$$P^1 = P_1^1 + P_2^1 + \ldots + P_n^1$$

$$P^j = P_1^j + P_2^j + \ldots + P_n^j$$

In other words the total power in symbolic expression (true as well as wattless) of a circuit or system is the sum of the powers of its individual components, in symbolic expression.

The first equation is obviously directly a result from the law of conservation of energy.

One result derived herefrom is for instance:

If in a generator supplying power to a system the current is out of phase with the E.M.F. so as to give the wattless power P^j, the current can be brought into phase with the generator E.M.F., or the load on the generator made non-inductive by inserting anywhere in the circuit an apparatus producing the wattless power $-P^j$, that is, compensation for wattless currents in a system takes place regardless of the location of the compensating device.

Obviously between the compensating device and the source of wattless currents to be compensated for, wattless currents will flow, and for this reason it may be advisable to bring the compensator as near as possible to the circuit to be compensated.

(*e.*) Like power, torque is a double frequency vector product also, of magnetism and M.M.F. or current, and thus can be treated in the same way.

In an induction motor, the torque is the product of the magnetic flux in one direction into the component of secondary induced current in phase with the magnetic flux in time, but in quadrature position therewith in space, times the number of turns of this current, or since the induced E.M.F. is in quadrature and proportional to the magnetic flux and the number of turns, the torque of the induction motor is the product of the induced E.M.F. into the component of secondary current in quadrature therewith in time and space, or the product of the induced current into the component of induced E.M.F. in quadrature therewith in time and space.

Thus, if:

$E = e^1 + j\, e^{11}$ = induced E.M.F. in one direction in space.

$I = i^1 + j\, i^{11}$ = secondary current in the quadrature direction in space,

the torque is:

$$T = [E\, I]^j = e^{11}\, i^1 - e^1\, i^{11}.$$

By this equation the torque is given in watts, the meaning being that $T = [E\, I]^j$ is the power which would be exerted by the torque at synchronous speed, or the torque in synchronous watts.

The torque proper is then:

$$\frac{T}{2\, \pi\, N\, p}$$

where:

p = number of pairs of poles of the motor.

Numerous instances of the the application of this are given in my previous paper on the single-phase induction motor.

As a further instance, we may consider the case of two polyphase induction motors in concatenation; that is, two equal induction motors in which the secondary of the first motor is closed by the primary of the second motor, the motors being mechanically connected so as to run at the same speed.

In this case, let:

$N =$ frequency of main circuit,

$s =$ slip of the first motor from synchronism.

the frequency induced in the secondary of the first motor and thus impressed upon the primary of the second motor is, $s\,N$.

The speed of the first motor is $(1 - s)\,N$, thus the slip of the second motor, or the frequency induced in its secondary is

$$s\,N - (1 - s)\,N = (2\,s - 1)\,N.$$

Let:

$e =$ counter E.M.F. induced in the secondary of the second motor, reduced to full frequency.

$Z_0 = r_0 - j\,x_0 =$ primary self-inductive impedance.

$Z_1 = r_1 - j\,x_1 =$ secondary self-inductive impedance.

$Y = g + j\,b =$ primary exciting admittance of each motor all reduced to full frequency and to the primary by the ratio of turns.

It is then:

Second motor:

secondary induced E.M.F.:

$$e\,(2\,s - 1)$$

secondary current:

$$I_1 = \frac{e\,(2\,s - 1)}{r_1 - j\,(2\,s - 1)\,x_1} = e\,(a_1 + j\,a_2)$$

where:

$$a_1 = \frac{(2\,s - 1)\,r_1}{r_1^2 + (2\,s - 1)^2\,x_1^2} \qquad a_2 = \frac{(2\,s - 1)^2\,x_1}{r_1^2 + (2\,s - 1)^2\,x_1^2}$$

primary exciting current:

$$I_0 = e\,(g + j\,b)$$

thus, total primary current:

$$I_2 = I_1 + I_0 = e\,(b_1 + j\,b_2)$$

where:

$$b_1 = a_1 + g \qquad\qquad b_2 = a_2 + b$$

primary induced E.M.F.:

$$s\,e$$

primary impedance voltage:

$$I_2\,(r_0 - j\,s\,x_0)$$

thus, primary impressed E.M.F.:

$$E_2 = s\,e + I_2\,(r_0 - j\,s\,x_0) = e\,(c_1 + j\,c_2)$$

where:

$$c_1 = s + r_0\,b_1 + s\,x_0\,b_2 \qquad\qquad c_2 = r_0\,b_2 - s\,x_0\,b_1$$

First motor:
secondary current:

$$I_2 = e\,(b_1 + j\,b_2)$$

secondary induced E.M.F.:

$$E_3 = E_2 + I_2\,(r_1 - j\,s\,x_1) = e\,(d_1 + j\,d_2)$$

where:

$$d_1 = c_1 + r_1\,b_1 + s\,x_1\,b_2 \qquad\qquad d_2 = c_2 + r_1\,b_2 - s\,x_1\,b_1$$

primary induced E.M.F.:

$$E_4 = \frac{E_3}{s} = e\,(f_1 + j\,f_2)$$

where:

$$f_1 = \frac{d_1}{s} \qquad\qquad f_2 = \frac{d_2}{s}\ *$$

primary exciting current:

$$I_4 = E_4\,(g + j\,b)$$

total primary current:

$$I = I_2 + I_4 = e\,(g_1 + j\,g_2)$$

* At $s = 0$ these terms f_1 and f_2 become indefinite, and thus at and very near synchronism have to be derived by substituting the complete expressions for f_1 and f_2.

where:

$$g_1 = b_1 + g f_1 - b f_2 \qquad g_2 = b_2 + g f_2 + b f_1$$

primary impedance voltage:

$$I (r_0 - j x_0)$$

thus, primary impressed E.M.F.:

$$E_0 = E_4 + I (r_0 - j x_0) = e (h_1 + j h_2)$$

where:

$$h_1 = f_1 + r_0 g_1 + x_0 g_2 \qquad h_2 = f_2 + r_0 g_2 - x_0 g_1$$

or, absolute:

$$e_0 = e \sqrt{h_1^2 + h_2^2}$$

and:

$$e = \frac{e_0}{\sqrt{h_1^2 + h_2^2}}$$

Substituting now this value of e in the preceding gives the values of the currents and E.M.F.'s in the different circuits of the motor series.

In the polyphase induction motor, the induced E.M.F. in quadrature in space to the induced E.M.F., E is:

$$j E.$$

Thus, in the second motor, the torque is:

$$T_2 = [j e I_1]^j = [e I_1]^1 = e^2 a_1$$

hence, its power output:

$$P_3 = (1 - s) T_2 = (1 - s) e^2 a_1$$

The power input is:

$$P_2 = [E_2 I_2] = [E_2 I_2]^1 + j [E_2 I_2]^j$$

$$= e^2 [(c_1 + j c_2) (b_1 + j b_2)]$$

hence, the efficiency:

$$\frac{P_3}{P_2^1} = \frac{(1 - s) e^2 a_1}{[E_2 I_2]^1} = \frac{(1 - s) a_1}{c_1 b_1 + c_2 b_2}$$

the power factor:

$$\frac{P_2^1}{Q} = \frac{[E_2\, I_2]^1}{E_2\, I_2} = \frac{c_1\, b_1 + c_2\, b_2}{\sqrt{(c_1^2 + c_2^2)\ (b_1^2 + b_2^2)}}$$

etc.

In the first motor:

the torque is:

$$T_1 = [E_4\, I_2]^1 = e^2\, [(f_1 + j\, f_2)\ (b_1 + j\, b_2)]^1$$
$$= e^2\, (f_1\, b_1 + f_2\, b_2)$$

the power output:

$$P_4 = T_1\, (1 - s)$$
$$= e^2\, (1 - s)\ (f_1\, b_1 + f_2\, b_2)$$

the power input:

$$P_1 = [E_0\, I] = e^2\ [(h_1 + j\, h_2)\, (g_1 + j\, g_2)]$$
$$= [E_0\, I]^1 + j\, [E_0\, I]^j$$

Thus, the efficiency:

$$\frac{P_4}{[E_0\, I]^1 - [E_2\ I_2]^1} = \frac{(1 - s)\, (f_1\, b_1 + f_2\, b_2)}{(h_1\, g_1 + h_2\, g_2) - (c_1\, b_1 + c_2\, b_2)}$$

the power factor of the whole system:

$$\frac{P_1}{E_0\, I} = \frac{h_1\, g_1 + b_2\, g_2}{\sqrt{(h_1^2 + h_2^2)\ (g_1^2 + g_2^2)}}$$

the power factor of the first motor:

$$\frac{P_1 - P_2}{E_0\, I - E_2\, I_2} = \frac{(h_1\, g_1 + h_2\, g_2) - (c_1\, b_1 + c_2\, b_2)}{\sqrt{(h_1^2 + h_2^2)\, (g_1^2 + g_2^2)} - \sqrt{c_1^2 + c_2^2)\, (b_1^2 + b_2^2)}}$$

the total efficiency of the system:

$$\frac{P_1 + P_3}{[E_0\ I]^1} = \frac{(1 - s)\ (f_1 + b_1 + f_2\, b_2 + a_1)}{h_2\, g_1 + h_2\, g_2}$$

etc.

As instance are given in Fig. 1. the curves of total torque, of torque of the second motor, and of current, for the range of slip from $s = +1.5$ to $s = -.7$ for a pair of induction motors in concatenation, of the constants:

$$Z_0 = Z_1 = .1 - .3\,j$$

$$Y = .01 + .1\,j.$$

FIG. 2 gives the curves of total torque and of current, from test of a motor of similar constants, for the range from

$$s = +1 \text{ to } s = 0.$$

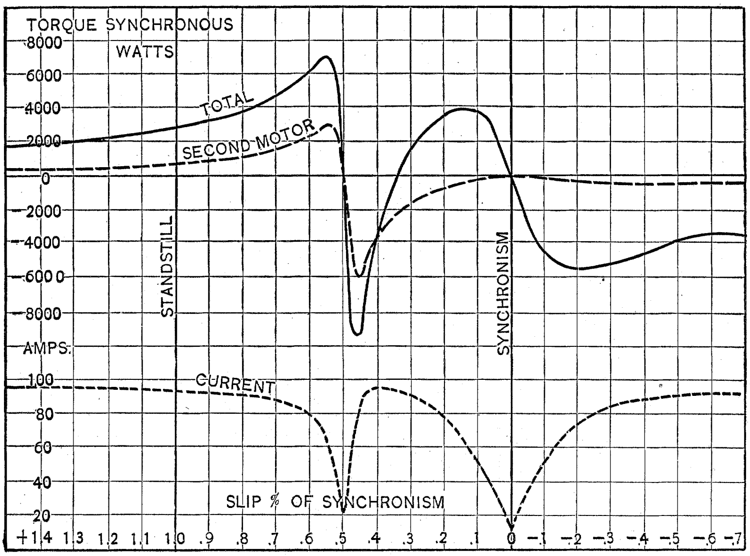

FIG. 1. Concatenation of Induction Motors. Speed Curves.

$Z = .1 - .3\,j \quad Y = .01 + .1\,j$

As seen, there are two ranges of positive torque for the whole system, one below half synchronism, and one from about $\frac{2}{3}$ to full synchronism, and two ranges of negative torque, or generator action of the motor, from half to two-third synchronism, and above full synchronism.

With higher resistance in the secondary of the second motor, the second range of positive torque of the system disappears more or less and the torque curves become as shown in Fig. 3.

PART II. *

(*a.*) The vector representation:

$$A = a^1 + j\, a^{11} = a\,(\cos \alpha + j \sin \alpha)$$

of the alternating wave:

$$A = a_0 \cos\,(\varphi - \alpha)$$

applies to the sine wave only.

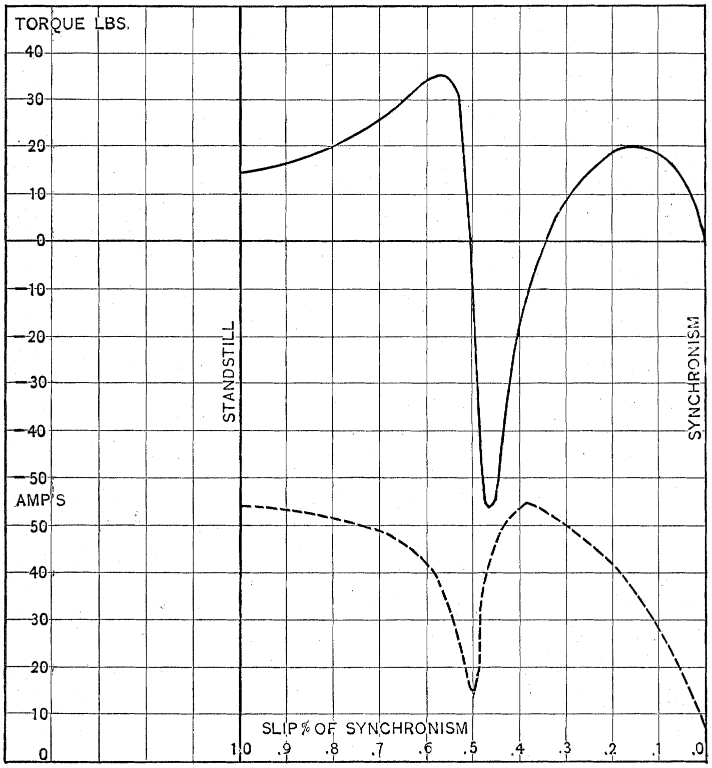

FIG. 2. Concatenation of Induction Motor. Speed Curves of two $I - 6 - 5 - 1200 - 110$.

The general alternating wave, however, contains an infinite series of terms, of odd frequencies:

$$A = A_1 \cos\,(\varphi - \alpha_1) + A_3 \cos\,(3\,\varphi - \alpha_3) + A_5 \cos\,(5\,\varphi - \alpha_5) + .$$

thus can not directly be represented by one complex imaginary vector quantity.

The replacement of the general wave by its equivalent sine wave, that is a sine wave of equal effective intensity and equal power, while sufficiently accurate in many cases, completely fails in other cases, especially in circuits containing capacity, or in circuits containing periodically (and in synchronism with the wave) varying resistance or reactance (as alternating arcs, reaction machines, synchronous induction motors, oversaturated magnetic circuits etc.)

Since however, the individual harmonics of the general alter-

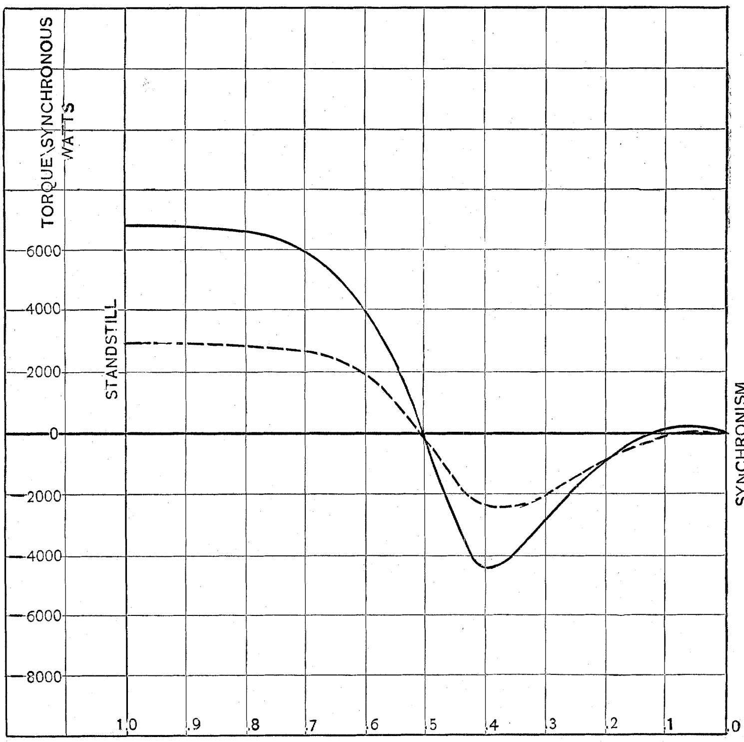

FIG. 3. Concatenation of Induction Motors Speed Curves.
$Z = .1 - .3j \quad Y = .01 + .1j$
Res. in Secondary of Second Motor.

nating wave are independent of each other, that is, all products of different harmonics vanish, each term can be represented by a complex symbol, and the equations of the general wave then are the resultants of those of the individual harmonics.

This can be represented symbolically, by combining in one formula symbolic representations of different frequencies, thus:

$$A = \sum_{1}^{\infty}{}_{2n-1} \left(a_n^{1} + j_n\, a_n^{11}\right)$$

where :

$$j_n = \sqrt{-1}$$

and the index of the j_n merely denotes that the j's of different indices n, while algebraically identical, physically represent different frequencies, and thus can not be combined.

The general wave of E.M.F. is thus represented by :

$$E = \sum_{1}^{\infty}{}_{2n-1} \left(e_n^{1} + j_n\, e_n^{11}\right)$$

the general wave of current by :

$$I = \sum_{1}^{\infty}{}_{2n-1} \left(i_n^{1} + j_n\, i_n^{11}\right)$$

If :

$$Z_1 = r - j\,(x_m + x_0 + x_c)$$

is the impedance of the fundamental harmonic, where :

x_m is that part of the reactance which is proportional to the frequency (inductance, etc.)

x_0 is that part of the reactance which is independent of the frequency (mutual induction, synchronous motion, etc.)

x_c is that part of the reactance which is inverse proportional to the frequency (capacity, etc).

the impedance for the nth harmonic is :

$$Z = r - j_n \left(n\, x_m + x_0 + \frac{x_c}{n}\right)$$

This term can be considered as the general symbolic expression of the impedance of a circuit of general wave shape.

Ohm's law, in symbolic expression, assumes for the general alternating wave the form :

$I = \dfrac{E}{Z}$ or :

$$\sum_{1}^{\infty}{}_{2n-1} \left(i_n^{1} + j_n i_n^{11}\right) = \sum_{1}^{\infty}{}_{2n-1} \frac{e_n^{1} + j_n e_n^{11}}{r - j_n\left(n x_m + x_0 + \dfrac{x_c}{n}\right)}$$

$E = I\,Z$ or:

$$\sum_1^{\infty}{}_{2n-1}\,(e_n^1 + j_n\,e_n^{11}) = \sum_1^{\infty}{}_{2n-1}\left[r - j_n\left(n x_m + x_0 + \frac{x_c}{n}\right)\right](i_n^1 + j_n i_n^{11})$$

$$Z = \frac{E}{I} \text{ or:}$$

$$Z = r - j_n\left(n\,x_m + x_0 + \frac{x_c}{n}\right) = \frac{e_n + j_n\,e_n^{11}}{i_n^1 + j_n\,i_n^{11}}$$

The symbols of multiplication and division of the terms E, I, Z, thus represent not algebraic operation, but multiplication and division of corresponding terms of E, I, Z, that is terms of the same index n, or, in algebraic multiplication and division of the series E, I, all compound terms, that is terms containing two different n, vanish.

(*b*.) The effective value of the general wave:

$$a = A_1 \cos(\varphi - a_1) + A_3 \cos(3\,\varphi - a_3) + A_5 \cos(5\,\varphi - a_5) + \ldots$$

is the square root of the sum of mean squares of individual harmonics:

$$\mathrm{A} = \sqrt{\tfrac{1}{2}\left\{A_1^2 + A_3^2 + A_5^2 + \ldots\right\}}$$

since, as discussed above, the compound terms, of two different indices n, vanish, the absolute value of the general alternating wave:

$$A = \sum_1^{\infty}{}_{2n-1}\,\frac{a_n^1 + j_n\,a_n^{11}}{b_n^1 + j_n\,b_n^{11}}$$

is thus:

$$\mathrm{A} = \sqrt{\sum_1^{\infty}{}_{2n-1}\,\frac{{a_n^1}^2 + {a_n^{11}}^2}{{b_n^1}^2 + {b_n^{11}}^2}}$$

which offers an easy means of reduction from symbolic to absolute values.

Thus, the absolute value of the E.M.F.

$$E = \sum_1^\infty {}_{2n-1} (e_n^1 + j_n e_n^{11})$$

is:

$$\mathrm{E} = \sqrt{\sum_1^\infty {}_{2n-1} ({e_n^1}^2 + {e_n^{11}}^2)}$$

the absolute value of the current:

$$I = \sum_1^\infty {}_{2n-1} (i_n^1 + j_n i_n^{11})$$

is:

$$\mathrm{I} = \sqrt{\sum_1^\infty {}_{2n-1} ({i_n^1}^2 + {i_n^{11}}^2)}$$

(*c*.) The double frequency power (torque, etc.) equation of the general alternating wave has the same symbolic expression as with the sine wave:

$$\begin{aligned} P &= [E I] \\ &= P^1 + j P^j \\ &= [E I]^1 + j [E I]^j \\ &= \sum_1^\infty {}_{2n-1} (e_n^1 i_n^1 + e_n^{11} i_n^{11}) + \sum_1^\infty {}_{2n-1} j_n (e_n^{11} i_n^1 - e_n i_n^{11}) \end{aligned}$$

where:

$$P^1 = [E I]^1 = \sum_1^\infty {}_{2n-1} (e_n^1 i_n^1 + e_n^{11} i_n^{11})$$

$$P^j = [E I]^j = \sum_1^\infty {}_{2n-1} \frac{j_n}{j} (e_n^{11} i_n^1 - e_n^1 i_n^{11})$$

The j_n enters under the summation sign of the "wattless power P^j, so that the wattless powers of the different harmonics can not be algebraically added.

Thus:

The total "true power" of a general alternating current circuit is the algebraic sum of the powers of the individual harmonics.

The total "wattless power" of a general alternating current circuit is not the algebraic, but the absolute sum of the wattless powers of the individual harmonics.

Thus, regarding the wattless power as a whole, in the general alternating circuit no distinction can be made between lead and lag, since some harmonics may be leading, others lagging.

The apparent power, or total volt-amperes, of the circuit is:

$$Q = \mathrm{E\,I} = \sqrt{\sum_1^\infty {}_{2n-1} \left({e_n^1}^2 + {e_n^{11}}^2\right) \sum_1^\infty {}_{2n-1} \left({i_n^1}^2 + {i_n^{11}}^2\right)}$$

The power factor of the circuit is:

$$p = \frac{P^1}{Q} = \frac{\sum_1^\infty {}_{2n-1} \left(e_n^1 \dot{i}_n^1 + e_n^{11} \dot{i}_n^{11}\right)}{\sqrt{\sum_1^\infty {}_{2n-1} \left({e_n^1}^2 + {e_n^{11}}^2\right) \sum_1^\infty {}_{2n-1} \left({i_n^1}^2 + {i_n^{11}}^2\right)}}$$

The term "inductance factor" however, has no meaning any more, since the wattless powers of the different harmonics are not directly comparable.

The quantity:

$$q_0 = \sqrt{1 - p^2}$$

has no physical significance, and is not $\frac{\text{wattless power}}{\text{total apparent power}}$

The term:

$$\frac{P^j}{\mathrm{E\,I}}$$

$$= \sum_1^\infty {}_{2n-1} \frac{e_n^{11} \dot{i}_n^1 - e_n^1 \dot{i}_n^{11}}{\mathrm{E\,I}}$$

$$= \sum_1^\infty {}_{2n-1} q_n$$

where:

$$q_n = \frac{e_n^{11} \dot{i}_n^1 - e_n^1 \dot{i}_n^{11}}{\mathrm{E\,I}}$$

consists of a series of inductance factors q_n of the individual harmonics.

As a rule,

$$p^2 + q^2 < 1$$

for the general alternating wave, that is q differs from

$$q_0 = \sqrt{1 - p^2}$$

The complex quantity:

$$U = \frac{P}{Q} = \frac{[E\,I]}{\mathrm{E\,I}} = \frac{[E\,I]^1 + j\,[E\,I]^j}{\mathrm{E\,I}}$$

$$= \frac{\sum_1^\infty{}_{2n-1}\,(e_n^{\,1}\,i_n^{\,1} + e_n^{\,11}\,i_n^{\,11}) + \sum_1^\infty{}_{2n-1}\,j_n\,(e_n^{\,11}\,i_n^{\,1} - e_n^{\,1}\,i_n^{\,11})}{\sqrt{\sum_1^\infty{}_{2n-1}\,(e_n^{\,1\,2} + e_n^{\,11\,2})\,\sum_1^\infty{}_{2n-1}\,(i_n^{\,1\,2} + i_n^{\,11\,2})}}$$

$$= p + \sum_1^\infty{}_{2n-1}\,j_n\,q_n$$

takes in the circuit of the general alternating wave the same position as power factor and inductance factor with the sine wave.

$$U = \frac{P}{Q} \text{ may be called the 'circuit factor."}$$

It consists of a real term p, the power factor, and a series of imaginary terms $j_n\,q_n$, the inductance factors of the individual harmonics.

The absolute value of the circuit factor:

$$u = \sqrt{p^2 + \left(\sum_1^\infty{}_{2n-1}\,q_n\right)^2}$$

as a rule, is < 1.

Some applications of this symbolism will explain its mechanism and its usefulness more fully.

1*st Instance:* Let the E.M.F.:

$$E = \sum_1^5{}_{2n-1}\,(e_n^{\,1} + j_n\,e_n^{\,11})$$

be impressed upon a circuit of the impedance:

$$Z = r - j_n \left(n\, x_m - \frac{x_c}{n} \right)$$

that is, containing resistance r, inductive reactance x_m and capacity reactance x_c in series.

Let:

$$e_1^{1} = 720 \qquad e_1^{11} = 540$$

$$e_3^{1} = 283 \qquad e_3^{11} = -283$$

$$e_5^{1} = -104 \qquad e_5^{11} = 138$$

or:

$$e_1 = 900 \qquad \tan \omega_1 = .75$$

$$e_3 = 400 \qquad \tan \omega_3 = -1$$

$$e_5 = 173 \qquad \tan \omega_5 = -1.33$$

It is thus in symbolic expression:

$$Z_1 = 10 + 80\, j_1 \qquad z_1 = 80.6$$

$$Z_3 = 10 \qquad z_3 = 10$$

$$Z_5 = 10 - 32\, j_5 \qquad z_5 = 33.5$$

and, E.M.F.:

$$E = (720 + 540\, j_1) + (283 - 283\, j_3) + (-104 + 138\, j_5)$$

or absolute:

$$E = 1000$$

and current:

$$I = \frac{E}{Z} = \frac{720 + 540\, j_1}{10 + 80\, j_1} + \frac{283 - 283\, j_3}{10} + \frac{-104 + 138\, j_5}{10 - 32\, j_5}$$

$$= (7.76 - 8.04\, j_1) + (28.3 - 28.3\, j_3) + (-4.86 - 1.73\, j_5)$$

or, absolute:

$$I = 41.85$$

of which is of fundamental frequency: $I_1 = 11.15$

" " " " triple " $I_3 = 40$

" " " " quintuple " $I_5 = 5.17$

The total apparent power of the circuit is:

$$Q = \mathrm{E\,I} = 41{,}850$$

The true power of the circuit is:

$$P^1 = [E\,I]^1 = 1240 + 16{,}000 + 270$$

$$= 17{,}510$$

the wattless power:

$$j\,P^j = j\,[E\,I]^j = 10{,}000\,j_1 - 850\,j_5$$

thus, the total power:

$$P = 17{,}510 + 10{,}000\,j_1 - 850\,j_5$$

That is, the wattless power of the first harmonic is leading that of the third harmonic zero, and that of the fifth harmonic lagging.

$$17{,}510 = \mathrm{I}^2\,r, \text{ as obvious.}$$

The circuit factor is:

$$U = \frac{P}{Q} = \frac{[E\,I]}{\mathrm{E\,I}}$$

$$= .418 + .239\,j_1 - .0203\,j_5$$

or, absolute:

$$u = \sqrt{.418^2 + .2593^2}$$

$$= .492$$

The power factor is:

$$p = .418$$

The inductance factor of the first harmonic is: $q_1 = .239$, that of the third harmonic $q_3 = 0$, and of the fifth harmonic $q_5 = -.0203$.

Considering the waves as replaced by their equivalent sine waves, from the sine wave formula:

$$p^2 + q^2 = 1$$

the inductance factor would be:

$$q_0 = .914$$

and the phase angle:

$$\tan \omega = \frac{q_0}{p} = \frac{.914}{.418} = 2.8, \qquad \omega = 65.4^0$$

giving apparently a very great phase displacement, while in reality, of the 41.85 amperes total current, 40 amperes (the current of the third harmonic) are in phase with their E.M.F.

We thus have here a case of a circuit with complex harmonic waves which can not be represented by their equivalent sine waves. The relative magnitudes of the different harmonics in the wave of current and of E.M.F. differ essentially, and the circuit has simultaneously a very low power factor and a very low inductance factor, that is a low power factor exists without corresponding phase displacement, the circuit factor being less than one-half.

Such circuits for instance are those including alternating arcs, reaction machines, synchronous induction motors, reactances with over-saturated magnetic circuit, high potential lines in which the maximum difference of potential exceeds the voltage at which brush discharges begin, etc. Such circuits can not correctly, and in many cases not even approximately, be treated by the theory of the equivalent sine waves, but require the symbolism of the complex harmonic wave.

2nd Instance: A condenser of capacity $C_0 = 20$ M.F. is connected into the circuit of a 60-cycle alternator giving a wave of the form:

$$e = E(\cos \varphi - .10 \cos 3\varphi - .08 \cos 5\varphi + .06 \cos 7\varphi)$$

or, in symbolic expression:

$$E = e\,(1_1 - .10_3 - .08_5 + .06_7)$$

The synchronous impedance of the alternator is:

$$Z_0 = r_0 - j_n\, n\, x_0 = .3 - 5\, n\, j_n$$

What is the apparent capacity C of the condenser (as calculated from its terminal volts and amperes), when connected directly with the alternator terminals, and when connected thereto through various amounts of resistance and inductive reactance.

The capacity reactance of the condenser is:

$$x_c = \frac{10^6}{2\,\pi\,N\,C_0} = 132 \text{ ohms.}$$

or, in symbolic expression:

$$+j_n\,\frac{x_c}{n} = \frac{132}{n}\,j_n$$

Let:

$Z_1 = r - j_n\,n\,x$ = impedance inserted in series with the condenser.

The total impedance of the circuit is then:

$$Z = Z_0 + Z_1 + j_n\,\frac{x_c}{n}$$

$$= (.3 + r) - j_n\left([5 + x]\,n - \frac{132}{n}\right)$$

The current in the circuit is:

$$I = \frac{E}{Z}$$

$$= e\left[\frac{1}{(.3+r)-j(x-132)} - \frac{.1}{(.3+r)-j_3(3x-29)}\right.$$

$$\left. - \frac{.08}{(.3+r)-j_5(5x-1.4)} + \frac{.06}{(.3+r)-j_7(7x+16.1)}\right]$$

and the E.M.F. at the condenser terminals:

$$E_1 = j_n\,\frac{x_c\,I}{n}$$

$$= e\left[\frac{132\,j_1}{(.3+r)-j_1(x-132)} - \frac{4.4\,j_3}{(.3+r)-j_3(3x-29)}\right.$$

$$\left. - \frac{2.11\,j_5}{(.3+r)-j_5(5x-1.4)} + \frac{1.13\,j_7}{(.3+r)-j_7(7x+16.1)}\right]$$

thus the apparent capacity reactance of the condensers:

$$x_1 = \frac{\mathrm{E}_1}{\mathrm{I}}$$

and the apparent capacity:

$$C = \frac{10^6}{2 \pi N x_1}$$

(*a.*) $x = 0$: Resistance r in series with the condenser. Reduced to absolute values, it is:

$$\frac{1}{x_1^2} = \frac{\frac{1}{(.3+r)^2+17424}+\frac{.01}{(.3+r)^2+841}+\frac{.0064}{(.3+r)^2+1.96}+\frac{.0036}{(.3+r)^2+259}}{\frac{17424}{(.3+r)^2+17424}+\frac{19.4}{(.3+r)^2+841}+\frac{4.45}{(.3+r)^2+1.96}+\frac{1.28}{(.3+r)^2+259}}$$

(*b.*) $r = 0$: Inductive reactance x in series with the condenser. Reduced to absolute values, it is:

$$\frac{1}{x_1^2} = \frac{\frac{1}{.09+(x-132)^2}+\frac{.01}{.09+(3x-29)^2}+\frac{.0064}{.09+(5x-1.4)^2}+\frac{.0036}{.09+(7x+16.1)^2}}{\frac{17424}{.09+(x-132)^2}+\frac{19.4}{.09+(3x-29)^2}+\frac{4.45}{.09+(5x-1.4)^2}+\frac{1.28}{.09+(7x+16.1)^2}}$$

From $\frac{1}{x_1^2}$ are derived the values of apparent capacity:

$$C = \frac{10^6}{2 \pi N x_1}$$

and plotted in Fig. 4 for values of r and x respectively varying from 0 to 22 ohms.

As seen, with neither additional resistance nor reactance in series to the condenser, the apparent capacity with this generator wave is 84 M.F., or 4.2 times the true capacity, and gradually decreases with increasing series resistance, to $C = 27.5$ M.F. $=$ 1.375 times the true capacity at $r = 13.2$ ohms or $\frac{1}{10}$ the true capacity reactance, with $r = 132$ ohms or with an additional resistance equal to the capacity reactance, $C = 20.5$ M. F. or only 2.5% in excess of the true capacity C_0, and at $r = \infty$, $C = 20.3$ M. F. or 1.5% in excess of the true capacity.

With reactance x but no additional resistance r in series, the apparent capacity C rises from 4.2 times the true capacity at $x = 0$, to a maximum of 5.03 times the true capacity or $C =$

100.6 M. F. at $x = .28$, the condition of resonance of the fifth harmonic, then decreases to a minimum of 27 M. F. or 35% in excess of the true capacity, rises again to 60.2 M. F. or 3.01 times the true capacity at $x = 9.67$, the condition of resonance with the third harmonic, and finally decreases, reaching 20 M. F. or the true capacity at $x = 132$ or an inductive reactance equal to the capacity reactance, then increases again to 20.2 M.F. at $x = \infty$.

This rise and fall of the apparent capacity is within certain limits independent of the magnitude of the higher harmonics of the generator wave of E. M. F., but merely depends upon their presence, and it thus follows that the true capacity of a condenser cannot even approximately be determined by measuring volts and amperes if there are any higher harmonics present in the

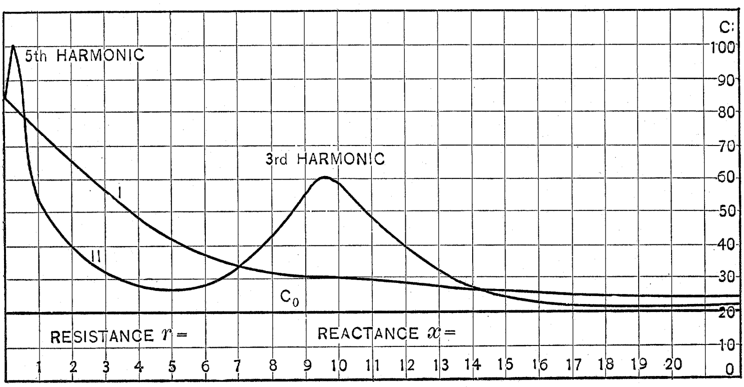

FIG. 4. Capacity $C_0 = 20\ mf$ in circuit of Generator $\mathfrak{E} = E\,(1 - .1 - .08 + .06)$ of impedance $Z_0 = .3 - 5\,j_n\,n$ with resistance $r\,(I)$ or reactance $x\,(II)$ in series.

generator wave, except by inserting a very large resistance or reactance in series to the condenser.

3rd Instance: An alternating current generator of the wave:

$$E_0 = 2000\,[1_1 + .12_3 - .23_5 - .13_7]$$

and of synchronous impedance:

$$Z_0 = .3 - 5\,n\,j_n$$

feeds over a line of impedance:

$$Z_1 = 2 - 4\,n\,j_n$$

a synchronous motor of the wave:

$$E_1 = 2250\,[(\cos\omega + j_1 \sin\omega) + .24\,(\cos 3\,\omega + j_3 \sin 3\,\omega)]$$

and of synchronous impedance:

$$Z_2 = .3 - 6\,n\,j_n$$

The total impedance of the system is then:

$$Z = Z_0 + Z_1 + Z_2$$
$$= 2.6 - 15\,n\,j_n$$

thus the current:

$$I = \frac{E_0 - E}{Z}$$

$$= \frac{2000 - 2250\cos\omega - 2250 j_1 \sin\omega}{2.6 - 15\,j_1} + \frac{240 - 540\cos 3\omega - 540 j_3 \sin 3\omega}{2.6 - 45\,j_3}$$

$$- \frac{460}{2.6 - 75\,j_5} - \frac{260}{2.6 - 105\,j_7}$$

$$= (a_1^1 + j_1 a_1^{11}) + (a_3^1 + j_3 a_3^{11}) + (a_5^1 + j_5 a_5^{11}) + (a_7^1 + j_7 a_7^{11})$$

where:

$a_1^1 = 22.5 - 25.2\cos\omega + 146\sin\omega$ $\quad a_1^{11} = 130 - 146\cos\omega - 25.2\sin\omega$

$a_3^1 = .306 - .69\cos 3\omega + 11.9\sin 3\omega$ $\quad a_3^{11} = 5.3 - 11.9\cos 3\omega - .69\sin 3\omega$

$a_5^1 = -.213$ $\quad a_5^{11} = -6.12$

$a_7^1 = -.061$ $\quad a_7^{11} = -2.48$

or, absolute:

1st. harmonic:

$$a_1 = \sqrt{{a_1^1}^2 + {a_1^{11}}^2}$$

3rd. harmonic:

$$a_3 = \sqrt{{a_3^1}^2 + {a_3^{11}}^2}$$

5th. harmonic:

$$a_5 = 6.12$$

7th. harmonic:

$$a_7 = 2.48$$

$$\mathrm{I} = \sqrt{a_1^2 + a_3^2 + a_5^2 + a_7^2}$$

while the total current of higher harmonics is:

$$I_0 = \sqrt{a_3^2 + a_5^2 + a_7^2}$$

The true input of the synchronous motor is:

$$P_1 = [E_1 I]^1$$

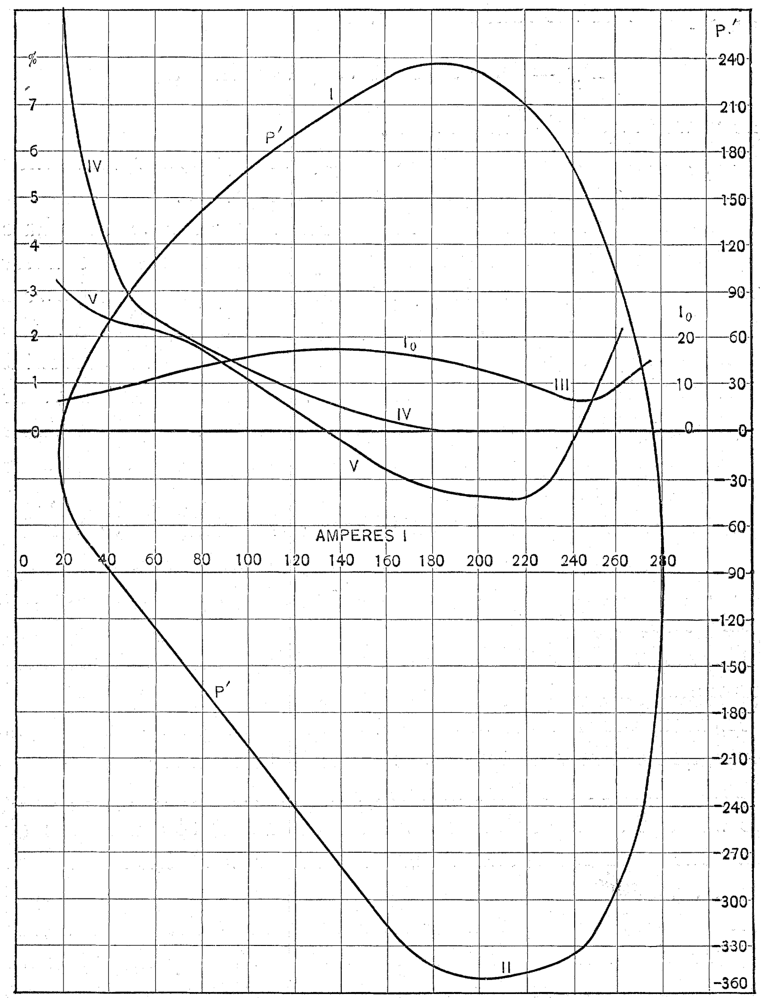

FIG. 5. Synchronous Motor.

$\mathfrak{E}_1 = (\cos.\ \omega + j_1 \sin.\ \omega) + .24\ (\cos.\ 3\ \omega + j_3 \sin 3\ \omega.)$

Operated from Generator.

$\mathfrak{E}_0 = 2000\ (1 + .12 - .23 - .13.)$

Over total impedance.

$Z_n = 2.6 - 15\ i_n\ n.$

$$= (2250\ a_1^1 \cos \omega + 2250 a_1^{11} \sin \omega) + (540 a_1^1 \cos 3\omega + 540 a_3^{11} \sin 3\omega)$$

$$= P_1^1 + P_3^1$$

$$P_1^1 = 2250\ (a_1^1 \cos \omega + a_1^{11} \sin \omega)$$

is the power of the fundamental wave,

$$P_3^1 = 540\ (a_3^1 \cos 3\ \omega + a_3^{11} \sin 3\ \omega)$$

the power of the third harmonic.

The 5th and 7th harmonics do not give any power, since they are not contained in the synchronous motor wave.

Substituting now different numerical values for ω the phase angle between generator E.M.F. and synchronous motor counter E.M.F., corresponding values of the currents I I_0, and the powers P_1, P_1^1, P_3^1 are derived. These are plotted in Fig. 5 with the total current I as abcissae. To each value of the total current I correspond two values of the total power P^1, a positive value plotted as Curve I —synchronous motor—and a negative value plotted as Curve II — alternating current generator—. Curve III gives the total current of higher frequency I_0, Curve IV, the difference between the total current and the current of fundamental frequency, I — a_1, in percentage of the total current I, and *V* the power of the third harmonic, P_3^1, in percentage of the total power P^1.

Curves III, IV and V correspond to the positive or synchronous motor part of the power curve P^1.

As seen, the increase of current due to the higher harmonics is small, and entirely disappears at about 180 amperes.

The power of the third harmonic is positive, that is, adds to the work of the synchronous motor up to about 140 amperes or near the maximum output of the motor and then becomes negative.

It follows herefrom that higher harmonics in the E.M.F. waves of generators and synchronous motors do not represent a mere waste of current, but may contribute more or less to the output of the motor. Thus at 75 amperes total current, the percentage of increase of power due to the higher harmonic is equal to the increase of current, or in other words the higher harmonics of current do work with the same efficiency as the fundamental wave.