

## STEP Support Programme

### Hints and Partial Solutions for Assignment 15

#### Warm-up

- 1 (i) It is best to start simplifying from the bottom. You should end up with  $x$ .
- (ii) The simplest way to show that  $x = 15$  is a root is to set  $x = 15$  in the equation and check that this gives 0. Factorising out some common factors helps with the arithmetic:

$$\begin{aligned}
 & 15^4 - 18 \times 15^3 + 35 \times 15^2 + 180 \times 15 - 450 \\
 &= 15 (15^3 - 18 \times 15^2 + 35 \times 15 + 180 - 30) \\
 &= 15^2 (15^2 - 18 \times 15 + 35 + 12 - 2) \\
 &= 15^2 (225 - 270 + 35 + 12 - 2) \dots
 \end{aligned}$$

Having established that 15 is a root of the equation, you can write

$$x^4 - 18x^3 + 35x^2 + 180x - 450 \equiv (x - 15)(x^3 - 3x^2 - 10x + 30)$$

more or less by inspection, without doing the full long division.

Alternatively, instead of substituting  $x = 15$  into the equation, you could do the long division (dividing the polynomial by  $(x - 15)$ ) carefully, remembering to show that the last subtraction does indeed give 0 (so that  $x - 15$  is indeed a factor); and you should state “therefore  $x = 15$  is a root” at the end — this is what you were asked to show, so you should indicate clearly that you realise that you have actually shown it and not just stopped.

The roots are  $x = 15$ ,  $x = 3$  and  $x = \pm\sqrt{10}$ .

- (iii) For part (a) you should get  $1 \times 2 \times 3 \times 4 = 24$ .

For the other two parts write out the first few terms, then write  $\dots$  once you have seen the pattern, then write the last one or two terms. You should see that lots of things cancel.

Answers: (b)  $\frac{1}{n+1}$  and (c)  $\frac{g(n)}{g(0)}$ .



## Preparation

- 2 (i) (a) This is a periodic sequence with period 2 (values are  $1, -2, 1, -2, \dots$ ).  
 (b) Start by doing what was suggested:

$$\begin{aligned} l &= 6 - \frac{4}{l} \\ \Rightarrow l^2 - 6l + 4 &= 0 \\ \Rightarrow l &= 3 \pm \sqrt{5} \end{aligned}$$

There appear to be two limits! You have to do a bit of work to find out which of the two limits is the relevant one (it depends on the value of  $u_1$ ). For both of the above values of  $l$ , if  $u_1 = l$  then  $u_2 = l$  and the whole sequence is constant, but if the value of  $u_1$  is not equal to one of the two limits then the sequence will always tend to  $3 + \sqrt{5}$ : this is a *stable* limit and  $3 - \sqrt{5}$  is an *unstable* limit. Even if you use a starting point very close to  $3 - \sqrt{5}$  (such as 0.763932) the sequence will still converge to the other limit. You can use a spreadsheet to see this happening.<sup>1</sup>

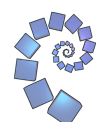
- (c) This sequence converges to the limit  $2 - \sqrt{2}$ .  
 (d) This sequence diverges.
- (ii) This is a periodic sequence with period 6 (the values are  $4, 1, -3, -4, -1, 3, \dots$ ).
- (iii) (a)  $u_2 = 4 - b$  and  $u_3 = 16 - 9b + b^2$   
 (b) Here you need to solve  $16 - 9b + b^2 = 2$ . There are two solutions.  
 (c) When  $b = 2$  the sequence is constant (all the terms are the same) but when  $b = 7$  the sequence is periodic with period 2.
- (iv) This is a way to estimate the square root of 2, and we were expecting you to use a calculator or spreadsheet here. Try comparing  $u_5$  with what your calculator gives you for  $\sqrt{2}$ .

Note that solving  $l = \frac{1}{2} \left( \frac{2}{l} + l \right)$  gives 2 possible values of  $l$ , but as  $u_1 > 0$ , and  $u_{k+1} > 0$  if  $u_k > 0$  then the whole sequence is positive and it will tend to  $\sqrt{2}$  (rather than  $-\sqrt{2}$ ).

This iterative formula was derived by rearranging  $x^2 = 2$  as follows:

$$\begin{aligned} x^2 &= 2 \\ \Rightarrow 2x^2 &= 2 + x^2 \\ \Rightarrow 2x &= \frac{2}{x} + x \\ \Rightarrow x &= \frac{1}{2} \left( \frac{2}{x} + x \right). \end{aligned}$$

<sup>1</sup>The reason why one limit is unstable is connected to the gradient of  $y = 6 - \frac{4}{x}$  at the limit. Try researching ‘‘Cobweb diagrams’’ for examples of why this happens.



This suggests using the sequence

$$u_{n+1} = \frac{1}{2} \left( \frac{2}{u_n} + u_n \right)$$

will give successively better approximations for  $\sqrt{2}$ . There are other ways to rearrange the equation  $x^2 = 2$  to get  $x = \dots$ , some will give a sequence which converges to  $\sqrt{2}$  but others will converge to  $-\sqrt{2}$  and some might not converge at all.

- (v) Solving  $l = \frac{1}{3} \left( \frac{a}{l^2} + 2l \right)$  will give the limit as  $\sqrt[3]{a}$  and so if you set  $a = 7$  you will be able to find an approximation for  $\sqrt[3]{7}$ . Six iterations should be enough to find  $\sqrt[3]{7}$  correct to 8 decimal places.

## The STEP question

- 3 (i) In each case we start with  $u_1 = 2$ .
- (a) Here we can put  $u_2 = u_1 = 2$  and solve for  $k$  to get  $k = 20$ .
- (b) Here we need to find an expression for  $u_3$  in terms of  $u_1$ . Then setting  $u_3 = u_1 = 2$  gives  $k^2 - 20k = 0$ .  
We already know that  $k = 20$  gives the constant sequence, so the value that gives a sequence with period 2 is  $k = 0$ .  
*We are taking the definition of period  $m$  to mean that the sequence repeats itself after  $m$  terms, but does not repeat itself after any number of terms smaller than  $m$ .*
- (c) Here we need  $u_5 = u_1 = 2$ , so start by finding an expression for  $u_5$  in terms of  $u_1$ , then set both equal to 2. This will give the quartic  $k^4 - 20k^3 - 72k^2 + 1440k = 0$ . We already know that two of the roots must be  $k = 0$  and  $k = 20$ , so we can factorise the quartic. Remembering that we only want the values of  $k$  for which the sequence has period 4 (and not period 2 or constant) we have  $k = \pm 6\sqrt{2}$ .
- (ii) If  $u_n \geq 2$  we have  $u_{n+1} = 37 - \frac{36}{u_n} \geq 37 - \frac{36}{2} = 19$  and of course  $19 \geq 2$ . Therefore, as  $u_1 \geq 2$ , we have  $u_n \geq 2$  for all  $n$ .  
To find the limit, solve  $l = 37 - \frac{36}{l}$ . There are two solutions, but only one is greater than 2.



## Warm down

- 4 (i) Creating a dump 100 miles from Cairo will solve this one. You will be able to leave 100 miles worth of fuel here.
- (ii) If you can set up a dump 400 miles from the destination with enough petrol in it, you can use part (i). You need to create a dump 60 miles from Cairo twice, leaving 180 miles worth of fuel each time. When you reach the dump for the third time you will have travelled 5 lots of 60 miles, have 240 miles worth of fuel in the car and a dump of 360 miles worth. Put 460 miles worth into the car and then the situation is just like part (i).

To make a longer trip, you have to set up more frequent dumps. If the range of the car with a full tank is  $R$ , then the best strategy turns out to be having dumps at distances  $R$ ,  $R + R/3$ ,  $R + R/3 + R/5$ ,  $R + R/3 + R/5 + R/7$ , etc, from the destination. Since the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots$$

(related to the so-called *harmonic* series  $\sum \frac{1}{n}$ ) does not converge (it gets bigger without limit the more terms you add), there is no limit to the width of the desert you can cross like this.

