

STEP Support Programme

Assignment 2

Warm-up

- 1 (i) Simplify $(2x - 3)^2 - (x - 1)^2$, giving your answer in factorised form. Check your answer by evaluating it for $x = 1$ and $x = 2$.

- (ii) Simplify

$$\frac{x}{x^2 - y^2} - \frac{y}{(x - y)^2} - \frac{1}{x + y}.$$

Hence find the possible values of x and y for which $\frac{x}{x^2 - y^2} - \frac{y}{(x - y)^2} - \frac{1}{x + y} = 0$.

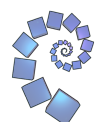
- (iii) Show that

$$\sqrt{1 + x^2} - x = \frac{1}{\sqrt{1 + x^2} + x}.$$

Deduce that if x is very large, then $\sqrt{1 + x^2} - x$ is approximately equal to $\frac{1}{2x}$.

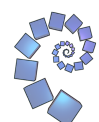
- (iv) Simplify $(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$. Hence find the solutions to $x^4 + 1 = 0$.

Your answer will involve $\sqrt{-1}$ (written as i).



Preparation

- 2 (i) Sketch the line $y = x + 1$ for $-2 \leq x \leq 2$.
What is the greatest value of $x + 1$ in this range?
- (ii) Sketch the line $y = -2x + c$ for $-2 \leq x \leq 2$.
Show that the greatest value of $-2x + c$ in this range is $4 + c$. What is the least value?
- (iii) Sketch $y = mx + 1$ for $-2 \leq x \leq 2$ in the cases $m > 0$, $m = 0$ and $m < 0$.
What are the greatest and least values of $mx + 1$ in each case?
- (iv) Sketch the curve (parabola) $y = (x - 1)^2$ for $-2 \leq x \leq 2$.
What are the greatest and least values of $(x - 1)^2$ in this range?
Be careful here: the minimum value is **not at one of the end points.**
- (v) Sketch the curve $y = (x - 3)^2$. What are the greatest and least values of $(x - 3)^2$ for $-2 \leq x \leq 2$?
- (vi) Write the expression $x^2 - 8x + 21$ in the form $(x + a)^2 + b$. Hence sketch the curve $y = x^2 - 8x + 21$ and find the greatest and least values of $x^2 - 8x + 21$ in the range $0 \leq x \leq 5$.
- (vii) Sketch the curve $y = x^2 + 2kx$ for $-2 \leq x \leq 2$, where $-2 < k < 2$.
What are the greatest and least values of $x^2 + 2kx$ for $-2 \leq x \leq 2$?
What would your answers be if $k > 2$?
Use the same techniques as in part (vi) to help you sketch the curve.



The STEP question

- 3** (i) Find the greatest and least values of $bx + a$ for $-10 \leq x \leq 10$, distinguishing carefully between the cases $b > 0$, $b = 0$ and $b < 0$.
- (ii) Find the greatest and least values of $cx^2 + bx + a$, where $c \geq 0$, for $-10 \leq x \leq 10$, distinguishing carefully between the cases that can arise for different values of b and c .

Discussion

This question has some features that are very typical of STEP questions.

First is the use of letters, a , b and c in this case, rather than numbers to make the equations less specific. The correct term for these letters is *parameters*, whereas x is a *variable*. In each case you will have to give your answers in terms of the given parameters.

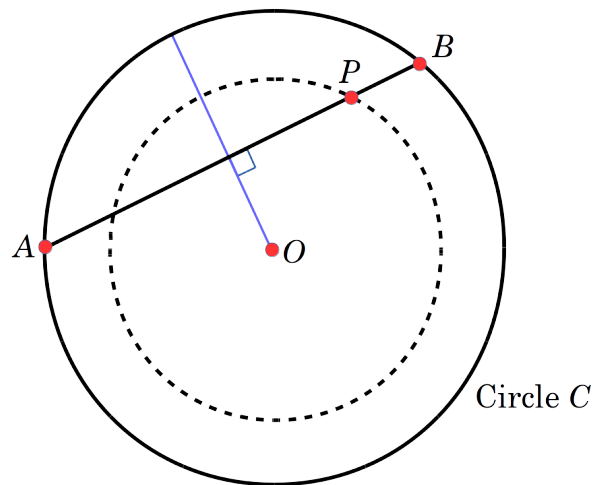
Second, although you are not told to draw sketches, you should do so: it is much easier to work on the different cases if you have sketches in front of you. Think back to question **2** to help you.



Warm down

- 4 The diagram shows a circle C with centre O , and a rod AB the ends of which can slide round the circle C (so that AB is a chord of C). The radius of the circle is R and the length of the rod is $2a$.

As the rod slides round C the point P , which is a fixed distance b from the centre of the rod, traces out a circle with centre O of radius r .



Show that the area between the two circles is $\pi(a^2 - b^2)$.

The surprising thing about this result is that it is independent of the radius of C (assuming that it is greater than a) and depends only on the length of the stick and the position of P on the stick. It doesn't matter how big the circle is, the area between the two circles is always $\pi(a^2 - b^2)$.

If you had known that the answer was independent of R , can you think of an easy way (by choosing R) of obtaining the result?

Even more surprising is the fact that the result holds even when C is not a circle, but is any closed curve round which the rod can slide smoothly. This is Holditch's theorem, proved in about 1840, and not much seen until it was used as a STEP question in 2010.

The lengths R and r are given here for your convenience: they are required for the calculation but do not appear in the answer. In a STEP question, it would have been up to you to decide what is needed for the calculation. And you might have had to draw the diagram yourself.

