

Independent/Dependent Variables (Dr. Pellumb Killogjeri)

Two random variables X and Y are independent if $P(X=x, Y=y) = P(X=x)P(Y=y)$, for all x, y . In general, if two random variables are independent then,

$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$, for the two sets A and B .

The other way, we can write: two random variables are independent if **either of the following statements are true:**

1. $P(x|y) = P(x)$, for all values of X and Y .
2. $P(x \cap y) = P(x) * P(y)$, for all values of X and Y .

This means that the random variable X doesn't have an effect on the other random variables Y or vice-versa, in an experiment. The other way, it doesn't affect the probability of another event happening. We can tell if two random variables are independent by looking at their individual probabilities. **If those probabilities don't change when the events meet, then those variables are independent.** Another way of saying this is that: if the two variables are correlated, then they are *not* independent. They are dependent.

Known that $\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$. Hence, we conclude:

If X and Y are statistically independent then $E[XY] = E[X]E[Y]$ hence, the covariance is zero. However, the converse is not generally true: it is possible for $E(XY) = E(X)E(Y)$, that is $\text{cov}(X, Y) = 0$, even though X and Y are dependent.

Conditional Independence

Let A , B and C be discrete random variables. We say that A and B are conditionally independent given C if, $P\{A = a, B = b / C = c\} = P\{A = a / C = c\} \cdot P\{B = b / C = c\}$ for every value of C . The conditional independence is just ordinary independence in a conditional. The three random variables A , B and C must be defined on the same probability space for this to make any sense. For instance, they may be random vectors. The left-hand side may be written as a product of two terms, one depending on the set of variable A only, and one depending on the set of variable B .

*** Conditional independence neither implies independence, nor is it implied by independence. Two events can be conditionally independent but not unconditionally independent. Also, we can have two events that are independent but not conditionally independent, given an event.

Example 1 Consider rolling a die and let be:

$A = (1,3)$, $B = (3,5,6)$, $C = (1,5)$. The probability space is $H = \{1,2,3,4,5,6\}$.
Then, we have: $P(A) = 1/3$, $P(B) = 1/2$, $P(A \cap B) = 1/6 = 1/2 \times 1/3 = P(A)P(B)$.
Hence A and B are independent.

$$\text{Now, } P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{1/6}{1/3} = 1/2. \quad \text{Also, } P(B/C) = 1/2 \quad P(A \cap B/C) = P(\{3\}/C) = 0$$

Thus way, $P(A \cap B/C) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} = P(A/C) \cdot P(B/C)$, which means A and B are not conditionally independent given C.

Example 2. As above consider the events: $A = (5,3)$, $B = (1,3,5,4)$ $C = (3,4,5)$.

The probability space is $H = \{1,2,3,4,5,6\}$. Then, we have: $P(A) = 1/3$, $P(B) = 2/3$, $P(A \cap B) = 2/6 \neq 1/3 \times 2/3 = P(A)P(B)$. Hence A and B are not independent.

$$\text{Now, } P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{1/3}{1/2} = 2/3. \quad \text{Also, } P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{1/2}{1/2} = 1 \quad P(A \cap B/C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{1/3}{1/2} = \frac{2}{3} = P(A/C) P(B/C), \text{ which means A and B are conditionally independent given C.}$$

Example 3 Suppose W, X, Y, Z are four random variables whose joint probability distribution factorises as follows: $P(W, X, Y, Z) = P(W|Z) P(Z|X, Y) P(X) P(Y)$.

Which of the following statements are true? Explain your answers.

- i) X and Y are independent, assuming we know nothing about W and Z.
- ii) X and W are conditionally independent, given that we know the value of Z.
- iii) X and Y are conditionally independent, given that we know the value of W.

Answer: *If we assume that X and W are conditionally independent, given that we know the value of Z, we would have*

$$P(W, X|Z) = P(W|Z)P(X|Z), \text{ meaning that (ii) is not true.}$$

If we assume that X and y are conditionally independent, given that we know the value of W, we would have

$$P(Y, X|W) = P(X|W)P(Y|W), \text{ meaning that (iii) is not true.}$$

Definitely, X and Y are independent, assuming we know nothing about W and Z.

The conditional independence of W with X and Y given Z (whether, conditional independence with X and conditional independence with Y, or just conditional independence of $(X \cap Y)$ is defined as:

$$P(W, X, Y|Z) = P(W|Z)P(X, Y|Z)$$

Since X and Y are not conditionally independent given Z, we also note that $P(X, Y|Z) \neq P(X|Z)P(Y|Z)$.