

STEP Support Programme

Assignment 22

Warm-up

1 This question is about the product rule for differentiating a product of two functions.

(i) Use a rough sketch to show that (for any function f that can be differentiated)

$$f(x+h) \approx f(x) + hf'(x) \quad (\dagger)$$

when h is “small”.

(ii) The function g is defined by $g(x) = f_1(x)f_2(x)$, where f_1 and f_2 are two given (differentiable) functions. Use the definition $g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$, then (\dagger) , to show that

$$g'(x) = f_1'(x)f_2(x) + f_1(x)f_2'(x).$$

2 The exponential function e^x is defined by the infinite series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad (*)$$

(which converges for all x).

There are various definitions of the function e^x , and you may well know a different definition from the one above. For example, you can define it as the inverse function to the natural logarithm $\ln x$, which is $\log_e x$ (e being a certain number). Or you could define it as simply e^x , where e is the certain number again (though in this definition, you might worry about the meaning of e^x when x is not an integer).¹

For this question, forget any definitions you know except for $(*)$.

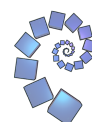
(i) Use definition $(*)$ to find $\frac{d}{dx}(e^x)$.

(ii) Use definition $(*)$ to find $\frac{d}{dx}(e^{kx})$ where k is a constant.

(iii) Let $f(x) = xe^x$. Show, using the product rule, that $f'(x) = (x+1)e^x$.

Can you get this result from the definition $(*)$ without using the product rule?

¹ In the usual notation (for A-levels, etc), the exponential function is written e^x , using roman type face e to show that it is a function rather than a number, and we follow that convention.



(iv) Use the product rule to show that

$$\frac{d}{dx} (e^{ax} e^{bx}) = (a + b)e^{ax} e^{bx}.$$

Since you are told that e^x is defined by (*), you cannot use any rules of indices here!

Starting with this result, show that $e^x e^{-x} = 1$ (which means that you have now shown that $e^{-x} = \frac{1}{e^x}$).

Use this result and definition (*) to show that $xe^x \rightarrow 0$ as $x \rightarrow -\infty$.

Preparation

- 3 (i) Find the range of values of x for which $3x^2 + x - 2 < 0$.
- (ii) Sketch the curve $y = e^x$.
- (iii) This part concerns the curve $y = (x - 3)e^x$.
- (a) Differentiate $(x - 3)e^x$ and hence find the coordinates of the stationary point of the curve $y = (x - 3)e^x$. Use the sign of $\frac{d^2y}{dx^2}$ to determine the nature of the stationary point.
- (b) Find the coordinates of the intersections of the curve with the axes. Determine the values of x for which $(x - 3)e^x$ is negative.
- (c) Sketch the curve $y = (x - 3)e^x$. You may assume that $xe^x \rightarrow 0$ as $x \rightarrow -\infty$.
- (d) Find the values of k for which the equation $(x - 3)e^x = k$ has two roots. Find the values of k for which the equation has one root.



- (iv) This part concerns the curve $y = \sin(x^2)$.
- (a) Sketch the curve $y = \sin x$ for $-4\pi \leq x \leq 4\pi$.
- (b) Find the first four non-negative values of x for which $\sin(x^2) = 0$.
- (c) If $f(x) = \sin(x^2)$, express $f(-a)$ in terms of $f(a)$.
- (d) Sketch the curve $y = \sin(x^2)$ for $-4 \leq x \leq 4$.

The STEP question

- 4 (i) Sketch the curve $y = e^x(2x^2 - 5x + 2)$.

Hence determine how many real values of x satisfy the equation $e^x(2x^2 - 5x + 2) = k$ in the different cases that arise according to the value of k .

You may assume that $x^n e^x \rightarrow 0$ as $x \rightarrow -\infty$ for any integer n .

- (ii) Sketch the curve $y = e^{x^2}(2x^4 - 5x^2 + 2)$.

Discussion

When sketching a curve, make sure you consider turning points, intersections with the axes, and the behaviour as $x \rightarrow \pm\infty$. You may be sure of the nature of the turning points without having to calculate the second derivative (though you might calculate a second derivative just to be confirm that your sketch is right).

The key to the second part is to work out how the two curves are related.



Warm down

5 Notation: for any polyhedron (i.e. three-dimensional solid whose surface consists of a collection of polygonal faces, joined at their edges), the number of faces is F , the number of edges is E and the number of vertices is V .

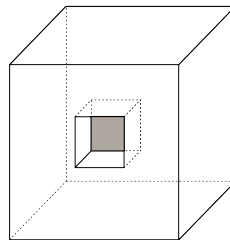
(i) Write down F , E and V for a tetrahedron (a pyramid with a triangular base). Calculate $F - E + V$.

(ii) Repeat part (i) for cube.

(iii) A regular icosahedron has 20 faces, each of which is an equilateral triangle. What is E ? What is V ? **Don't just write down the answers; provide brief justification.**

Calculate $F - E + V$ for an icosahedron.

(iv) The diagram below shows a cube with a small cubical hole dug into one face. Calculate $F - E + V$.



Discussion

Euler's formula $V - E + F = 2$ holds for *convex* polyhedra (ones where any two points on the surface are connected by a straight line that lies entirely within or on the surface of the polyhedron).

Other three dimensional shapes satisfy $V - E + F = \chi$, where χ is the *Euler characteristic*. For example, a *torus* (ring doughnut shape) had $\chi = 1$. There is lots of information out there on proofs of Euler's formula and different characteristics, so have a search!

