

## STEP Support Programme

### Hints and Partial Solutions for Assignment 4

#### Warm-up

- 1 (i) This question was asking you to prove that the angle at the centre of a circle is twice the angle at the circumference. When drawing a diagram try not to draw a “special” triangle, i.e. try not to make  $APB$  equilateral or isosceles (unless of course you have been told that this is true!). The triangle  $APO$  must be isosceles as  $AO$  and  $OP$  are both radii, which means that you can say things like “ $\angle APO = \angle PAO = x$ ” etc. which means that  $\angle AOP = 180^\circ - 2x$ . If  $\angle BPO = y$  then you should find that  $\angle APB = x + y$  and  $\angle AOB = 2(x + y)$ .
- (ii) In this case  $APBO$  does not form an arrowhead (sometimes called a *delta*) and two of the lines cross. You should add in a line  $OP$  in order to find some isosceles triangles. In my proof I ended up with  $\angle APB = x - y$  and  $\angle AOB = 2(x - y)$ .

Can you prove the case where  $\angle APB$  is obtuse? You may wish to prove that opposite angles in a cyclic quadrilateral are *supplementary*<sup>1</sup> first.

#### Preparation

- 2 (i) Two points to watch out for here.
- (a) You have to be careful about wording. In this part of the question,  $x$  could lie in one of two ranges, so you should write  $x < a$  **or**  $x > b$  (say) rather than  $x < a$  **and**  $x > b$ , as any single value of  $x$  cannot be in both ranges at once.
- (b) You have to be careful to use the appropriate type of inequality — if the question involves **strict** inequalities ( $<$ ,  $>$ ) then the answer should involve strict inequalities, and similarly for **weak** inequalities ( $\leq$ ,  $\geq$ ).

Answer:  $x < -1$  or  $x > 4$ .

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<sup>1</sup>Supplementary means that the angles add up to  $180^\circ$ , so we can say that “the angles in a triangle are supplementary”. Angles that add up to  $90^\circ$  are called *complementary*.



- (ii) You don't know that the roots are going to be integers; but if they are integers there are not many possibilities (they can only be factors of 6), so it might have been worth simply testing these until you have found the three roots. Alternatively (and if the other roots had not been integers this is the approach you would have had to use) you can divide the cubic by the known factor  $(x - 3)$  and then factorise the resulting quadratic.

Don't forget to answer the question — it asks you to find the roots so you do actually need to write “ $x = \dots$ ” as well as writing the cubic as a product of three linear factors. If you had divided the cubic by  $(x - 3)$  then you will probably have written the cubic as a product of three linear factors first — you still need to write down the three roots.

Final answer:  $x \leq -2$  or  $1 \leq x \leq 3$ .

- (iii) Once you have factorised the quadratic  $x^2 - 3xy + 2y^2$  into two brackets you know that bracket or the other (or both) must be zero, which gives the equations of two straight lines —  $y = x$  and  $y = \frac{1}{2}x$ . Sketch these and then you are done — nothing more complicated needed.
- (iv) You should already have a diagram showing the two relevant lines from the previous part of the question.

There are two main approaches here.

- (a) You can write the expression as the product of two linear factors and then consider the signs of each bracket, which will result in two inequalities for each case (for example,  $x - 2y \geq 0$  and  $x - y \leq 0$  for the first bracket to be positive and the second bracket negative).
- (b) A perhaps simpler line of attack is note that the two straight lines drawn in part (iii) divide the plane into 4 regions. You then only need to determine the sign of the quadratic expression in each of the 4 regions, for example by calculating its value at a single point in each region. So, for example, using the point  $(3, 2)$  we have

$$x^2 - 3xy + 2y^2 = 3^2 - 3 \times 3 \times 2 + 2 \times 2^2$$

which is negative, so the region in which this point lies is one of the required regions.

Be careful to pick points which lie **within** the regions — the point  $(1, 1)$  is no good as it lies on one of the boundary lines.

Note that there are only 4 regions — the  $x$  and  $y$  axes are **not** boundaries of regions. It is not a problem if you test in 8 regions rather than 4; you are just making life harder for yourself!



## The STEP question

- 3 (i) As in the last question, you have to be careful to distinguish between **strict** or **weak** inequalities. The three roots of  $x^3 - 4x^2 - x + 4 = 0$  are all integers.

Answer:  $-1 \leq x \leq 1$  or  $x \geq 4$ .

- (ii) You can realise that  $x = y$  is a solution and then divide throughout by  $(x - y)$ , but the simplest approach is to use your answer to part (i) and essentially replace 1 with  $y$ .

Answer:  $y = x$ ,  $y = \frac{1}{4}x$  and  $y = -x$ .

- (iii) You can consider where the product of the three factors was positive (i.e. where all three are positive, or where two are negative and one positive), or test points in all the 6 regions that the three lines create to see where the inequality is true. Both ways are fine **but** you must fully justify your answer. The second approach is simpler and probably easier to explain fully.

## Warm down

- 4 (i) You need to think very carefully about the question. It says that **if** there is an even number **then** the other side is a vowel. It does not follow that if there is a vowel on one side then there must be an even number on the other side; furthermore if there is an odd number on one side there can be any letter on the other side.

It is fairly obvious that we must check the card showing the 6. There is one other card that we must check, and this card could show the statement to be false. Think about the different implications there might be depending on what is on the second sides of the cards.

In the first version of this assignment we neglected to include the information that the cards had a number on one side and a letter on the other. If we didn't know this then we would actually have to turn over three cards — the only one we would not have had to check is the “E”. (Why?)

Answer: We must check “6” to see if there is a vowel on the other side. The other card we must check is “Q”, as if this has an even number on the other side it will disprove my claim.

- (ii) If you are stuck you can start by trying to work out how to time 30 minutes. Or you can google it, but please not before you have had a good think about it.

