

## STEP Support Programme

### Hints and Partial Solutions for Assignment 11

#### Warm-up

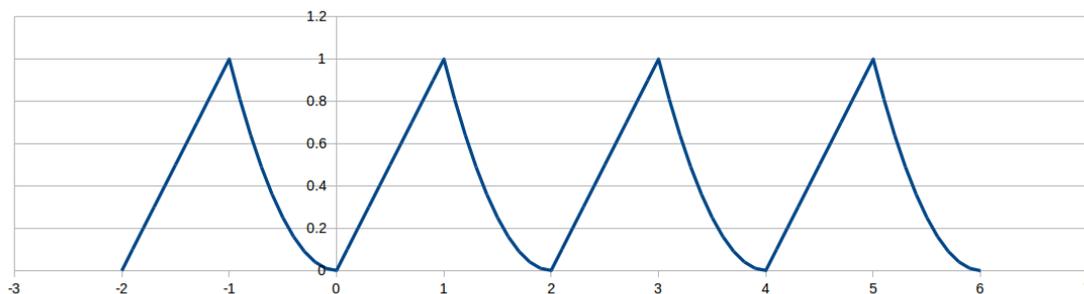
- 1 (i)  $T(n)$ , or  $T_n$  as it is sometimes written, is the  $n$ th *triangular number*, so called because you need  $T(n)$  billiard (or snooker or pool) balls to make an equilateral triangular shape of side  $n$  balls. You can see from the given recurrence formula that  $T(n)$  is the sum of the first  $n$  integers.

Nothing to do with the question, but if you arrange the balls in a right-angled triangle, you can prove by means of a picture that

$$T(n) + T(n - 1) = n^2.$$

You can also fit two lots of  $T(n)$  together to make a rectangle, and use this to find a formula for  $T(n)$ . You could use this to show that  $T(14) = 105$ .

- (ii) This function is periodic, with period 2. It looks a bit like saw teeth, as shown below.



This graph is continuous (you can draw it without taking the pencil off the paper), but the gradient of the graph is not continuous.



## Preparation

2 (i) Writing

$$4^x = (2^2)^x = 2^{2x} = (2^x)^2 = y^2,$$

gives  $y^2 - 7y - 8 = 0$  and, apparently, two solutions for  $y$ . But only one solution is relevant, because  $y = 2^x$  which has to be positive. Answer:  $x = 3$  (substitute back into the original equation to make sure it works!).

(ii) You need to be very careful if you use squaring when solving an equation. It often means that extra “solutions” are created.

As a very simple example, consider the equation:

$$x + 1 = 2$$

which has just the one solution,  $x = 1$ .

However, if our first step towards solving the question was to square both sides (admittedly a pretty daft thing to do), we would get:

$$\begin{aligned} (x + 1)^2 &= 4 \\ \implies x^2 + 2x + 1 &= 4 \\ \implies x^2 + 2x - 3 &= 0 \\ \implies (x + 3)(x - 1) &= 0 \\ \implies x = -3 \text{ or } x = 1, \end{aligned}$$

so this way we seem to get two solutions. Of course, only one of them satisfies the original equation  $x + 1 = 2$ .

If you square both sides of an equation, you must make sure that you check your solutions in the **original** equation to make sure that they are indeed solutions.

To solve  $\sqrt{3x-5} - \sqrt{x+6} = 1$ , you could you square both sides (or take  $\sqrt{x+6}$  to the other side and then square both sides). You will get a cross term (note that  $(\sqrt{A} - \sqrt{B})^2 \neq A - B$  or even  $A + B$ ) so there is still a square root in the squared equation. However, there is only one square root term, so you can isolate this on one side of the equation and square again. This should result in the equation  $x^2 - 13x + 30 = 0$  which has two solutions, of which **only one** is a solution of the original equation!



## The STEP question

**3** Substitutions work quite nicely for all three equations.

In the first case you should end up with  $y = \frac{-3 \pm \sqrt{11}}{2}$ . Then you should note that as  $y = \sqrt{x}$ , which is positive (or zero) by definition, only the positive solution for  $y$  makes sense, and  $x = \left(\frac{-3 + \sqrt{11}}{2}\right)^2 = \dots$

There is one root for part **(ii) (a)**, start by substituting  $y = \sqrt{x+2}$  and you should obtain  $y^2 + 10y - 24 = 0$ . Using the fact that  $y \geq 0$  will give you one possibly value of  $y$  which you can then use to find the value of  $x$ . Substituting this back into the original equation should be reassuring.

There are two roots for part **(ii) (b)**, start by substituting  $y = \sqrt{2x^2 - 8x - 3}$  (or you can do two substitutions,  $y = 2x^2 - 8x - 3$  and  $z = \sqrt{y}$ ). You should end up with  $x = 2 \pm \sqrt{10}$ . You can substitute these back into the equation, but this time you might like to weigh up the value of the reassurance this will give and the time you would need to spend doing it.

Squaring can be used as an alternative to substitution in order to eliminate the square roots, but you end up with an unpleasant quartic in part **(ii)(b)**. If you take this approach, you must check your solutions satisfy the original equation as “false solutions” will be created by squaring.

## Warm down

**4 (i)** Noting that  $2^{m+1} + 2^m = 2^m(2 + 1)$ , and similarly for the other side then we have

$$3 \times 2^m = 8 \times 3^n.$$

Since each number can be written as a *unique* product of primes, we can deduce from this what  $m$  and  $n$  are.

If you prefer, you can divide throughout by 3 and 8 to get:

$$2^{m-3} = 3^{n-1}$$

and then as one side is a power of 2 and the other is a power of 3, the indices on each side must be equal to 0.

**(ii)** One way to approach this is to use two substitutions, such as  $3^x = a$  and  $5^x = b$ . You will then end up with the equation  $15a^2 - 34ab + 15b^2 = 0$ . You can then find two possible relationships between  $a$  and  $b$ , which can be used to find two possible values of  $x$ . One condition is  $5a = 3b$  which gives  $5 \times 3^x = 3 \times 5^x$  which implies that  $x = 1$ . The other condition will result in the solution  $x = -1$ .

