Solutions to the Olympiad Cayley Paper

C1. A train travelling at constant speed takes five seconds to pass completely through a tunnel which is 85 m long, and eight seconds to pass completely through a second tunnel which is 160 m long. What is the speed of the train?

Solution

Let the train have length l metres.

When the train passes completely through the tunnel, it travels the length of the tunnel plus its own length.

Therefore the first fact tells us that the train travels $85 + l$ metres in five seconds, and the second fact tells us that it travels $160 + l$ metres in eight seconds.

By subtraction, we see that the train travels a distance of $(160 + l) - (85 + l)$ metres in $8 - 5$ seconds, that is, it travels 75 metres in 3 seconds, which corresponds to a speed of 25 m/s.

C2. The integers a, b, c, d, e, f and g, none of which is negative, satisfy the following five simultaneous equations:

$$\begin{aligned} a + b + c &= 2 \\ b + c + d &= 2 \\ c + d + e &= 2 \\ d + e + f &= 2 \\ e + f + g &= 2. \end{aligned}$$

What is the maximum possible value of $a + b + c + d + e + f + g$?

Solution

All the equations are satisfied when $a = d = g = 2$ and $b = c = e = f = 0$. In that case we have

$$\begin{aligned} a + b + c + d + e + f + g &= 2 + 0 + 0 + 2 + 0 + 0 + 2 \\ &= 6. \end{aligned}$$

Therefore the maximum possible value of $a + b + c + d + e + f + g$ is at least 6. However, from the first and fourth equations we obtain

$$\begin{aligned} a + b + c + d + e + f + g &= (a + b + c) + (d + e + f) + g \\ &= 2 + 2 + g \\ &= 4 + g. \end{aligned}$$

But the fifth equation is $e + f + g = 2$ and the integers e and f are non-negative, so that g is at most 2. Thus $4 + g$ cannot exceed 6, and thus $a + b + c + d + e + f + g$ cannot exceed 6.

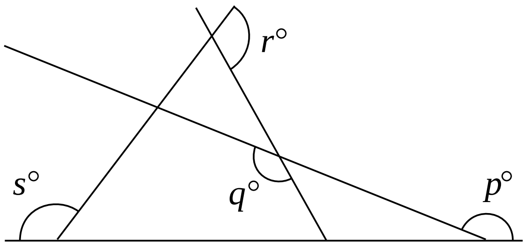
Hence the maximum possible value of $a + b + c + d + e + f + g$ is indeed 6.

Remark

It is possible to use the same method but with different equations in the second part.

C3. Four straight lines intersect as shown.

What is the value of $p + q + r + s$?

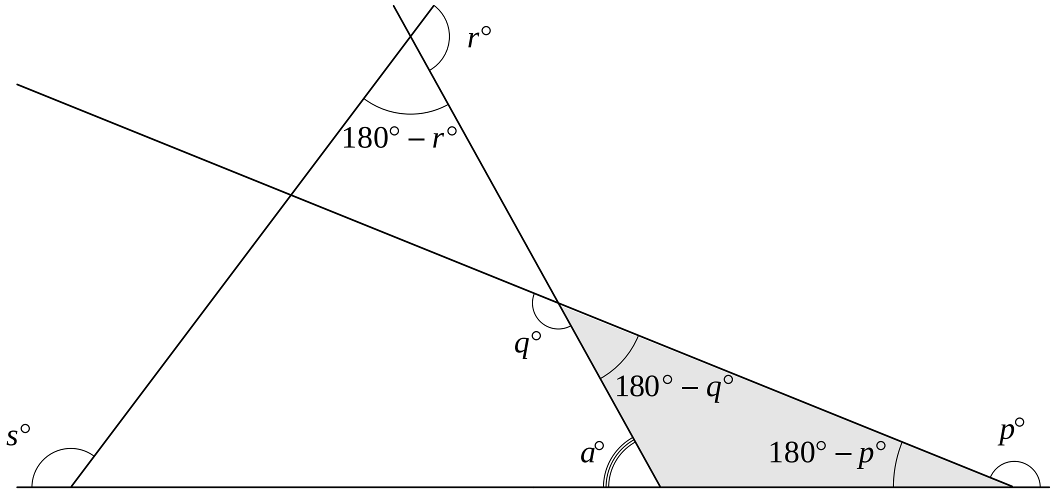


Solution

We present two of the many possible methods of solving this problem.

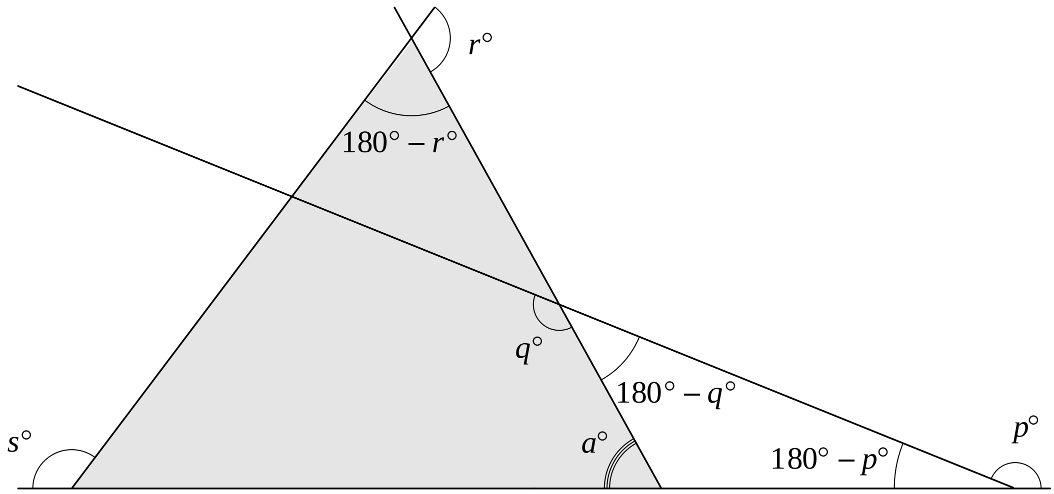
Method 1

We start by using ‘angles on a straight line add up to 180°’ to mark on the diagram three supplementary angles.



Now we know two of the interior angles in the shaded triangle, and hence can use ‘an exterior angle of a triangle is the sum of the two interior opposite angles’ to obtain $a = (180 - p) + (180 - q) = 360 - p - q$.

We now have expressions for two of the interior angles of the triangle shown shaded in the next figure.



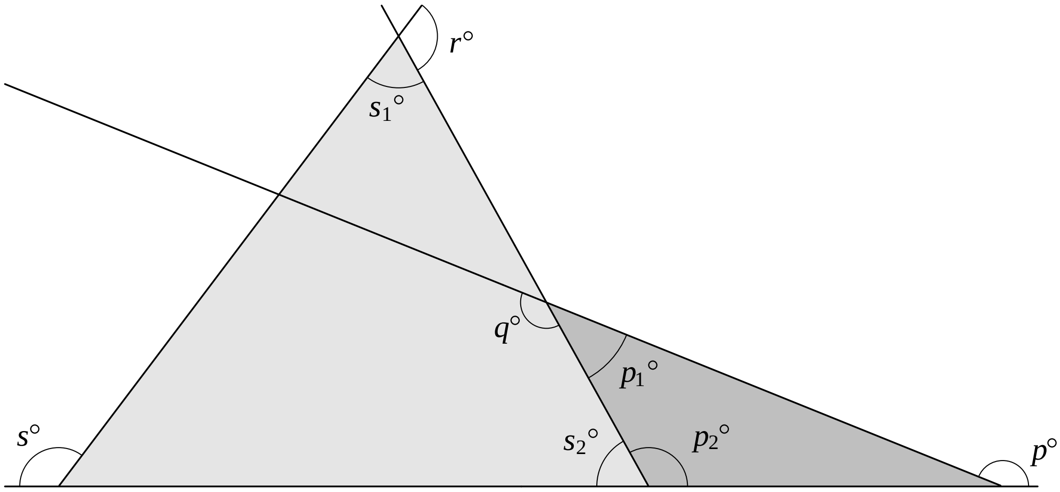
Therefore, once again using ‘an exterior angle of a triangle is the sum of the two interior opposite angles’, we obtain

$$s = (360 - p - q) + (180 - r),$$

which we may rearrange to give $p + q + r + s = 540$.

Method 2

We start by using 'an exterior angle of a triangle is the sum of the two interior opposite angles' in the two triangles shown shaded in the following figure.



We obtain $s = s_1 + s_2$ and $p = p_1 + p_2$.

Now we notice that we have a pair of angles together at each of three points, so we use 'angles on a straight line add up to 180°' three times to get

$$p_1 + q = 180$$

$$r + s_1 = 180$$

$$\text{and} \qquad p_2 + s_2 = 180.$$

Adding these three results together, we obtain $p_1 + q + r + s_1 + p_2 + s_2 = 540$, from which it follows that $p + q + r + s = 540$.

C4. Ten balls, each coloured green, red or blue, are placed in a bag. Ten more balls, each coloured green, red or blue, are placed in a second bag.

In one of the bags there are at least seven blue balls and in the other bag there are at least four red balls. Overall there are half as many green balls as there are blue balls.

Prove that the total number of red balls in both bags is equal to either the total number of blue balls in both bags or the total number of green balls in both bags.

Solution

Let r, g and b respectively be the numbers of red, green and blue balls that there are in total. From the given information, it follows that $b \geqslant 7$ and $r \geqslant 4$. We also know that $b = 2g$. Since the total number of balls is 20, we have $r + g + b = 20$. Replacing b by $2g$, we see that

$$r = 20 - 3g. \tag{1}$$

We now consider possible values of g in turn.

Case A: $g \leqslant 3$

In this case $2g \leqslant 6$, so that $b \leqslant 6$ because $b = 2g$. But 6 is less than the specified minimum number of blue balls. So $g \leqslant 3$ is not possible.

Case B: $g = 4$

In this case $b = 8$ because $b = 2g$, and $r = 8$ from equation (1). Therefore the total number of red balls is equal to the total number of blue balls.

Case C: $g = 5$

In this case $r = 5$ from equation (1), and $b = 10$. Therefore the total number of red balls is equal to the total number of green balls.

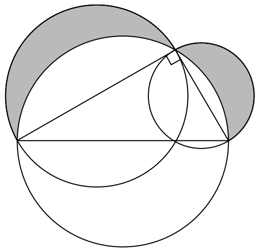
Case D: $g \geqslant 6$

In this case $20 - 3g \leqslant 2$ from equation (1). Thus $r \leqslant 2$, which is less than the specified minimum number of red balls. So $g \geqslant 6$ is not possible.

Hence in all possible cases the numbers of balls are as we want.

C5. The diagram shows a right-angled triangle and three circles. Each side of the triangle is a diameter of one of the circles. The shaded region R is the region inside the two smaller circles but outside the largest circle.

Prove that the area of R is equal to the area of the triangle.



Solution

Let T be the area of the triangle, let S_1 be the area of the semicircle whose diameter is the hypotenuse, and let S_2 and S_3 be the areas of the two semicircles with the shorter sides of the triangle as diameters.

We can view the region in the figure above the hypotenuse in two different ways: either as a semicircle with diameter the hypotenuse plus the shaded region R, or as the triangle plus the two smaller semicircles. This gives us the equation

$$(\text{area of } R) + S_1 = T + S_2 + S_3. \tag{1}$$

Let the hypotenuse of the triangle have length c and let the shorter sides have lengths a and b. Then the semicircles on those sides have radii $\frac{1}{2}c$, $\frac{1}{2}a$ and $\frac{1}{2}b$ respectively. Hence the areas of the semicircles are $S_1 = \frac{1}{8}\pi c^2$, $S_2 = \frac{1}{8}\pi a^2$ and $S_3 = \frac{1}{8}\pi b^2$ respectively. But $a^2 + b^2 = c^2$ by Pythagoras' Theorem, so that we have

$$S_1 = S_2 + S_3. \tag{2}$$

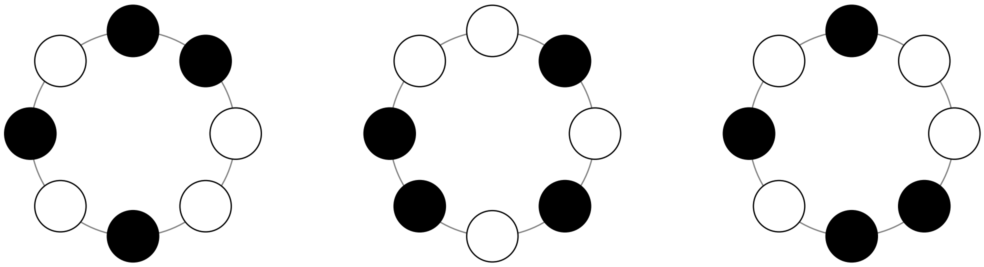
Subtracting equation (2) from equation (1) gives us area of $R = T$, which is what we needed.

C6. I have four identical black beads and four identical white beads.

Carefully explain how many different bracelets I can make using all the beads.

Solution

Firstly, we note the importance of the fact that we are dealing with bracelets: a bracelet can be turned over, as well as turned round. This means that the bracelets shown in the following figures are *not* counted as different.



We need to deal with the possibilities in a systematic fashion, to ensure that we count everything.

Our first step is to consider the runs of consecutive black beads. There are five possibilities: 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. Here the notation $3 + 1$, for example, means that there is a run of three consecutive black beads and a separate run of just one black bead. We consider each case in turn.

Case A: 4

In this case there is only one gap for the white beads, so the four white beads are also consecutive. Just one bracelet is possible, as shown in Figure 1.

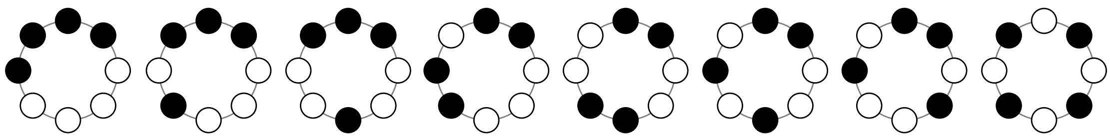


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5 Figure 6 Figure 7 Figure 8

Case B: $3 + 1$

In this case there are two gaps for the white beads, so there are two possibilities for consecutive runs of white beads: $3 + 1$ or $2 + 2$. Therefore two bracelets are possible, shown in Figure 2 and Figure 3.

Case C: $2 + 2$

Once again there are two gaps for the white beads, so there are two possibilities for consecutive runs of white beads: $3 + 1$ or $2 + 2$. Therefore two bracelets are possible, shown in Figure 4 and Figure 5.

Case D: $2 + 1 + 1$

In this case there are three gaps for the white beads. So there is only one possibility for consecutive runs of white beads: $2 + 1 + 1$. However, two bracelets are possible, shown in Figure 6 and Figure 7, depending on whether the group of two black beads is adjacent to the group of two white beads, or not.

Case E: $1 + 1 + 1 + 1$

Now there are four gaps for the white beads. So there is only one possibility for consecutive runs of white beads: $1 + 1 + 1 + 1$. So only one bracelet is possible, shown in Figure 8.

Therefore exactly eight different bracelets can be made.