

## STEP Support Programme

### Hints and Partial Solutions for Assignment 12

#### Warm-up

- 1 (i) You don't need a particularly fancy argument here.  $n(n+1)$  is the product of two consecutive integers, so one of them is divisible by 2. Hence  $n(n+1)$  is divisible by 2.
- $n(n+1)(n+2)$  is the product of three consecutive integers, so one of them is a multiple of 3 and one (at least) is divisible by 2. The product is therefore divisible by 6.
- (ii)  $n^3 - n = (n-1)n(n+1)$ , and so is the product of three consecutive integers. Therefore it is divisible by 6, as above.
- (iii)  $n^5 - n^3 = n^3(n-1)(n+1)$ . From part (ii) it is divisible by three. Then if  $n$  is even,  $n^3$  is divisible by 8, and if  $n$  is odd,  $n-1$  and  $n+1$  are consecutive even numbers — which means one of them is a multiple of 4 and the other is a multiple of 2 (and therefore the product is a multiple of 8).
- More formally we could let  $n = 2k + 1$ , then we have  $(2k+1)^3 \times 2k \times (2k+2) = (2k+1)^3 \times 4k(k+1)$ . Since we know that  $k(k+1)$  is even we have another factor of 2 and are done.
- (iv)  $2^{2n} - 1 = (2^n - 1)(2^n + 1)$ . Note that you cannot say “this is the product of two consecutive odd numbers, so one of them must be divisible by 3” — consider 5 and 7!
- Instead consider  $(2^n - 1)2^n(2^n + 1)$  which is the product of three consecutive integers, so must be divisible by 3. But  $2^n$  is **not** divisible by 3, so one of  $2^n - 1$  and  $2^n + 1$  must be divisible by 3.
- (v) There are various ways to proceed, but perhaps the simplest is to note that  $n - 1 = 3k$ , so  $n = 3k + 1$ . Substitute and expand  $(3k + 1)^3 - 1$ .



## Preparation

- 2 (i) (a) Answer:  $\frac{2}{(x-1)(x+1)(x+2)}$ .
- (b) Note you can cancel out some  $m$ s in the second batch of fractions as a first step.  
Answer:  $\frac{m-1}{m+1}$ .

- (ii) Remember that the probabilities for the second sweet depend upon what happened for the first sweet (they are **conditional**). Answer:  $\frac{17}{35}$ .

It makes no difference to the probability if you take both sweets in one or, or one after the other. If doing one after the other then a tree diagram is useful, if taking them simultaneously then arrangements are probably the way to go.

- (iii) Again, the probabilities change depending on what happened with the first sweet. If the first was an apple sour, and you remove it, you are then left with  $a-1$  apple sour and a total of  $a+b-1$  sweets.

Answers: (a)  $\frac{a}{a+b}$  and (b)  $\frac{b}{a+b} \times \frac{b-1}{a+b-1} \times \frac{b-2}{a+b-2}$ .

- (iv) (a) The easiest approach here is to find the probability that they have all forgotten and subtract from 1. Answer:  $\frac{37}{64}$ .
- (b) Answer:  $\frac{1}{4}$ .
- (c) The order is not particularly important, the important bit is that one particular child has not got goggles — we could have said “Philip” has forgotten his goggles whilst the other two have them. Answer:  $\frac{3}{4} \times (\frac{1}{4})^2 = \frac{3}{64}$ . If asked to find the probability that one child forgets goggles whilst the other two have them this probability would be multiplied by 3.



## The STEP question

3 (i) As long as you have a £1 coin person first, then you can give change to the one £2 coin person whenever they arrive. The answer is therefore  $\frac{m}{m+1}$ .

(ii) Here the options that work are (omitting the pound signs):

- 1, 1, anything (as then you have two £1 coins for change)
- 1, 2, 1, anything

This gives  $P = \frac{m}{m+2} \times \frac{m-1}{m+1} + \frac{m}{m+2} \times \frac{2}{m+1} \times \frac{m-1}{m}$  which gives the required result.

You can consider the options that **don't** work instead (which will be the sequences starting “2,” and “1, 2, 2,”).

(iii) Here the options that work are:

- 1, 1, 1, anything
- 1, 1, 2, 1, anything
- 1, 1, 2, 2, 1, anything
- 1, 2, 1, 1, anything
- 1, 2, 1, 2, 1, anything

You need to be sure you have covered all the possible options, and explain in your proof why this is so. Working systematically helps.

The fractions look a bit of a mess, but there are lots of common factors in the numerators and denominators that you can factorise out as a first step.

This was a question from the Probability and Statistics section of a STEP paper. Note that it doesn't actually need anything other than GCSE probability and algebraic fraction manipulation, but does require you to read the question carefully, work logically and have fluent algebraic manipulation skills.



## Warm down

- 4 Start by calling the person in part **(ii)** Yolanda ( $Y$ ), the person in part **(iii)** Xander ( $X$ ), and the person in part **(iv)** Zoe ( $Z$ ). You can then put all the information from the first 5 parts of the question in a table:

Race	1st	2nd	3rd	4th	5th
There	X		Z	A	Y
Back	Y	Z	X	B	

You now know that Anna and Bryan are **not** any of  $X$ ,  $Y$  and  $Z$ . The missing space for the race to the tree must be Bryan and the missing space on the race back must be Anna. Then use the information from the last two parts to work out which of  $X$ ,  $Y$  and  $Z$  must be Charlie.

Final answer:

Race	1st	2nd	3rd	4th	5th
There	E	B	C	A	D
Back	D	C	E	B	A

