

STEP Support Programme

Hints and Partial Solutions for Assignment 7

Warm-up

You can check your graphs by plotting them using [Desmos](#).

- 1 (i) (a) For $y = x + \frac{1}{x}$, there are two turning points. For $y = x - \frac{1}{x}$, the gradient is positive for all (non-zero) x so there are no stationary points (the graph heads “upwards” at each point).

- (b) For both graphs, $y \rightarrow +\infty$ as $x \rightarrow +\infty$ and $y \rightarrow -\infty$ as $x \rightarrow -\infty$, but we can do a little better than that. As $x \rightarrow \pm\infty$, the $\pm\frac{1}{x}$ term becomes negligible, so we can say that $y \rightarrow x$ as $x \rightarrow \pm\infty$, in both cases.

To describe this behaviour, we say that $y = x$ is an *asymptote* — which means that the gap between the graph and the line $y = x$ approaches zero as $x \rightarrow \pm\infty$. It is good to draw the asymptotic line on the graph, and we usually draw it as a dotted line.

- (c) The behaviour as $x \rightarrow 0$ depends on whether x is positive or negative. Consider the first graph:

$$y = x + \frac{1}{x}.$$

As x tends to 0 from the positive direction, y tends to $+\infty$ (as $x + \frac{1}{x} > 0$ when $x > 0$), but as x tends to 0 from the negative direction, y tends to $-\infty$. We can write this very conveniently as:

$$\text{as } x \rightarrow 0_+, y \rightarrow +\infty \text{ and as } x \rightarrow 0_-, y \rightarrow -\infty.$$

- (d) Neither graph intersects the y -axis (which would require $x = 0$). For the second graph $y = 0$ when $x = \pm 1$, so it intersects the x -axis at these values of x . For the first graph, there are no intersections with the x -axis since there are no (real) values of x that satisfy the equation $y = 0$.

- (ii) (a) Multiplying throughout by x is a **bad** idea: when x is negative the inequality sign will be reversed, which makes it rather complicated. Instead draw the line $y = 2$ onto your graph of $y = x + \frac{1}{x}$.

Since the inequality is strict, you must be careful to exclude $x = 1$ (which gives equality) from your set of values. You can write “ $0 < x < 1$ or $x > 1$ ” or “ $x > 0$, $x \neq 1$ ” (or something else equivalent).

- (b) Replace the inequality sign with an equal sign and solve the resulting equation. Then sketch the line $y = \frac{3}{2}$ onto your graph to see where the inequality holds, leading to $\frac{1}{2} \leq x < 0$ or $x \geq 2$.



Preparation

- 2** **(i)** You can eliminate c by considering $(\alpha^2 + b\alpha + c) - (\beta^2 + b\beta + c) = 0$. When dividing by $(\alpha - \beta)$ it is good practice to say that you can do so because it is given that $\alpha \neq \beta$. If $\alpha \neq \beta$ were not given, then you would need to consider the case $\alpha - \beta = 0$ separately. Substituting b back into one of the equations gives $c = \alpha\beta$.
- (ii)** This part is working in the opposite direction to part **(i)**, i.e. it is starting with $x^2 + bx + c \equiv (x - \alpha)(x - \beta)$.
- (iii)** $\alpha = 2, \beta = 5$ or the reverse.
- (iv)** Comparing the given result and what you are trying to show should suggest that substituting $x = 0, 1$ and -1 is probably a good idea.
- (v)** $\alpha = 0, -2, 2, -4, 4, -6, 14$ or -16 . Since $\alpha\beta\gamma = -16$ we can rule out $0, -6$ and 14 .
 You also have $(1 - \alpha)(1 - \beta)(1 - \gamma) = 9$, which means that the only possible values of α are $-2, 2$ and 4 .
 The final answer is $\alpha = -2, \beta = 2, \gamma = 4$.

The STEP question

- 3** To obtain the three given, results substitute in $x = 0, x = 1$ and $x = -1$. Do be careful to write $(-1)^n$ and not -1^n . The third equation is $(k_1 - 1)(k_2 - 1) \cdots (k_n - 1) = a_n - a_{n-1} + a_{n-2} + \cdots + (-1)^{n-1}a_1 + (-1)^n$.

Looking at specific cases (such as $n = 3$ and $n = 4$) can help you generalise.

To find the roots, it is easiest to start with $(k_1 - 1)(k_2 - 1)(k_3 - 1)(k_4 - 1) = 175 = 1 \times 5 \times 5 \times 7$. There are 12 different possibilities for $k_1 - 1$ (and hence for k_1).

We also have $k_1 k_2 k_3 k_4 = 576 = 1 \times 2^6 \times 3^2$. This means that k_1 can only have factors of 1, 2 and 3, and cannot be 0. You can use this to eliminate 5 values of k_1 .

You can then use $(k_1 + 1)(k_2 + 1)(k_3 + 1)(k_4 + 1) = 1323$ to eliminate everything but 3 values.

It is a very good idea to check your answer by expanding the brackets and show that this is the given quartic.

Please note that the question asks for the **roots** so the answer is **not** 2, 6, 6, 8 (which are the values of k_i).



Warm down

4 This problem is known as “Bachet’s Weights Problem”, after Claude Gaspard Bachet de Méziriac (1581-1638) who published it in 1624, but it is thought to date back to Fibonacci in 1202, making it one of the earliest problems in integer partitions.

(i) (a) If we are only putting weights on one side, then we obviously need a weight of 1oz, and we need either another 1oz or a 2oz to make 2oz. With the 2oz we can make 1oz, 2oz and 3oz (whereas if we had picked a second 1oz we would only be able to make 1oz or 2oz). If we then add a 4oz we can make 1oz, 2oz, 3oz, 4oz, 5oz, 6oz and 7oz etc.

(b) 1oz, 2oz, 4oz, 8oz and 16oz.

(c) Each weight is either in the pan or not, so for each weight there are two options and with n weights there are $2 \times 2 \times \dots \times 2 = 2^n$ options in total. Note that the weights have to be carefully picked if you are going to achieve 2^n *different* weights; for example, if you chose 1oz, 2oz, 3oz and 6oz, then two different combinations, $(1 + 2 + 3)$ oz or 6oz, give the *same* weight.

There is a connection with binary numbers here — the number 10 can be written as 1010, which is one 8, no 4’s, one 2 and no 1’s. The number 31 is 11111 — i.e. one 16, one 8, one 4, one 2 and one 1. The 1’s and 0’s correspond to whether a weight is in the pan or not.

If you have the weights $1, 2, 4, \dots, 2^{n-1}$ then you can make all the numbers up to $1 + 2 + 4 + \dots + 2^{n-1} = 2^n - 1$.

(ii) (a) If we have the weights 1oz and 3oz, then we can obviously weigh 1oz and 3oz, if we put the two together in the same pan then we can weigh 4oz and if they are in different pans then we can weigh the difference; i.e. 2oz.

(b) There are **three** options for each weight this time: left pan, right pan or no pan, so there are 3^n different arrangements of the weights.

However we **cannot** actually weigh 3^n **different** weights.

To find the total number of distinct non-zero weights we can weigh, start by considering the 3^n possible distributions of our weights in the two pans. Take off 1 for the case where no weights are in either pan, which leaves us with $3^n - 1$ distributions. Then for each distribution, there is a “mirror image” where the weights swap pans. Both a given distribution and its mirror distribution weigh the same, so we need to divide by 2 giving at most $\frac{3^n - 1}{2}$ distinct weights.

(c) Guided by the previous parts, we try 1, 3, 9 and 27 oz, which works (and is in fact the only possibility with four weights).

Bachet proposed that the weights of $1, 3, 3^2, \dots, 3^n - 1$ ounces enable one to weigh any number of weights from 1oz to $(1 + 3 + 3^2 + \dots + 3^{n-1}) = \frac{1}{2}(3^n - 1)$ but he did not prove that this was the least possible number of weights. Major Percy MacMahon (1854-1929) proved this and that this solution is unique (so, for example, there is no other set of 4 weights that can weigh all integer weights from 1oz to 40oz).

