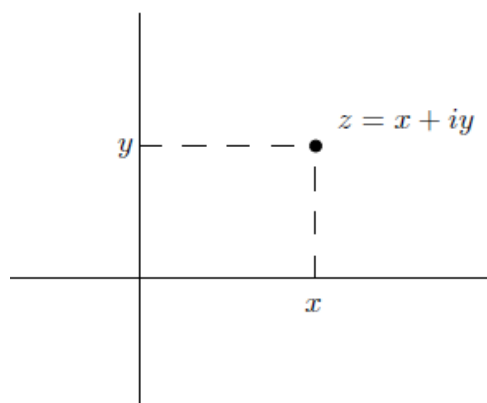


Complex Numbers

A complex number is an expression of the form $x + iy$ where $x, y \in \mathbb{R}$. (Here $i = \sqrt{-1}$ so that $i^2 = -1$). We denote the set of complex numbers by \mathbb{C} . We can represent \mathbb{C} as the Argand diagram or complex plane by drawing the point $x+iy \in \mathbb{C}$ as the point with co-ordinates (x, y) in the plane \mathbb{R}^2 (see Figure, below).



Pay attention that,

$$z = x + iy$$

↑ ↙
real part imaginary part.

If $z = x + iy$, $x, y \in \mathbb{R}$,

the real part of $z = \Re(z) = \text{Re}(z) = x$

the imaginary part of $z = \Im(z) = \text{Im}(z) = y$.

eg. $z = 3 + 4i$

$$\Re(z) = 3$$

$$\Im(z) = 4.$$

If $z = x + iy$, then \bar{z} (“ z bar”) is given by

$$\bar{z} = x - iy$$

and is called the *complex conjugate* of z .

eg. If $z = 3 + 4i$, then $\bar{z} = 3 - 4i$.

Application of complex numbers

Example. Solve $x^2 - 2x + 3 = 0$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2\sqrt{-2}}{2} = 1 \pm \sqrt{2}i.$$

*****Addition/subtraction and multiplication/division of complex numbers**

If $a + ib, c + id \in \mathbb{C}$ then we can add and multiply them as follows

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc).$$

$$(2 + 3i) + (4 + i) = 6 + 4i.$$

$$(8 - 3i) - (-2 + 4i) = 10 - 7i.$$

$$(2 + 3i)(1 + 2i) = 2 + 4i + 3i - 6 = -4 + 7i$$

$$(3 - 2i)(3 + 2i) = 9 - (2i)^2 = 9 + 4 = 13$$

Notice that,

when we multiply two complex conjugates, we get a real number.

$$\frac{1}{a + ib} = \frac{1}{a + ib} \frac{a - ib}{a - ib} = \frac{a - ib}{a^2 - i^2b^2} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}.$$

$$\frac{2+3i}{1+4i} = \frac{2+3i}{1+4i} \times \frac{1-4i}{1-4i} = \frac{(2+3i)(1-4i)}{(1+4i)(1-4i)} = \frac{2-8i+3i-12i^2}{1-(4i)^2} = \frac{14-5i}{17}$$

Properties

Let $z, w \in \mathbb{C}$. Then

(i) $|z| = 0$ if and only if $z = 0$;

(ii) $|zw| = |z||w|$;

(iii) $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ if $z \neq 0$;

(iv) $|z + w| \leq |z| + |w|$;

(v) $||z| - |w|| \leq |z - w|$.

Prove only (iv), first note that if $z = x + iy$ then

$$\operatorname{Re}(z) = x \leq \sqrt{x^2 + y^2} = |z|.$$

Thus we have,

$$\begin{aligned} |z + w|^2 &= (z + w)\overline{(z + w)} \\ &= (z + w)(\bar{z} + \bar{w}) \\ &= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \\ &= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w \\ &= |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}) \text{ using Exercise 1.5(iv)} \\ &\leq |z|^2 + |w|^2 + 2|z\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||\bar{w}| \\ &= |z|^2 + |w|^2 + 2|z||w| \\ &= (|z| + |w|)^2. \end{aligned}$$

The Argand diagram or the complex plane.

We say that $z \in \mathbb{C}$ is *real* if $\operatorname{Im}(z) = 0$ and we say that $z \in \mathbb{C}$ is *imaginary* if $\operatorname{Re}(z) = 0$. In the complex plane, the set of real numbers corresponds to the x -axis (which we will often call the *real axis*) and the set of imaginary numbers corresponds to the y -axis (which we will often call the *imaginary axis*).

If $z = x + iy$, $x, y \in \mathbb{R}$ then we define $\bar{z} = x - iy$ to be the *complex conjugate* of z .

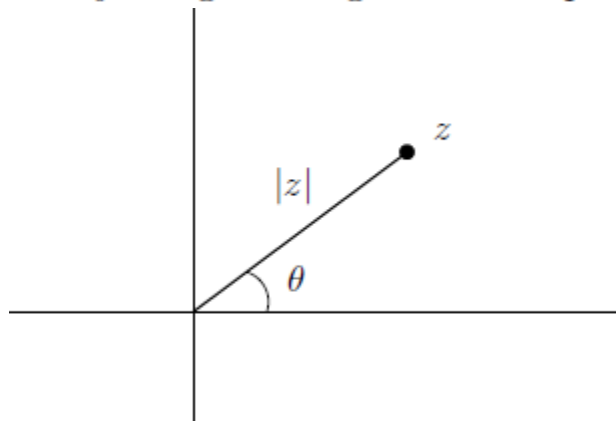
Let $z = x + iy$, $x, y \in \mathbb{R}$. The *modulus* (or *absolute value*) of z is

$$|z| = \sqrt{x^2 + y^2} \geq 0.$$

(If z is real then this is just the usual absolute value.) It is straightforward to check that $|\bar{z}| = |z|$ and that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Let $z \neq 0$. If we plot the point z in the complex plane then $|z|$ denotes the length of the vector joining the origin 0 to the point z .



The angle θ is called the argument of z and we write $\theta = \arg z$.

We have that $\tan \theta = y/x$. Note that θ is

not uniquely determined: if we replace θ by $\theta + 2n\pi$, $n \in \mathbb{Z}$, then we get the same point. However, there is a unique value of θ such that $-\pi < \theta \leq \pi$; this is called the *principal value* of $\arg z$. We write $\text{Arg}(z)$ for the principal value of the argument of z .

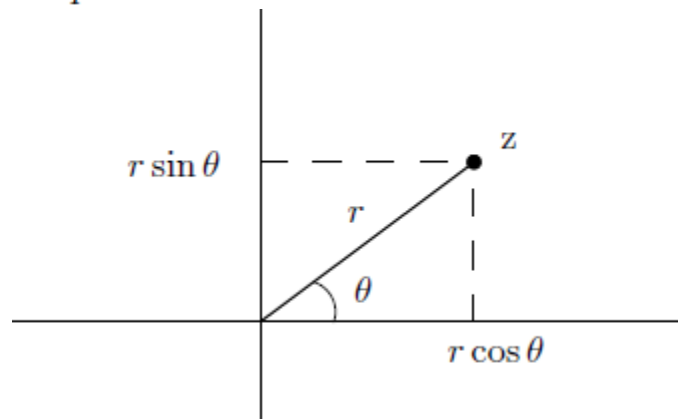
Let $z \in \mathbb{C}$. We can represent z in polar co-ordinates as follows.

$$z = x + iy$$

Then $x = r \cos \theta$ and $y = r \sin \theta$.

θ is the argument of z and $r = \sqrt{x^2 + y^2} = |z|$.

We call (r, θ) the *polar co-ordinates* of z and write $z = r(\cos \theta + i \sin \theta)$.



If z has polar co-ordinates (r, θ) then the real part of z is $r \cos \theta$ and the imaginary part of z is $r \sin \theta$, and conversely.

Example 1. Find x, y if $(3 + 4i)^2 - 2(x - iy) = x + iy$.

$$\begin{aligned} \text{Left hand side (LHS)} &= 9 - 16 + 24i - 2x + i2y \\ &= -7 - 2x + i(24 + 2y) \end{aligned}$$

Follows that,

$$\therefore -7 - 2x = x$$

$$3x = -7$$

$$x = -\frac{7}{3}$$

$$\& 24 + 2y = y$$

$$y = -24$$

Example 2. Find x, y if $\frac{x}{1+i} + \frac{y}{2-i} = 2 + 4i$.

$$\text{LHS} = \frac{x}{1+i} + \frac{y}{2-i}$$

$$\begin{aligned}
&= \frac{x}{1+i} \times \frac{1-i}{1-i} + \frac{y}{2-i} \times \frac{2+i}{2+i} \\
&= \frac{x(1-i)}{1+1} + \frac{y(2+i)}{4+1} \\
&= \frac{x(1-i)}{2} + \frac{y(2+i)}{5}
\end{aligned}$$

$$\text{Now } \frac{x(1-i)}{2} + \frac{y(2+i)}{5} = 2 + 4i.$$

$$\therefore 5x(1-i) + 2y(2+i) = 20 + 40i$$

$$5x - i5x + 4y + i2y = 20 + 40i$$

$$5x + 4y + i(-5x + 2y) = 20 + 40i$$

Equating real and imaginary part,

$$5x + 4y = 20$$

$$-5x + 2y = 40$$

————— Solving simultaneously,

$$6y = 60$$

$$y = 10$$

$$\& \therefore x = -4. \quad \square$$

EXERCISES:

1.

Write the following expressions in the form $x + iy$, $x, y \in \mathbb{R}$:

$$(i) (3 + 4i)^2; \quad (ii) \frac{2 + 3i}{3 - 4i}; \quad (iii) \frac{1 - 5i}{3i - 1}; \quad (iv) \frac{1 - i}{1 + i} - i + 2; \quad (v) \frac{1}{i}.$$

2

Given the complex numbers $u = 1 + 3i$ and $v = 3 + 4i$, find the complex numbers:

a $(u - v)^2$

b $(3u - 2v)^2$.

3.

Given the complex numbers $u = 3 - i$ and $v = 4 - 3i$, find the complex numbers:

a u^{-1}

b $\frac{u}{v}$.

4.

Given the complex numbers $u = 1 + 3i$ and $v = 3 + 4i$, find the complex number $\frac{u + v}{u - v}$.

5.

By writing $z = x + iy$ find all solutions of the following equations:

$$(i) z^2 = -5 + 12i; \quad (ii) z^2 + 4z + 12 - 6i = 0.$$

6.

Let $z, w \in \mathbb{C}$. Show that (i) $\overline{z \pm w} = \bar{z} \pm \bar{w}$, (ii) $\overline{z\bar{w}} = \bar{z}w$, (iii) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$ if $z \neq 0$, (iv) $z + \bar{z} = 2\operatorname{Re}(z)$, (v) $z - \bar{z} = 2i\operatorname{Im}(z)$.

7.

Find the values of x and y if $4x - 2iy + 3ix - 4y = -6 - i$.

8.

Find the values of x and y if $(x + yi)(3 - 2i) = 6 - i$.

9.

Find the complex number z if $\frac{z - i}{z + i} = 2 + i$.

10.

Determine the complex number z if $\frac{5(z + 2i)}{z - 2} = 11 - 2i$.

11.

Find $\operatorname{Im}\left(\frac{25}{4 - 3i} + i^{77}\right)$.

12.

Find $\operatorname{Re}\left(\frac{10}{1 + 3i} + i^{96}\right)$.

13.

Simplify :

$$a \frac{10i^3}{\sqrt{-25}} \quad b \frac{-7i^6}{\sqrt{-49}} \quad c \frac{\sqrt{-18}}{3i^9} \quad d \frac{-\sqrt{-75}}{i^{10}} \quad e \frac{-\sqrt{-72}}{4i^6}$$

14.

Solve each of the following for z .

a $z^2 - 49 = 0$

b $z^2 + 49 = 0$

c $4z^2 + 9 = 0$

d $3z^2 + 25 = 0$

e $2z^2 + 81 = 0$

f $az^2 + b = 0$ if $ab > 0$