

STEP Support Programme

Hints and Partial Solutions for Assignment 14

Warm-up

- 1 (i) You can replace b with $-b$ to get the second result (or if you are uncomfortable with this you can let $b = -c$). Expand your answer for $a^3 + b^3$ to check your factorisation was correct.

(ii) (a) $x^3 + 5^3 = (x + 5)(x^2 - 5x + 25)$.

- (b) You may get slightly different answers depending on whether your first step is

$$(x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3)$$

(difference of two squares) or

$$(x^2)^3 - (y^2)^3 = (x^2 - y^2)(x^4 + x^2y^2 + y^4)$$

(using the result of part (i)).

The expression can be written as $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$, which is easier to spot if you take the first approach above.

- (iii) The sum of the geometric progression is $\frac{1 - t^5}{1 - t}$ (or equivalent). This means that we have $(1 - t)(1 + t + t^2 + t^3 + t^4) = 1 - t^5$.

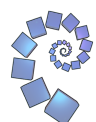
To deduce the factorisation for $a^5 - b^5$, use the substitution $t = \frac{b}{a}$ and then multiply by b^5 . For $a^5 + b^5$ you can replace b with $-b$ as in part (i).

- (iv) Multiplying out the brackets gives $x^4 + y^4$.

Then, taking $y = 1$, you get

$$x^4 + 1 = (x^2 + 1 - \sqrt{2}x)(x^2 + 1 + \sqrt{2}x).$$

Setting this equal to zero gives two quadratic equations which you can solve to find the four fourth roots of -1 . Clearly, they will be complex (not real), since $x^4 \geq 0$ if x is real. They can be written as $x = \frac{\pm\sqrt{2}(1 \pm i)}{2}$.



Preparation

2 (i) You should get $\frac{3}{2}$.

(ii) This is a GP of the form $\sum r^k$, so for convergence you need to show that $|r| < 1$, which you can do by noting (or proving) that $\sqrt{3} < 2$. The sum to infinity is $3(2 + \sqrt{3})$.

The last bit says “write down”, which suggests that no real work is required. You can do this by just replacing $\sqrt{3}$ with $-\sqrt{3}$. A quick sanity check tells us that each term in the last series is smaller than the corresponding term in the first series, so you would expect the sum to be smaller.

(iii) By factorising as suggested you get $(x - y)(x + y) = z$ i.e. $z(x + y) = z$. This means that either $z = 0$ or $x + y = 1$. It is important that you consider the case $z = 0$ separately and don't just divide by z .

It turns out that $x^2 = -2$ when $z = 0$, so there are no real solutions in this case. In fact, the question does not ask for *real* solutions, so you should give this one as well ($x = \pm i\sqrt{2}$, $y = \pm i\sqrt{2}$, $z = 0$).

If $z \neq 0$, then we have $x + y = 1$ and so we have $x(1 - x) = -2$. The solutions in this case are $x = 2, y = -1, z = 3$ and $x = -1, y = 2, z = -3$.



The STEP question

- 3** (i) For $n = 0$, $F_0 = 0$ so $a + b = 0$. Thus $b = -a$ and $F_n = a(\lambda^n - \mu^n)$. The other equations ($n = 1, 2, 3$) are:

$$\begin{aligned} a(\lambda - \mu) &= 1 \\ a(\lambda^2 - \mu^2) &= 1 \\ a(\lambda^3 - \mu^3) &= 2. \end{aligned}$$

You can then use the result from question 1(i) to get the required result (noting that $\lambda \neq \mu$). You can also use the first two equations to get $\lambda + \mu = 1$.

When solving the equations, it is helpful to notice that as $a > 0$ and $a(\lambda - \mu) = 1$ then $\lambda > \mu$.

The final answers are:

$$a = \frac{1}{\sqrt{5}}, \quad b = -\frac{1}{\sqrt{5}}, \quad \lambda = \frac{1 + \sqrt{5}}{2}, \quad \mu = \frac{1 - \sqrt{5}}{2}.$$

- (ii) If you noticed that the sequence is the Fibonacci sequence, then you should easily calculate that $F_6 = 8$ using the implicit formula $F_{n+2} = F_{n+1} + F_n$. However you have to show this by using $F_n = a(\lambda^n - \mu^n)$, and as the answer is known (or easily found by other means) you must be careful to fully justify your answer. It is helpful to factorise the brackets, i.e. write it as:

$$F_6 = \frac{1}{\sqrt{5}} \times \frac{1}{2^6} \left[(1 + \sqrt{5})^6 - (1 - \sqrt{5})^6 \right].$$

You could use $x^6 - y^6$ to help simplify $(1 + \sqrt{5})^6 - (1 - \sqrt{5})^6$, but it is probably simpler to use a binomial expansion.

- (iii) Note that the sum can be written as $\sum_{n=0}^{\infty} \frac{a(\lambda^n - \mu^n)}{2^{n+1}}$ or $\frac{a}{2} \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^n - \frac{a}{2} \sum_{n=0}^{\infty} \left(\frac{\mu}{2}\right)^n$. It is probably easier to find the infinite sums in terms of a , λ and μ before substituting for them. The final answer is 1.



Warm down

4 If we start on the first day with:

Group 1 : ABC

Group 2 : DEF

Group 3 : GHI

then on subsequent days A , B and C must be in separate groups, say A in group 1, B in group 2 and C in group 3. D must then go with A , B and C in turn etc.

One solution is:

Day 1	Day 2	Day 3	Day 4
ABC	ADG	AFH	AEI
DEF	BEH	BDI	BFG
GHI	CFI	CEG	CDH

