

## STEP Support Programme

### Hints and Partial Solutions for Assignment 10

#### Warm-up

- 1 (i) Note that  $AP = \tan \alpha$  and  $PC = \tan \beta$ . You can also show that  $AB = \frac{1}{\cos \alpha}$ . The result  $\sin(90^\circ - \beta) = \cos \beta$  will be helpful.

Do be careful with brackets, write

$$\cos \alpha \cos \beta \times (\tan \alpha + \tan \beta)$$

rather than

$$\cos \alpha \cos \beta \times \tan \alpha + \tan \beta.$$

You can find  $\sin(\alpha - \beta)$  from  $\sin(\alpha + (-\beta))$  using the previous result, along with  $\sin(-\beta) = -\sin \beta$  and  $\cos(-\beta) = \cos \beta$ .

You can find  $\cos(\alpha + \beta)$  from  $\cos(\alpha + \beta) = \sin(90^\circ - (\alpha + \beta)) = \sin((90^\circ - \alpha) - \beta)$  and similarly for  $\cos(\alpha - \beta)$ .

Note that when you have the expression for  $\sin(\alpha + \beta)$ , you can derive  $\sin(\alpha - \beta)$  and  $\cos(\alpha \pm \beta)$  with hardly any extra work. With these you can also derive  $\tan(\alpha \pm \beta)$  fairly painlessly.

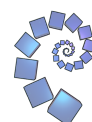
- (ii)  $\sin 75^\circ = \sin(45^\circ + 30^\circ)$  and  $\sin 15^\circ = \sin(45^\circ - 30^\circ)$ . At the risk of repeating ourselves: when a question asks for **values** then decimal approximations are not what are wanted.

$$\text{Answers: } \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}} \text{ and } \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

- (iii) You will need expressions for  $\sin 2A$  and  $\cos 2A$ , but these can be found from the results in part (i), for example  $\sin 2A = \sin(A + A) = \dots$ . You will probably also need to use  $\sin^2 A + \cos^2 A = 1$ .

It is a good idea to check your answer, for example by substituting  $A = 30^\circ$ .

$$\text{Answer: } \cos 3A = 4 \cos^3 A - 3 \cos A.$$



## Preparation

- 2 (i) Things will be much easier if you write  $1 - \frac{1}{2} = \frac{1}{2}$  etc. before you try to multiply. The prime factors of 120 are 2, 3, and 5. The final answer is 32.
- (ii) Note that if  $a$  is a *positive* integer then  $a - 1$  is an integer satisfying  $a - 1 \geq 0$ . If you expand you get  $x^a - x^{a-1}$ , and since  $x$  and  $a$  are integers and  $a - 1 \geq 0$  this is an integer subtracted from an integer, hence is an integer!
- (iii)  $39600 = 2^4 \times 3^2 \times 5^2 \times 11$  and  $52920 = 2^3 \times 3^3 \times 5 \times 7^2$  so the HCF is  $2^3 \times 3^2 \times 5 = 360$ .
- (iv) These sorts of statements can be a bit confusing. I usually re-order them in my head, so  
 “ $a = b$  **if**  $a^2 = b^2$ ” becomes “**if**  $a^2 = b^2$  **then**  $a = b$ ” which I can now see is false  
 and  
 “ $a = b$  **only if**  $a^2 = b^2$ ” becomes “**only if**  $a^2 = b^2$  **is it the case that**  $a = b$ ” which I can now see is true.
- (a) True:  $ab$  even  $\Leftrightarrow a$  and  $b$  both even.  
 (b) False as  $ab$  would be even if just one of  $a$  and  $b$  were even, so  $ab$  even  $\not\Rightarrow a$  and  $b$  both even.  
 (c) False: either  $a = b$  or  $a = -b$  if  $a^2 = b^2$ , so  $a = b \not\Leftrightarrow a^2 = b^2$ .  
 (d) True:  $a = b \Rightarrow a^2 = b^2$ .  
 (e) True: equilateral  $\Leftrightarrow$  three equal sides.  
 (f) True: equilateral  $\Rightarrow$  three equal sides.
- (v) (a) True.  
 (b) False. An odd number is prime **if** it is three, but this is not an **only if**.  
 (c) False.  $x = 3$  only if  $x^2 - 9 = 0$  (i.e. this is not an **if**).  
 (d) False. It is certainly the case that the triangle is right-angled **if**  $a^2 + b^2 = c^2$ , but this is not a necessary condition (it is not **only if**): the triangle is also right-angled if  $a^2 = b^2 + c^2$  (i.e.  $a$  is the hypotenuse).  
 (e) True.



## The STEP question

- 3 (i) (a)  $f(12) = 4$  and  $f(180) = 48$ .
- (b) If you use the hint provided and write  $N = p_1^{a_1} \times p_2^{a_2} \times \dots$  then you should find that  $f(N) = p_1^{a_1-1}(p_1 - 1)p_2^{a_2-1}(p_2 - 1) \dots$ . You can then explain why this is an integer.
- (ii) (a) A simple counterexample (remember the simpler the better!) would be  $f(12) \neq f(2) \times f(6)$ . You show it is indeed a counterexample by calculating the three values of the function.
- (b) This result is true, and can be shown just by using the definitions such as  $f(p) = p \left(1 - \frac{1}{p}\right)$ . When  $p$  and  $q$  are not distinct ( $p = q$ ), the result is not true.
- (c) Try  $f(4)$  and  $f(15)$ . The statement  $f(p)f(q) = f(pq)$  holds as long as  $p$  and  $q$  are *co-prime*, that is that they have no prime factors in common. When  $p$  and  $q$  have a prime factor in common then  $f(pq) \neq f(p)f(q)$ .
- (iii) Start by writing  $f(p^m) = p^m \left(1 - \frac{1}{p}\right) = p^{m-1}(p - 1)$ . Then express 146410 as a product and compare this to  $p^{m-1}(p - 1)$  so that you get  $p^{m-1}(p - 1) = 11^4 \times 10$ . You should then be able to spot what  $p$  and  $m$  are.

## Warm down

- 4 (i) Leave answers in fractions (unless specifically asked for a decimal approximation). Answer:  $\frac{144}{13}$ .
- (ii) There is nothing wrong with the negative solution (nothing in the question says that the numbers have to be positive) so put down both solutions. Answer:  $(7, 4)$  or  $(-4, -7)$ .
- (iii) Two of the terms simplify nicely, but the other two do not.  
Answer:  $ax + by - a^{\frac{1}{4}}b^{\frac{2}{5}}x^{\frac{1}{3}}y^{\frac{1}{6}} - a^{\frac{3}{4}}b^{\frac{3}{5}}x^{\frac{2}{3}}y^{\frac{5}{6}}$ .
- (iv) Start by multiplying numerator and denominator by  $\sqrt{3} - 1$ .  
Simplifying  $\sqrt{12 + 6\sqrt{3}} \times (\sqrt{3} - 1)$  can be little complicated. One way to proceed it to take the  $\sqrt{3} - 1$  inside the square root, so simplify  $\sqrt{(12 + 6\sqrt{3}) \times (\sqrt{3} - 1)^2}$ . Everything should simplify nicely to give  $\sqrt{3}$ .

