

## Mathematics problems

- 1) A small block of mass  $m$  slides on the surface of a second larger inclined block of mass  $M$ , as shown in Figure 1. This second block is on a flat horizontal surface, on which it is free to slide, and makes an incline with the surface at an angle  $\theta$ . At time  $t = 0$  the system is at rest.

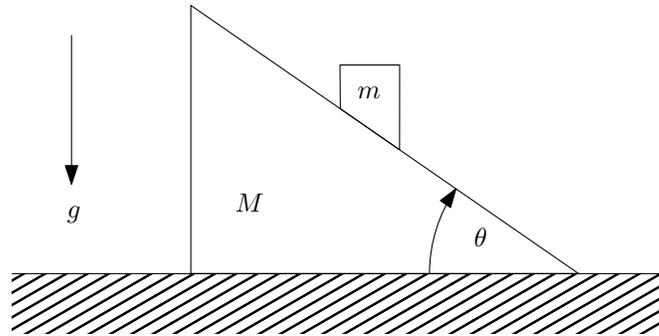


Figure 1: Two sliding blocks.

- a) Find the equations of motion of the two blocks, and write the equations for the acceleration of the two blocks in the horizontal and vertical directions.
  - b) Calculate the size of the internal forces between the two blocks.
  - c) Find the acceleration of the smaller block relative to its starting point.
  - d) Comment on your results in the limit  $\frac{M}{m} \gg 1$ .
- 2) A uniform chain of length  $L$  and mass  $M$  falls through a frictionless hole in the table, accelerating under gravity  $g$ , as shown in Figure 2. Initially the chain is at rest, stretched out with a length  $h$  hanging from the hole.

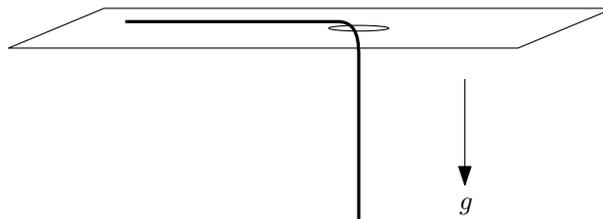


Figure 2: A falling chain.

- a) How long will it take for the chain to fall from the table?
  - b) The same chain is then picked up and suspended vertically above the surface of a set of measuring scales, and released from rest to fall onto the scales. What will the scales read when half of the chain has fallen onto the scales?
- 3) The integral of the Gaussian function is  $\int_{\mathbb{R}} \exp(-x^2) dx = \sqrt{\pi}$ , (c.f. the probability density function of the normal distribution). We define a constant  $\alpha > 0$  where  $\alpha \in \mathbb{R}^+$ .
- a) Find  $\int_{\mathbb{R}} x^2 \exp(-x^2) dx$  and  $\int_{\mathbb{R}} x^4 \exp(-x^2) dx$ , and hence derive an expression for  $\int_{\mathbb{R}} x^{2n} \exp(-x^2) dx$  where  $n \in \mathbb{N}$ .

- b) Find  $\int_{\mathbb{R}^+} x \exp(-x^2) dx$ ,  $\int_{\mathbb{R}^+} x^3 \exp(-x^2) dx$ , and  $\int_{\mathbb{R}^+} x^5 \exp(-x^2) dx$ , and hence derive an expression for  $\int_{\mathbb{R}^+} x^{2n+1} \exp(-x^2) dx$  where  $n \in \mathbb{N}$ .
- 4) We recall the factorial function of a positive integer  $n$  is defined by  $n! := n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ , with the convention  $0! := 1$ .
- a) Show  $n! \equiv \int_{\mathbb{R}^+} x^n \exp(-x) dx$ .
- b) By defining the function  $f$  through  $\exp(f(x)) := x^n \exp(-x)$ , build a second order quadratic model of  $f$  around its maximum using a Taylor series expansion.
- c) Using the previous result, find an approximate formula for  $\log(n!)$  correct to  $\mathcal{O}(\log(n))$  for  $n \gg 1$ . (You will have proved Stirling's formula).
- 5) A canon is on an inclined slope. The slope is at an incline to the horizontal with an angle  $\alpha$ , and the canon is at a relative incline to the surface with an angle  $\theta$ , and can fire either up or down the incline, as shown in Figure 3. The canon fires projectiles at a speed  $v$ .

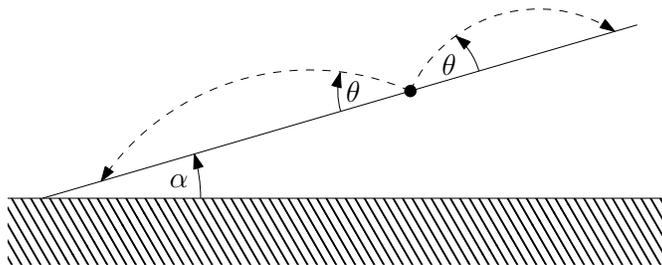


Figure 3: An inclined canon.

- a) Find expressions for the maximum ranges up and down the plane, and show that the ratio of these is  $\frac{1-\sin(\alpha)}{1+\sin(\alpha)}$ .
- b) Find the projectile's distance up the plane when  $\alpha = \theta = 30^\circ$ .
- 6) A rocket has an initial mass  $M$ , of which half is fuel. The rocket is launched vertically upwards at  $t = 0$ . The rocket burns fuel at a mass rate  $\alpha$ , and this fuel is ejected from the back of the rocket at a speed  $v$  relative to the rocket. Assume the gravitational field of the rocket is constant throughout the rocket's flight.
- a) Find the equation of motion of the rocket, and show that the rocket cannot leave the launch pad at ignition unless  $\alpha v > Mg$ .
- b) Given the rocket leaves the launch pad at ignition, at what time does the fuel burn-out?
- c) What is the velocity of the rocket at the burn-out time?
- d) What is the maximal height reached by the rocket?
- 7) A builder is at a latitude  $\lambda$  on the northern hemisphere and hangs a plumb-bob.
- a) Find an expression for the deflection of the plumb bob from the local vertical.
- b) Given the radius of the earth is approximately 6'400km, evaluate this deflection for  $\lambda = 51.5^\circ$  (the latitude of London) and comment on its size.
- 8) Drops of water are thrown off of a half submerged wheel, which rotates at an angular velocity  $\omega$  and radius  $a$ , as shown in Figure 4.
- a) Show that for drops to be thrown off of the wheel  $\omega > \sqrt{\frac{g}{a}}$ .
- b) Provided the ceiling is at a height  $h$  above the water, show that for the ceiling to remain dry  $\omega$  must satisfy  $(a\omega)^2 < gh + g\sqrt{h^2 - a^2}$ .
- 9) A twice differentiable function  $f(\cdot) \in C^2$ , has an unknown root  $x^*$  such than  $f(x^*) = 0$ , (there may be multiple  $x^*$ , cf. the fundamental theorem of calculus).

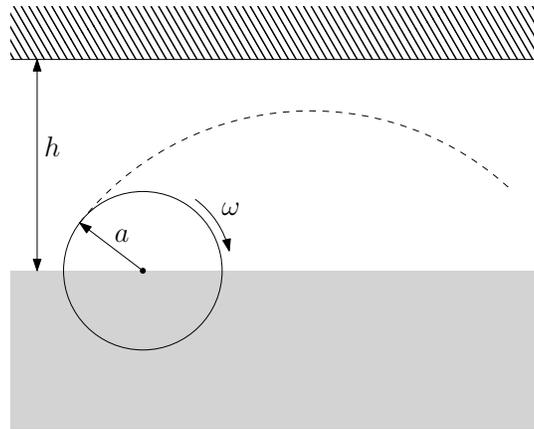


Figure 4: Water drops being thrown off of a rotating wheel.

- a) Consider a value  $x \approx x^*$ , and write the Taylor expansion of  $f$  in the neighbourhood of  $x$  up to the first  $n$  terms.
- b) Write the Taylor expansion of  $f$  in the neighbourhood of  $x^*$ .
- c) By considering the appropriate Taylor expansion up to first order, compute an approximate value of  $x^*$ , and hence construct an iterative scheme for the root of  $f$ . (This is called the Newton-Raphson method).
- d) Construct an iterative scheme using the second order Taylor expansion. Comment on any differences between the this schemes and the scheme from (9.c).
- e) Consider the limit of the scheme from (9.d) as  $f'' \rightarrow 0$ . Comment on the behaviour of the iterative scheme, and explain why one of the iterative updating schemes diverges.
- f) Using such a scheme, find the roots of:
  - i)  $f(x) := x$ .
  - ii)  $f(x) := x^2$ .
  - iii)  $f(x) := x^2 - x - 1$ , (cf. the golden ratio).
  - iv)  $f(x) := x^2 - 2$ , ( $\sqrt{2}$ ).
- g) Numerous circumstances can cause these numerical schemes to go wrong. Discuss.
- h) The Netwon-Raphson method is extremely useful and is the example *par excellence* of a numerical scheme for root finding (and also optimisation), and has application from extremely complex and technical applications to the most fundamental numerical schemes. An example which demonstrates how applicable and ubiquitous this method is can be found in floating point arithmetic. Consider the pseudo code in Code 1.

```

1 double x, y;
2 x = 4.0;      /* We chose 4.0 as a simple example, as 1.0 / 4.0 is 0.25. */
3 y = 1.0 / x;

```

Code 1: An example of floating-point (`double`) division in C.

- i) Rewrite the assignment of  $y$  as the root of a polynomial function  $f(y; x)$ , where  $x$  acts as a parameter in  $f$ .
- ii) Construct the appropriate Newton-Raphson scheme. (NB - This should not require division, only multiplication, addition, and subtraction).
- iii) By considering initial starting points of  $x \in \{0, 0.1, 0.2, \dots, 0.8, 0.9, 1.0\}$ , evaluate the first 10 values for each starting point. Comment on the results.
- iv) By drawing a cobweb diagram (a Verhulst diagram), comment on appropriate starting points for the scheme.

- v) Using this scheme, compute  $\frac{1}{7}$  to 10 decimal places, ( $\frac{1}{6} = 0.1\dot{6}$ ,  $\frac{1}{8} = 0.125$ , and  $\frac{1}{7} \approx 0.14285714285714285\dots$ ). Compare this scheme to the bisection method and to brute force traditional long division.
- vi) For a  $n$ -bit floating point number with  $a$ -bits and  $b$ -bits representing the significant figures and the exponent in binary respectively, discuss how you might provide a good initial guess.
- 10) Consider a soldier ant which can walk along the edges of a square. The ant starts at position  $O$  and every second he walks along one of the edges adjacent to him. The edge along which he walks is randomly chosen with an equal probability for any direct. At the opposite corner of the square  $A$  is an anteater which will eat the ant should the ant walk to position  $A$ . The layout is depicted in Figure 5a.

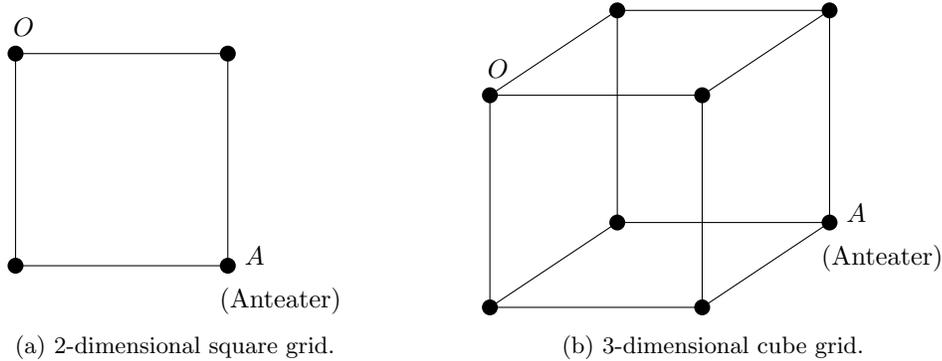


Figure 5: Layouts of the different dimensional grids along which the ant can traverse.

- a) For ants starting at position  $O$ , how long are they expected to survive for before being eaten once dropped at  $O$ ?
- b) For an ant which can move along the edges of a cube, as depicted in Figure 5b, what is the expected survival time?
- c) For the 3-dimensional cube structure. find the survival times for ants which begin:
- 2 edges away from the anteater.
  - 1 edge away from the anteater.
- 11) For a differentiable function  $f(\cdot) \in C$  where the value of it and all its derivatives are known at some position  $x$ , for a point  $-1 \leq \delta \leq 1$  away from  $x$  (at position  $x + \delta$ ), the Taylor series approximation is given by Equation (1).

$$f(x + \delta) = f(x) + \frac{df}{dx}\Big|_x \delta + \frac{d^2f}{dx^2}\Big|_x \frac{\delta^2}{2!} + \frac{d^3f}{dx^3}\Big|_x \frac{\delta^3}{3!} + \dots \quad (1)$$

By using a Taylor series expansion evaluate the following:

- $\sqrt{0.01}$
- $\sin(0.2)$
- $\log(1.1)$
- $\exp(1)$

There are several methods to numerically compute  $\pi$ . By first considering the Taylor series expansions of  $(1 - x^2)^{-1}$ , evaluate the Taylor series expansion of  $\arctan(x)$ . By considering the expansion at  $x = 1$ , use this to produce a series approximation for  $\pi$ .

- 12) The Fibonacci sequence is constructed by summing together the two previous terms in the sequence to form the next term. By convention the sequence begins with  $1, 1, \dots$  and the third term of the series is then  $1 + 1 = 2$ , and the next term  $1 + 2 = 3$ , and so on. The  $n$ -th term in the sequence is denoted by  $F_n$ . The first few terms in the sequence are

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

- a) If  $\varphi := \frac{1+\sqrt{5}}{2}$  and  $\psi := \frac{1-\sqrt{5}}{2}$  show:
- $\psi = -\frac{1}{\varphi}$
  - $\psi + \varphi = 1$
- b) By supposing the sequence takes the form  $F_n \propto \lambda^n$  (where  $\lambda \in \mathbb{R}$ ), derive an expression for  $F_n$  in terms of  $\psi$  and  $\varphi$ .

One method to compute the  $n$ -th term in the sequence is to implement a recursive algorithm. An example of such an algorithm is presented in Code 2, (which is written in C).

```

1 int fibonacci_recursive(int n)
2 {
3     if (n == 0)
4     {
5         return 1;
6     }
7     else if (n == 1)
8     {
9         return 1;
10    }
11    else
12    {
13        return fibonacci_recursive(n - 1) + fibonacci_recursive(n - 2);
14    }
15 }

```

Code 2: Recursive function in C to compute the Fibonacci numbers.

- For some integer  $n \gg 1$ , how does the number of function calls to `fibonacci_recursive()` scale?
  - By considering a binomial tree of function calls, show the number of function calls has an upper bound  $\mathcal{O}(2^n)$ .
  - Form a better scaling bound in terms of  $\phi$ .
- 13) Which is bigger,  $\sin(\cos(x))$  or  $\cos(\sin(x))$ , and does this relation hold over  $x \in \mathbb{R}$ , and do the two curves ever become equal or intersect?
- 14) A car dealership is performing a competition, of which 3 people have been entered, and the entrants are unable to communicate or plan a co-operation with each other prior to the competition. Each entrant has to choose a strictly positive integer (1, 2, 3...), and whoever has the lowest unique choice wins the car. This competition is held only once, and if no-one wins the car then the competition is not repeated.
- What should each player do?
  - How would this change if the game was repeated until someone won the car?
  - How would this change if the players could communicate prior to the selection of the integers?
- 15) Two masked robbers are caught by the police after a bank robbery, where one robbed the bank and the other was the get-away driver. They are being interrogated in separate holding cells. Their solicitors explain the following to each of them in turn:
- If they both plead guilty they will both receive 20 years each in jail.
  - If they both plead innocent they will both receive 5 years each in jail, as neither can be fully attributed to the crime.
  - If one pleads guilty and the other pleads innocent, the guilty defendant will receive 25 years in jail, and the innocent defendant only 2 years in jail.
- How should each robber plead?
  - Would this change if they could communicate before they are asked their pleas?

Another thief is also being trailed 100 separate times for 100 smaller crimes, for each of which they used the same accomplice. If this thief is also explained the same jail sentences for each of their crimes, but with days instead of years.

- c) how should they plead?
  - d) A final thief is also being trailed 100 separate times for 100 smaller crimes, for each of which they used a new accomplice. How should they plead?
- 16) An irrational number is one that cannot be expressed as a fraction of two integers.
- a) Show  $\sqrt{2}$  is irrational.
  - b) Show that an irrational number to the power of another irrational number can be rational.
- 17) Estimate how many people need to be in a room before there is a 50% or better chance that there is a shared birthday. How many people need to be in a room if we not only match the birthday but also the year of birth? (The relations  $\log(1+x) \approx x - \frac{1}{2}x^2$  for  $|x| \ll 1$  and  $\log(n!) \approx n - n \log(n)$  for  $n \gg 1$  might be useful).
- 18) Estimate the following:
- a) The natural walking speed of a human being.
  - b) The number of atoms in the sun.
  - c) How long does the sun take to set? (The sun is approximately 96 million miles from the earth and is 1 million miles in diameter).
- 19) A circle has the property that its width is a constant regardless of its orientation. Is a circle the only shape with such a property. By considering the maximal height of an equilateral triangle rolling on flat surface, construct a polygon with a constant width, and show this is non-circular. Can the same argument be extended to a square or a pentagon?
- 20) Romeo and Juliet have a dynamic and ever changing love-hate relationship. Let  $r(t)$  denote Romeo's affection for Juliet, where  $r > 0$  corresponds to infatuation and love,  $r = 0$  corresponds to indifference, and  $r < 0$  corresponds to irritation and hatred. Similarly  $j(t)$  denotes Juliet's feelings towards Romeo. Without solving any equations, can any stable or meaningful relationships form under the following scenarios:
- a) Romeo is a fickle lover, and dislikes Juliet's affection. The more she loves him the more he begins to dislike her. Conversely, if she is disinterested in Romeo, he then craves her attention and begins to idolise her. Juliet however is less complicated, and just echoes Romeo's interests. If Romeo likes her then she begins to like him, and if he dislikes her then she dislikes him. We map this to the dynamics  $\dot{r} = -j$  and  $\dot{j} = r$ .

By solving these equations, predict their romance cycle.

A more general model for the relationships are summarised by the equations  $\dot{r} = ar + bj$  and  $\dot{j} = cr + dj$ , where  $a$  and  $b$  are real constants specifying Romeo's romantic style, and similarly  $c$  and  $d$  specify Juliet's. These fall into several styles:

**Eager-beaver**  $a > 0$  and  $b > 0$ .

**Narcissist**  $a > 0$  and  $b < 0$ .

**Cautious**  $a < 0$  and  $b > 0$ .

**Reclusive**  $a < 0$  and  $b < 0$ .

What behaviour will be seen if an eager-beaver and a recluse are paired up, or a narcissist and a cautious lover?

- 21) Four ants are placed at the positions  $(\pm 1, \pm 1)$ , as shown in Figure 6. Each ant travels at a constant speed  $v$  towards the next ant. Do the ants ever catch up with the ant they're chasing? If so, how long does this take? Compute an expression for the centripetal force acting on the ants, and the amount the ants go around the origin  $O$ .

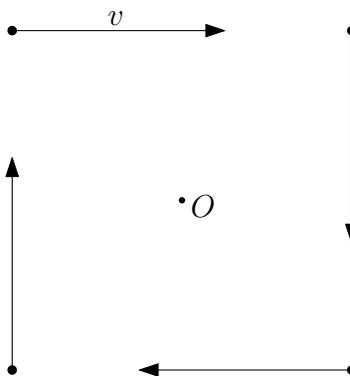


Figure 6: Four ants chasing each other.

- 22) During a winter's morning it first starts snowing at a constant rate and continues snowing indefinitely. That day a snow plough starts to clear the snow, and is capable of getting through a fixed volume of snow at a constant rate. The snow plough starts at 12 noon and travels 2 miles in the first hour, and 1 mile in the next hour. At what time did it start snowing?
- 23) Find the formulae for the following series:
- The geometric series  $\sum_{k=0}^n r^k$ , where  $|r| < 1$ .
  - The arithmetic series:
    - $\sum_{k=1}^n k$
    - $\sum_{k=1}^n k^2$
    - $\sum_{k=1}^n k^3$
    - $\sum_{k=1}^n k^4$
  - Describe how you would find the formulae for the arithmetic series of the 10<sup>th</sup> powers, without computing the formulae all the lower order arithmetic series.
  - The series  $1 + 2x + 3x^2 + 4x^3 + \dots$ , and  $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$ .
- 24) Water is flowing through a pipe with a constriction, with a volume  $V$  of water leaving the pipe each second, as shown in Figure 7. The wide end of the pipe has radius  $R$  and the narrow part  $r$ , with flow speeds of  $w$  and  $u$  respectively. Derive formulae for the flow speeds and determine where the water is flowing fastest, and the ratio of the slowest to the fastest flow. Where is the pressure greatest?

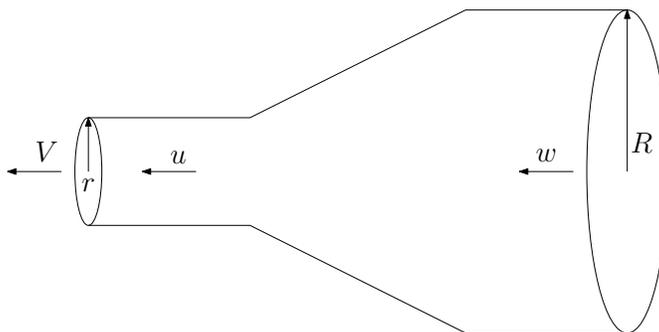


Figure 7: Water flowing through a pipe.

- 25) Concrete is very strong when compressed, and very weak when stretched. Steel is very easily bent, but is difficult to stretch. If a concrete beam is bent as in Figure 8, what will happen, and how can steel be incorporated to remedy this?
- 26) A4 paper is popular because when folded in along the long edge (producing A5 paper), the new piece of paper maintains the same aspect ratio. What is the aspect ratio of A4 paper?

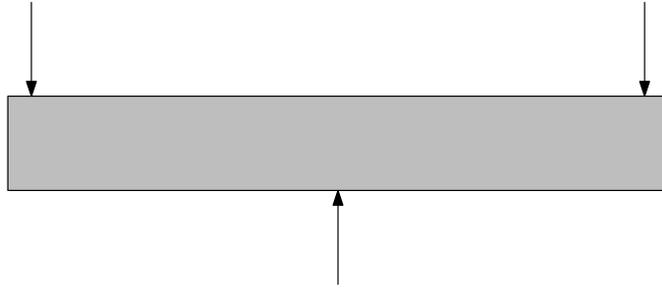


Figure 8: A concrete beam

Square sheets of paper are stacked together as shown Figure 9. What is the aspect ratio of this sheet after very many of these combinations?

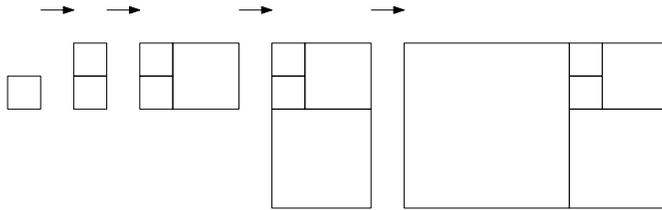


Figure 9: Joining together square sheets

- 27) The moment generating function  $M_X(t)$  of a random variable  $X$  is defined as  $M_X(t) := \mathbb{E}(\exp(tX))$ .
- For two independent random variables  $X$  and  $Y$  and a real scalar constant  $\lambda$  derive expressions for  $M_{\lambda X}(t)$  and  $M_{X+Y}(t)$  in terms of  $M_X(t)$  and  $M_Y(t)$ .
  - Find the moment generating function of a Gaussian random variable.
  - For  $N$  independently and identically distributed random variables, show that their mean converges to a Normally distributed random variable, where you can assume that convergence in moment generating functions implies convergence. (This will have proved the central limit theorem).
  - By considering the probability density function for a standard uniform random variable, show that the integral  $\int_0^1 \frac{1}{1+x^2} dx$  can be written as an expectation.
  - Using your convergence result, and 100 random samples, produce an estimate for  $\pi$  and comment on the accuracy of your estimate.
  - By considering  $\int_{\mathbb{R}} \exp(-x^2) dx$ , produce another estimate for  $\pi$ .
  - We can also estimate  $e$  by noting that  $e = \mathbb{E}(\xi)$  where  $\xi$  is a random variable defined as the minimal number of random uniforms such that their summation exceeds the value one. Using this definition estimate  $e$ .
- 28) Suppose we are trying to fit a linear model  $y = bx$  to some data (i.e. double the input and you'll double the output), as depicted in Figure 10. Suppose we have  $N$  observations  $\{x_i, y_i\}_{i=1}^N$ , where we wish to predict  $Y$  based on  $X$ , where we assume  $X$  is known without any uncertainty, but some form of error is produced when mapping  $X$  to the corresponding  $Y$ . Given the data points, produce and estimate  $\hat{b}$  of  $b$ . (You may assume that the error introduced is additive and independently and identically distributed and following a Gaussian distribution with a constant variance).

Can you predict any of the following relationships?

- Size of a species' skull and its corresponding brain.
- Size of a species' brain and its intelligence.
- Size of a human's brain and its intelligence (measured using IQ).
- The weight of a person and the size of their skull.

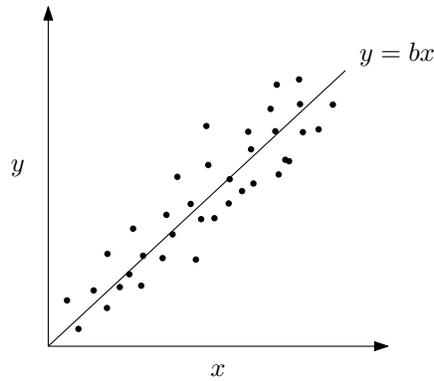


Figure 10: An example of a liner model.

- e) The age of a person and their weight.
- f) The gender of a person and their body weight.

With these relationships in mind, comment on the difference between causation and correlation, and the dangers of getting these confused.

- 29) There are  $N$  numbered boxes lined up from left to right, and inside of each box is a number. The numbers inside the boxes are in ascending order. All the boxes start off closed, and only 2 boxes can ever be open at the same time. After a box has been opened, 1 minute must elapse before the next box is opened. You are not allowed to write down or remember the numbers of any previously opened boxes, nor which numbers were inside these boxes. Your challenge is to determine the two boxes whose contents add up to some known number (which you are allowed to remember), or if it is even possible. If there are very many boxes, what is the optimal strategy for completing this task, and how does this scale with  $N$ ? What would be the case if the boxes weren't sorted into ascending order?
- 30) There are  $N$  numbered tiles shuffled in a line on a table. Your task is to sort them into an ascending line. You can only compare two tiles at any one time, where you must be holding one tile in each hand to perform a comparison. Making this comparison takes 1 second. You can only hold one tile in each hand at any one point in time. Putting a tile down takes 1 second, and picking one up takes one second, irrespective of the locations from where they are picked up and put down. How long does it take to sort the tiles?
- 31) There are 8 numbers inside a list, and the task is to form the total summation, and a list detailing the cumulative summation. To achieve this there are 8 processors available, numbered 1–8. A single processor can only ever perform arithmetic between two numbers, and has enough memory to store only 2 numbers. Each processor can only perform a single task each second. If there is only one processor available, how long does it take to form the total summation, and how long for the cumulative summation?

If we now allow there to be all 8 processors operating at the same time, then at each second a processor can only send information to one other processor, and a processor can only receive information from one other processor. A processor can both send and receive some information at the same time. How should these processors be coordinated and instructed to perform and produce the total summation and cumulative summation. How do these times compare to the case when only one processor was available.

Suppose each processor costs £1 for every second that elapses. If there were 1024 numbers to add together and 1024 processors available, what would be the quickest method, and what would be the most cost effective use of processors to achieve the required tasks?

- 32) Consider a block that slides down a frictionless inclined plane, and a cylinder rolling down an inclined plane (without skidding/sliding), as shown in Figure 11. If both of these descend a height  $h$ , and then slide/roll along the horizontal surface, which will be travelling faster? If the cylinder instead has all of its mass concentrated at the edge, what will be the relative change in speed?
- 33) If a stick is randomly broken into 3 pieces, with the locations of the break points being equi-probable, what is the probability that the three pieces can form a triangle?

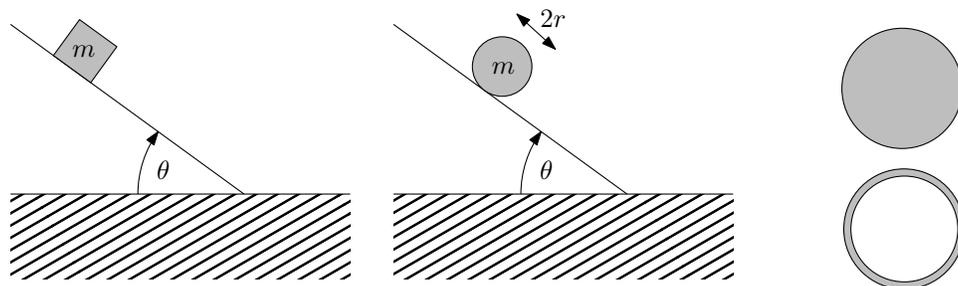


Figure 11: A sliding block and a rolling cylindrical mass, and a cylinder with its mass evenly distributed, and one with its mass concentrated at the edge.