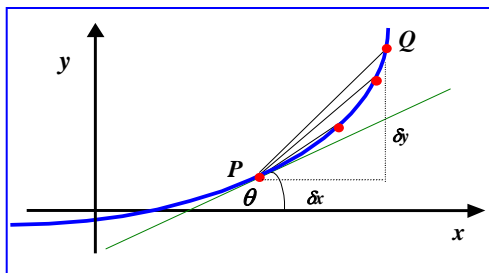


Chapter 1: DIFFERENTIATION I

Consider two neighbouring points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on a curve given by $y = f(x)$. The gradient of the chord PQ is: $\frac{\delta y}{\delta x}$.



As Q approaches P , the gradient of the chord becomes more and more like the gradient of the tangent to the curve at P , that is, $\tan \theta$. In this way, we can

find the gradient of the tangent to a curve at any point. It is called differentiation from first principles and it is formalised mathematically by Example 1 below.

(Note: This approach is attributed to Isaac Barrow (1630-1677, England), a forerunner and lecturer to Isaac Newton (1643-1727, England).

Example 1 Find the gradient at point $P(x, y)$ to the curve $y = x^2$.

Let $Q(x + \delta x, y + \delta y)$ be a neighbouring point on the curve

for P $y = x^2$ (1)

for Q $y + \delta y = (x + \delta x)^2$ (2)

From (1) and (2) (i.e. expand (2) and substitute for y using (1)),

$$\delta y = x^2 + 2x\delta x + (\delta x)^2 - x^2$$

Hence the gradient of the chord PQ is: $\frac{\delta y}{\delta x} = 2x + \delta x$.

As $\delta x \rightarrow 0$, Q approaches P and the gradient of the chord becomes the gradient of the tangent, which is written as.

$$\lim_{\delta x \rightarrow 0} \left(\frac{\delta y}{\delta x} \right) = \frac{dy}{dx} = 2x$$

(Note: It is important to write $\frac{dy}{dx}$ for the gradient and not $\frac{\partial y}{\partial x}$

which means something entirely different in differentiation.)

Example 2 Find $\frac{dy}{dx}$ if $y = \frac{2}{x}$.

Let x increase by a small increment δx causing a corresponding increment in y of δy

Thus $y + \delta y = \frac{2}{x + \delta x}$

Hence $\delta y = \frac{2}{x + \delta x} - \frac{2}{x}$

$$\Rightarrow \delta y = \frac{2x - 2x - 2\delta x}{x(x + \delta x)}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{-2}{x^2 + x\delta x}$$

As $\delta x \rightarrow 0$ $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{x^2}$$

By doing several such differentiations the following relationship is suggested for functions of the type $y = 3x^4, -2x^{-5}, \frac{1}{2}x^7 \dots$

$$\text{If } y = ax^n \Rightarrow \frac{dy}{dx} = anx^{n-1}$$

Example 3 Apply this formula to:

$$y = 4x^6 \Rightarrow \frac{dy}{dx} = 24x^5$$

$$y = x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$y = x^{-3} \Rightarrow \frac{dy}{dx} = -3x^{-4}$$

$$y = 2 \Rightarrow \frac{dy}{dx} = 0 \text{ (i.e. a horizontal line has zero gradient)}$$

$$y = x + 3 \Rightarrow \frac{dy}{dx} = 1 \text{ (i.e. a straight line } y = mx + c \text{ has a gradient } m)$$

Function of a function

Suppose that u is a function of x (i.e. $u = u(x)$) and that y is a function of u (i.e. $y = y(u)$). Let an increment in x of δx cause a corresponding increase in u of δu and let this cause an increase in y of δy . By ordinary fractions:

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x}$$

As $\delta x \rightarrow 0$ $\frac{\delta y}{\delta u} \rightarrow \frac{dy}{du}$, $\frac{\delta u}{\delta x} \rightarrow \frac{du}{dx}$ and $\frac{\delta y}{\delta x} \rightarrow \frac{dy}{dx}$,

$$y = y(u) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 4 If $y = (x^2 + 1)^5$ find $\frac{dy}{dx}$.

Note that without using a substitution one would simply expand the right hand side (RHS). Whilst this can be done here it would be very laborious for say $y = (x^2 + 1)^{30}$.

Let $u = x^2 + 1 \Rightarrow y = u^5$

Thus $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = 5u^4$

$$\frac{dy}{dx} = 5u^4 \cdot 2x$$

$$\frac{dy}{dx} = 10(x^2 + 1)^4 x$$

$$y = u^n \Rightarrow \frac{dy}{dx} = n u^{n-1} \frac{du}{dx}$$

In the examples considered so far we have differentiated with respect to (w.r.t.) x . This need not always be the case.

E.g. differentiate $(x^2 + 1)^4$ w.r.t. $x^2 + 1$ gives $4(x^2 + 1)^3$.

In effect, $u = x^2 + 1$ so $y = u^4$ and we are differentiating w.r.t. u .

Using this idea, the function of a function types may be dealt with more concisely.

Example 5 Differentiate $(x^2 + x)^7$ with respect to x .

First differentiate with respect to $x^2 + x$ (i.e. think of $x^2 + x$ as the 'new' 'x') and then multiply by the differentiation of $x^2 + x$ with respect to x .

ANS $7(x^2 + x)^6 (2x + 1)$

You may sometimes have to differentiate more than twice. In fact you keep differentiating in this way and only stop when you differentiate w.r.t. x .